



## A note on one-dimensional time fractional ODEs

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## ABSTRACT

In this note, we prove or re-prove several important results regarding one dimensional time fractional ODEs following our previous work Feng et al. [15]. Here we use the definition of Caputo derivative proposed in Li and Liu (2017) [5,7] based on a convolution group. In particular, we establish generalized comparison principles consistent with the new definition of Caputo derivatives. In addition, we establish the full asymptotic behaviors of the solutions for  $D_c^\gamma u = Au^\rho$ . Lastly, we provide a simplified proof for the strict monotonicity and stability in initial values for the time fractional differential equations with weak assumptions.

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## 1. Introduction

The fractional calculus in time has been used widely in physics and engineering for memory effect, viscoelasticity, porous media etc. [1–5]. There is a huge amount of literature discussing time fractional differential equations. For instance, one can find some results in [3,6] using the classic Caputo derivatives. In this paper, we study the following time fractional ODE:

$$D_c^\gamma u = f(t, u), \quad u(0) = u_0, \quad (1.1)$$

for  $\gamma \in (0, 1)$  and  $f$  measurable. Here  $D_c^\gamma u$  is the generalized Caputo derivative introduced in [7,8]. As we will see later, this generalized definition is theoretically more convenient, since it allows us to take advantage of the underlying group structure.

As in [7], we use the following distributions  $\{g_\beta\}$  as convolution kernels for  $\beta \in (-1, 0)$ :

$$g_\beta(t) = \frac{1}{\Gamma(1 + \beta)} D(\theta(t)t^\beta).$$

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Here  $\theta(t)$  is the standard Heaviside step function,  $\Gamma(\cdot)$  is the gamma function, and  $D$  means the distributional derivative on  $\mathbb{R}$ . Indeed,  $g_\beta$  can be defined for  $\beta \in \mathbb{R}$  (see [7]) so that  $\{g_\beta : \beta \in \mathbb{R}\}$  forms a convolution group. In particular, we have

$$g_{\beta_1} * g_{\beta_2} = g_{\beta_1 + \beta_2}. \quad (1.2)$$

Here since the support of  $g_{\beta_i}$  ( $i = 1, 2$ ) is bounded from left, the convolution is well-defined. Now we are able to give the generalized definition of fractional derivatives:

**Definition 1.1** ([7,8]). Let  $0 < \gamma < 1$ . Consider  $u \in L_{\text{loc}}^1[0, T]$ . Given  $u_0 \in \mathbb{R}$ , we define the  $\gamma$ -th order generalized Caputo derivative of  $u$ , associated with initial value  $u_0$ , to be a distribution in  $\mathcal{D}'(-\infty, T)$  with support in  $[0, T]$ , given by

$$D_c^\gamma u = g_{-\gamma} * ((u - u_0)\theta(t)).$$

If  $\lim_{t \rightarrow 0+} \frac{1}{t} \int_0^t |u(s) - u_0| ds = 0$ , we call  $D_c^\gamma u$  the Caputo derivative of  $u$ .

As in [7], if the function  $u$  is absolutely continuous, the generalized definition reduces to the classical definition. However, the generalized definition is theoretically useful because it reveals the underlying group structure (see [Proposition 1.1](#)).

**Definition 1.2.** Let  $T > 0$ . A function  $u \in L_{\text{loc}}^1[0, T]$  is a weak solution to [\(1.1\)](#) on  $[0, T]$  with initial value  $u_0$ , if  $f(t, u(t)) \in \mathcal{D}'(-\infty, T)$  and the equality holds in the distributional sense. We call a weak solution  $u$  a strong solution if (i).  $\lim_{t \rightarrow 0+} \frac{1}{t} \int_0^t |u(s) - u_0| ds = 0$ ; (ii). both  $D_c^\gamma u$  and  $f(t, u(t))$  are locally integrable on  $[0, T]$ .

