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Partial regularity of weak solutions to a PDE system with cubic nonlinearity

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Abstract

In this paper we investigate regularity properties of weak solutions to a PDE system that arises in the study of biological transport networks. The system consists of a possibly singular elliptic equation for the scalar pressure of the underlying biological network coupled to a diffusion equation for the conductance vector of the network. There are several different types of nonlinearities in the system. Of particular mathematical interest is a term that is a polynomial function of solutions and their partial derivatives and this polynomial function has degree three. That is, the system contains a cubic nonlinearity. Only weak solutions to the system have been shown to exist. The regularity theory for the system remains fundamentally incomplete. In particular, it is not known whether or not weak solutions develop singularities. In this paper we obtain a partial regularity theorem, which gives an estimate for the parabolic Hausdorff dimension of the set of possible singular points.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ and T a positive number. Set $\Omega_T = \Omega \times (0, T)$. We study the behavior of weak solutions of the system

$$-\text{div}\,[(I + m \otimes m)\nabla p] = S(x) \quad \text{in } \Omega_T, \tag{1.1}$$

$$\partial_t m - D^2 \Delta m - E^2 (m \cdot \nabla p)\nabla p + |m|^{2(\gamma-1)} m = 0 \quad \text{in } \Omega_T \tag{1.2}$$

for given function $S(x)$ and physical parameters D, E, γ with properties:

(H1) $S(x) \in L^q(\Omega),\ q > \frac{N}{2}$; and
(H2) $D, E \in (0, \infty), \gamma \in (\frac{1}{2}, \infty)$.

This system has been proposed by Hu and Cai ([10], [11]) to describe natural network formulation. Then the scalar pressure function $p = p(x, t)$ follows Darcy's law, while the vector-valued function $m = m(x, t)$ is the conductance vector. The function $S(x)$ is the time-independent source term. Values of the parameters D, E, and γ are determined by the particular physical applications one has in mind. For example, $\gamma = 1$ corresponds to leaf venation [10]. Of particular physical interest is the initial boundary value problem: in addition to (1.1) and (1.2) one requires

$$m(x, 0) = m_0(x), \quad x \in \Omega, \tag{1.3}$$

$$p(x, t) = 0, \quad m(x, t) = 0, \quad (x, t) \in \Sigma_T \equiv \partial\Omega \times (0, T), \tag{1.4}$$

at least in a suitably weak sense; here the initial data should satisfy

$$m_0(x) = 0 \quad \text{on } \partial\Omega.$$

The existence of weak solutions of this initial boundary value problem was proved by Haskovec, Markowich, and Perthame [8]. However, the regularity theory remains fundamentally incomplete. In particular, it is not known whether or not weak solutions develop singularities.

Let us call a point $(x, t) \in \Omega_T$ singular if m is not Hölder continuous in any neighborhood of (x, t); the remaining points will be called regular points. By a partial regularity theorem, we mean an estimate for the dimension of the set S of singular points. It is well-known that weak solutions to even uniformly elliptic systems of partial differential equations are not regular everywhere. We refer the reader to [6] for counter examples. Thus it is only natural to seek partial regularity theorems for these weak solutions. The system under our consideration exhibits a rather peculiar nonlinear structure. The first equation in the system degenerates in the t-variable and the elliptic coefficients there are singular in the sense that they are not uniformly bounded above a priori, while the second equation contains the term $(m \cdot \nabla p)\nabla p$, which is a cubic nonlinearity. Thus the classical partial regularity argument developed in ([6], [1]) does not seem to be applicable here. Our system does resemble the so-call thermistor problem considered in ([16–18]). The key difference is that the elliptic coefficients in the preceding papers and also in [6] are assumed to be bounded and continuous functions of solutions. As a result, the modulus of continuity can be taken to be a bounded, continuous, and concave function. This fact is essential to the arguments in both [16] and [6]. Our elliptic coefficients here are quadratic in m, and thus a new proof must be developed.

Definition. A pair (m, p) is said to be a weak solution if:

(D1) $m \in L^\infty(0,T; \left(W_0^{1,2}(\Omega)\cap L^{2\gamma}(\Omega)\right)^N)$, $\partial_t m \in L^2(0,T; \left(L^2(\Omega)\right)^N)$, $p \in L^\infty(0,T; W_0^{1,2}(\Omega))$, $m\cdot\nabla p \in L^\infty(0,T;L^2(\Omega))$;
(D2) $m(x,0) = m_0$ in $C([0,T]; \left(L^2(\Omega)\right)^N)$;
(D3) (1.1) and (1.2) are satisfied in the sense of distributions.

A result in [8] asserts that (1.1)–(1.4) has a weak solution provided that, in addition to assuming $S(x) \in L^2(\Omega)$ and (H2), we also have

(H3) $m_0 \in \left(W_0^{1,2}(\Omega)\cap L^{2\gamma}(\Omega)\right)^N$.

Note that the question of existence in the case where $\gamma = \frac{1}{2}$ is addressed in [9]. In this case the term $|m|^{2(\gamma-1)}m$ is not continuous at $m = 0$. It must be replaced by the following function

$$g(x,t) = \begin{cases} |m|^{2(\gamma-1)}m & \text{if } m \neq 0, \\ \in [-1,1]^N & \text{if } m \neq 0. \end{cases}$$

Partial regularity relies on local estimates [6]. One peculiar feature about our problem (1.1)–(1.4) is that certain important global estimates have no local versions. This is another source of difficulty for our mathematical analysis. We are ready to state our main result:

Theorem 1.1. *Let (H1)–(H3) be satisfied. Assume that $N \le 3$. Then the initial boundary value problem (1.1)–(1.4) has a weak solution on Ω_T whose singular set S satisfies*

$$\mathcal{P}^{N+\varepsilon}(S) = 0 \tag{1.5}$$

for each $\varepsilon > 0$.

Here $\mathcal{P}^s$, $s \ge 0$, denotes the s-dimensional parabolic Hausdorff measure. Recall that the s-dimensional parabolic Hausdorff measure of a set $E \subset \mathbb{R}^N \times \mathbb{R}$ is defined as follows:

$$\mathcal{P}^s(E) = \sup_{\varepsilon>0} \inf\{\sum_{j=0}^\infty r_j^s : \cup_{j=0}^\infty Q_{r_j}(z_j) \supset E, r_j < \varepsilon\},$$

where $Q_{r_j}(z_j)$ are parabolic cylinders with geometric centers at $z_j = (y_j, \tau_j)$, i.e., one has

$$Q_{r_j}(z_j) = B_{r_j}(y_j) \times (\tau_j - \frac{1}{2}r_j^2, \tau_j + \frac{1}{2}r_j^2)$$

with

$$B_{r_j}(y_j) = \{x \in \mathbb{R}^N : |x - y_j| < r_j\}.$$

It is not difficult to see that $\mathcal{P}^s$ is an outer measure, for which all Borel sets are measurable; on its σ-algebra of measurable sets, $\mathcal{P}^k$ is a Borel regular measure (cf. [5], Chap. 2.10). If $\mathcal{P}^s(E) < \infty$, then $\mathcal{P}^{s+\varepsilon}(E) = 0$ for each $\varepsilon > 0$. We define the parabolic Hausdorff dimension $\dim_{\mathcal{P}} E$ of a set E to be

$$\dim_{\mathcal{P}} E = \inf\{s \in \mathbb{R}^+ : \mathcal{P}^s(E) = 0\}.$$

Then Theorem 1.1 says that

$$\dim_{\mathcal{P}} S \le N. \tag{1.6}$$

Hausdorff measure $\mathcal{H}^s$ is defined in an entirely similar manner, but with $Q_{r_j}(z_j)$ replaced by an arbitrary closed subset of $\mathbb{R}^N \times \mathbb{R}$ of diameter at most r_j. (One usually normalizes $\mathcal{H}^s$ for integer s so that it agrees with surface area on smooth s-dimensional surfaces.) Clearly,

$$\mathcal{H}^k(X) \le c(k)\mathcal{P}^k(X) \quad \text{for each } X \subset \mathbb{R}^N \times \mathbb{R}. \tag{1.7}$$

To characterize the singular set S, we will need to invoke the following known result.

Lemma 1.1. *Let $f \in L^1_{loc}(\Omega_T)$ and for $0 \le s < N + 2$ set*

$$E_s = \{z \in \Omega_T : \limsup_{\rho \to 0^+} \rho^{-s} \int_{Q_\rho(z)} |f| dxdt > 0\}.$$

Then $\mathcal{P}^s(E_s) = 0$.

The proof of this lemma is essentially contained in [1].

A key observation about our weak solutions in the study of partial regularity is the following proposition.

Proposition 1.1. *Let* (H1)–(H3) *be satisfied and (p, m) be a weak solution to (1.1)–(1.4). Then we have*

$$p \in C([0, T]; L^2(\Omega)). \tag{1.8}$$

The proof of this proposition will be given at the end of Section 2.

Let (m, p) be a weak solution. In view of ([3],[16]), to establish Theorem 1.1, we will need to define a suitable scaled energy $E_r(z)$ for our system. For this purpose, let $z = (y, \tau) \in \Omega_T$, $r > 0$ with $Q_r(z) \subset \Omega_T$ and pick

$$0 < \beta < \min\{2 - \frac{N}{q}, 1\}, \tag{1.9}$$

where q is given as in (H1). We consider the following quantities:

$$p_{y,r}(t) = \fint_{B_r(y)} p(x,t)dx = \frac{1}{|B_r(y)|} \int_{B_r(y)} p(x,t)dx, \tag{1.10}$$

$$m_{z,r} = \fint_{Q_\rho(z)} m(x,t)dxdt, \tag{1.11}$$

$$A_r(z) = \frac{1}{r^N} \max_{t\in[\tau-\frac{1}{2}r^2,\tau+\frac{1}{2}r^2]} \int_{B_r(y)} (p(x,t) - p_{y,r}(t))^2 dx. \tag{1.12}$$

The right choice for $E_r(z)$ seems to be

$$E_r(z) = \frac{1}{r^{N+2}} \int_{Q_r(z)} |m - m_{z,r}|^2 dxdt + A_r(z) + r^{2\beta}. \tag{1.13}$$

The last term in $E_r(z)$ accounts for the non-homogeneous term $S(x)$ in (1.1). Due to the fact that the first equation (1.1) does not have the $\partial_t p$ term, we are forced to use the term $A_r(z)$ instead of $\frac{1}{r^{N+2}} \int_{Q_r(z)} |p - p_{z,r}|^2 dxdt$ in $E_r(z)$. This will cause two problems: one is that in our application of the classical blow-up argument ([3],[6],[16]), the resulting blow-up sequence is not compact in the desired function space; the other is the characterization of the singular set S. That is, it is not immediately clear how one can describe the set

$$\Omega_T \setminus \{z \in \Omega_T : \lim_{r\to 0} A_r(z) = 0\} \tag{1.14}$$

in terms of the parabolic Hausdorff measure. (Note that this issue is rather simple in the context of [16].) To overcome these two problems, we find a suitable decomposition of p. This enables us to show that the lack of compactness in the blow-up sequence does not really matter. To be more specific, we obtain that the blow-up sequence can be decomposed into the sum of two other sequences, one of which converges strongly while the terms of the other are very smooth in the space variables, and this is good enough for our purpose. This idea was first employed in [16]. However, as we mentioned earlier, the nature of our mathematical difficulty here is totally different. A similar decomposition technique can also be used to derive the parabolic Hausdorff dimension of the set in (1.14).

The key to our development is this assertion about energy:

Proposition 1.2. *Let the assumptions of Theorem 1.1 hold. For each $M > 0$ there exist constants $0 < \varepsilon, \delta < 1$ such that*

$$|m_{z,r}| \le M \;\; and \;\; E_r(z) \le \varepsilon \tag{1.15}$$

imply

$$E_{\delta r}(z) \le \frac{1}{2} E_r(z) \tag{1.16}$$

for all $z \in \Omega_T$ and $r > 0$ with $Q_r(z) \subset \Omega_T$.

