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Dynamic Pricing with Stochastic Reference Price Effect

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Abstract
We study a dynamic pricing problem of a firm facing stochastic reference price effect. Randomness is incorporated in the formation of reference prices to capture either consumers' heterogeneity or exogenous factors that affect consumers' memory processes. We apply the stochastic optimal control theory to the problem and derive an explicit expression for the optimal pricing strategy. The explicit expression allows us to obtain the distribution of the steady-state reference price. We compare the expected steady-state reference price to the steady-state reference price in a model with deterministic reference price effect, and we find that the former one is always higher. Our numerical study shows that the two steady-state reference prices can have opposite sensitivity to the problem parameters and the relative difference between the two can be very significant.

Keywords Reference price effect · Dynamic pricing · Stochastic optimal control

Mathematics Subject Classification 93E20 · 49L20

This paper is dedicated to Professor Yin-Yu Ye in celebration of his 70th birthday.

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1 Introduction

In an effort to better capture the relationship between demand and prices in a market with repeated purchases, the concept of reference price has been developed and examined through extensive empirical studies in the economics and marketing literature (see [1], for a review). It argues that consumers form price expectations, called reference price, from information such as prices they observed in their past purchasing occasions. Consumers then make their purchasing decisions based on the relative magnitude of the reference price and the current selling price. A purchasing instance is perceived by consumers as gains or losses depending on whether the selling price is considered as discounts or surcharges relative to the reference price. Gains induce consumers to buy while losses deter them from purchasing.

Since the notion of reference price is a subjective construction, which cannot be directly measured from transaction records, a large amount of literature is devoted to the formation of reference prices. Briesch et al. [2] provided a comprehensive review of different reference price models and empirically compare them using scanner panel data for various product categories. They found that in four categories the memory-based reference price model, in which reference price is assumed to be a weighted average of past encountered prices, performs the best.

The empirical evidences of reference price effect have motivated researchers to study how a firm should set its price when facing reference price-dependent demand. Greenleaf [3] analyzed the impact of reference price effect on a single-period promotion. Specifically, the author argued how the reference price effect creates a trade-off between additional short-term profits and a better long-term prospect. In a discrete-time framework, Kopalle et al. [4] numerically demonstrated several structural properties of the optimal pricing strategies under asymmetric reference price effect. Popescu and Wu [5] formally proved that when demands are loss/gain neutral or loss-averse, a constant pricing strategy is optimal in the long run, while [6] showed in a special case of gain-seeking demands that a cyclic skimming pricing strategy is optimal in the long run. A constant pricing strategy was also found to be optimal with loss-averse demands but under a different reference price model based on peak-end rule in Nasiry and Popescu [7]. Chen et al. [8] further developed efficient algorithms to compute the optimal prices. In a continuous-time framework, Fibich et al. [9] provided an explicit solution to the optimal prices using optimal control theory and using the explicit expression they also arrive at the conclusion that constant prices are optimal in the long run.

Most of the previous literature on reference price models and the corresponding dynamic pricing problems assumes a deterministic reference price model. That is, if a firm can figure out consumers' initial reference price, and was given a price path the firm can perfectly predict all the future reference prices of consumers, who are assumed to be homogeneous over time. However, there are two common features of the market that a deterministic reference price model does not capture. First, a consumer population is rarely homogeneous over time. Indeed, it was pointed out in Wang [10] that different consumer groups often visit the store at different times and it was found in Krishnamurthi et al. [11] that different consumer groups, say brand loyal consumers and brand switchers, can make different purchasing decisions. Second, even if the firm is facing a single homogeneous consumer group over time, there are many exogenous factors like advertisement activities and competitors' prices that can influence consumers' memory processes. For instance, it was argued in Rajendran and Tellis [12] that consumers' reference price may also be affected by contextual effects, i.e., other prices consumers observe at the time of purchase.

In this paper, we try to incorporate consumers' temporal heterogeneity as well as the exogenous shocks to describe the more complex consumer behavior by modeling the evolution of reference prices as a random process. We assume that at any given time point, the firm is able to observe the incoming type of consumers group as well as the exogenous factors like advertising activities, competitors prices, etc. That is, the firm can figure out the current reference price of consumers. However, given a price path, due to heterogeneity and exogenous shocks, the firm is unable to perfectly predict future reference prices. Instead, it only has the knowledge of the distributions of the future reference prices.

We analyze a dynamic pricing problem with the above stochastic reference price model in a continuous-time framework. Applying stochastic optimal control theory, we derive an explicit expression for the optimal pricing strategy. The reference prices under the optimal pricing strategy converge in distribution to a steady-state reference price whose distribution is given explicitly. The expected steady-state reference price is compared to the steady-state reference price in the deterministic reference price model analyzed by Fibich et al. [9]. Interestingly, we find that the expected steady-state reference price is always higher than its deterministic counterpart. Our results suggest that in the long run it is always beneficial for the firm to have, on average, a higher price compared to the deterministic model in order to deal with the uncertainties in consumers' future reference prices. Our numerical study further shows that as consumers adapt to the new price information at a faster rate or the magnitude of reference price effects decreases, the deterministic steady-state reference price always grows while the expected steady-state reference price can become smaller; i.e., they have opposite sensitivity to problem parameters.

The remainder of this paper is organized as follows. In Sect. 2, we introduce the model of stochastic reference price effect in a continuous-time framework. Explicit expressions to the optimal pricing strategy as well as the distribution of the steady-state reference price are obtained in Sect. 3. Section 4 compares in detail the expected steady-state reference price to the steady-state reference price in the deterministic reference price model. Finally, we conclude in Sect. 5.