By the group property [\(1.2\)](#), we have

**Proposition 1.1** ([7]). Suppose  $f \in L_{\text{loc}}^\infty([0, \infty) \times \mathbb{R}; \mathbb{R})$ . Fix  $T > 0$ . Then,  $u(t) \in L_{\text{loc}}^1[0, T]$  with initial value  $u_0$  is a strong solution of [\(1.1\)](#) on  $(0, T)$  if and only if  $\lim_{t \rightarrow 0+} \frac{1}{t} \int_0^t |u(s) - u_0| ds = 0$  and it solves the following integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s, u(s)) ds, \quad \forall t \in (0, T). \quad (1.3)$$

Using this integral formulation, the following has been shown in [7].

**Proposition 1.2.** Suppose  $f : [0, \infty) \times (\alpha, \beta) \rightarrow \mathbb{R}$  is continuous and locally Lipschitz continuous in  $u$ . For any given initial value  $u_0 \in (\alpha, \beta)$ , there is a unique strong solution, which either exists globally on  $[0, \infty)$  or approaches the boundary of  $(\alpha, \beta)$  in finite time. Moreover, this solution is continuous on the interval of existence.

Below in [Section 2](#), we will establish some generalized comparison principles consistent with the new definition of Caputo derivatives. In [Section 3](#), we establish the full asymptotic behaviors of the solutions for  $D_c^\gamma u = Au^p$ . In [Section 4](#), we provide a new proof for the strict monotonicity and stability in initial values with weak assumptions.

## 2. Generalized comparison principles

The comparison principles are important in the analysis of time fractional PDEs (See [9]). There are many versions of comparison principles proved in literature using various definitions of Caputo derivatives. In [7], the authors assumed  $f(t, \cdot)$  to be non-decreasing. In [10, Lemma 2.6],  $f(t, \cdot)$  was assumed to be non-increasing. In [11, Theorem 2.3], there is no assumption on the monotonicity of  $f(t, \cdot)$ , but the function  $v$  is assumed to be  $C^1$  so that the pointwise value of  $D_c^\gamma v$  can be defined. Combining these ideas and establishing a crucial lemma (Lemma 2.1), we prove some generalized comparison principles in this section. Similar to [7], we define the inequality in the distributional sense:

**Definition 2.1.** Let  $U$  be an open interval. We say  $f \in \mathcal{D}'(U)$  is a nonpositive (nonnegative) distribution if for any  $\varphi \in C_c^\infty(U)$  with  $\varphi \geq 0$ , we have  $\langle f, \varphi \rangle \leq 0$  ( $\langle f, \varphi \rangle \geq 0$ ). We say  $f_1 \leq f_2$  in the distributional sense for  $f_1, f_2 \in \mathcal{D}'(U)$ , if  $f_1 - f_2$  is nonpositive. We say  $f_1 \geq f_2$  in the distributional sense if  $f_1 - f_2$  is nonnegative.

In order to prove the comparison principle, we first prove the following auxiliary lemma:

**Lemma 2.1.** Suppose  $u \in L_{\text{loc}}^1[0, T)$  and  $\lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t |u(s) - u_0| ds = 0$ . If there exists a function  $f \in L_{\text{loc}}^1(0, T)$  such that on interval  $(0, T)$  we have in the distributional sense that  $D_c^\gamma u \leq f$ , then for any given  $A \in \mathbb{R}$ , we have in the distributional sense

$$D_c^\gamma(u - A)^+ \leq \chi(u \geq A)f, \quad \text{on } (0, T).$$

**Proof.** First, recall the following result in [7, Proposition 3.11]: if  $u \in C[0, T) \cap C^1(0, T)$  and  $u \mapsto E(u)$  is  $C^1$  and convex, we have

$$D_c^\gamma E(u) \leq E'(u)D_c^\gamma u.$$

Now let us consider  $\eta \in C_c^\infty(-1, 0)$  with  $\eta \geq 0$  and  $\int \eta dt = 1$ . Define  $\eta^\epsilon(t) = \frac{1}{\epsilon} \eta(\frac{t}{\epsilon})$  and  $u^\epsilon = \eta^\epsilon * u$ . As showed in [7, Proposition 3.11],  $u^\epsilon(0) \rightarrow u_0$  and  $u^\epsilon(t) \rightarrow u(t)$  in  $L_{\text{loc}}^1[0, T)$ .