The proof of this proposition is given in Section 4. It relies on the decomposition of the function p we mentioned earlier. An immediate consequence of this proposition is:

Corollary 1.1. *Let the assumptions of Theorem 1.1 hold. To each $M > 0$ there corresponds a pair of numbers δ_1, ε_1 in $(0, 1)$ such that whenever*

$$|m_{z,r}| < \frac{M}{2} \text{ and } E_r(z) < \varepsilon_1 \tag{1.17}$$

we have

$$E_{\delta_1^k r}(z) \le \left(\frac{1}{2}\right)^k \varepsilon_1 \quad \text{for each positive integer } k. \tag{1.18}$$

Proof. We essentially follow the proof of Corollary 3.8 in [16] (also see [6]). Let $M > 0$ be given. By Proposition 1.2, there exist $0 < \varepsilon, \delta < 1$ such that (1.15) and (1.16) hold. We claim that we can take

$$\delta_1 = \delta, \tag{1.19}$$

$$\varepsilon_1 = \min\left\{\varepsilon, \left(\frac{M\delta^{N+2}(\sqrt{2}-1)\sqrt{\omega_N}}{2\delta^{N+2} + 2\sqrt{2}}\right)^2\right\}, \tag{1.20}$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$. To see this, let (1.17) hold. Obviously, (1.18) is satisfied for $k = 1$. Now for each positive integer j suppose (1.18) is true for all $k \le j$. We will show that it is also true for $k = j + 1$. To this end, we integrate the inequality

$$|m_{z,\delta^i r} - m_{z,\delta^{i-1} r}| \le |m_{z,\delta^i r} - m(x,t)| + |m(x,t) - m_{z,\delta^{i-1} r}|$$

over $Q_{\delta^i r}(z)$ to derive

$$\begin{aligned} |m_{z,\delta^i r} - m_{z,\delta^{i-1} r}| &\le \fint_{Q_{\delta^i r}(z)} |m_{z,\delta^i r} - m(x,t)| dxdt \\ &\quad + \frac{1}{\delta^{N+2}} \fint_{Q_{\delta^{i-1} r}(z)} |m(x,t) - m_{z,\delta^{i-1} r}| dxdt \\ &\le \left(\fint_{Q_{\delta^i r}(z)} |m_{z,\delta^i r} - m(x,t)|^2 dxdt \right)^{\frac{1}{2}} \\ &\quad + \frac{1}{\delta^{N+2}} \left(\fint_{Q_{\delta^{i-1} r}(z)} |m(x,t) - m_{z,\delta^{i-1} r}|^2 dxdt \right)^{\frac{1}{2}} \end{aligned}$$

$$\leq \left(\frac{1}{\omega_N}E_{\delta^i r}(z)\right)^{\frac{1}{2}} + \frac{1}{\delta^{N+2}}\left(\frac{1}{\omega_N}E_{\delta^{i-1}r}(z)\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{1}{\omega_N}\left(\frac{1}{2}\right)^i\varepsilon_1\right)^{\frac{1}{2}} + \frac{1}{\delta^{N+2}}\left(\frac{1}{\omega_N}\left(\frac{1}{2}\right)^{i-1}\varepsilon_1\right)^{\frac{1}{2}},$$

$$i = 1, \cdots, j. \tag{1.21}$$

Subsequently, we have

$$|m_{z,\delta^j r}| \leq m_{z,r} + \sum_{i=1}^{j}|m_{z,\delta^i r} - m_{z,\delta^{i-1}r}|$$

$$\leq \frac{M}{2} + \sum_{i=1}^{j}\left(\frac{1}{\omega_N}\left(\frac{1}{2}\right)^i\varepsilon_1\right)^{\frac{1}{2}} + \sum_{i=1}^{j}\frac{1}{\delta^{N+2}}\left(\frac{1}{\omega_N}\left(\frac{1}{2}\right)^{i-1}\varepsilon_1\right)^{\frac{1}{2}}$$

$$\leq \frac{M}{2} + \frac{\delta^{N+2}+\sqrt{2}}{\delta^{N+2}(\sqrt{2}-1)\sqrt{\omega_N}}\sqrt{\varepsilon_1} \leq M. \tag{1.22}$$

By Proposition 1.2, (1.18) holds for $k = j+1$. This completes the proof. □

This corollary combined with the argument in ([6], p. 86) asserts that there exist $c = c(\delta_1, \varepsilon_1, r) \in (0,1)$, $\gamma = \gamma(\delta_1) > 0$ such that

$$E_\rho(z) \leq c\rho^\gamma \quad \text{for all } 0 < \rho \leq r. \tag{1.23}$$

Obviously, $m_{z,r}$, $⨍_{Q_r(z)}|m - m_{z,r}|^2 dxdt$ are both continuous functions of z. By Proposition 1.1, $E_r(z)$ is also a continuous function of z. Thus whenever (1.17) holds for some $z = z_0$ there is an open neighborhood O of z_0 over which (1.17) remains true. As a result, (1.23) is satisfied on O. This puts us in a position to apply a result in [12]. To state the result, we define, for $\mu \in (0,1)$,

$$[m]_{\mu,O} = \sup\left\{\frac{|m(x,t)-m(y,\tau)|}{\left(|x-y|+|t-\tau|^{\frac{1}{2}}\right)^\mu} : (x,t),(y,\tau) \in O\right\}.$$

Parabolic Hölder spaces can be characterized by the following version of Campanato's theorem ([12], Theorem 1).

Lemma 1.2. *Let $u \in L^2(\Omega_T)$. If there exist $\alpha \in (0,1)$ and $R_0 > 0$ such that*

$$⨍_{Q_\rho(z)} |u - u_{z,\rho}|^2 dxdt \leq A^2\rho^{2\alpha}$$

for all z in an open subset O of Ω_T and all $\rho \leq R_0$ with $Q_\rho(z) \subset \Omega_T$, then we have

$$[m]_{\alpha,O} \le c(N)A.$$

That is, u is Hölder continuous in O.

To describe the singular set S, we set

$$R = \{z = (y,\tau) \in \Omega_T : \sup_{r>0} |m_{z,r}| < \infty,\ \lim_{r\to 0} E_r(z) = 0\}. \tag{1.24}$$

Here and in what follows $\lim_{r\to 0}$ means $\lim_{r\to 0^+}$ because we always have $r > 0$. If $z \in R$, we take $M > 2\sup_{r>0} |m_{z,r}|$. By Corollary 1.1, there exist $\delta_1, \varepsilon_1 \in (0,1)$ such that (1.17) and (1.18) hold. We can find a r such that

$$E_r(z) < \varepsilon_1.$$

For the same r we obviously have

$$|m_{z,r}| < \frac{M}{2}.$$

Consequently, m is Hölder continuous in a neighborhood of z. That is, R is a set of regular points. Obviously, R is an open set.

Note that since we have the term $A_r(z)$ instead of $\frac{1}{r^{N+2}}\int_{Q_r(z)} |p - p_{z,r}|^2 dxdt$ in $E_r(z)$ Proposition 1.2 does not imply that p is locally Hölder continuous in the space–time domain R. The difference between the two quantities can be seen from the following calculation:

$$\begin{aligned}
\fint_{Q_\rho(z)} |p - p_{z,r}|^2 dxdt &= \frac{1}{r^2}\int_{\tau-\frac{1}{2}r^2}^{\tau+\frac{1}{2}r^2} \fint_{B_r(y)} |p - p_{z,r}|^2 dxdt \\
&\le \frac{2}{r^2}\int_{\tau-\frac{1}{2}r^2}^{\tau+\frac{1}{2}r^2} \fint_{B_r(y)} |p - p_{y,r}(t)|^2 dxdt \\
&\quad + \frac{2}{r^2}\int_{\tau-\frac{1}{2}r^2}^{\tau+\frac{1}{2}r^2} |p_{y,r}(t) - p_{z,r}|^2 dt \\
&\le \frac{2}{\omega_N} A_r(z) + \frac{2}{r^2}\int_{\tau-\frac{1}{2}r^2}^{\tau+\frac{1}{2}r^2} |p_{y,r}(t) - p_{z,r}|^2 dt.
\end{aligned} \tag{1.25}$$

Obviously, the last term above causes the problem. Of course, for each $t = t_0$, $p(x, t_0)$ is locally Hölder continuous in x in $R \cap \{t = t_0\}$.

To estimate the parabolic Hausdorff dimension of the singular set $S \subseteq \Omega_T \setminus R$, we have the following proposition.

Proposition 1.3. *Let* (H1)–(H3) *hold and* (p, m) *be a weak solution. Then we have*

$$dim_{\mathcal{P}}(\Omega_T \setminus R) = N. \tag{1.26}$$

The proof of this proposition relies on almost the same decomposition of p as that in the proof of Proposition 1.2. The details will be given in Section 3.

Thus Theorem 1.1 is a consequence of Propositions 1.1–1.3. The rest of the paper is organized as follows. In Section 2, we develop some new global estimates. They serve as a motivation for our local estimates. The section will end with the proof of Proposition 1.1. In Section 3, we will first establish some local estimates and then proceed to prove Proposition 1.3. Section 4 is devoted to the proof of Proposition 1.2. Note that the three propositions are independent, and thus the order of their proofs is not important.

2. Global estimates

In this section, we first summarize the main a priori estimates already established in [8]. Then we present our new global estimates. The proof of Proposition 1.1 is given at the end.

To begin with, we use $p(x, t)$ as a test function in (1.1) to obtain

$$\int_\Omega |\nabla p|^2 dx + \int_\Omega (m \cdot \nabla p)^2 dx = \int_\Omega S(x) p dx. \tag{2.1}$$

Here and in what follows we suppress the dependence of p, m on (x, t) for simplicity of notation if no confusion arises. Let $\tau \in (0, T)$, $\Omega_\tau = \Omega \times (0, \tau)$. Take the dot product of both sides of (1.2) with m, integrate the resulting equation over Ω_τ, and thereby yield

$$\begin{aligned} &\frac{1}{2}\int_\Omega |m(x,\tau)|^2 dx + D^2 \int_{\Omega_\tau} |\nabla m|^2 dxdt \\ &\quad - E^2 \int_{\Omega_\tau} (m \cdot \nabla p)^2 dxdt + \int_{\Omega_\tau} |m|^{2\gamma} dxdt = \frac{1}{2}\int_\Omega |m_0|^2 dx, \end{aligned} \tag{2.2}$$

where $|\nabla m|^2 = |\nabla \otimes m|^2 = \sum_{i,j=1}^N (\frac{\partial m_j}{\partial x_i})^2$. Multiply through (2.1) by $2E^2$, integrate over $(0, \tau)$, and then add it to (2.2) to arrive at

$$\begin{aligned} &\frac{1}{2}\int_\Omega |m(x,\tau)|^2 dx + D^2 \int_{\Omega_\tau} |\nabla m|^2 dxdt + E^2 \int_{\Omega_\tau} (m \cdot \nabla p)^2 dxdt \\ &\quad + \int_{\Omega_\tau} |m|^{2\gamma} dxdt + 2E^2 \int_{\Omega_\tau} |\nabla p|^2 dx d\tau \\ &= \frac{1}{2}\int_\Omega |m_0|^2 dx + 2E^2 \int_{\Omega_\tau} S(x) p dxdt. \end{aligned} \tag{2.3}$$

Take the dot product of (1.2) with $\partial_t m$ and integrate the resulting equation over Ω to obtain

$$\int_\Omega |\partial_t m|^2 dx + \frac{D^2}{2}\frac{d}{dt}\int_\Omega |\nabla m|^2 dx \\ - E^2 \int_\Omega (m\cdot\nabla p)\nabla p \partial_t m dx + \frac{1}{2\gamma}\frac{d}{dt}\int_\Omega |m|^{2\gamma} dx = 0. \tag{2.4}$$

Use $\partial_t p$ as a test function in (1.1) to derive

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla p|^2 dx + \int_\Omega (m\cdot\nabla p) m \nabla \partial_t p dx = \int_\Omega S(x)\partial_t p dx. \tag{2.5}$$

Multiply through this equation by $-E^2$ and add the resulting one to (2.4) to obtain

$$\int_\Omega |\partial_t m|^2 dx + \frac{D^2}{2}\frac{d}{dt}\int_\Omega |\nabla m|^2 dx - \frac{E^2}{2}\frac{d}{dt}\int_\Omega |(m\cdot\nabla p)^2 dx \\ - \frac{E^2}{2}\frac{d}{dt}\int_\Omega |\nabla p|^2 dx + \frac{1}{2\gamma}\frac{d}{dt}\int_\Omega |m|^{2\gamma} dx = -E^2\int_\Omega S(x)\partial_t p dx. \tag{2.6}$$

Differentiate (2.1) with respect to t, multiply through the resulting equation by E^2, then add it to the above equation, and thereby deduce

$$\int_{\Omega_\tau} |\partial_t m|^2 dxdt + \frac{D^2}{2}\int_\Omega |\nabla m(x,\tau)|^2 dx + \frac{E^2}{2}\int_\Omega (m\cdot\nabla p)^2 dx \\ + \frac{E^2}{2}\int_\Omega |\nabla p|^2 dx + \frac{1}{2\gamma}\int_\Omega |m|^{2\gamma} dx \\ = \frac{D^2}{2}\int_\Omega |\nabla m_0|^2 dx + \frac{E^2}{2}\int_\Omega (m_0\cdot\nabla p_0)^2 dx + \frac{1}{2\gamma}\int_\Omega |m_0|^{2\gamma} dx \\ + \frac{E^2}{2}\int_\Omega |\nabla p_0|^2 dx, \tag{2.7}$$

where p_0 is the solution of the boundary value problem

$$-\mathrm{div}[(I + m_0\otimes m_0)\nabla p_0] = S(x), \quad \text{in } \Omega, \tag{2.8}$$

$$p_0 = 0 \quad \text{on } \partial\Omega. \tag{2.9}$$

Local versions of (2.1) and (2.3) will be established in Section 3. Unfortunately, they are not enough to yield a partial regularity result. Naturally, one tries to seek a local version of (2.7). But this cannot be done because we have no control over $\partial_t p$. To partially circumvent this, we have developed some new estimates.

