Multiplex Conductance and Gossip Based Information Spreading in Multiplex Networks

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Abstract—In this network era, not only people are connected, different networks are also coupled through various interconnections. This kind of network of networks, or multilayer networks, has attracted research interest recently, and many beneficial features have been discovered. However, quantitative study of information spreading in such networks is essentially lacking. Despite some existing results in single networks, the layer heterogeneity and complicated interconnections among the layers make the study of information spreading in this type of networks challenging. In this work, we study the information spreading time in multiplex networks, adopting the gossip (random-walk) based information spreading model. A new metric called *multiplex conductance* is defined based on the multiplex network structure and used to quantify the information spreading time in a general multiplex network in the idealized setting. Multiplex conductance is then evaluated for some interesting multiplex networks to facilitate understanding in this new area. Finally, the tradeoff between the information spreading efficiency improvement and the layer cost is examined to explain the user's social behavior and motivate effective multiplex network designs.

Index Terms—Information spreading, multiplex networks, gossip algorithm, multiplex conductance

1 Introduction

In the election year, one of the most important tasks for presidential candidates is to disseminate their words and opinions to voters in a fast and efficient manner. The underlying research problem on information spreading has already received great interest and been extensively studied in a single network. However, with the continuous advancement of modern technology, the ways that the candidates can exploit to promote their influence are no longer limited to the campaign tour; radio networks, TV networks, telephone networks, and Internet have all been utilized for their purposes. Especially, with the phenomenal popularity of social networks, all candidates have utilized their Facebook and Twitter accounts to post and spread their political agenda, through which their words can be shared and disseminated in an unprecedented range and scale. Therefore, with increasingly complicated interconnections and interactions, various kinds of communication networks and social media have formed a new network structure that enables people to spread and receive information simultaneously through multiple channels and platforms. Recently, multilayer network models have been introduced to facilitate relevant studies on emerging inter-connected complex networks [2–5]. In this work, we take a first step to investigate information spreading in a special type of multilayer networks, termed multiplex networks, for which all layers share the same set of nodes. In practice, the same set of nodes may correspond to individuals who can communicate through multiple networks or platforms, and duplicates of the same node may represent different communication devices or social accounts a person may have. Arguably, this somewhat simplified version of multilayer networks already captures many interesting multi-scale and multi-component features, and serves as a good starting point for our intended study.

As a common tool for studying information spreading in the single network, the compartmental epidemic model (e.g., Susceptible-Infected-Removed (SIR) or Susceptible-Infected-Susceptible (SIS) model) has been utilized to discover how the multiplex network structure affect the information spreading process [6–11]. It can be used to study a spreading process happening on all layers of a multiplex network. Cozzo et al. [9] proposed a contact-based epidemic-like information spreading model in multiplex networks. It further demonstrated that the critical point for the network having a constant portion of informed nodes is determined by the layer whose contacting probability matrix has the largest eigenvalue. In [7], Zhao et al. considered a spreading process on a two-layer multiplex network with a certain similarity between these two layers and showed that a positive degree-degree correlation between two layers may lead to a smaller infection size in the end. Spreading not only exists on intra-layer links but also on inter-layer ones. The effect of the layer-switching cost on spreading processes have been studied in [8], where different effective infection rates in the SIR model are considered for intra- and inter-layer links. It is shown that when both layers have the same average degree, a lower layer-switching cost, which is determined by the difference between intra- and inter-layer infection rates, can trigger the epidemics more easily (lower the epidemic threshold) while it suppresses the final epidemic size. Multiple spreading processes can also be addressed simultaneously through multiplex network modeling. In [6, 10], Granell et al. studied the coupled processes of awareness and infection on multiplex networks. By adopting the heterogeneous mean-field approximation, they quantified the effect of using the information spreading

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process to stop the epidemic process, where the immunological information spreading process in the social network will help deter the epidemic spreading process happening in the physical world. Wei et al. in [11] instead focused on two cooperative spreading processes on a multiplex network and proved that the cooperation can promote the spreading progress.