2 Model

We first introduce, in a continuous-time framework, the exponential smoothing model, a widely used reference price model in the literature (see, for instance, [9]) to describe the evolution of consumers' reference prices. Given a price path $p(t)$ and an initial reference price r_0, the reference price at time t is given by:

$$r(t) = \mathrm{e}^{-\alpha t}\left[r_0 + \alpha \int_0^t \mathrm{e}^{\alpha s} p(s)\mathrm{d}s\right], \quad t \geqslant 0, \tag{2.1}$$

where $\alpha > 0$ is interpreted as the "memory factor." The larger the memory factor α, the faster consumers incorporate new price information. Alternatively, one can rewrite (2.1) in differential form as

$$\begin{cases} \mathrm{d}r = \alpha[p(t) - r(t)]\mathrm{d}t, \\ r(0) = r_0. \end{cases} \tag{2.2}$$

The intuition behind this differential form is quite clear: Reference price starts at an initial value r_0, and at a constant rate α, it would drift to close the gap $p(t) - r(t)$. The resulting $r(t)$ is a deterministic process. That is, given an initial reference price r_0 and a price path $p(t)$, a firm is able to determine perfectly the reference price at any given time.

In a real market, however, a firm may encounter heterogeneous consumer groups at different points of time and consumers' memory processes can be affected by various exogenous factors like advertisements, price information of other products, etc. In other words, in reality, a firm only has a general knowledge about the trend or drift of the reference price process but is unable to predict perfectly the future reference prices. In our study, we try to incorporate consumers' temporal heterogeneity as well as exogenous shocks to describe the more complex behavior of the consumer population. Specifically, we extend the above reference price dynamics using a stochastic differential equation(SDE) (see [13], for a reference on the topic of SDE) to model the reference price evolution process:

$$\mathrm{d}r(t) = \alpha[p(t) - r(t)]\mathrm{d}t + \sigma\sqrt{r(t)}\mathrm{d}W(t), \tag{2.3}$$

where $W(t)$ denotes a standard Wiener process and reference price $r(t)$ is now a stochastic process. At any given time, it yields a probability distribution over all possible reference prices.

There are two main considerations in our choice of models. From a modeling perspective, we want a model that can give a good approximation in terms of capturing consumers' heterogeneity as well as exogenous factors. To model consumer heterogeneity, incorporating randomness is a common practice used in economics and marketing (see [14], for instance). One possible way is to assume α to be random. However, it is easy to see that if the price is a predetermined constant, i.e., $p(t) = p$, for all $t \geqslant 0$, the variance of $r(t)$ will go to zero as $t \to \infty$. That is, the firm can eliminate such heterogeneity in consumers' reference prices by employing a constant pricing strategy. While this could be plausible in some scenario, we believe, in general, variability in consumers' perception of prices should persist under commonly seen pricing strategies. On the other hand, variability in reference prices always exists (unless $p(t) = 0$ for all t) in (2.3). In addition, (2.3) has the nice property that the probability of $r(t)$ going negative is always zero. To model exogenous factors, one usually adds a random shock to represent those exogenous factors. The square-root diffusion process (2.3) has the additional merit of allowing a reference price level-dependent variance. It predicts that the variance of the $r(t)$ gets smaller as $r(t)$ itself becomes smaller.

From an analysis perspective, the square-root diffusion process (2.3) can provide analytical tractability and has found applications ranging from term-structure modeling [15] to option pricing [16]. In our application, in particular, it enables a closed-form solution and results in a simple steady-state distribution. As a result, we are able to compare analytically the expected steady state to the steady state derived from the deterministic reference price model.

Note that for a predetermined price path $p(t)$, $\mathrm{d}E[r(t)] = \alpha[p(t) - E[r(t)]]\mathrm{d}t$. That is, if the firm pre-commits to a price path that is independent of the realization of randomness, then the evolution of the expected reference prices coincides with the deterministic model (2.2) used in Fibich et al. [9]. We illustrate in Fig. 1 a sample path of (2.3) as well as $E[r(t)]$ under a constant pricing strategy with two price levels: the high price $p_{\mathrm{H}} = 0.92$ and the low price $p_{\mathrm{L}} = 0.29$, respectively. In Fig. 1, we take the initial reference price $r_0 = \frac{p_{\mathrm{H}}+p_{\mathrm{L}}}{2} = 0.605$, $\alpha = 0.5$ and $\sigma = 0.2$. One can see that $r(t)$ has a higher variance under p_{H} than under p_{L} which reflects the square-root diffusion term in (2.3).

Given the above dynamics, we introduce the dynamic pricing problem. The demand rate function is given as:

$$D(r, p) = b - ap - \eta \cdot (p - r). \tag{2.4}$$

The first part of this represents a normal linear demand function, and the second part is the reference price effect. $\eta > 0$ controls the magnitude of this effect. When $p(t) < r(t)$, consumers perceive the deal as a gain and demand would rise. On the contrary, when $p(t) > r(t)$, consumers perceive it as a loss and demand would fall. We remark here that for tractability, we have assumed the demand to be loss/gain neutral;

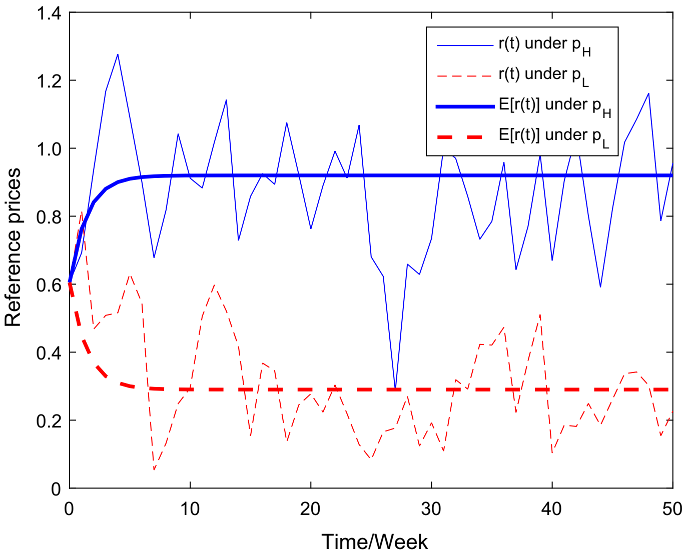

Fig. 1 $E[r(t)]$ and sample paths of $r(t)$ under $p_{\mathrm{H}} = 0.92$ and $p_{\mathrm{L}} = 0.29$, respectively