Proposition 2.1. *Let (H1) and (H2) be satisfied and* (m, p) *a weak solution of (1.1)–(1.4). Then:*

(C1) There is a positive number $c = c(\Omega, N)$ *such that* $\|p\|_{\infty,\Omega_T} \equiv ess\, sup_{\Omega_T}|p| \le c\|S(x)\|_{q,\Omega}$*, where* $\|\cdot\|_{q,\Omega}$ *denotes the norm in* $L^q(\Omega)$*. We shall write* $\|\cdot\|_s$ *for* $\|\cdot\|_{s.\Omega}$ *for simplicity;*
(C2) For each $K > 0$ *we can choose* $\beta \in (0, 1)$ *suitably small such that*

$$\begin{aligned}&\int_\Omega \int_0^{|m(x,\tau)|^2} [(s-K^2)^+ + K^2]^\beta ds dx + \int_{\Omega_\tau} v^\beta |\nabla m|^2 dxdt \\ &\quad + \int_{\Omega_\tau} v^{\beta-1}|\nabla v|^2 dxdt + \int_{\Omega_\tau} |m|^{2\gamma} v^\beta dxdt \\ &\quad + \int_{\Omega_\tau} v^\beta |\nabla p|^2 dxdt + \int_{\Omega_\tau} v^\beta (m\cdot\nabla p)^2 dxdt \\ &\le c\int_{\Omega_\tau} |S(x)| v^\beta dxdt + \int_\Omega \int_0^{|m_0|^2} [(s-K^2)^+ + K^2]^\beta ds dx + c \ \text{for all } \tau \in (0,T),\end{aligned}$$

where

$$v = (|m|^2 - K^2)^+ + K^2 \ge K^2. \tag{2.10}$$

By the Sobolev embedding theorem, we have

$$m \in L^\infty(0, T; L^{\frac{2N}{N-2}}(\Omega)).$$

Thus the first integral on the right-hand side of the above inequality is finite.

Proof. The proof of (C1) is standard. See, e.g., ([2], p. 131). For the reader's convenience, we shall reproduce the proof here. Let κ be a positive number to be determined. Write

$$\kappa_n = \kappa - \frac{\kappa}{2^n}, \quad A_n(t) = \{x \in \Omega : p(x,t) > \kappa_n\}, \quad n = 0, 1, 2, \cdots.$$

Use $(p-\kappa_n)^+$ as a test function in (1.1) to deduce

$$\begin{aligned}&\int_\Omega |\nabla(p-\kappa_n)^+|^2 dx + \int_\Omega |(m\cdot\nabla(p-\kappa_n)^+)^2 dx \\ &= \int_\Omega S(x)(p-\kappa_n)^+ dx \\ &\le \left(\int_{A_n(t)} |S(x)|^{\frac{2N}{N+2}}\right)^{\frac{N+2}{2N}} \|(p-\kappa_n)^+\|_{\frac{2N}{N-2}} \\ &\le c\|S(x)\|_q |A_n(t)|^{\frac{N+2}{2N}-\frac{1}{q}} \|\nabla(p-\kappa_n)^+\|_2,\end{aligned} \tag{2.11}$$

from whence follows

$$|A_{n+1}(t)| \le c\|S(x)\|_q^{\frac{2N}{N-2}} \frac{2^{\frac{2Nn}{N-2}}}{\kappa^{\frac{2N}{N-2}}} |A_n(t)|^{1+\frac{2}{N-2}(2-\frac{N}{q})}. \tag{2.12}$$

By (H1), we have $\alpha \equiv \frac{2}{N-2}(2-\frac{N}{q}) > 0$. This enables us to apply Lemma 4.1 in ([2], p. 12) to obtain

$$|A_\infty(t)| = 0, \quad \text{provided that } \kappa = c\|S(x)\|_q \text{ for some } c = c(\Omega, N).$$

This implies (C1).

Let $K > 0, \beta > 0$ be given and v be defined as in (2.10). For $L > K$, define

$$\theta_L(s) = \begin{cases} L^2 & \text{if } s \ge L^2, \\ s & \text{if } K^2 < s < L^2, \\ K^2 & \text{if } s \le K. \end{cases} \tag{2.13}$$

Set $v_L = \theta_L(|m|^2)$. Then the function $v_L^\beta m$ is a legitimate test function for (1.2). Upon using it, we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega \int_0^{|m|^2} [\theta_L(s)]^\beta ds dx + D^2 \int_\Omega v_L^\beta |\nabla m|^2 dx \\ &+ \frac{D^2\beta}{2}\int_\Omega v_L^{\beta-1}|\nabla v_L|^2 + \int_\Omega |m|^{2\gamma} v_L^\beta dx \\ &= E^2 \int_\Omega v_L^\beta (m\cdot\nabla p)^2 dx. \end{aligned} \tag{2.14}$$

In the derivation of the third term above, we have used the fact that

$$\nabla v_L = 0 \ \text{ on the set where } |m|^2 > L^2 \text{ or } |m|^2 < K^2. \tag{2.15}$$

Use $v_L^\beta p$ as a test function in (1.1) to deduce

$$\begin{aligned} &\int_\Omega v_L^\beta |\nabla p|^2 dx + \int_\Omega v_L^\beta (m\cdot\nabla p)^2 dx \\ &= -\int_\Omega \nabla p p \beta v_L^{\beta-1} \nabla v_L dx \\ &\quad - \int_\Omega (m\cdot\nabla p) m p \beta v_L^{\beta-1}\nabla v_L dx + \int_\Omega S(x) v_L^\beta p dx \end{aligned}$$

$$
\begin{aligned}
&\leq \varepsilon\beta \int_\Omega v_L^{\beta-1}|\nabla v_L|^2 dx + c(\varepsilon)\beta \int_\Omega v_L^{\beta-1} p^2 |\nabla p|^2 dx \\
&\quad + \varepsilon \int_\Omega v_L^{\beta}(m\cdot\nabla p)^2 dx + c(\varepsilon)\beta^2 \int_\Omega v_L^{\beta-2}|m|^2 p^2 |\nabla v_L|^2 dx \\
&\quad + \int_\Omega S(x) v_L^{\beta} p dx, \quad \varepsilon > 0.
\end{aligned} \tag{2.16}
$$

By virtue of (2.15), we have that $v_L^{\beta-2}|m|^2|\nabla v_L|^2 = v_L^{\beta-1}|\nabla v_L|^2$. Remember that $\beta \in (0, 1)$. This gives $v_L^{\beta-1} p^2 \leq \|p\|_\infty^2 K^{2(\beta-1)}$. Multiply through the above inequality by $2E^2$, add the resulting inequality to (2.14), thereby obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \int_0^{|m|^2} [\theta_L(s)]^\beta ds dx + \int_\Omega v_L^\beta |\nabla m|^2 dx \\
&\quad + \beta \int_\Omega v_L^{\beta-1}|\nabla v_L|^2 dx + \int_\Omega |m|^{2\gamma} v_L^\beta dx \\
&\quad + \int_\Omega v_L^\beta |\nabla p|^2 dx + \int_\Omega v_L^\beta (m\cdot\nabla p)^2 dx \\
&\leq c\beta \frac{\|p\|_\infty^2}{K^2} \int_\Omega v_L^\beta |\nabla p|^2 dx + c\beta^2 \|p\|_\infty^2 \int_\Omega v_L^{\beta-1}|\nabla v_L|^2 dx \\
&\quad + \int_\Omega S(x) v_L^\beta p dx.
\end{aligned} \tag{2.17}
$$

Choosing β sufficiently small so that the second term on the right-hand in the above inequality can be absorbed into the third term on the left-hand side there, integrating the resulting inequality with respect to t, and then taking $L \to \infty$ yields (C2). The proof is complete. □

It turns out that a local version of (C2) is possible only if $N \leq 3$. This accounts for the restriction on the space dimension in Theorem 1.1.

At the end of this section, we present the proof of Proposition 1.1.

Proof of Proposition 1.1. It is easy to see that $m(x,t) \in C([0,T]; (L^2(\Omega))^N)$. By the proof of Lemma 2.3 in [19], we can conclude that for each $t \in [0,T]$ there is a unique weak solution $p = p(x,t)$ in the space $W_0^{1,2}(\Omega)$ to (1.1) with $m(x,t)\cdot\nabla p(x,t) \in L^2(\Omega)$. Fix a t^* in $[0,T]$. Let $\{t_j\}$ be a sequence in $[0,T]$ with the property

$$
t_j \to t^*. \tag{2.18}
$$

Set $m_j = m(x,t_j)$ and denote by p_j the solution of (1.1) with m being replaced by m_j. Obviously, we have

$$m_j \to m^* \equiv m(x,t^*) \quad \text{strongly in } \left(L^2(\Omega)\right)^N \text{ as } j \to \infty. \tag{2.19}$$

We claim that we also have

$$p_j \to p^* \equiv p(x,t^*), \text{ the solution of (1.1) corresponding to } t = t^*,$$
$$\text{strongly in } L^2(\Omega) \text{ as } j \to \infty, \tag{2.20}$$

and this will be enough to imply the proposition. To see this, note that $m_j \otimes m_j \nabla p_j = (m_j \cdot \nabla p_j) m_j$, and thus we have the equation

$$-\text{div}(\nabla p_j + (m_j \cdot \nabla p_j) m_j) = S(x) \quad \text{in } \Omega. \tag{2.21}$$

Using p_j as a test function, we can easily derive

$$\int_\Omega |\nabla p_j|^2 dx + \int_\Omega (m_j \cdot \nabla p_j)^2 dx \le c \int_\Omega |S(x)|^2 dx. \tag{2.22}$$

Thus we may assume that

$$p_j \rightharpoonup p \quad \text{weakly in } W_0^{1,2}(\Omega) \text{ and strongly in } L^2(\Omega) \tag{2.23}$$

(passing to a subsequence if need be). This together with (2.19) implies

$$m_j \cdot \nabla p_j \rightharpoonup m^* \cdot \nabla p \quad \text{weakly in } L^1(\Omega), \text{ and therefore also weakly in } L^2(\Omega).$$

Subsequently, we have

$$(m_j \cdot \nabla p_j) m_j \rightharpoonup (m^* \cdot \nabla p) m^* \quad \text{weakly in } \left(L^1(\Omega)\right)^N.$$

Thus we can take $j \to \infty$ in (2.21) to obtain

$$-\text{div}(\nabla p + (m^* \cdot \nabla p) m^*) = S(x) \quad \text{in } \Omega. \tag{2.24}$$

The solution to this equation is unique in $W_0^{1,2}(\Omega)$, and therefore $p = p^*$ and the whole sequence $\{p_j\}$ tends to p^* strongly in $L^2(\Omega)$. The proof is complete. □

3. Local estimates

In this section we begin with a derivation of local versions of (2.1) and (2.3). Then we proceed to prove Proposition 1.3.