Denote  $E(u) = (u - A)^+$  and define  $E^\delta(u) = (E * \eta^\delta)(u)$ . Clearly,  $(E^\delta)'(u) = \eta^\delta * \chi(u \geq A)$  is nonnegative and increasing, which implies that  $E^\delta$  is a convex increasing function. Then, we have

$$D_c^\gamma E^\delta(u^\epsilon) \leq (E^\delta)'|_{u^\epsilon} D_c^\gamma u^\epsilon. \quad (2.1)$$

It is not hard to see  $\limsup_{\epsilon \rightarrow 0} (E^\delta)'|_{u^\epsilon} D_c^\gamma u^\epsilon \leq (E^\delta)'|_u f(t)$ . Since  $E^\delta(u^\epsilon)$  converges to  $E^\delta(u)$  in  $L_{\text{loc}}^1$  and  $E^\delta(u^\epsilon(0))$  converges to  $E^\delta(u_0)$ , according to Definition 1.1,  $D_c^\gamma E^\delta(u^\epsilon) \rightarrow D_c^\gamma E^\delta(u)$  as distributions. Moreover, notice that the inequality is preserved in the distributional sense (Definition 2.1). We have  $D_c^\gamma E^\delta(u) \leq (E^\delta)'|_u f(t)$ . Taking  $\delta \rightarrow 0$ , similarly we have  $D_c^\gamma E^\delta(u)$  converges as distributions to  $D_c^\gamma(u - A)^+$ . Then the right hand side of (2.1) converges to  $\chi(u \geq A)f(t)$ , and the inequality is preserved in the distributional sense.  $\square$

As is well-known, if  $u \in H^1(0, T)$ ,  $D(u - A)^+ = \chi(u - A)Du$ . Since Caputo derivative is nonlocal, the equality is no longer true in general. However, we have similar inequalities and Lemma 2.1 provides an answer.

**Corollary 2.1.** Suppose  $u(t)$  is a locally integrable function with  $\lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t |u(s) - u_0| ds = 0$ . Let  $A \in \mathbb{R}$  and  $t_1 \in (0, T)$  is a Lebesgue point. If  $u \leq A$  for a.e.  $t \leq t_1$ , and on the interval  $(t_1, T)$  we have  $D_c^\gamma u \leq 0$  in the distributional sense, then we have  $u \leq A$ , a.e.  $(0, T)$ .

Let  $u^\epsilon$  be the mollification in the proof of [Lemma 2.1](#). Consider  $v^\epsilon = u^\epsilon - \frac{C(\epsilon)\theta(t)}{\Gamma(1+\gamma)}t^\gamma$  such that  $v^\epsilon \leq A$  for  $t \in [0, t_1 + \epsilon]$ .  $C(\epsilon) \rightarrow 0$  since  $t_1$  is a Lebesgue point. Applying [Lemma 2.1](#),  $D_c^\gamma(v^\epsilon - A)^+ \leq \chi(t \geq t_1 + \epsilon)(D_c^\gamma u^\epsilon - C(\epsilon)) \leq \chi(t \geq t_1 + \epsilon)(D_c^\gamma u^\epsilon - \eta_\epsilon * D_c^\gamma u)$ . Taking  $\epsilon \rightarrow 0$  yields  $D_c^\gamma(u - A)^+ \leq 0$ . The details are left to readers. Now several versions of comparison principles can be stated as follows:

**Theorem 2.1.**

(i) Suppose  $u_i \in L_{\text{loc}}^1[0, T]$  with  $\lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t |u_i(s) - u_{i,0}| ds = 0$  ( $i = 1, 2$ ). Suppose  $u_1(t) \leq u_2(t)$  on  $[0, t_1]$  for a Lebesgue point  $t_1$ , and the  $\gamma$ -th Caputo derivatives of  $u_1, u_2$  on  $[0, t_1]$  are locally integrable. Define

$$h_i(t) = u_{i,0} + \frac{1}{\Gamma(\gamma)} \int_0^{t \wedge t_1} (t-s)^{\gamma-1} D_c^\gamma u_i(s) ds, \quad i = 1, 2.$$

Then,  $h_1(t) \leq h_2(t)$  for all  $t \in [0, T]$ . Moreover, assume there exists a measurable function  $f(t, u)$  such that (i)  $f(\cdot, u_i(\cdot))$  ( $i = 1, 2$ ) is locally integrable on  $[t_1, T]$ ; (ii)  $f(t, \cdot)$  is non-decreasing on  $[t_1, T]$ ; (iii)  $D_c^\gamma u_1 \leq f(t, u_1)$  and  $D_c^\gamma u_2 \geq f(t, u_2)$  in the distributional sense on  $(t_1, T)$ , then  $u_1 \leq u_2$  a.e. on  $[0, T]$ .