i.e., market's responses to gain and loss are of the same magnitude. While both loss-averse and gain-seeking behaviors are observed in the literature (see [1] for a review), it was also pointed out in Bell and Lattin [17] that such behavior asymmetries are not a universal phenomenon and may be a consequence of not taking consumer heterogeneity into account. In particular, Bell and Lattin [17] found that in five of the 11 product categories, one can not reject the hypothesis of loss/gain neutral after incorporating consumer heterogeneity. Therefore, we believe the loss/gain neutral assumption along with our reference price dynamics (2.3), which accounts for consumer heterogeneity, can provide a good approximation in some scenarios.

Given the demand rate, revenue would accumulate at the following rate:

$$F(r, p) = (p - c)D(r, p) = (p - c)[b - ap - \eta(p - r)], \tag{2.5}$$

where c is the marginal cost. With an initial condition $r(0) = r_0$, our goal is to maximize the total discounted profit over an infinite horizon:

$$\begin{aligned} V(r_0) &= \max_{p(t)} E\left[\int_0^\infty \mathrm{e}^{-\gamma t} F(r(t), p(t))\mathrm{d}t\right], \\ \text{s.t.} \quad \mathrm{d}r(t) &= \alpha[p(t) - r(t)]\mathrm{d}t + \sigma\sqrt{r(t)}\mathrm{d}W(t), \end{aligned} \tag{2.6}$$

where γ is the discount factor.

3 Explicit Solution and Steady State

We adopt a dynamic programming approach. The Hamilton–Jacobi–Bellman (HJB) equation to problem (2.6) can then be written as

$$\gamma V(r) = \max_p \left\{F(r, p) + \alpha(p - r)\frac{\mathrm{d}V(r)}{\mathrm{d}r} + \frac{\sigma^2}{2}r\frac{\mathrm{d}^2V(r)}{\mathrm{d}r^2}\right\}. \tag{3.1}$$

Readers are referred to Miranda and Fackler [18], for instance, for an intuitive derivation of the HJB equation (3.1). We denote $p^*(r)$ to be the optimal solution to (3.1) and $r^*(t)$ to be the reference price path under $p^*(r)$ which satisfies the SDE

$$\mathrm{d}r^*(t) = \alpha[p^*(r^*(t)) - r^*(t)]\mathrm{d}t + \sigma\sqrt{r^*(t)}\mathrm{d}W(t).$$

Note here that we are seeking a state feedback solution $p^*(r)$ since, as we have mentioned in Introduction, we assume that the firm has the ability to measure or observe the realization of consumers' reference price and can set a price accordingly.

Alternatively, if the firm cannot observe the realization of consumers' reference price, then he can only choose a predetermined price path $\{p(t)\}$ such that $E\left[\int_0^\infty \mathrm{e}^{-\gamma t} F(r(t), p(t))\mathrm{d}t\right]$ is maximized and we call the solution $\{p_{\text{open-loop}}(t)\}$ in this case as the open-loop solution. Clearly, by Fubini's theorem and the linearity of $F(r, p)$ in r, we have

$$E\left[\int_0^\infty \mathrm{e}^{-\gamma t} F(r(t), p(t))\mathrm{d}t\right] = \int_0^\infty \mathrm{e}^{-\gamma t} F(E[r(t)], p(t))\mathrm{d}t,$$

where

$$\mathrm{d}Er(t) = \alpha[p(t) - Er(t)]\mathrm{d}t.$$

That is, finding the optimal open-loop solution reduces to the deterministic model studied in Fibich et al. [9] and we denote the expected value under the open-loop solution as $V_{\text{open-loop}}(r)$.

In comparison, we give in the following proposition an explicit expression to the optimal state feedback solution $p^*(r)$.

Proposition 3.1 *The optimal solution $p^*(r)$ to the HJB equation (3.1) is given by*

$$p^*(r) = \frac{\eta + 2\alpha Q}{2(a+\eta)} r + \frac{\alpha R + b}{2(a+\eta)} + \frac{c}{2}, \tag{3.2}$$

where Q and R are given by

$$Q = \frac{\gamma}{2\alpha^2}(a+\eta) + \frac{2a+\eta}{2\alpha} - \frac{a+\eta}{2\alpha^2}\Delta,$$

$$R = \left[\frac{b + c(a+\eta)}{\alpha} + \frac{\sigma^2(a+\eta)}{\alpha^2}\right]\frac{\gamma - \Delta}{\gamma + \Delta} + \left[b + ca + \frac{\sigma^2(2a+\eta)}{2\alpha}\right]\frac{2}{\gamma + \Delta}$$

and Δ is

$$\Delta = \sqrt{\gamma^2 + 2\alpha\frac{2a(\gamma+\alpha) + \gamma\eta}{\eta + a}}.$$

A few monotonic properties are immediate from the explicit expression in (3.2). First, it is easy to verify that $Q > 0$ and consequently $p^*(r)$ is increasing in r. This confirms the intuition that when consumers have a higher reference price, the firm can take advantage of this by pricing at a higher level. In addition, it is easy to see that the slope $(\eta + 2\alpha Q)/(2(a+\eta)) < 1$. With some algebraic calculations, one can verify that for fixed reference price r, $p^*(r)$ is increasing in the marginal cost c and the market size b, which is again quite intuitive. Lastly, since the constant R is increasing in σ^2, the optimal price is also increasing in σ^2. This provides a new insight that it is always beneficial for the firm to price higher if the firm is facing larger uncertainties in consumers' future reference prices. To simplify our notations, we let $c = 0$ for the rest of the section; all our results generalize to the case for $c > 0$.