Let $z = (y, \tau) \in \Omega_T$, $r > 0$ with $Q_r(z) \subset \Omega_T$ be given. Pick a C^∞ function ξ on $\mathbb{R}^{N+1}$ satisfying

$$\xi = 1 \quad \text{on } Q_{\frac{1}{2}r}(z),$$
$$\xi = 0 \quad \text{off } Q_r(z),$$
$$0 \leq \xi \leq 1 \quad \text{on } Q_r(z),$$
$$|\partial_t \xi| \leq \frac{c}{r^2},$$
$$|\nabla \xi| \leq \frac{c}{r}.$$

Note that $m \otimes m \nabla p = (m \cdot \nabla p)m$. Keep this in mind, while using $\xi^2(p - p_{y,r}(t))$ as a test function in (1.1), to obtain

$$\begin{aligned} &\int_{B_r(y)} |\nabla p|^2 \xi^2 dx + \int_{B_r(y)} (m \cdot \nabla p)^2 \xi^2 dx \\ &\leq \frac{c}{r^2} \int_{B_r(y)} |p - p_{y,r}(t)|^2 dx + \frac{c}{r^2} \int_{B_r(y)} |m|^2 |p - p_{y,r}(t)|^2 dx \\ &\quad + \int_{B_r(y)} |S(x)| \xi^2 |p - p_{y,r}(t)| dx. \end{aligned} \tag{3.1}$$

Set $M_0 = \text{ess sup}_{\Omega_T} |p(x,t)|$. Then the fourth integral in (3.1) can be estimated as follows:

$$\begin{aligned} \int_{B_r(y)} |m|^2 |p - p_{y,r}(t)|^2 dx &\leq 2 \int_{B_r(y)} |m - m_{z,r}|^2 |p - p_{y,r}(t)|^2 dx \\ &\quad + 2|m_{z,r}|^2 \int_{B_r(y)} |p - p_{y,r}(t)|^2 dx \\ &\leq 8M_0^2 \int_{B_r(y)} |m - m_{z,r}|^2 dx \\ &\quad + 2|m_{z,r}|^2 \int_{B_r(y)} |p - p_{y,r}(t)|^2 dx. \end{aligned} \tag{3.2}$$

We apply Poincaré's inequality to the last integral in (3.1) to yield

$$\begin{aligned} \int_{B_r(y)} |S(x)| \xi^2 |p - p_{y,r}(t)| dx &\leq \left(\int_{B_r(y)} |S(x)|^{\frac{2N}{N+2}} \right)^{\frac{N+2}{2N}} \\ &\quad \cdot \left(\int_{B_r(y)} |\xi(p - p_{y,r}(t))|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{2N}} \end{aligned}$$

$$\leq \left(\int_{B_r(y)} |S(x)|^{\frac{2N}{N+2}} \right)^{\frac{N+2}{2N}}$$

$$\cdot \left(\frac{c}{r^2} \int_{B_r(y)} |p - p_{y,r}(t)|^2 dx + \int_{B_r(y)} \xi^2 |\nabla p|^2 dx \right)^{\frac{1}{2}}$$

$$\leq \varepsilon \int_{B_r(y)} \xi^2 |\nabla p|^2 dx + \frac{c\varepsilon}{r^2} \int_{B_r(y)} |p - p_{y,r}(t)|^2 dx$$

$$+ c(\varepsilon) r^{N+2-\frac{2N}{q}} \tag{3.3}$$

for each $\varepsilon > 0$. Use (3.3) and (3.2) in (3.1), choose ε sufficiently small in the resulting inequality, and thereby arrive at

$$\int_{B_r(y)} |\nabla p|^2 \xi^2 dx + \int_{B_r(y)} (m \cdot \nabla p)^2 \xi^2 dx$$

$$\leq \frac{c(1 + |m_{z,r}|^2)}{r^2} \int_{B_r(y)} |p - p_{y,r}(t)|^2 dx + \frac{c}{r^2} \int_{B_r(y)} |m - m_{z,r}|^2 dx$$

$$+ c r^{N+2-\frac{2N}{q}}. \tag{3.4}$$

Now we use $(m - m_{z,r})\xi^2$ as a test function in (1.2) to obtain

$$\frac{d}{dt} \int_{B_r(y)} \frac{1}{2} |m - m_{z,r}|^2 \xi^2 dx + c \int_{B_r(y)} |\nabla m|^2 \xi^2 dx + \int_{B_r(y)} |m|^{2\gamma} \xi^2 dx$$

$$\leq \frac{c}{r^2} \int_{B_r(y)} |m - m_{z,r}|^2 dx + E^2 \int_{B_r(y)} (m \cdot \nabla p)^2 \xi^2 dx$$

$$+ c|m_{z,r}|^2 \int_{B_r(y)} |\nabla p|^2 \xi^2 dx + m_{z,r} \int_{B_r(y)} |m|^{2(\gamma-1)} m \xi^2 dx. \tag{3.5}$$

In view of the interpolation inequality ([7], p. 145), we have

$$\left| m_{z,r} \int_{B_r(y)} |m|^{2(\gamma-1)} m \xi^2 dx \right| \leq \varepsilon \int_{B_r(y)} |m|^{2\gamma} \xi^2 dx + c(\varepsilon) |m_{z,r}|^{2\gamma} r^N, \quad \varepsilon > 0. \tag{3.6}$$

Substitute (3.6) into (3.5), choose ε so small in the resulting inequality that the second integral in (3.6) can be absorbed into the third term in (3.5), then integrate with respect to t to yield

$$
\begin{aligned}
&\max_{t\in[\tau-\frac{1}{8}r^2,\tau+\frac{1}{8}r^2]}\int_{B_{\frac{r}{2}}(y)}\frac{1}{2}|m-m_{z,r}|^2dx\\
&+c\int_{Q_{\frac{r}{2}}(z)}|\nabla m|^2dxdt+\int_{Q_{\frac{r}{2}}(z)}|m|^{2\gamma}dxdt\\
&\leq c(|m_{z,r}|^2+1)\left(\int_{Q_r(z)}|\nabla p|^2\xi^2dxdt+\int_{Q_r(z)}(m\cdot\nabla p)^2\xi^2dxdt\right)\\
&+\frac{c}{r^2}\int_{Q_r(z)}|m-m_{z,r}|^2dxdt+c|m_{z,r}|^{2\gamma}r^{N+2}.
\end{aligned}
\tag{3.7}
$$

We are ready to prove Proposition 1.3.

Proof of Proposition 1.3. For each $\varepsilon>0$ we consider the set

$$
H_\varepsilon=\{z\in\Omega_T:\lim_{r\to 0}\frac{1}{r^{N+\varepsilon}}\int_{Q_r(z)}\left(|m|^d+|\partial_t m|^2+|\nabla m|^2+|\nabla p|^2+(m\cdot\nabla p)^2\right)dxdt=0\},
\tag{3.8}
$$

where $d=\frac{2N}{N-2}$ if $N\neq 2$ and any number bigger than $2+\frac{8}{N}$ if $N=2$. On account of Lemma 1.1, we have

$$
\mathcal{P}^{N+\varepsilon}(\Omega_T\setminus H_\varepsilon)=0. \tag{3.9}
$$

Thus it is enough for us to show

$$
H_\varepsilon\subset R, \tag{3.10}
$$

where R is defined in (1.24). We divide the proof of this into several claims. □

Claim 3.1. *If $z=(y,\tau)\in H_\varepsilon$, then we have*

$$
\sup_{r>0}|m_{z,r}|<\infty. \tag{3.11}
$$

Proof. We follow the argument given in ([6], p. 104). That is, we calculate

$$
\left|\frac{d}{d\rho}m_{z,\rho}\right|=\left|\frac{d}{d\rho}\fint_{Q_1(0)}m(y+\zeta\rho,\tau+\rho^2\omega)d\zeta d\omega\right|
$$

$$
\begin{aligned}
&= \left| \fint_{Q_1(0)} \left(\nabla m(y+\zeta\rho, \tau+\rho^2\omega)\zeta + \partial_\omega m(y+\zeta\rho, \tau+\rho^2\omega) 2\rho\omega \right) d\zeta d\omega \right| \\
&\le \fint_{Q_\rho(z)} |\nabla m| dxdt + 2 \fint_{Q_\rho(z)} |\partial_t m \rho| dxdt \\
&\le c \left(\frac{1}{\rho^{N+2}} \int_{Q_\rho(z)} |\nabla m|^2 dxdt \right)^{\frac{1}{2}} + c \left(\frac{1}{\rho^{N}} \int_{Q_\rho(z)} |\partial_t m|^2 dxdt \right)^{\frac{1}{2}} \\
&= \frac{1}{\rho^{1-\frac{\varepsilon}{2}}} \left[\left(\frac{1}{\rho^{N+\varepsilon}} \int_{Q_\rho(z)} |\nabla m|^2 dxdt \right)^{\frac{1}{2}} + c \left(\frac{1}{\rho^{N-2+\varepsilon}} \int_{Q_\rho(z)} |\partial_t m|^2 dxdt \right)^{\frac{1}{2}} \right] \\
&\le \frac{c}{\rho^{1-\frac{\varepsilon}{2}}}. \qquad (3.12)
\end{aligned}
$$

Here and in the remainder of the proof of Proposition 1.3 the constant c may depend on ε and z. It immediately follows that

$$
\begin{aligned}
\left| m_{z,\rho_1} - m_{z,\rho_2} \right| &\le \left| \int_{\rho_1}^{\rho_2} \left| \frac{d}{d\rho} m_{z,\rho} \right| d\rho \right| \\
&\le c \left| \rho_1^{\frac{\varepsilon}{2}} - \rho_2^{\frac{\varepsilon}{2}} \right|. \qquad (3.13)
\end{aligned}
$$

Thus the claim follows. □

Claim 3.2. *If $z \in H_\varepsilon$, then*

$$
\fint_{Q_\rho(z)} |m - m_{z,r}|^2 dxdt \le c r^\varepsilon. \qquad (3.14)
$$

Proof. Note that

$$
m_{z,r} = \frac{1}{r^2} \int_{\tau-\frac{1}{2}r^2}^{\tau+\frac{1}{2}r^2} \fint_{B_r(y)} m(x,t) dxdt = \frac{1}{r^2} \int_{\tau-\frac{1}{2}r^2}^{\tau+\frac{1}{2}r^2} m_{y,r}(t) dt.
$$

That is, $m_{z,r}$ is the average of $m_{y,r}(t)$ over $[\tau - \frac{1}{2}r^2, \tau + \frac{1}{2}r^2]$. Subsequently, we have

$$|m_{y,r}(t)-m_{z,r}| \le \int_{\tau-\frac{1}{2}r^2}^{\tau+\frac{1}{2}r^2} \left|\frac{d}{d\omega} m_{y,r}(\omega)\right| d\omega$$
$$\le r \left(\int_{\tau-\frac{1}{2}r^2}^{\tau+\frac{1}{2}r^2} \fint_{B_r(y)} |\partial_\omega m|^2 dx d\omega \right)^{\frac{1}{2}}, \tag{3.15}$$

from whence follows

$$\frac{1}{r^2}\int_{\tau-\frac{1}{2}r^2}^{\tau+\frac{1}{2}r^2} |m_{y,r}(t)-m_{z,r}|^2 dt \le \frac{c}{r^{N-2}} \int_{Q_r(z)} |\partial_t m|^2 dx dt. \tag{3.16}$$

In view of Poincaré's inequality ([4], p. 141), we have

$$\fint_{B_r(y)} |m-m_{y,r}(t)|^2 dx \le cr^2 \fint_{B_r(y)} |\nabla m|^2 dx. \tag{3.17}$$

We compute

$$\fint_{Q_\rho(z)} |m-m_{z,r}|^2 dx dt = \frac{1}{r^2}\int_{\tau-\frac{1}{2}r^2}^{\tau+\frac{1}{2}r^2} \fint_{B_r(y)} |m-m_{z,r}|^2 dx dt$$
$$\le \frac{2}{r^2}\int_{\tau-\frac{1}{2}r^2}^{\tau+\frac{1}{2}r^2} \fint_{B_r(y)} |m-m_{y,r}(t)|^2 dx dt$$
$$+\frac{2}{r^2}\int_{\tau-\frac{1}{2}r^2}^{\tau+\frac{1}{2}r^2} |m_{y,r}(t)-m_{z,r}|^2 dt$$
$$\le \frac{c}{r^N}\int_{Q_r(z)} |\nabla m|^2 dx dt + \frac{c}{r^{N-2}}\int_{Q_r(z)} |\partial_t m|^2 dx dt$$
$$\le cr^{\varepsilon}. \tag{3.18}$$

This completes the proof. □

Claim 3.3. *Let $z \in H_\varepsilon$. Then for each $\alpha \in (0, \min\{\frac{2}{N-2}, \frac{4}{N}\}]$ there is a positive number c such that*

$$\max_{t\in[\tau-\frac{1}{2}r^2,\tau+\frac{1}{2}r^2]} \fint_{B_r(y)} |m-m_{y,r}(t)|^{2+\alpha} dx \le cr^{\frac{(2\alpha+2)\varepsilon}{d}}. \tag{3.19}$$

Proof. It follows from (3.18) and (3.7) that

$$\max_{t\in[\tau-\frac{1}{2}r^2,\tau+\frac{1}{2}r^2]} \fint_{B_r(y)} |m-m_{y,r}(t)|^2 dx \le cr^{\varepsilon}. \tag{3.20}$$

Note that the corollary in ([15], p. 144) is not applicable here. We offer a direct proof. To this end, we estimate from Poincaré's inequality that

$$\begin{aligned}
&\frac{1}{r^2}\int_{\tau-\frac{1}{2}r^2}^{\tau+\frac{1}{2}r^2} \fint_{B_r(y)} |m-m_{y,r}(t)|^{\frac{4}{N}+2} dxdt \\
&\le \frac{1}{r^2}\int_{\tau-\frac{1}{2}r^2}^{\tau-\frac{1}{2}r^2} \left(\fint_{B_r(y)} |m-m_{y,r}(t)|^2 dx\right)^{\frac{2}{N}} \left(\fint_{B_r(y)} |m-m_{y,r}(t)|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}} dt \\
&\le c\left(\max_{t\in[\tau-\frac{1}{2}r^2,\tau+\frac{1}{2}r^2]} \fint_{B_r(y)} |m-m_{y,r}(t)|^2 dx\right)^{\frac{2}{N}} \\
&\quad \cdot \int_{\tau-\frac{1}{2}r^2}^{\tau-\frac{1}{2}r^2} \fint_{B_r(y)} |\nabla m|^2 dxdt \le cr^{\varepsilon+\frac{2\varepsilon}{N}}.
\end{aligned} \tag{3.21}$$