(ii) Suppose  $u_i \in L_{\text{loc}}^1[0, T]$  with  $\lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t |u_i(s) - u_{i,0}| ds = 0$  ( $i = 1, 2$ ). If  $u_1 \leq u_2$  on  $[0, t_1]$  for a Lebesgue point  $t_1$  and  $D_c^\gamma(u_1 - u_2) \leq f(t, u_1) - f(t, u_2)$  as distributions on  $(t_1, T)$ , with  $f(t, \cdot)$  being non-increasing on  $(t_1, T)$  and  $f(\cdot, u_i(\cdot))$  ( $i = 1, 2$ ) being locally integrable on  $[t_1, T]$ , then  $u_1 \leq u_2$  a.e. on  $[0, T]$ .

(iii) Suppose  $u(t)$  is a continuous function on  $[0, T]$ . If  $u(t_1) = \sup_{0 \leq s \leq t_1} u(s)$  for some  $t_1 \in (0, T]$  and  $f(t) = D_c^\gamma u(t)$  is a continuous function, then  $f(t_1) \geq 0$ .

**Proof.** (i) Clearly,  $D_c^\gamma h_i = D_c^\gamma u_i$  for  $t \leq t_1$  and  $D_c^\gamma h_i = 0$  for  $t > t_1$ . Let  $u = h_1 - h_2$ ,  $A = 0$  in [Corollary 2.1](#), we find  $h_1 \leq h_2$ . On  $[t_1, T]$ , we have

$$u_1(t) \leq h_1(t) + \frac{1}{\Gamma(\gamma)} \int_{t_1}^t (t-s)^{\gamma-1} f(s, u_1) ds, \quad u_2(t) \geq h_2(t) + \frac{1}{\Gamma(\gamma)} \int_{t_1}^t (t-s)^{\gamma-1} f(s, u_2) ds.$$

As  $h_1(t) \leq h_2(t)$  and  $f(t, \cdot)$  is non-decreasing, one has  $u_1(t) \leq u_2(t)$  (see [\[7, Theorem 4.10\]](#)).

(ii) Apply [Lemma 2.1](#) for  $u_1 - u_2$  and  $A = 0$ . (The proof is similar as in [Corollary 2.1](#).)  
 (iii) Consider  $u^\epsilon(t) = u(t) + \frac{\epsilon\theta(t)}{\Gamma(1+\gamma)}t^\gamma$ , where  $\epsilon > 0$ . Then,  $t_1$  is the unique maximizer of  $u^\epsilon$  on  $[0, t_1]$ . Let  $f^\epsilon = D_c^\gamma u^\epsilon = f + \epsilon$ . It suffices to show

$$f^\epsilon(t_1) \geq 0, \quad \forall \epsilon > 0. \quad (2.2)$$

Otherwise, there is an  $\epsilon_0 > 0$  such that  $f^{\epsilon_0}(t_1) < 0$ . Since  $f^{\epsilon_0}$  is continuous, we can find  $\delta > 0$  such that on  $[t_1 - \delta, t_1]$   $f^{\epsilon_0}$  is negative and  $u^{\epsilon_0}(t) \leq u^{\epsilon_0}(t_1 - \delta)$  for  $t \leq t_1 - \delta$ . Applying [Corollary 2.1](#), we have  $u^{\epsilon_0}(t) \leq u^{\epsilon_0}(t_1 - \delta)$  for  $t \in [t_1 - \delta, t_1]$ , which is a contradiction. Taking  $\epsilon \rightarrow 0$  then gives the result.  $\square$

**Remark 2.1.** Though the conditions here are weaker under the new definition of Caputo derivative, (ii) is essentially [\[10, Lemma 2.6\]](#) and (iii) is well-known for  $C^1$  functions (see, for example [\[11,12\]](#)).