Substituting (3.2) into the reference price dynamics (2.3), we have the following SDE characterizing the evolution of $r^*(t)$:

$$\begin{aligned}\mathrm{d}r^*(t) &= \alpha\left[\frac{2\alpha Q - 2a - \eta}{2(a+\eta)} r^*(t) + \frac{\alpha R + b}{2(a+\eta)}\right]\mathrm{d}t + \sigma\sqrt{r^*(t)}\mathrm{d}W(t)\\ &:= \lambda(\mu - r^*(t)) + \sigma\sqrt{r^*(t)}\mathrm{d}W(t),\end{aligned} \tag{3.3}$$

where

$$\lambda = \alpha \frac{2a + \eta - 2\alpha Q}{2(a + \eta)}, \quad \mu = \alpha \frac{\alpha R + b}{2\lambda(a + \eta)}.$$

Interestingly, under the optimal pricing strategy, the reference price dynamics (3.3) is again a square-root diffusion process. Similar to the previous literature, we are interested in the long-run behavior of the optimal prices as well as the resulting reference price path. Specifically, what will $r^*(t)$ be as t goes to infinity? Proposition 3.2 gives a complete answer to this question.

Proposition 3.2 *The optimal reference price path $r^*(t)$ converges in distribution to the steady state, denoted as R_s^*. The density of R_s^* is*

$$f_{R_s^*}(r) = \frac{(2\lambda/\sigma^2)^{2\lambda\mu/\sigma^2}}{\Gamma(2\lambda\mu/\sigma^2)} r^{2\lambda\mu/\sigma^2 - 1} \mathrm{e}^{-2r\lambda/\sigma^2},$$

where $\Gamma(\cdot)$ is the Gamma function. That is, R_s^* follows a Gamma distribution with a shape parameter $2\lambda\mu/\sigma^2$ and a rate parameter $2\lambda/\sigma^2$.

Proposition 3.2 not only claims the convergence to a steady state, but also gives an explicit expression for the steady-state distribution in terms of problem parameters. Our result differs from previous literature in the sense that the steady state R_s^* is a random variable rather than a deterministic value. This confirms our motivation in modeling consumer heterogeneity: Even under optimal pricing strategy, variability in consumers' reference prices still persists.

Figure 2 illustrates the steady-state distributions under different levels of α. In Fig. 2, we have fixed $c = 0$, $b = 10$, $a/b = 0.8$, $\eta/b = 0.5$, $\gamma = 0.01$ and $\sigma^2 = 0.2$. One can see that as α grows, the spread of the distribution shrinks. Intuitively, this is due to the fact that as α grows, the drift term in (2.3) will have a relatively stronger effect compared to the diffusion term and result in less variance. In other words, if consumers in the population adapts to the new price information at a faster rate, then the variability in their perception of the fair prices can be reduced.

Using Proposition 3.2, we can easily compute the expected steady-state reference price as well as the variance of steady-state reference price. Their explicit expressions are summarized in the following corollary.

Corollary 3.3 *The expected steady-state reference price $r_s^* = E[R_s^*]$ is given by*

$$r_s^* = \mu = r_D^* + \frac{\sigma^2}{2a(\gamma + \alpha) + \gamma\eta} \left[\frac{a + \eta}{\alpha} \left(\frac{\gamma}{2} - \frac{\Delta}{2} \right) + \frac{2a + \eta}{2} \right],$$

where r_D^* is the steady state in the deterministic problem ($\sigma^2 = 0$):

$$r_D^* = \frac{(\gamma + \alpha)b}{2a(\gamma + \alpha) + \gamma\eta}.$$

The variance of steady-state reference price is given by

$$\mathrm{var}(R_s^*) = \frac{\mu}{2\lambda}\sigma^2.$$

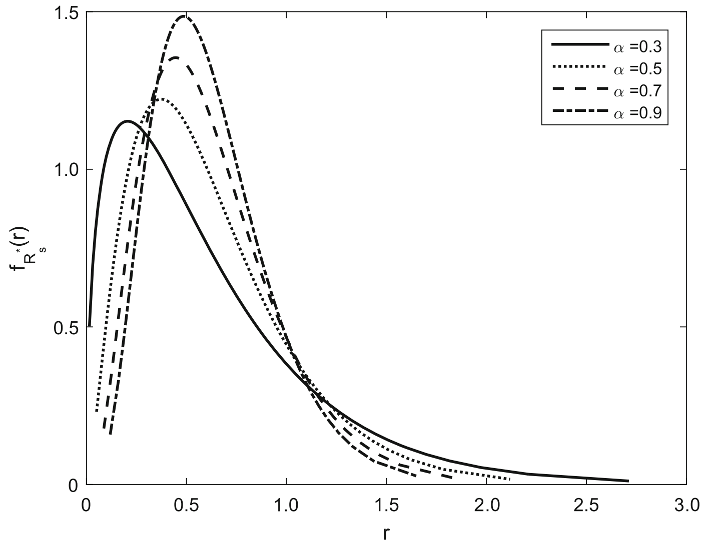

Fig. 2 Shape of $f_{R_s^*}(r)$ under different α

We remark here that r_D^* is exactly the steady state derived by Fibich et al. [9] in the deterministic reference price model. Clearly, when $\sigma = 0$, our model reduces to the deterministic model in Fibich et al. [9] and r_s^* agrees with their solution. When $\sigma > 0$, on the other hand, it is easy to verify that $r_s^* > r_D^*$. That is, the expected steady-state reference price is always higher than the steady-state reference price when there is no randomness. This result is in sharp contrast with the intuition developed in some previous pricing literature. Recall in Fig. 1 that a higher price induces a higher variability in reference price and consequently higher variability in demands. Such variability in demands are undesirable in many settings. For example, in a joint inventory and pricing setting, by comparing the optimal price with the riskless price (the price obtained from deterministic demands), the optimal price is always set in a way such that variability in demands is reduced [19]. In our dynamic pricing problem, however, the firm does not need to worry about the risk of mismatch between supply and demand and demand variability will not be a concern. On the contrary, it will bring more opportunities to the firm since higher variability in reference prices will allow the firm to take advantage of the possible high reference price level.