Let $\alpha \in (0, \min\{\frac{2}{N-2}, \frac{4}{N}\}]$ be given. For $t \in [\tau-\frac{1}{2}r^2, \tau+\frac{1}{2}r^2]$ set

$$f_r(t) = \fint_{B_r(y)} |m-m_{y,r}(t)|^{2+\alpha} dx.$$

Observe that

$$|m-m_{y,r}(t)|^{2+2\alpha} \le 2^{2\alpha+1}\left(|m|^{2+2\alpha} + \fint_{B_r(y)} |m|^{2+2\alpha} dx\right), \tag{3.22}$$

$$\left|\frac{d}{dt} m_{y,r}(t)\right|^2 \le \left(\fint_{B_r(y)} |\partial_t m| dx\right)^2 \le \fint_{B_r(y)} |\partial_t m|^2 dx. \tag{3.23}$$

Keeping these two inequalities in mind, we calculate that

$$\left|\frac{d}{dt}f_r(t)\right| = (2+\alpha)\left|\ \fint_{B_r(y)} |m-m_{y,r}(t)|^{\alpha}(m-m_{y,r}(t))\cdot\frac{d}{dt}(m-m_{y,r}(t))dt\right|$$
$$\le c\fint_{B_r(y)} |m-m_{y,r}(t)|^{\alpha+1}|\partial_t m - \frac{d}{dt}m_{y,r}(t)|dx$$
$$\le c\left(\ \fint_{B_r(y)} |m-m_{y,r}(t)|^{2\alpha+2}dx\right)^{\frac{1}{2}}\left(\ \fint_{B_r(y)} |\partial_t m - \frac{d}{dt}m_{y,r}(t)|^2dx\right)^{\frac{1}{2}}$$
$$\le c\left(\ \fint_{B_r(y)} |m|^{2\alpha+2}dx\right)^{\frac{1}{2}}\left(\ \fint_{B_r(y)} |\partial_t m|^2dx\right)^{\frac{1}{2}}. \tag{3.24}$$

Note that $2+2\alpha \le d$, where d is given in (3.8). We estimate

$$\max_{t\in[\tau-\frac{1}{2}r^2,\tau+\frac{1}{2}r^2]} f_r(t) \le \max_{t\in[\tau-\frac{1}{2}r^2,\tau+\frac{1}{2}r^2]}\left| f_r(t) - \frac{1}{r^2}\int_{\tau-\frac{1}{2}r^2}^{\tau+\frac{1}{2}r^2} f_r(\omega)d\omega\right|$$
$$+\frac{1}{r^2}\int_{\tau-\frac{1}{2}r^2}^{\tau+\frac{1}{2}r^2} f_r(\omega)d\omega$$
$$\le \int_{\tau-\frac{1}{2}r^2}^{\tau+\frac{1}{2}r^2}\left|\frac{d}{dt}f_r(t)\right|dt + \frac{1}{r^2}\int_{\tau-\frac{1}{2}r^2}^{\tau+\frac{1}{2}r^2} f_r(t)dt$$
$$\le c\left(\frac{1}{r^N}\int_{Q_r(z)} |m|^{2\alpha+2}dx\right)^{\frac{1}{2}}\left(\frac{1}{r^N}\int_{Q_r(z)} |\partial_t m|^2dx\right)^{\frac{1}{2}}$$
$$+\frac{1}{r^2}\int_{\tau-\frac{1}{2}r^2}^{\tau+\frac{1}{2}r^2}\fint_{B_r(y)} |m-m_{y,r}(t)|^{\alpha+2}dxdt$$
$$\le c\left(\frac{1}{r^N}\int_{Q_r(z)} |m|^{d}dx\right)^{\frac{2\alpha+2}{2d}}\left(\frac{1}{r^N}\int_{Q_r(z)} |\partial_t m|^2dx\right)^{\frac{1}{2}}$$

$$+\left(\frac{1}{r^2}\int_{\tau-\frac{1}{2}r^2}^{\tau+\frac{1}{2}r^2}\fint_{B_r(y)}|m-m_{y,r}(t)|^{\frac{4}{N}+2}dxdt\right)^{\frac{2+\alpha}{2+\frac{4}{N}}}$$

$$\le cr^{\frac{(2\alpha+2)\varepsilon}{d}}+cr^{\varepsilon(1+\frac{\alpha}{2})}\le cr^{\frac{(2\alpha+2)\varepsilon}{d}}. \tag{3.25}$$

The proof is complete. □

Claim 3.4. *If $z\in H_\varepsilon$, then there is $\varepsilon_1>0$ such that*

$$A_r(z)\le cr^{\varepsilon_1}. \tag{3.26}$$

Obviously, this claim implies (3.10).

Proof. Let $z=(y,\tau)\in H_\varepsilon$ be given. Fix $r>0$ with $Q_r(z)\subset\Omega_T$. Set

$$w_r=m-m_{y,r}(t). \tag{3.27}$$

Note that

$$\begin{aligned}m\otimes m&=(m-m_{y,r}(t))\otimes m+m_{y,r}(t)\otimes(m-m_{y,r}(t))\\&\quad+m_{y,r}(t)\otimes m_{y,r}(t)\\&=w_r\otimes m+m_{y,r}(t)\otimes w_r+m_{y,r}(t)\otimes m_{y,r}(t).\end{aligned}$$

Thus p satisfies the system

$$\begin{aligned}&-\mathrm{div}[(I+m_{y,r}(t)\otimes m_{y,r}(t))\nabla p]\\&\quad=\mathrm{div}\,[(m\cdot\nabla p)w_r]+\mathrm{div}\left[(w_r\cdot\nabla p)m_{y,r}(t)\right]+S(x)\quad\text{in } Q_r(z).\end{aligned} \tag{3.28}$$

Here we have used the fact that $(w_r\otimes m)\nabla p=(m\cdot\nabla p)w_r$. We decompose p into $\eta+\phi$ on $Q_r(z)$ as follows: η is the solution of the problem

$$-\mathrm{div}\left[(I+m_{y,r}(t)\otimes m_{y,r}(t))\nabla\eta\right]=0$$
$$\text{in } B_r(y),\ t\in[\tau-\frac{1}{2}r^2,\tau+\frac{1}{2}r^2], \tag{3.29}$$

$$\eta=p\quad\text{on }\partial B_r(y),\ \ t\in[\tau-\frac{1}{2}r^2,\tau+\frac{1}{2}r^2], \tag{3.30}$$

while ϕ is the solution of the problem

$$-\mathrm{div}\left[(I+m_{y,r}(t)\otimes m_{y,r}(t))\nabla\phi\right]=\mathrm{div}\,[(m\cdot\nabla p)w_r]+\mathrm{div}\left[(w_r\cdot\nabla p)m_{y,r}(t)\right]$$
$$+S(x)\ \text{ in } B_r(y),\ t\in[\tau-\frac{1}{2}r^2,\tau+\frac{1}{2}r^2], \tag{3.31}$$

$$\phi=0\quad\text{on }\partial B_r(y),\ t\in[\tau-\frac{1}{2}r^2],\tau+\frac{1}{2}r^2 \tag{3.32}$$

Recall from (3.15) that

$$
\begin{aligned}
|m_{y,r}(t)| &\le |m_{y,r}(t) - m_{z,r}| + |m_{z,r}| \\
&\le cr\left(\frac{1}{r^N}\int_{Q_r(z)} |\partial_t m|^2 dxdt\right)^{\frac{1}{2}} + |m_{z,r}|.
\end{aligned} \tag{3.33}
$$

By Theorem 2.1 in ([6], p. 78), there is a positive number c depending only on $\sup_{r>0}|m_{y,r}(t)|$ such that

$$
\fint_{B_\rho(y)} |\eta - \eta_{y,\rho}|^2 dx \le c\left(\frac{\rho}{R}\right)^2 \fint_{B_R(y)} |\eta - \eta_{y,R}|^2 dx \tag{3.34}
$$

for all $0 < \rho \le R \le r$ and $t \in [\tau - \frac{1}{2}r^2, \tau + \frac{1}{2}r^2]$. On the other hand, another classical regularity result [13] for linear elliptic equations with continuous coefficients asserts that for each $s \in (1, \infty)$ there is a positive number c with the property

$$
\begin{aligned}
\|\nabla\phi\|_s &\le c\|(m\cdot\nabla p)w_r\|_s + c\|(w_r\cdot\nabla p)m_{y,r}(t)\|_s \\
&\quad + c\|S(x)\|_{\frac{sN}{s+N}}, \quad t \in [\tau - \frac{1}{2}r^2, \tau + \frac{1}{2}r^2].
\end{aligned} \tag{3.35}
$$

Note that the constant c here is also independent of r. We remark that in general the above inequality is not true for $s = 1$. This is why Claim 3.3 is crucial to our development. Obviously, if we replace $m_{z,r}$ by $m_{y,r}(t)$ in (3.4), the resulting inequality still holds. This implies

$$
\max_{t\in[\tau-\frac{1}{2}r^2,\tau+\frac{1}{2}r^2]} \frac{1}{r^{N-2}} \int_{B_r(y)} \left(|\nabla p|^2 + (m\cdot\nabla p)^2\right) dx \le c. \tag{3.36}
$$

We can easily find a $s \in (1, 2)$ so that

$$
\frac{2s}{2-s} = 2 + \frac{4(s-1)}{2-s} \le 2 + \min\{\frac{2}{N-2}, \frac{4}{N}\}. \tag{3.37}
$$

We estimate

$$
\begin{aligned}
&\max_{t\in[\tau-\frac{1}{2}r^2,\tau+\frac{1}{2}r^2]} \frac{1}{r^{N-s}} \int_{B_r(y)} |(m\cdot\nabla p)w_r|^s dx \\
&\le \left(\max_{t\in[\tau-\frac{1}{2}r^2,\tau+\frac{1}{2}r^2]} \frac{1}{r^{N-2}} \int_{B_r(y)} (m\cdot\nabla p)^2 dx\right)^{\frac{s}{2}}
\end{aligned}
$$

$$
\cdot\left(\max_{t\in[\tau-\frac{1}{2}r^2,\tau+\frac{1}{2}r^2]}\frac{1}{r^N}\int_{B_r(y)}|w_r|^{\frac{2s}{2-s}}dx\right)^{\frac{2-s}{2}}
$$

$$
\leq c\left(\max_{t\in[\tau-\frac{1}{2}r^2,\tau+\frac{1}{2}r^2]}\frac{1}{r^N}\int_{B_r(y)}|w_r|^{\frac{2s}{2-s}}dx\right)^{\frac{2-s}{2}}
$$

$$
\leq cr^{\frac{(2-s)(2\alpha+2)\varepsilon}{2d}}, \tag{3.38}
$$

where $\alpha=\frac{4(s-1)}{2-s}$. Similarly, we have

$$
\max_{t\in[\tau-\frac{1}{2}r^2,\tau+\frac{1}{2}r^2]}\frac{1}{r^{N-s}}\int_{B_r(y)}|(w_r\cdot\nabla p)m_{y,r}(t)|^s dx
$$

$$
\leq\left(\max_{t\in[\tau-\frac{1}{2}r^2,\tau+\frac{1}{2}r^2]}\frac{1}{r^{N-2}}\int_{B_r(y)}|\nabla p|^2 dx\right)^{\frac{s}{2}}
$$

$$
\cdot\left(\max_{t\in[\tau-\frac{1}{2}r^2,\tau+\frac{1}{2}r^2]}\frac{1}{r^N}\int_{B_r(y)}|w_r|^{\frac{2s}{2-s}}dx\right)^{\frac{2-s}{2}}
$$

$$
\leq c\left(\max_{t\in[\tau-\frac{1}{2}r^2,\tau+\frac{1}{2}r^2]}\frac{1}{r^N}\int_{B_r(y)}|w_r|^{\frac{2s}{2-s}}dx\right)^{\frac{2-s}{2}}
$$

$$
\leq cr^{\frac{(2-s)(2\alpha+2)\varepsilon}{2d}}, \tag{3.39}
$$

$$
\frac{1}{r^{N-s}}\left(\int_{B_r(y)}|S(x)|^{\frac{Ns}{N+s}}dx\right)^{\frac{s+N}{N}}
$$

$$
\leq\frac{1}{r^{N-s}}\left(\int_{B_r(y)}|S(x)|^q dx\right)^{\frac{s}{q}}r^{s+N-\frac{Ns}{q}}\leq cr^{s(2-\frac{N}{q})}. \tag{3.40}
$$

To summarize, we have

$$
\max_{t\in[\tau-\frac{1}{2}r^2,\tau+\frac{1}{2}r^2]}\frac{1}{r^{N-s}}\int_{B_r(y)}|\nabla\phi|^s dx\leq cr^{\min\{\frac{(2-s)(2\alpha+2)\varepsilon}{2d},\ s(2-\frac{N}{q})\}}. \tag{3.41}
$$