Now, we establish a generalized Grönwall inequality (or another version of comparison principle), consistent with the new definition of Caputo derivative. The main construction is inspired by [\[11\]](#).

**Theorem 2.2.** Suppose  $f(t, u)$  is continuous and locally Lipschitz in  $u$ . Let  $v(t)$  be a continuous function. If  $D_c^\gamma v \leq f(t, v)$  in the distributional sense, and  $D_c^\gamma u = f(t, u)$ , with  $v_0 \leq u_0$ . Then,  $v \leq u$  on the common interval. Similarly, if we have  $D_c^\gamma v \geq f(t, v)$  as distributions and  $v_0 \geq u_0$ , then  $v \geq u$  on the common interval.

**Proof.** We only prove the first claim (the proof for the other is similar). By [Proposition 1.2](#),  $D_c^\gamma u = f(t, u)$  with initial value  $u(0) = u_0$  has a unique solution on the interval  $[0, T_b]$ , where  $T_b$  is the largest time of existence. Moreover,  $u$  is continuous on  $[0, T_b]$ .

Fix  $T \in (0, T_b)$ . Pick  $M$  large enough so that  $u(t)$  and  $v(t)$  fall into  $[0, T] \times [-M, M]$ . Let  $L$  be the Lipschitz constant of  $f(t, \cdot)$  for the region  $[0, T] \times [-2M, 2M]$ . Consider

$$v^\epsilon = v - \epsilon w.$$

Here  $w = E_\gamma(2Lt^\gamma)$  is the solution to  $D_c^\gamma w = 2Lw$  with initial value 1, where  $E_\gamma(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\gamma+1)}$  is the Mittag-Leffler function [\[13,14\]](#). Clearly, if  $\epsilon$  is sufficiently small,  $v^\epsilon$  falls into  $[0, T] \times [-2M, 2M]$ . Then, we find that in the distributional sense

$$D_c^\gamma v^\epsilon = D_c^\gamma v - \epsilon 2Lw \leq f(t, v) - \epsilon 2Lw \leq f(t, v^\epsilon) - \epsilon Lw.$$

We claim that for all such small  $\epsilon$ ,

$$v^\epsilon(t) \leq u(t), \forall t \in [0, T]. \quad (2.3)$$

If not, define

$$t_1 = \sup\{t \in (0, T] : v^\epsilon(s) \leq u(s), \forall s \in [0, t]\}.$$

Since  $v^\epsilon(0) = v_0 - \epsilon < u_0$ , by continuity we have  $t_1 > 0$ . By assumption, [\(2.3\)](#) is not true, and we have  $t_1 < T$ . Consequently, there exists  $\delta_1 > 0$ , such that  $v^\epsilon(t_1) = u(t_1)$  and  $v^\epsilon(t) > u(t)$  for  $t \in (t_1, t_1 + \delta_1)$ . Moreover,

$$D_c^\gamma(v^\epsilon - u) \leq f(t, v^\epsilon) - \epsilon Lw - f(t, u).$$

By continuity, for some  $\delta_2 \in (0, \delta_1)$ ,  $D_c^\gamma(v^\epsilon - u)$  is a nonpositive distribution on the interval  $(t_1, t_1 + \delta_2)$ . By [Corollary 2.1](#), we have  $v^\epsilon(t) \leq u(t)$  for  $t \in (t_1, t_1 + \delta_2)$ , which is a contradiction. Hence, [\(2.3\)](#) is true. Taking  $\epsilon \rightarrow 0$  in [\(2.3\)](#) yields the result on  $[0, T]$ . Since  $T$  is arbitrary, the result is true.  $\square$

### 3. Asymptotic behaviors for a class of fractional ODEs

In this section, we study the solution curves to the following autonomous fractional ODEs:

$$D_c^\gamma u = Au^p, \quad u(0) = u_0 > 0. \quad (3.1)$$

The monotonicity of the solutions to [\(3.1\)](#) and some partial results for the asymptotic behaviors have been established in our previous work [\[15\]](#). The asymptotic behaviors of the solutions for the  $A < 0, p > 0$  case have also been discussed in [\[10, Theorem 7.1\]](#). However, the discussion on all the range of  $A$  and  $p$  is not complete. Here, we will give a complete description on asymptotic behaviors of the solution curves.