However, the compartmental epidemic model mainly focuses on the macroscopic network behavior. More specifically, information is diffused from any informed/infected node to its neighbors in a broadcasting-like manner; the independent cascade model and linear threshold model widely used in the study of social networks [12] also belong to this category. This type of models is predominantly used to describe group behaviors and determine the final size of informed nodes. However, this class of models is by no means the only one considered in literature. First, as a tradeoff for mathematical elegance, some simplified assumptions are made about node behaviors and the underlying communication structure. In practice, broadcasting is not always adopted for information spreading due to various concerns including excessive resource consumption and privacy. Actually, in many scenarios, one-to-one communications happen naturally between two individuals through phone calls, text messages, emails, and other directed communications. This type of information spreading is best captured by another popular model, random-walk based model, or gossip model [13]. The gossip based algorithms and designs possess many advantages: simple in implementation, efficient in resource utilization, and robust to network dynamics, thus becoming an appealing architectural solution for many problems in large-scale dynamic networks. One particular advantage of the gossip model, as compared to the aforementioned information diffusion models, is to allow the quantitative study of the information spreading speed. In summary, our work adopts the gossip (random-walk) based information spreading model, which is considered as better capturing the personal communications behaviors in many scenarios, reflecting more details of the underlying communication dynamics and network structures, and can facilitate the quantification of the information spreading time. To the best of our knowledge [14–18], the gossip based information spreading model has not been explored for multiplex networks.

In this work, the gossip-based information spreading time is found to be closely connected to a newly defined metric, *multiplex conductance*. Specifically, our contributions can be summarized as follows:

- The *informed probability* model is proposed to facilitate the study of information spreading in multiplex networks.
- A new metric, *multiplex conductance*, Φ_{mp} is defined based on the multiplex network structure, and it is shown that $\Theta(\Phi_{\text{mp}}^{-1} \cdot \log n)$ is a good lower bound for information spreading time in a general multiplex network when the layer number is large.
- Multiplex conductance of some interesting multiplex networks is evaluated to shed light on this burgeoning research field.
- The tradeoff between the cost of additional layers and the improvement of information spreading efficiency is discussed from both the user's and the network designer's aspect.

The rest of the paper is organized as follows. The system model, the gossip algorithm, and the definition of information spreading time for multiplex networks are introduced in Section 2. Section 3 presents the main theoretical results for information spreading time in multiplex networks and the evaluation of multiplex conductance. In Section 4, some discussion on the trade-off between the layer cost and the improvement of information spreading efficiency is given. Section 5 concludes the work.

2 Problem Formulation

In this section we introduce the network and system models as well as the definition of information spreading time.

2.1 Basic Models

1) *Multiplex Network*: A multilayer network is modeled by a family of graphs $\{G_m \triangleq (V_m, E_m)\}_{m=1}^{M}$ that constitute the layers of this complex system, together with the interlayer connections represented by $E_{\alpha\beta}$, for any two different layers G_α and G_β. In this study, we will focus on *multiplex networks*, for which all layers share the same set of nodes, i.e., $V_1 = V_2 = ... = V = [n]$, and interlayer connections exist *only* between the duplicates of the same node at different layers, i.e., $E_{\alpha\beta} = \{(v_\alpha, v_\beta); v \in V\}$ for all $\alpha \neq \beta$, where v_α is the duplicate of node v in layer α.

2) *Synchronous Time Model*: In this work, the synchronous time model is adopted, i.e., all nodes in the network take action simultaneously at discrete time steps. This is a common model used for studying gossip-based information spreading [19].