It follows from Poincaré's inequality that

$$\left(\fint_{B_r(y)} |\phi - \phi_{y,r}(t)|^{\frac{Ns}{N-s}} dx \right)^{\frac{N-s}{Ns}} \leq cr \left(\fint_{B_r(y)} |\nabla\phi|^s dx \right)^{\frac{1}{s}} = c \left(\frac{1}{r^{N-s}} \int_{B_r(y)} |\nabla\phi|^s dx \right)^{\frac{1}{s}}. \tag{3.42}$$

Remember that $\|\phi\|_\infty \leq \|\eta\|_\infty + \|p\|_\infty \leq 2\|p\|_\infty$. Hence we can always find a positive number $\varepsilon_1 \in (0, 2)$ so that

$$\max_{t \in [\tau - \frac{1}{2}r^2, \tau + \frac{1}{2}r^2]} \fint_{B_r(y)} |\phi - \phi_{y,r}(t)|^2 dx \leq cr^{\varepsilon_1}. \tag{3.43}$$

For $0 < \rho \leq r$ we derive from (3.34) and (3.43) that

$$\begin{aligned} &\int_{B_\rho(y)} |p - p_{y,\rho}(t)|^2 dx \\ &\leq 2 \int_{B_\rho(y)} |\eta - \eta_{y,\rho}(t)|^2 dx + 2 \int_{B_\rho(y)} |\phi - \phi_{y,\rho}(t)|^2 dx \\ &\leq c \left(\frac{\rho}{r}\right)^{N+2} \int_{B_r(y)} |\eta - \eta_{y,r}(t)|^2 dx + 2 \int_{B_r(y)} |\phi - \phi_{y,r}(t)|^2 dx \\ &\leq c \left(\frac{\rho}{r}\right)^{N+2} \int_{B_r(y)} |p - p_{y,r}(t)|^2 dx + cr^{N+\varepsilon_1}. \end{aligned} \tag{3.44}$$

Here we have used the fact that $\int_{B_\rho(y)} |\phi - \phi_{y,\rho}(t)|^2 dx$ is an increasing function of ρ. We set

$$\sigma(r) = \max_{t \in [\tau - \frac{1}{2}r^2, \tau + \frac{1}{2}r^2]} \int_{B_r(y)} |p - p_{y,r}(t)|^2 dx.$$

We easily infer from (3.44) that

$$\sigma(\rho) \leq c \left(\frac{\rho}{r}\right)^{N+2} \sigma(r) + cr^{N+\varepsilon_1} \tag{3.45}$$

for all $0 < \rho \leq r$. This puts us in a position to apply Lemma 2.1 in ([6], p. 86), from whence follows

$$\sigma(\rho) \leq c \left(\frac{\rho}{r}\right)^{N+\varepsilon_1} \sigma(r) + c\rho^{N+\varepsilon_1} \tag{3.46}$$

for all $0 < \rho \leq r$. This gives the claim. □

4. Proof of Proposition 1.2

In this section we present the proof of Proposition 1.2. We would like to remark that the proof of this proposition is more challenging than that of Proposition 1.3 mainly because we do not have a local estimate for $\partial_t m$ or a local L^∞ estimate for p. This also causes us to impose the restriction $N \le 3$. Note that this restriction is not needed in Propositions 1.1 and 1.3.

Proof of Proposition 1.2. We argue by contradiction. Suppose that the proposition is false. Then for some $M > 0$ (1.15) and (1.16) fail to hold no matter how we pick numbers ε, δ from the interval $(0, 1)$. In particular, we can choose a sequence $\{\varepsilon_k\} \subset (0, 1)$ with the property

$$\varepsilon_k \to 0 \quad \text{as } k \to 0. \tag{4.1}$$

The selection of δ from $(0, 1)$ is more delicate, and it will be made clear later. Let δ be chosen as below. For each k there exist cylinders $Q_{r_k}(z_k) \subset \Omega_T$ such that

$$|m_{z_k,r_k}| \le M \quad \text{and} \quad E_{r_k}(z_k) \le \varepsilon_k, \tag{4.2}$$

whereas

$$E_{\delta r_k}(z_k) > \frac{1}{2} E_{r_k}(z_k), \quad k = 1, \cdots. \tag{4.3}$$

Set

$$\lambda_k^2 = E_{r_k}(z_k).$$

Then (4.1) asserts

$$\lambda_k \to 0 \quad \text{as } k \to \infty.$$

We rescale our variables to the unit cylinder $Q_1(0)$, as follows. If $z = (y, \tau) \in Q_1(0)$, write

$$\psi_k(y, \tau) = \frac{p(y_k + r_k y, \tau_k + r_k^2 \tau) - p_{y_k,r_k}(\tau_k + r_k^2 \tau)}{\lambda_k}, \tag{4.4}$$

$$n_k(y, \tau) = m(y_k + r_k y, \tau_k + r_k^2 \tau), \tag{4.5}$$

$$w_k(y, \tau) = \frac{n_k(y, \tau) - m_{z_k,r_k}}{\lambda_k}. \tag{4.6}$$

We can easily verify

$$\max_{\tau \in [-\frac{1}{2}, \frac{1}{2}]} \int_{B_1(0)} \psi_k^2(y, \tau) dy = \frac{1}{\lambda_k^2} A_{r_k}(z_k) \le 1,$$

$$\int_{Q_1(0)} |w_k(y, \tau)|^2 dy d\tau = \frac{1}{\lambda_k^2 r_k^{N+2}} \int_{Q_{r_k}(z_k)} |m(x, t) - m_{z_k,r_k}|^2 dx dt \le 1,$$

but

$$\frac{1}{\delta^{N+2}}\int_{Q_\delta(0)}|w_k-(w_k)_{0,\delta}|^2dyd\tau+\frac{1}{\delta^N}\max_{\tau\in[-\frac{1}{2}\delta^2,\frac{1}{2}\delta^2]}\int_{B_\delta(0)}|\psi_k-(\psi_k)_{0,\delta}(\tau)|^2dy$$
$$+\frac{\delta^{2\beta}r_k^{2\beta}}{\lambda_k^2}>\frac{1}{2}. \tag{4.7}$$

Here and in what follows we suppress the dependence of ψ_k, w_k, n_k on (y,τ) for simplicity of notation. Our plan is to show that the lim sup of the left-hand side of the above inequality as $k\to\infty$ can be made smaller than $\frac{1}{2}$ if we adjust δ to be small enough, and thus the desired contradiction follows.

We easily see from the definition of λ_k that

$$\frac{\delta^{2\beta}r_k^{2\beta}}{\lambda_k^2}\le\delta^{2\beta}. \tag{4.8}$$

To analyze the first two terms in (4.7), we first conclude from the proof in [3] that $\psi_k(y,\tau)$, $w_k(y,\tau)$ satisfy the system

$$-\Delta\psi_k-\operatorname{div}[(n_k\cdot\nabla\psi_k)n_k]=\frac{r_k^2}{\lambda_k}S(y_k+r_ky)\equiv F_k(y)\quad\text{in } Q_1(0), \tag{4.9}$$

$$\partial_t w_k-D^2\Delta w_k-E^2\lambda_k(n_k\cdot\nabla\psi_k)\nabla\psi_k+\frac{r_k^2}{\lambda_k}|n_k|^{2(\gamma-1)}n_k=0\quad\text{in } Q_1(0). \tag{4.10}$$

We can infer from (3.4) that

$$\int_{Q_{\frac{1}{2}}(0)}|\nabla\psi_k|^2dyd\tau+\int_{Q_{\frac{1}{2}}(0)}|n_k\cdot\nabla\psi_k|^2dyd\tau\le c. \tag{4.11}$$

Similarly, we can derive from (3.7) that

$$\max_{\tau\in[-\frac{1}{8},\frac{1}{8}]}\int_{B_{\frac{1}{2}}(0)}|w_k|^2dy+\int_{Q_{\frac{1}{2}}(0)}|\nabla w_k|^2dyd\tau+\frac{r_k^2}{\lambda_k^2}\int_{Q_{\frac{1}{2}}(0)}|n_k|^{2\gamma}dyd\tau\le c+c\frac{r_k^2}{\lambda_k^2}\le c. \tag{4.12}$$

Consequently, we have

$$\int_{Q_{\frac{1}{2}}(0)}\left|\frac{r_k^2}{\lambda_k}|n_k|^{2\gamma-1}\right|^{\frac{2\gamma}{2\gamma-1}}dyd\tau=\lambda_k^{\frac{2\gamma}{2\gamma-1}}\left(\frac{r_k^2}{\lambda_k^2}\right)^{\frac{1}{2\gamma-1}}\frac{r_k^2}{\lambda_k^2}\int_{Q_{\frac{1}{2}}(0)}|n_k|^{2\gamma}dyd\tau$$
$$\to 0\ \text{ as } k\to 0. \tag{4.13}$$

This together with (4.10), (4.11), and (4.12) implies that the sequence $\{\partial_\tau w_k\}$ is bounded in $L^2(-\frac{1}{8},\frac{1}{8};W^{-1,2}(B_{\frac{1}{2}}(0)))+L^1(Q_{\frac{1}{2}}(0))$. By a well-known result in [14], w_k is precompact in $L^2(Q_{\frac{1}{2}}(0))$. Passing to subsequences if necessary, we have

$$m_{z_k,r_k}\to a, \tag{4.14}$$

$$n_k=\lambda_k w_k+m_{z_k,r_k}\to a \quad \text{strongly in } L^2(Q_1(0)), \tag{4.15}$$

$$w_k\to w \quad \text{strongly in } L^2(Q_{\frac{1}{2}}(0))$$
$$\text{and weakly in } L^2(-\tfrac{1}{8},\tfrac{1}{8};W^{1,2}(B_{\frac{1}{2}}(0))), \tag{4.16}$$

$$\psi_k\to\psi \quad \text{and weakly in } L^2(-\tfrac{1}{8},\tfrac{1}{8};W^{1,2}(B_{\frac{1}{2}}(0))). \tag{4.17}$$

In view of (4.11) and (4.13), we can send k to infinity in (4.10) to obtain

$$\partial_\tau w-D^2\Delta w=0 \ \text{ in } Q_{\frac{1}{2}}(0) \tag{4.18}$$

in the weak, and therefore classical sense. It follows from (4.15) and (4.17) that

$$n_k\nabla\psi_k\rightharpoonup a\nabla\psi \quad \text{weakly in } L^1(Q_{\frac{1}{2}}(0)), \tag{4.19}$$

and therefore weakly in $L^2(Q_{\frac{1}{2}}(0))$ due to (4.11). This, in turns, implies

$$(n_k\nabla\psi_k)n_k\rightharpoonup a\nabla\psi a \quad \text{weakly in } L^1(Q_{\frac{1}{2}}(0)). \tag{4.20}$$

We estimate the last term in (4.9) as follows

$$\begin{aligned}\int_{B_1(0)}|F_k|^q dy&=\frac{r_k^{2q}}{\lambda_k^q}\int_{B_1(0)}|S(y_k+r_k y)|^q dy\\&=\frac{r_k^{2q-N}}{\lambda_k^q}\int_{B_{r_k}(y_k)}|S(x)|^q dx\\&\le c\frac{r_k^{\beta q}}{\lambda_k^q}r_k^{q(2-\frac{N}{q}-\beta)}\le cr_k^{q(2-\frac{N}{q}-\beta)}\to 0.\end{aligned} \tag{4.21}$$

The last step is due to (1.9). We are ready to let k go to infinity in (4.9), thereby obtaining

$$-\mathrm{div}\,[(I+a\otimes a)\nabla\psi]=0 \ \text{ in } Q_{\frac{1}{2}}(0). \tag{4.22}$$

Remember that a is a constant vector. By the classical regularity theory for linear elliptic equations, there exist $c>0,\alpha\in(0,1)$ determined only by M and N with the property

$$\max_{\tau\in[-\frac{1}{2}]\delta^2,\frac{1}{2}\delta^2]} \fint_{B_\delta(0)} |\psi-\psi_{0,\delta}(\tau)|^2 dy \le \max_{\tau\in[-\frac{1}{2}]\delta^2,\frac{1}{2}\delta^2]} c\delta^{2\alpha} \fint_{B_{\frac{1}{2}}(0)} |\psi-\psi_{0,\frac{1}{2}}(\tau)|^2 dy \le c\delta^{2\alpha} \tag{4.23}$$

for all $\delta \le \frac{1}{4}$. Subsequently,

$$\frac{1}{\delta^N} \max_{\tau\in[-\frac{1}{2}]\delta^2,\frac{1}{2}\delta^2]} \int_{B_\delta(0)} |\psi-\psi_{0,\delta}(\tau)|^2 dy \le c\delta^{2\alpha} \tag{4.24}$$

for all $0<\delta\le\frac{1}{4}$. It is also well-known (see, e.g., Claim 1 in [17]) that there exist $c>0$, $\alpha\in(0,1)$ determined only by N, D such that

$$\fint_{Q_\delta(0)} |w-w_{0,\delta}|^2 dy d\tau \le c\delta^{2\alpha} \fint_{Q_{\frac{1}{2}}(0)} |w-w_{0,\frac{1}{2}}|^2 dy d\tau \le c\delta^{2\alpha} \tag{4.25}$$

for all $0<\delta\le\frac{1}{4}$.