4 Numerical Study

This section numerically examines the impact of stochastic reference price on the expected steady-state reference prices and the optimal values. We provide insights into the magnitude of markups that the firm should employ in order to deal with stochastic reference price effect and the value of acquiring consumers' reference price information.

4.1 Comparison of Steady States

The key insight from the results developed in Sect. 3 is that in the long run it is always beneficial for the firm to have, on average, a markup based on the steady-state price from the deterministic model in order to deal with uncertainties in consumers' future reference prices. In this subsection, we explore how such markup varies according to the problem parameters and quantify the magnitude of the markup.

Figure 3 illustrates the gap between r_s^* and r_D^* under a range of values of α and different levels of η/b with other parameters fixed at the same values in Fig. 2. One can see that the gap decreases as α increases and η/b decreases. As α grows, consumers adapt to the new price at a faster rate and in the extreme case it adjusts to the current price instantaneously. Such decrease in the average gap between reference price and price reduces the (stochastic) reference price effects and consequently results in a smaller difference between r_s^* and r_D^*. Similarly, when η/b is small, reference price effects play a minor role and in the limiting case when η/b approaches zero, both r_s^* and r_D^* get closer to the static price, the optimal price under the static demand model (the demand model without reference price effects, i.e., $\eta = 0$), and consequently their gap goes to zero.

More interestingly, the expected steady-state reference price r_s^* and its deterministic counterpart r_D^* can have different behaviors relative to some problem parameters. When reference price effects are significant (η/b is large), r_s^* is decreasing in α while r_D^* is increasing in α. It is easy to see the monotonicity of r_D^*. Since the static price is always higher than r_D^*, as α increases, the model more closely resembles the static

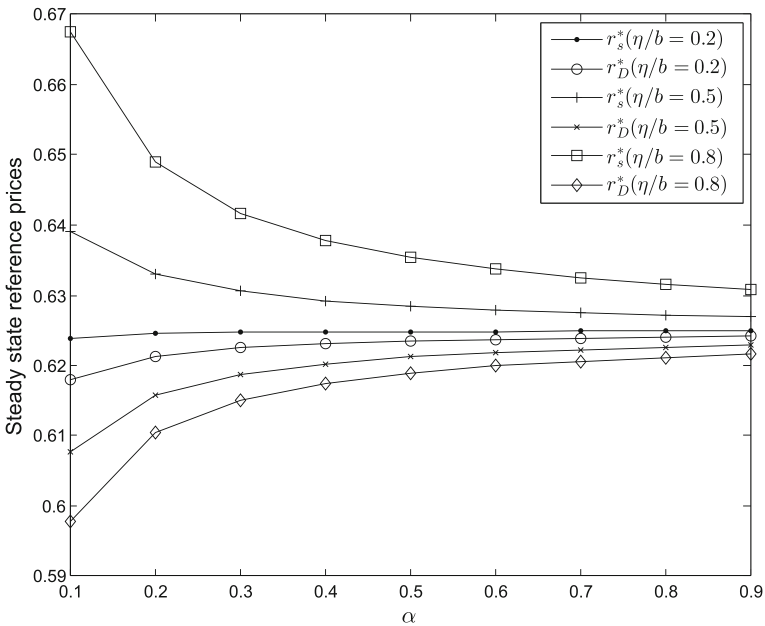

Fig. 3 Comparisons of r_s^* and r_D^*

demand model and as a result, r_D^* increasingly approaches the static price. The opposite direction of r_s^* is less obvious. One possible explanation is that when α becomes larger, the benefit of having larger variations in reference price decreases. Similar explanations apply to the sensitivity of r_s^* and r_D^* to η/b. As η/b becomes smaller, the effects of reference price gradually vanish and r_D^* increases to the static price while r_s^* decreases to it.

In addition to the qualitative description, in the following we quantitatively measure the magnitude of the gap between r_s^* and r_D^*. Specifically, we use the relative price change: $\frac{1}{\sigma^2}\frac{r_s^*-r_D^*}{r_D^*}$, to measure the percentage change in the steady-state reference price, per unit of variance σ^2. We fix c at 0 and b at 10 and choose different levels for a/b, η/b and α. In addition, two levels of discount factor γ are used: 0.01 and 0.05, and the corresponding results are listed in Tables 1 and 2, respectively.

A few observations are immediate from these tables. First of all, the relative change in steady-state reference price can be very significant in many scenarios. Keep in mind that these are "per unit of σ^2" figures, so the actual relative changes in steady-state reference price are these numbers multiplied by σ^2. Similar to the sensitivity in Fig. 3, we find that when a/b is large, relative price change becomes less significant. Moreover, relative price change decreases when γ gets larger, or when future profit is discounted more. In the limiting case, when γ gets arbitrarily large, future profit is discounted so much that essentially we would be dealing with a single-period problem. In a single-period problem, the relative price change would become 0 as reference price effect itself would vanish.