By [Proposition 1.2](#), the strong solution  $u$  to [\(3.1\)](#) exists on  $[0, T_b]$  for  $T_b \in (0, \infty]$ . If  $T_b < \infty$ , either  $\lim_{t \rightarrow T_b^-} u(t) = 0$  or  $\lim_{t \rightarrow T_b^-} u(t) = \infty$ . We give a complete description regarding the solutions curves to [\(3.1\)](#):

**Theorem 3.1.** *Consider [\(3.1\)](#). If  $A = 0$ , then  $u(t) = u_0$ . If  $A > 0$ , then all the solutions are strictly increasing on  $(0, T_b)$ . If  $A < 0$ , then all solutions are strictly decreasing before they touch 0.*

- (i) Suppose  $A > 0$ . If  $p > 1$ , then  $T_b < \infty$  and  $u(t) \sim \left[ \frac{\Gamma(\frac{p\gamma}{p-1})}{A\Gamma(\frac{\gamma}{p-1})} \right]^{\frac{1}{p-1}} (T_b - t)^{-\frac{\gamma}{p-1}}$ , as  $t \rightarrow T_b^-$ . If  $p = 1$ , then  $u(t) = u_0 E_\gamma(At^\gamma)$ . If  $p < 1$ , then there exist  $c_1 > 0$  and  $c_2 > 0$  such that  $c_1 t^{\frac{\gamma}{1-p}} \leq u(t) \leq c_2 t^{\frac{\gamma}{1-p}}$ ,  $t \geq 1$ .
- (ii) Suppose  $A < 0$ . If  $p < 0$ , the solution curve touches  $u = 0$  in finite time where the right hand side blows up. If  $p = 0$ , then  $u = u_0 + Ag_{1+\gamma}$ . If  $p > 0$ , then  $T_b = \infty$ , and there exist  $c_1 > 0, c_2 > 0$  such that  $c_1 t^{-\frac{\gamma}{p}} \leq u(t) \leq c_2 t^{-\frac{\gamma}{p}}$ ,  $t \geq 1$ .

**Proof.** The  $A = 0$  or  $p = 0$  cases are trivial. The monotonicity has been proved in [15]. The  $A > 0, p > 1$  case has also been discussed there. Indeed, there is also an accurate estimate of  $T_b$  in [15]. The  $p = 1$  case is trivial. The  $A < 0, p > 0$  case has been discussed in [10, Theorem 7.1]. In fact, they established a version of comparison principle and used a subsolution and a supersolution to get  $c_1 t^{-\frac{\gamma}{p}} \leq u(t) \leq c_2 t^{-\frac{\gamma}{p}}$ ,  $t \geq 1$ . For the case  $A < 0, p < 0$ , since the solution is decreasing, we have  $D_c^\gamma u \leq A u_0^p < 0$  before  $u$  touches zero. Hence, the claim follows.

Now, we establish the results for  $A > 0, p < 1$  case. First, let us construct the sub-solution as follows:

$$\omega(t) = \begin{cases} u_0, & t \in [0, t_0], \\ a t^{\frac{\gamma}{1-p}}, & t \geq t_0. \end{cases}$$

Here  $a > 0$  is to be determined and  $t_0$  is determined by  $a t_0^{\frac{\gamma}{1-p}} = u_0$ . Clearly,  $\omega$  is absolutely continuous on any finite interval. For  $t < t_0$ ,  $D_c^\gamma \omega = 0 \leq A \omega^p$ . For  $t \geq t_0$ , we have

$$D_c^\gamma \omega = \frac{a\gamma}{(1-p)\Gamma(1-\gamma)} \int_{t_0}^t \frac{\tau^{\frac{\gamma}{1-p}-1}}{(t-\tau)^\gamma} d\tau < \frac{a\gamma B(\frac{\gamma}{1-p}, 1-\gamma)}{(1-p)\Gamma(1-\gamma)} t^{\frac{\gamma p}{1-p}} = \frac{a\Gamma(\gamma/(1-p)+1)}{\Gamma(\gamma p/(1-p)+1)} t^{\frac{p\gamma}{1-p}},$$

where  $B(\cdot, \cdot)$  is the Beta function. Clearly, if we choose  $a > 0$  such that  $\frac{a\Gamma(\gamma/(1-p)+1)}{\Gamma(\gamma p/(1-p)+1)} \leq A a^p$ , then  $D_c^\gamma \omega \leq A \omega^p$ . Such  $a$  exists because  $p < 1$ .