If we could pass to the limit in (4.7), this would result in the desired contradiction. What prevents us from doing so is the lack of compactness of the sequence $\{\psi_k\}$ in the t-variable. To circumvent this problem, we fix a suitably small number $\frac{1}{16}\ge\delta_0>0$ and consider the decomposition $\psi_k=\eta_k+\phi_k$ on $Q_{\delta_0}(0)$, where η_k is the solution of the problem

$$-\mathrm{div}\left[(I+m_{z_k,r_k}\otimes m_{z_k,r_k})\nabla\eta_k\right]=0 \quad \text{in } B_{\delta_0}(0),\ \tau\in[-\frac{1}{2}\delta_0^2,\frac{1}{2}\delta_0^2], \tag{4.26}$$

$$\eta_k=\psi_k \quad \text{on } \partial B_{\delta_0}(0),\quad \tau\in[-\frac{1}{2}\delta_0^2,\frac{1}{2}\delta_0^2], \tag{4.27}$$

while ϕ_k is the solution of the problem

$$-\mathrm{div}\left[(I+m_{z_k,r_k}\otimes m_{z_k,r_k})\nabla\phi_k\right]=\lambda_k\mathrm{div}((n_k\cdot\nabla\psi_k)w_k)+\lambda_k\mathrm{div}((w_k\cdot\nabla\psi_k)m_{z_k,r_k}) + F_k \quad \text{in } B_{\delta_0}(0),\ \tau\in[-\frac{1}{2}\delta_0^2,\frac{1}{2}\delta_0^2], \tag{4.28}$$

$$\phi_k=0 \quad \text{on } \partial B_{\delta_0}(0),\ \tau\in[-\frac{1}{2}\delta_0^2,\frac{1}{2}\delta_0^2]. \tag{4.29}$$

We will show that $\{\phi_k\}$ is precompact in $L^\infty(-\frac{1}{2}\delta_0^2,\frac{1}{2}\delta_0^2;L^2(B_{\delta_0}(0)))$, and this is enough for our purpose in spite of the fact that $\{\eta_k\}$ may not be precompact in the preceding function space. To see this, we first infer from (3.4) that

$$\max_{\tau\in[-\frac{1}{32},\frac{1}{32}]}\left(\int_{B_{\frac{1}{4}}(0)}|\nabla\psi_k|^2dy+\int_{B_{\frac{1}{4}}(0)}(n_k\cdot\nabla\psi_k)^2dy\right)\le c+\max_{\tau\in[-\frac{1}{8},\frac{1}{8}]}\int_{B_{\frac{1}{2}}(0)}|w_k|^2dy\le c. \tag{4.30}$$

Using $\eta_k-\psi_k$ as a test function in (4.26) yields

$$\max_{\tau\in[-\frac{1}{2}\delta_0^2,\frac{1}{2}\delta_0^2]}\int_{B_{\delta_0}(0)}|\nabla\eta_k|^2dy \le \max_{\tau\in[-\frac{1}{2}\delta_0^2,\frac{1}{2}\delta_0^2]} c\int_{B_{\delta_0}(0)}|\nabla\psi_k|^2dy \le c. \tag{4.31}$$

Note that (4.26) is an uniformly elliptic equation with constant coefficients. The classical regularity theory asserts that there exist $c>0, \alpha\in(0,1)$ depending only on M, N such that

$$\begin{aligned}\frac{1}{\delta^N}\int_{B_\delta(0)}|\eta_k-(\eta_k)_{0,\delta}(\tau)|^2dy &\le c\delta^{2\alpha}\fint_{B_{\delta_0}(0)}|\eta_k-(\eta_k)_{0,\delta_0}(\tau)|^2dy\\ &\le c\delta^{2\alpha}\delta_0^2\fint_{B_{\delta_0}(0)}|\nabla\eta_k|^2dy\le c\delta^{2\alpha}\end{aligned} \tag{4.32}$$

for all $\delta\le\frac{1}{2}\delta_0$.

Now we turn our attention to the sequence $\{\phi_k\}$. We wish to show

$$\phi_k\to 0 \quad \text{strongly in } L^\infty(-\tfrac{1}{2}\delta_0^2,\tfrac{1}{2}\delta_0^2;L^2(B_{\delta_0}(0))). \tag{4.33}$$

This is where the subtlety of our analysis lies. We observe from (4.30) that

$$\max_{\tau\in[-\frac{1}{32},\frac{1}{32}]}\int_{B_{\frac{1}{4}}(0)}|\psi_k|^{\frac{2N}{N-2}}dy\le c. \tag{4.34}$$

In view of (4.31), $\{\phi_k\}$ also satisfies the above estimate. By the interpolation inequality ([6], p. 146)

$$\|\phi_k(\cdot,\tau)\|_2\le\varepsilon\|\phi_k(\cdot,\tau)\|_{\frac{2N}{N-2}}+c(\varepsilon)\|\phi_k(\cdot,\tau)\|_1,\quad \varepsilon>0, \tag{4.35}$$

it is sufficient for us to show

$$\max_{\tau\in[-\frac{1}{2}\delta_0^2,\frac{1}{2}\delta_0^2]}\int_{B_{\delta_0}(0)}|\phi_k(y,\tau)|dy\to 0 \ \text{ as } k\to\infty. \tag{4.36}$$

Note that the elliptic coefficients in (4.28) are constants. This puts us in a position to invoke the classical $W^{1,s}$ estimate for ϕ_k. That is, for each $s\in(1,\infty)$ there is a positive number c with the property

$$\|\nabla\phi_k\|_s\le c\lambda_k\|(n_k\cdot\nabla\psi_k)w_k\|_s+c\lambda_k\|((w_k\cdot\nabla\psi_k)m_{z_k,r_k}\|_s+c\|F_k\|_{\frac{sN}{s+N}}. \tag{4.37}$$

Remember that (4.37) does not hold for $s=1$. To find a $s>1$, we will show that there is a $\beta>0$ such that

$$\max_{\tau\in[-\frac{1}{2}\delta_0^2,\frac{1}{2}\delta_0^2]}\int_{B_{\delta_0}(0)}|w_k(y,\tau)|^{2(1+\beta)}dy\le c. \tag{4.38}$$

Obviously, this will imply that

$$\max_{\tau\in[-\frac{1}{2}\delta_0^2,\frac{1}{2}\delta_0^2]}\left(\|(n_k\cdot\nabla\psi_k)w_k\|_s+\|(w_k\cdot\nabla\psi_k)m_{z_k,r_k}\|_s\right)\le c \tag{4.39}$$

for some $s>1$. Consequently, the right-hand side of (4.37) goes to 0 as $k\to\infty$. To establish (4.38), we will develop a suitable local version of (C2) in Proposition 2.1. This effort is complicated by the fact that a local version of (C1) in the proposition is not available. The remaining part of this section will be dedicated to the proof of (4.38), which will be divided into two claims.

Claim 4.1. *We have:*

$$\int_{Q_{\frac{1}{8}}(0)}|\psi_k\nabla\psi_k|^2dyd\tau+\int_{Q_{\frac{1}{8}}(0)}(n_k\cdot\nabla\psi_k)^2|\psi_k|^2dyd\tau\le c. \tag{4.40}$$

Proof. Let ξ be a C^∞ function on $\mathbb{R}^N\times\mathbb{R}$ with the properties

$$\xi=0 \quad \text{outside } Q_1(0), \text{ and} \tag{4.41}$$

$$\xi\in[0,1] \quad \text{in } Q_1(0). \tag{4.42}$$

Upon using $\psi_k^3\xi^2$ as a test function in (4.9), we deduce

$$\begin{aligned}&\int_{B_1(0)}|\psi_k\nabla\psi_k|^2\xi^2dy+\int_{B_1(0)}(n_k\cdot\nabla\psi_k)^2\psi_k^2\xi^2dy\\&\le c\int_{B_1(0)}\psi_k^4|\nabla\xi|^2dy+c\int_{B_1(0)}|n_k|^2\psi_k^4|\nabla\xi|^2dy+\int_{B_1(0)}|F_k||\psi_k|^3\xi^2dy.\end{aligned} \tag{4.43}$$

Observe that

$$|\lambda_k\psi_k|\le c. \tag{4.44}$$

Subsequently, we have

$$\begin{aligned}|n_k|^2\psi_k^4&=|\lambda_kw_k+m_{z_k,r_k}|^2\psi_k^4\\&\le 2\lambda_k^2\psi_k^4|w_k|^2+c\psi_k^4\\&\le c\psi_k^2|w_k|^2+c\psi_k^4\\&\le c\psi_k^{\frac{2N}{N-2}}+c|w_k|^N+c\psi_k^4.\end{aligned} \tag{4.45}$$

We estimate from (4.12) and the Sobolev Embedding Theorem that

$$\begin{aligned}\int_{Q_{\frac{1}{2}}(0)} |w_k|^{2+\frac{4}{N}} dy d\tau &\leq \int_{-\frac{1}{8}}^{\frac{1}{8}} \left(\int_{B_{\frac{1}{2}}(0)} |w_k|^2 dy \right)^{\frac{2}{N}} \left(\int_{B_{\frac{1}{2}}(0)} |w_k|^{\frac{2N}{N-2}} dy \right)^{\frac{N-2}{N}} d\tau \\ &\leq c \left(\max_{\tau \in (-\frac{1}{8}, \frac{1}{8})} \int_{B_{\frac{1}{2}}(0)} |w_k|^2 dy \right)^{\frac{2}{N}} \\ &\quad \cdot \left(\int_{Q_{\frac{1}{2}}(0)} |\nabla w_k|^2 dy d\tau + \int_{Q_{\frac{1}{2}}(0)} |w_k|^2 dy d\tau \right) \\ &\leq c. \end{aligned} \tag{4.46}$$

Our assumption on the space dimension N implies

$$N \leq 2 + \frac{4}{N}, \quad \frac{2N}{N-2} > 4.$$

By virtue of (4.30), we obtain

$$\begin{aligned}\int_{B_{\frac{1}{4}}(0)} \psi_k^4 dy &\leq c \left(\int_{B_{\frac{1}{4}}(0)} \psi_k^{\frac{2N}{N-2}} dy \right)^{\frac{2(N-2)}{N}} \\ &\leq c \left(\int_{B_{\frac{1}{4}}(0)} |\nabla \psi_k|^2 + \int_{B_{\frac{1}{4}}(0)} |\psi_k|^2 dy \right)^2 \\ &\leq c \quad \text{for each } \tau \in [-\tfrac{1}{32}, \tfrac{1}{32}]. \end{aligned} \tag{4.47}$$

We finally arrive at

$$\int_{Q_{\frac{1}{4}}(0)} |n_k|^2 \psi_k^4 dy d\tau \leq c. \tag{4.48}$$

Recall that $q > \frac{N}{2}$. Then we have $\frac{2Nq}{(N+2)q-2N} \leq \frac{2N}{N-2}$. Keeping this in mind, we calculate from (4.21) that

$$\begin{aligned}\|F_k\psi_k\xi\|^2_{\frac{2N}{N+2}} &\le \|F_k\|^2_{q,B_1(0)}\|\psi_k\xi\|^2_{\frac{2Nq}{(N+2)q-2N}}\\ &\le c\|\psi_k\xi\|^2_{\frac{2N}{N-2}}\\ &\le c\int_{B_1(0)}|\nabla\psi_k|^2\xi^2dy + c\int_{B_1(0)}|\psi_k|^2|\nabla\xi|^2dy. \qquad (4.49)\end{aligned}$$

The last term in (4.43) can be estimated as follows

$$\begin{aligned}\int_{B_1(0)}|F_k||\psi_k|^3\xi^2dy &\le \|F_k\psi_k\xi\|_{\frac{2N}{N+2}}\|\psi_k^2\xi\|_{\frac{2N}{N-2}}\\ &\le c\|F_k\psi_k\xi\|_{\frac{2N}{N+2}}\|\nabla(\psi_k^2\xi)\|_2\\ &\le \delta\|\nabla(\psi_k^2\xi)\|_2^2 + c(\delta)\|F_k\psi_k\xi\|^2_{\frac{2N}{N+2}}, \quad \delta>0. \qquad (4.50)\end{aligned}$$

Substituting this and (4.45) into (4.43) and choosing δ suitably small in the resulting inequality yield

$$\begin{aligned}&\int_{B_1(0)}|\psi_k\nabla\psi_k|^2\xi^2dy + \int_{B_1(0)}(n_k\cdot\nabla\psi_k)^2\psi_k^2\xi^2dy\\ &\le c\int_{B_1(0)}\psi_k^4|\nabla\xi|^2dy + c\int_{B_1(0)}|n_k|^2w_k^4|\nabla\xi|^2dy + c\|F_k\psi_k\xi\|^2_{\frac{2N}{N+2}}. \qquad (4.51)\end{aligned}$$

Integrate this inequality over $[-\frac{1}{128},\frac{1}{128}]$, then choose ξ suitably, i.e., $\xi=1$ on $Q_{\frac{1}{8}}(0)$ and 0 outside $Q_{\frac{1}{4}}(0)$, and thereby obtain the claim. □

Fix $K>0$. Define

$$v_k = \left(|w_k|^2 - K^2\right)^+ + K^2.$$

Claim 4.2. *There is a $\beta>0$ such that*

$$\max_{\tau\in[-\frac{1}{512},\frac{1}{512}]}\int_{B_{\frac{1}{16}}(0)}\int_0^{|w_k|^2}[(s-K^2)^+ + K^2]^\beta dsdy \le c. \qquad (4.52)$$

Obviously, this claim implies (4.38).