4.2 The Value of Reference Price Information

In this subsection, we examine the value of reference price information. In seeking a state feedback solution $p^*(r)$, we have assumed that the firm has complete

Table 1 Relative price change with discount factor $\gamma = 0.01$

	$\eta/b = 0.2$			$\eta/b = 0.5$			$\eta/b = 0.8$		
α	0.1	0.3	0.5	0.1	0.3	0.5	0.1	0.3	0.5
$a/b = 0.2$	15%	5%	3%	63%	24%	15%	125%	48%	29%
$a/b = 0.5$	7%	3%	2%	36%	14%	8%	79%	29%	18%
$a/b = 0.8$	5%	2%	1%	26%	10%	6%	58%	22%	13%

Table 2 Relative price change with discount factor $\gamma = 0.05$

	$\eta/b = 0.2$			$\eta/b = 0.5$			$\eta/b = 0.8$		
α	0.1	0.3	0.5	0.1	0.3	0.5	0.1	0.3	0.5
$a/b = 0.2$	8%	4%	3%	35%	19%	13%	67%	37%	25%
$a/b = 0.5$	4%	2%	1%	21%	11%	7%	45%	24%	16%
$a/b = 0.8$	3%	1%	1%	15%	8%	5%	34%	18%	12%

information of consumers' current reference price. This can be achieved by either observing incoming type of consumers group and the exogenous factors or directly measuring reference prices through surveys or experiments. In either way, acquiring such information over time may be costly. Therefore, it is important to quantify the additional profit such information can provide. We remark that since the open-loop solution coincides with the solution under the deterministic model studied in Fibich et al. [9], our value of reference price information is equivalent as quantifying the profit gained from considering the stochastic reference price compared with that of the deterministic reference price model.

In Fig. 4, we illustrate the comparison between the value function $V(r)$ under the state feedback solution $p^*(r)$ and the value function $V_{\text{open-loop}}(r)$ under the open-loop solution $\{p_{\text{open-loop}}(t)\}$. Figure 4 uses the same parameter as Fig. 2 with $\alpha = 0.5$, and one can see that $V(r)$ is always higher than $V_{\text{open-loop}}(r)$.

To measure the benefit of having a state feedback solution, we compute the relative value change: $\frac{V(r)-V_{\text{open-loop}}(r)}{V_{\text{open-loop}}(r)}$ for different parameter configurations. Similar to Tables 1 and 2, we fix $r = 1.0$, $\sigma^2 = 0.2$ and choose different levels for a/b, η/b and α. The results are summarized in Tables 3 and 4, respectively, for $\gamma = 0.01$ and $\gamma = 0.05$. One can see that the sensitivity of the relative value change with respect to problem parameters is similar to that of the relative price change. That is, as the reference price effect becomes stronger relative to the direct price effect, i.e., η/b is larger and a/b is smaller, then the relative value change is higher. In addition, the relative value change will become more significant as consumers adapt to new price information at a slower rate and as the firm discounts less into the future.

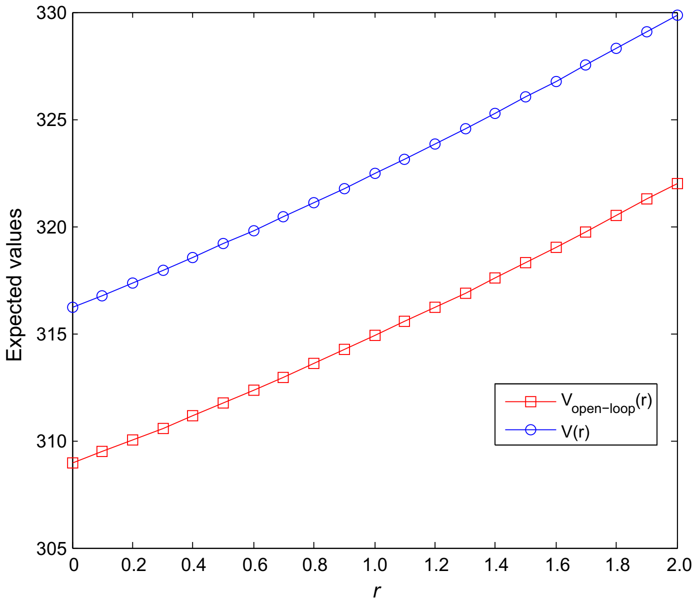

Fig. 4 Comparisons of $V(r)$ and $V_{\text{open-loop}}(r)$

Table 3 Relative value change with discount factor $\gamma = 0.01$

	$\eta/b = 0.2$			$\eta/b = 0.5$			$\eta/b = 0.8$		
α	0.1	0.3	0.5	0.1	0.3	0.5	0.1	0.3	0.5
$a/b = 0.2$	6%	2%	1%	27%	10%	6%	55%	20%	12%
$a/b = 0.5$	3%	1%	1%	16%	6%	3%	35%	12%	7%
$a/b = 0.8$	2%	1%	0%	12%	4%	2%	26%	9%	5%

Table 4 Relative value change with discount factor $\gamma = 0.05$

	$\eta/b = 0.2$			$\eta/b = 0.5$			$\eta/b = 0.8$		
α	0.1	0.3	0.5	0.1	0.3	0.5	0.1	0.3	0.5
$a/b = 0.2$	4%	2%	1%	16%	8%	5%	30%	15%	10%
$a/b = 0.5$	2%	1%	1%	12%	5%	3%	23%	10%	7%
$a/b = 0.8$	2%	1%	0%	10%	4%	2%	20%	8%	5%

From a managerial perspective, our results suggest that when the direct price effect dominates the reference price effect (in our example, more than twice of the reference price effect), then it is sufficient for the firm to employ an open-loop strategy without monitoring consumers' reference prices overtime. This justifies, in some scenarios, the study of deterministic reference price models in the literature (e.g., [5,9]). On the other hand, when reference price effect dominates the direct price effect, ignoring the realization of reference price information or, equivalently, treating reference price deterministically will result in significant loss in profit. In this case, firm should actively explore consumers' reference price level and adjust its prices accordingly.