For the super-solution, let us consider

$$v(t) = \begin{cases} u_0 + B_1 \frac{t^\gamma}{\Gamma(1+\gamma)}, & t \in [0, 1], \\ B_2 t^{\frac{\gamma}{1-p}}, & t \geq 1. \end{cases}$$

$B_2$  is determined by  $B_2 = u_0 + \frac{B_1}{\Gamma(1+\gamma)}$ . This choice of  $B_2$  makes  $v$  absolutely continuous on any finite interval. We now determine  $B_1$ . On  $[0, 1]$ , one has  $D_c^\gamma v = B_1$ . For  $t > 1$ , we have

$$D_c^\gamma v = \frac{B_1 \gamma}{B(1+\gamma, 1-\gamma)} \int_0^1 \frac{\tau^{\gamma-1}}{(t-\tau)^\gamma} d\tau + \frac{B_2}{\Gamma(1-\gamma)} \frac{\gamma}{1-p} \int_1^t \frac{\tau^{\frac{\gamma}{1-p}-1}}{(t-\tau)^\gamma} d\tau.$$

On  $[1, 2]$ , one has  $D_c^\gamma v > \frac{B_1 \gamma}{B(1+\gamma, 1-\gamma)} \int_0^1 \frac{\tau^{\gamma-1}}{(2-\tau)^\gamma} d\tau = B_1 C_1(\gamma)$ . For  $t > 2$ , we have

$$D_c^\gamma v > B_2 \frac{1}{\Gamma(1-\gamma)} \frac{\gamma}{1-p} t^{\frac{\gamma p}{1-p}} \int_{\frac{1}{t}}^1 \frac{\tau^{\frac{\gamma}{1-p}-1}}{(1-\tau)^\gamma} d\tau \geq B_2 t^{\frac{\gamma p}{1-p}} C_2(p, \gamma).$$

It is clear that there exists  $M_1(A, p, \gamma)$  such that as long as  $B_2 \geq M_1$ ,  $D_c^\gamma v \geq A v^p$  for  $t \geq 2$  since  $p < 1$ . For  $v$  to be a super-solution, one needs

$$u_0 + B_1 \frac{1}{\Gamma(1+\gamma)} \geq M_1, \quad B_1 \min(1, C_1(\gamma)) \geq A \max \left( u_0^p, \left( u_0 + \frac{B_1}{\Gamma(1+\gamma)} \right)^p 2^{\frac{p\gamma}{1-p}} \right).$$

Such  $B_1$  exists since  $p < 1$ . Hence, applying comparison principle Theorem 2.2 yields the result.  $\square$

#### 4. Strict monotonicity and stability in initial values

It is well-known that solution curves for well-behaved ODEs do not touch each other. However, for fractional ODEs, similar results are not trivial since the dynamics is non-Markovian. By the comparison principles (or generalized Grönwall inequality), if  $f(t, u)$  in (1.1) is continuous and locally Lipschitz in  $u$ ,  $u(0) < v(0)$  implies  $u(t) \leq v(t)$  for  $t \geq 0$ . However we do not have strict inequality. In [3, Theorem 6.12], the

strict inequality has been established following a series of contraction techniques. Using our new definition of Caputo derivative, we provide a new proof of that solutions are strict monotone in initial values, by assuming  $f \in L_{\text{loc}}^\infty$ .