Proof. Let ξ be given as in (4.41)–(4.42) and $\beta>0$. We may assume that $w_k\in L^\infty(\Omega_T)$ for each k. (Otherwise, we use the cut-off function in (2.13).) Then the function $v_k^\beta w_k\xi^2$ is a legitimate test function for (4.10). Upon using it, we derive

$$\frac{1}{2}\int\limits_{B_1(0)} v_k^{\beta}\xi^2\partial_\tau|w_k|^2dy + D^2\int\limits_{B_1(0)} v_k^{\beta}\xi^2|\nabla w_k|^2dy + \frac{D^2\beta}{2}\int\limits_{B_1(0)} v_k^{\beta-1}\xi^2|\nabla v_k|^2dy$$
$$+D^2\int\limits_{B_1(0)} v_k^{\beta}\nabla w_k w_k 2\xi\nabla\xi dy + \frac{r_k^2}{\lambda_k}\int\limits_{B_1(0)} |n_k|^{2(\gamma-1)}n_k v_k^{\beta}w_k\xi^2dy$$
$$= E^2\lambda_k\int\limits_{B_1(0)} (n_k\cdot\nabla\psi_k)\nabla\psi_k v_k^{\beta}w_k\xi^2dy. \tag{4.53}$$

Note that

$$\frac{r_k^2}{\lambda_k}\int\limits_{B_1(0)} |n_k|^{2(\gamma-1)}n_k v_k^{\beta}w_k\xi^2dy = \frac{r_k^2}{\lambda_k^2}\int\limits_{B_1(0)} |n_k|^{2(\gamma-1)}n_k v_k^{\beta}(n_k - m_{z_k,r_k})\xi^2dy$$
$$= \frac{r_k^2}{\lambda_k^2}\int\limits_{B_1(0)} |n_k|^{2\gamma}v_k^{\beta}\xi^2dy$$
$$-\frac{r_k^2}{\lambda_k^2}\int\limits_{B_1(0)} |n_k|^{2(\gamma-1)}n_k v_k^{\beta}m_{z_k,r_k}\xi^2dy$$
$$\geq \frac{1}{2}\frac{r_k^2}{\lambda_k^2}\int\limits_{B_1(0)} |n_k|^{2\gamma}v_k^{\beta}\xi^2dy - c\frac{r_k^2}{\lambda_k^2}\int\limits_{B_1(0)} v_k^{\beta}\xi^2dy. \tag{4.54}$$

Now we analyze the last term in (4.53) to obtain

$$\lambda_k\int\limits_{B_1(0)} (n_k\cdot\nabla\psi_k)\nabla\psi_k v_k^{\beta}w_k\xi^2dy = \int\limits_{B_1(0)} (n_k\cdot\nabla\psi_k)\nabla\psi_k v_k^{\beta}(n_k - m_{z_k,r_k})\xi^2dy$$
$$= \int\limits_{B_1(0)} (n_k\cdot\nabla\psi_k)^2 v_k^{\beta}\xi^2dy$$
$$-\int\limits_{B_1(0)} (n_k\cdot\nabla\psi_k)\nabla\psi_k v_k^{\beta}m_{z_k,r_k}\xi^2dy$$
$$\leq 2\int\limits_{B_1(0)} (n_k\cdot\nabla\psi_k)^2 v_k^{\beta}\xi^2dy$$
$$+c\int\limits_{B_1(0)} |\nabla\psi_k|^2 v_k^{\beta}\xi^2dy. \tag{4.55}$$

Combining the preceding three estimates gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\int\limits_{B_1(0)}\int\limits_0^{|w_k|^2}[(s-K^2)^+ + K^2]^\beta ds\xi^2 dy + D^2\int\limits_{B_1(0)} v_k^\beta\xi^2|\nabla w_k|^2 dy\\
&\quad+\frac{D^2\beta}{2}\int\limits_{B_1(0)} v_k^{\beta-1}\xi^2|\nabla v_k|^2 dy+\frac{r_k^2}{\lambda_k^2}\int\limits_{B_1(0)}|n_k|^{2\gamma}v_k^\beta\xi^2 dy\\
&\le c\int\limits_{B_1(0)}\int\limits_0^{|w_k|^2}[(s-K^2)^+ + K^2]^\beta ds\xi\partial_\tau\xi dy + c\int\limits_{B_1(0)} v_k^\beta|w_k|^2|\nabla\xi|^2 dy\\
&\quad+c\frac{r_k^2}{\lambda_k^2}\int\limits_{B_1(0)} v_k^\beta\xi^2 dy + 2E^2\int\limits_{B_1(0)}(n_k\cdot\nabla\psi_k)^2 v_k^\beta\xi^2 dy + c\int\limits_{B_1(0)}|\nabla\psi_k|^2 v_k^\beta\xi^2 dy. \qquad (4.56)
\end{aligned}
$$

To estimate the last two terms in the above inequality, we use $\psi_k v_k^\beta \xi^2$ as a test function in (4.9) to obtain

$$
\begin{aligned}
&\int\limits_{B_1(0)}|\nabla\psi_k|^2 v_k^\beta\xi^2 dy + \int\limits_{B_1(0)}\nabla\psi_k\psi_k\beta v_k^{\beta-1}\nabla v_k\xi^2 dy + \int\limits_{B_1(0)}\nabla\psi_k\psi_k v_k^\beta 2\xi\nabla\xi dy\\
&\quad+\int\limits_{B_1(0)}(n_k\cdot\nabla\psi_k)^2 v_k^\beta\xi^2 dy + \int\limits_{B_1(0)}(n_k\cdot\nabla\psi_k)n_k\psi_k\beta v_k^{\beta-1}\nabla v_k\xi^2 dy\\
&\quad+\int\limits_{B_1(0)}(n_k\cdot\nabla\psi_k)n_k\psi_k v_k^\beta 2\xi\nabla\xi dy\\
&=\int\limits_{B_1(0)} F_k\psi_k v_k^\beta\xi^2 dy. \qquad (4.57)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\left|\int\limits_{B_1(0)}\nabla\psi_k\psi_k\beta v_k^{\beta-1}\nabla v_k\xi^2 dy\right| &\le \frac{D^2}{16E^2}\beta\int\limits_{B_1(0)} v_k^{\beta-1}\xi^2|\nabla v_k|^2 dy\\
&\quad+c\beta\int\limits_{B_1(0)} v_k^{\beta-1}\xi^2|\nabla\psi_k\psi_k|^2 dy\\
&\le \frac{D^2}{16E^2}\beta\int\limits_{B_1(0)} v_k^{\beta-1}\xi^2|\nabla v_k|^2 dy\\
&\quad+c\beta K^{2(\beta-1)}\int\limits_{B_1(0)}\xi^2|\nabla\psi_k\psi_k|^2 dy. \qquad (4.58)
\end{aligned}
$$

Here we have used the fact that $v_k \geq K^2$ and $\beta < 1$. The fifth integral in (4.57) can be estimated as follows.

$$\left| \int_{B_1(0)} (n_k \cdot \nabla\psi_k) n_k \psi_k \beta v_k^{\beta-1} \nabla v_k \xi^2 dy \right| \leq \frac{D^2}{16E^2}\beta \int_{B_1(0)} v_k^{\beta-1}\xi^2 |\nabla v_k|^2 dy$$
$$+ c\beta \int_{B_1(0)} v_k^{\beta-1}\xi^2 n_k^2 \psi_k^2 (n_k \cdot \nabla\psi_k)^2 dy. \tag{4.59}$$

Remember

$$\begin{aligned} |n_k|^2\psi_k^2 &= |\lambda_k w_k + m_{z_k,r_k}|^2 \psi_k^2 \\ &\leq 2\lambda_k^2 |w_k|^2 \psi_k^2 + c\psi_k^2 \\ &\leq c|w_k|^2 + c\psi_k^2 \end{aligned} \tag{4.60}$$

and $v_k^{\beta-1}|w_k|^2 \leq v_k^\beta$. Consequently,

$$\left| \int_{B_1(0)} (n_k \cdot \nabla\psi_k) n_k \psi_k \beta v_k^{\beta-1} \nabla v_k \xi^2 dy \right| \leq \frac{D^2}{16E^2}\beta \int_{B_1(0)} v_k^{\beta-1}\xi^2 |\nabla v_k|^2 dy$$
$$+ c\beta \int_{B_1(0)} v_k^{\beta}\xi^2 (n_k \cdot \nabla\psi_k)^2 dy$$
$$+ c\beta K^{2(\beta-1)} \int_{B_1(0)} \xi^2 \psi_k^2 (n_k \cdot \nabla\psi_k)^2 dy. \tag{4.61}$$

Using the preceding estimates in (4.57)

$$\begin{aligned} &\int_{B_1(0)} |\nabla\psi_k|^2 v_k^\beta \xi^2 dy + (1 - c\beta) \int_{B_1(0)} (n_k \cdot \nabla\psi_k)^2 v_k^\beta \xi^2 dy \\ &\leq \frac{D^2}{8E^2}\beta \int_{B_1(0)} v_k^{\beta-1}\xi^2 |\nabla v_k|^2 dy + c\beta K^{2(\beta-1)} \int_{B_1(0)} \xi^2 |\nabla\psi_k \psi_k|^2 dy \\ &\quad + c\beta K^{2(\beta-1)} \int_{B_1(0)} \xi^2 \psi_k^2 (n_k \cdot \nabla\psi_k)^2 dy + c \int_{B_1(0)} v_k^\beta \psi_k^2 |\nabla\xi|^2 dy \\ &\quad + c \int_{B_1(0)} v_k^\beta |w_k|^2 |\nabla\xi|^2 dy + \int_{B_1(0)} F_k \psi_k v_k^\beta \xi^2 dy. \end{aligned} \tag{4.62}$$

Plugging this into (4.56) and choosing β suitably small in the resulting inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\int_{B_1(0)}\int_0^{|w_k|^2}[(s-K^2)^+ + K^2]^\beta ds\xi^2 dy + D^2\int_{B_1(0)} v_k^\beta \xi^2|\nabla w_k|^2 dy \\
&\quad + \frac{D^2\beta}{2}\int_{B_1(0)} v_k^{\beta-1}\xi^2|\nabla v_k|^2 dy + \frac{r_k^2}{\lambda_k^2}\int_{B_1(0)} |n_k|^{2\gamma} v_k^\beta \xi^2 dy \\
&\le c\int_{B_1(0)}\int_0^{|w_k|^2}[(s-K^2)^+ + K^2]^\beta ds\xi\partial_\tau\xi dy + c\int_{B_1(0)} v_k^\beta |w_k|^2|\nabla\xi|^2 dy \\
&\quad + c\frac{r_k^2}{\lambda_k^2}\int_{B_1(0)} v_k^\beta\xi^2 dy + c\beta K^{2(\beta-1)}\int_{B_1(0)}\xi^2|\nabla\psi_k\psi_k|^2 dy \\
&\quad + c\beta K^{2(\beta-1)}\int_{B_1(0)}\xi^2\psi_k^2(n_k\cdot\nabla\psi_k)^2 dy + c\int_{B_1(0)} v_k^\beta\psi_k^2|\nabla\xi|^2 dy \\
&\quad + \int_{B_1(0)} F_k\psi_k v_k^\beta\xi^2 dy. \qquad (4.63)
\end{aligned}$$

In view of (4.46), (4.47), and (4.49), if β is sufficiently small, we have

$$\int_{Q_{\frac{1}{2}}(0)}\int_0^{|w_k|^2}[(s-K^2)^+ + K^2]^\beta ds dy d\tau \le c,$$

$$\int_{Q_{\frac{1}{2}}(0)} v_k^\beta |w_k|^2 dy d\tau \le c,$$

$$\int_{B_{\frac{1}{2}}(0)} v_k^\beta\psi_k^2 dy \le c \text{ for } \tau\in[-\tfrac{1}{8},\tfrac{1}{8}],$$

$$\left|\int_{B_{\frac{1}{2}}(0)} F_k\psi_k v_k^\beta dy\right| \le c \text{ for } \tau\in[-\tfrac{1}{8},\tfrac{1}{8}].$$

Integrate (4.63) with respect to τ, choose ξ suitably, and remember Claim 1 to yield the desired result. □

This finishes the proof of Proposition 1.2. □

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