5 Conclusion

This paper studies a dynamic pricing problem under stochastic reference price effect. A stochastic differential equation is proposed to model the reference price evolution in order to capture the temporal heterogeneity in consumer groups as well as exogenous shocks that affect consumers' memory processes. The corresponding dynamic pricing problem is analyzed using stochastic optimal control theory. By solving the HJB equation, we are able to provide an explicit expression for the optimal pricing strategy and the distribution of the steady state. We find that, on average, the firm should always have a markup over the steady-state price under the deterministic model in order to deal with the uncertainties in future reference prices.

Our numerical results reveal that such markup can be very significant, and it is more valuable for the firm to acquire the information of consumers' reference price if reference price effect dominates the direct price effect, consumers adapt slower to new price information and the future profit is discounted less.

As pointed out earlier, Bell and Lattin [17] found that in some scenarios, loss/gain neutral assumption is appropriate if consumer heterogeneity is taken into consideration. However, there are other cases when loss-averse or gain-seeking behavior is found across different consumer groups, which can result in either loss-sensitive or gain-sensitive demands (see [20]). It would be interesting to see how our results can be generalized when there are asymmetric responses in demands.

While the continuous-time framework is convenient for deriving closed-form solutions, it is usually harder to implement in practice and hence a discrete-time counterpart of our model (or more specifically, the square-root-diffusion process (2.3)) is desirable. Note that if one applies the standard Euler-Maruyama discretization scheme (see, e.g., Chapter 9 in [21]) to (2.3), one would obtain

$$r_{t+1} = (1-\alpha)r_t + \alpha p_t + \sigma\sqrt{r_t}X_t,$$

where X_t, $t = 0, 1, \ldots$ are i.i.d. standard normal random variables. However, since X_t is unbounded from below, the reference prices r_t are not guaranteed to be nonnegative almost surely, and such reference price evolution is not well defined. There are different alternative discretization schemes proposed in the literature to rectify the above issue (see [22,23]). It is unclear at this point, however, which model is more appropriate in describing the evolution of reference prices in discrete time, and more future research is needed.

Finally, our observation that a higher demand variability is desirable may no longer hold if a joint inventory and pricing model is considered (see [24–28]). Recently, Chen et al. [29] have studied the joint inventory and pricing problem that incorporates reference price effect. However, their model assumes deterministic reference price effect and it would be valuable to see what the additional insights are if one considers stochastic reference price effect.

Appendix

Proof of Proposition 3.1

To solve the HJB equation (3.1), we start from solving a more general finite horizon problem. That is, let

$$V(r,t) = \max_{p(s)} \left[\int_t^{\mathrm{T}} \mathrm{e}^{-\gamma t} F(r(s), p(s))\mathrm{d}s \right]$$

be the value of optimal accumulated profit (profit-to-go function) from time t to the end of horizon T when the initial reference price is r. Notice that the value function in our problem (2.6) $V(r_0) = \lim_{T\to\infty} V(r_0, 0)$.

From standard theory in stochastic optimal control, $V(r,t)$ then satisfies the HJB equation

$$\gamma V(r,t) = \max_p \left[F(r,p) + \frac{\partial V(r,t)}{\partial t} + \alpha(p-r)\frac{\partial V(r,t)}{\partial r} + \frac{\sigma^2 r}{2}\frac{\partial^2 V(r,t)}{\partial r^2} \right]. \quad (6.1)$$

Using first-order condition in (6.1) and with a slight abuse of notation, we can solve p as

$$p^*(r) = \frac{c}{2} + \frac{b+\eta r}{2(a+\eta)} + \frac{\alpha}{2(a+\eta)} \times \frac{\partial V(r,t)}{\partial r}.$$

Substitute the above equation into (6.1), it follows

$$\frac{\sigma^2}{2} r \frac{\partial^2 V}{\partial r^2} + \frac{\partial V}{\partial t} - \gamma V + \frac{\alpha^2}{4(a+\eta)} \left(\frac{\partial V}{\partial r}\right)^2 + \left[\frac{\alpha c}{2} - \alpha r + \frac{\alpha(b+\eta r)}{2(a+\eta)}\right] \frac{\partial V}{\partial r}$$
$$- \frac{c(b+\eta r)}{2} + \frac{(b+\eta r)^2}{4(a+\eta)} + \frac{c^2(a+\eta)}{4} = 0.$$

Introducing a few new notations, this can be written concisely as:

$$\frac{\partial V}{\partial t} - \gamma V + Ar\frac{\partial^2 V}{\partial r^2} + B\left(\frac{\partial V}{\partial r}\right)^2 + (p_{10} + p_{11} r)\frac{\partial V}{\partial r} + p_{20} + p_{21} r + p_{22} r^2 = 0, \quad (6.2)$$

where

$$A = \frac{\sigma^2}{2},$$
$$B = \frac{\alpha^2}{4(a+\eta)},$$
$$p_{10} = \frac{\alpha c}{2} + \frac{\alpha b}{2(a+\eta)},$$
$$p_{11} = -\alpha + \frac{\alpha\eta}{2(a+\eta)},$$
$$p_{20} = -\frac{bc}{2} + \frac{b^2}{4(a+\eta)} + \frac{c^2(a+\eta)}{4},$$
$$p_{21} = -\frac{c\eta}{2} + \frac{b\eta}{2(a+\eta)},$$
$$p_{22} = \frac{\eta^2}{4(a+\eta)}.$$

If we assume function $V(r,t)$ has the following form:

$$V(r,t) = Q(t)r^2 + R(t)r + M(t), \quad (6.3)$$

then we get the following ordinary differential equations (ODEs):

$$\frac{\mathrm{d}Q}{\mathrm{d}t} - \gamma Q + 4BQ^2 + 2p_{11}Q + p_{22} = 0, \quad (6.4)$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} - \gamma R + 2AQ + 4BQR + 2p_{10}Q + p_{11}R + p_{21} = 0, \quad (6.5)$$

$$\frac{\mathrm{d}M}{\mathrm{d}t} - \gamma M + BR^2 + p_{10}R + p_{20} = 0, \tag{6.6}$$

with terminal condition $Q(T) = R(T) = M(T) = 0$.