The following lemma (a variant of [15, Lemma 3.4] or [16, Theorem 1]), is important:

**Lemma 4.1.** *Let  $r_\lambda(t) = -\frac{d}{dt}E_\gamma(-\lambda\Gamma(\gamma)t^\gamma)$  be the resolvent for kernel  $\lambda t^{\gamma-1}$  (in other words,  $r_\lambda(t) + \lambda \int_0^t (t-s)^{\gamma-1} r_\lambda(s) ds = \lambda t^{\gamma-1}$ ). Let  $T > 0$ . Assume  $h \in L^1[0, T]$ ,  $h > 0$  a.e., satisfying*

$$h(t) - \int_0^t r_\lambda(t-s)h(s)ds > 0, \text{ a.e., } \forall \lambda > 0.$$

Suppose  $v \in L^\infty[0, T]$ , then the integral equation

$$y(t) + \int_0^t (t-s)^{\gamma-1} v(s) y(s) ds = h(t) \quad (4.1)$$

has a unique solution  $y(t) \in L^1[0, T]$ . Moreover,  $y(t) > 0$ , a.e.

The proof is exactly the same as [15, Lemma 3.4], though we only assume  $v \in L^\infty[0, T]$  here. Next, we provide a new proof for the strict monotonicity in initial value. We also prove the stability of solutions with respect to initial values.

**Theorem 4.1.** *Assume that  $f(\cdot, \cdot) \in L_{\text{loc}}^\infty([0, \infty) \times \mathbb{R})$ . Moreover, assume for every compact set  $K$ , there is  $L_K > 0$  such that  $|f(t, u) - f(t, v)| \leq L_K |u - v|$  for a.e.  $(t, u), (t, v) \in K$ . Then, for a given initial value  $u_0$ , the solution in  $L_{\text{loc}}^\infty[0, T_b)$  is unique. Further, we have*

- Any two solutions  $u_i \in L_{\text{loc}}^\infty[0, T_b^i)$  ( $i = 1, 2$ ) with initial values  $u_{1,0} < u_{2,0}$  satisfy  $u_1(t) < u_2(t)$  on  $[0, \min(T_b^1, T_b^2))$ .
- For any  $T > 0$ ,  $M > 0$ , there exists  $C(M, T) > 0$  such that any two solutions with  $\|u_i\|_{L^\infty[0, T]} \leq M$  ( $i = 1, 2$ ) and initial values  $u_{1,0}, u_{2,0}$  satisfy

$$\|u_1 - u_2\|_{L^\infty[0, T]} \leq C(M, T) |u_{1,0} - u_{2,0}|.$$

**Proof.** Fix  $T \in (0, \min(T_b^1, T_b^2))$ . There exists  $K$  compact such that for a.e  $t \in [0, T]$ ,  $(t, u_i(t)) \in K$ . By Proposition 1.1, one has

$$u_i(t) = u_{i,0} + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s, u_i(s)) ds.$$

The boundedness of  $f(s, u_i(s))$  implies that  $u_i(t) \in C[0, T]$ . If  $u_{1,0} = u_{2,0}$ , by taking the difference,  $|u_1(t) - u_2(t)| \leq C \int_0^t (t-s)^{\gamma-1} |u_1(s) - u_2(s)| ds$  and the uniqueness therefore follows.

Now, assume  $u_{1,0} \neq u_{2,0}$ . Define  $y(t) = (u_2(t) - u_1(t))/(u_{2,0} - u_{1,0})$ , we have

$$y(t) + \int_0^t (t-s)^{\gamma-1} v(s) y(s) ds = 1, \text{ where } v(s) = -\frac{1}{\Gamma(\gamma)} \frac{f(s, u_2(s)) - f(s, u_1(s))}{u_2(s) - u_1(s)}.$$

If  $u_1(s) = u_2(s)$ , we define  $v(s) = 0$ . Note that  $|v| \leq L_K/\Gamma(\gamma)$  a.e. for  $t \in (0, T)$ . By setting  $h = 1$  in Lemma 4.1, one has

$$1 - \int_0^t r_\lambda(t-s) ds = E_\gamma(-\lambda\Gamma(\gamma)t^\gamma) > 0.$$

By Lemma 4.1,  $y(t) > 0$ . Since  $y$  is continuous, satisfying

$$y(t) \leq 1 + \int_0^t (t-s)^{\gamma-1} \|v\|_{L^\infty[0, T]} y(s) ds,$$

we have  $y(t) \leq C(\|v\|_{L^\infty}, T)$  by [15, Proposition 5]. This verifies the last claim.  $\square$

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