We first explicitly solve ODE (6.4) by rewriting it as:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -4B(Q - Q_1)(Q - Q_2),$$

where $Q_1 < Q_2$ are the two distinct roots of the equation:

$$4BQ^2 - (\gamma - 2p_{11})Q + p_{22} = 0.$$

Namely:

$$Q_1 = \frac{\gamma - 2p_{11} - \sqrt{(\gamma - 2p_{11})^2 - 16Bp_{22}}}{8B},$$
$$Q_2 = \frac{\gamma - 2p_{11} + \sqrt{(\gamma - 2p_{11})^2 - 16Bp_{22}}}{8B}.$$

Therefore,

$$\begin{aligned} &\frac{\mathrm{d}Q}{(Q - Q_1)(Q - Q_2)} = -4B\mathrm{d}t \\ \Rightarrow\quad &\frac{dQ}{Q_1 - Q_2}\left[\frac{1}{Q - Q_1} - \frac{1}{Q - Q_2}\right] = -4B\mathrm{d}t \\ \Rightarrow\quad &\ln\frac{Q - Q_1}{Q - Q_2} = -4B(Q_1 - Q_2)t + C \\ \Rightarrow\quad &\frac{Q - Q_1}{Q - Q_2} = D \cdot \mathrm{e}^{-4B(Q_1 - Q_2)t}, \end{aligned} \tag{6.7}$$

where C and $D = e^C$ are constants to be determined. By $Q(T) = 0$, we can solve

$$D = \frac{Q_1}{Q_2}\mathrm{e}^{4B(Q_1 - Q_2)T}.$$

Substitute D back into (6.7), it follows

$$Q(t) = \frac{Q_1\mathrm{e}^{4B(Q_1 - Q_2)T} - Q_1\mathrm{e}^{4B(Q_1 - Q_2)t}}{Q_1/Q_2\mathrm{e}^{4B(Q_1 - Q_2)T} - \mathrm{e}^{4B(Q_1 - Q_2)t}}. \tag{6.8}$$

With the expressions for $Q(t)$, expressions for $R(t)$ and $M(t)$ can then be obtained by solving (6.5) and (6.6). Consequently, $p^*(r)$ can be determined as well. One can easily verify using Theorem 4.1 in chapter VI of [30] that $p^*(r)$ solved in this way is indeed optimal and $V(r, t)$ is given by (6.3).

The solution to (3.1) is then obtained by letting $T \to \infty$. Since $Q_1 < Q_2$, we have $Q := Q_1 = \lim_{T\to\infty} Q(t)$, where by substituting the expressions for B, p_{11} and p_{22}

$$Q = \frac{\gamma}{2\alpha^2}(a+\eta) + \frac{2a+\eta}{2\alpha} - \frac{a+\eta}{2\alpha^2}\Delta$$

and Δ is given by:

$$\Delta = \sqrt{\gamma^2 + 2\alpha\frac{2a(\gamma+\alpha)+\gamma\eta}{\eta+a}}.$$

Correspondingly, one can also obtain $R := \lim_{T\to\infty} R(t)$ as

$$\begin{aligned} R &= \frac{2p_{10}Q + p_{21} + 2AQ}{\gamma - 4BQ - p_{11}} \\ &= \left[\frac{b}{\alpha} + \frac{\sigma^2(a+\eta)}{\alpha^2} + \frac{c(a+\eta)}{\alpha}\right]\frac{\gamma-\Delta}{\gamma+\Delta} + \left[b + ca + \frac{\sigma^2(2a+\eta)}{2\alpha} + \right]\frac{2}{\gamma+\Delta}. \end{aligned}$$

Similarly, $M := \lim_{T\to\infty} M(t)$ can be computed.

Now, we have explicitly solved (3.1), where $V(r) = Qr^2 + Rr + M$ and the optimal pricing strategy can then be obtained as (3.2).

Finally, we remark that both Q_1 and Q_2 are positive solutions to the Algebraic Riccati Equation (ARE): $4BQ^2 - (\Gamma - 2p_{11})Q + p_{22} = 0$, which has no negative solution. This deviates significantly from standard linear quadratic control theory and is the primal reason we need to solve from the finite horizon problem instead of solving ARE directly.

Proof of Proposition 3.2

Note that $r^*(t)$ follows (3.3), which is a square-root diffusion process. It is not difficult to show that $\lambda, \mu > 0$. For such square-root diffusion process, it is known that $r^*(t)$ converges in distribution to a steady state which follows a Gamma distribution with shape parameter $\frac{2\lambda\mu}{\sigma^2}$ and rate parameter $\frac{2\lambda}{\sigma^2}$ (see, for instance, [15]).

Proof of Corollary 3.3

By Proposition 3.2, R_s^* follows a Gamma distribution and its mean and variance can then be computed as

$$\begin{aligned} E[R_s^*] &= \frac{2\lambda\mu}{\sigma^2}\frac{\sigma^2}{2\lambda} = \mu, \\ \mathrm{var}(R_s^*) &= \frac{2\lambda\mu}{\sigma^2}\left(\frac{\sigma^2}{2\lambda}\right)^2 = \frac{\mu}{2\lambda}\sigma^2. \end{aligned}$$

Substituting the expressions for Q, R and λ into μ, with cumbersome algebraic manipulations, one can further obtain

$$\mu = \frac{(\gamma + \bar{\alpha})b}{2a(\gamma + \bar{\alpha}) + \gamma\eta} + \frac{\sigma^2}{2a(\gamma + \bar{\alpha}) + \gamma\eta} \left[\frac{a + \eta}{\bar{\alpha}} \left(\frac{\gamma}{2} - \frac{\Delta}{2} \right) + \frac{2a + \eta}{2} \right].$$

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