3) *Gossip Algorithm in Multiplex Network*: For the gossip algorithm in the single network, in each time slot, each node contacts one of its neighbors independently and uniformly at random. During each meaningful contact (for which exactly one node has the piece of information[1]), the message is successfully delivered in either direction (through the "push" or "pull" operation). Specifically, in each round, for the push operation, every informed node randomly chooses a neighbor and attempts to pass the information, while for the pull operation, every uninformed node randomly chooses a neighbor and attempts to grab the information. There are two key differences for information spreading in a multiplex network: First, the message can be spread on multiple layers simultaneously. Second, when a node gets informed at one layer, it will automatically become informed at all other layers. While these two assumptions are somewhat idealistic, we will use them in this study to explore the maximum potential for information spreading in multiplex networks. In particular, we will consider the following gossip algorithm for a multiplex network: Before the gossip process, it is assumed that all duplicates of the same node are synchronized. During a gossip step, each node and its duplicates contact one of its neighbors (not

[1] While that the classic single piece information spreading problem is focused in this work, most of our results can be naturally extended to the multi-piece information spreading scenario following existing literature [20].

necessarily the same in each layer) uniformly at random in all layers simultaneously. After each gossip step, the newly informed nodes (if exist) will broadcast the information to all their duplicates.

2.2 Information Spreading Time

The metric commonly used to measure the efficiency of gossip based information spreading is the information spreading time. Denote S_t as the informed node set at round t, with $S_0 = \{s\}$, for some arbitrary $s \in V$. The information spreading time in a network G of size n, $T_{spr}(G,\gamma)$, for some $\gamma > 0$, is defined as the stopping time by which all nodes are informed with probability $1 - O(n^{-\gamma})$ [19], i.e., $T_{spr}(G,\gamma) = \sup_{s\in V} \inf\{t : Pr(S_t \neq V | S_0 = \{s\}) \leq O(n^{-\gamma})\}$.

3 Main Results

Analyzing the information spreading process is difficult even in a single network due to the heterogeneous network topology and random gossip processes. To the best of our knowledge, the tightest upper bound for the information spreading time in a general single network in literature is $O(\Phi^{-1} \cdot \log n)$ [19, 21], where Φ is the corresponding graph conductance. The multiplex network structure introduces interconnections and interactions among layers, which further complicate the analysis. In this study, we slightly relax the problem and endeavor to find the information spreading time in a general multiplex network in an idealized setting. We are able to obtain a neat result in this setting (Theorem 2), which connects the information spreading time to a new metric for multiplex networks, *multiplex conductance* (Definition 5). In this section, the informed probability model is first introduced to facilitate the understanding of information spreading in multiplex networks. Then the corresponding information spreading time will be analyzed.

3.1 Informed Probability Model

In this part, the informed probability model is presented to offer some preliminary insights of information spreading in a multiplex network. In order to facilitate our analysis, the information spreading time is reformulated. First, the *informed probability* is defined as:

Definition 1. The informed probability $\mathbb{P}_{t,s}(u), t \in \mathbb{N}$ is defined as the probability that node u becomes informed by round $t-1$ (right before round t) given that the source node is s.

Based on the above definition, the information spreading time is redefined:

Definition 2. Given the informed probability sets $[\mathbb{P}_{t,s}(u)]_{u\in V}, t \in \mathbb{N}$ defined for information spreading in the network $G = (V,E)$ of size n starting at node s, the corresponding information spreading time can be defined as

$$T_{spr}(G,\gamma,s) = \inf\{t : \max_{u\in V}\{(1-\mathbb{P}_{t,s}(u))\} \leq O(n^{-\gamma-1})\}, \tag{1}$$

and the information spreading time of network G can be alternatively defined as

$$T_{spr}(G,\gamma) = \sup_{s\in V} T_{spr}(G,\gamma,s). \tag{2}$$

Remark 1. Given the information spreading time $T_{spr}(G,\gamma), \gamma > 0$, $(1 - \mathbb{P}_{T_{spr}(G,\gamma)}(u)) \leq O(n^{-\gamma-1})$ holds for any source node $s \in V$, and any other node $u \in V$. The equivalence between this new definition of information spreading time and the original one can be shown by the following inequalities:

$$\begin{aligned}
&Pr(not\ all\ nodes\ are\ informed) \\
&= Pr(\bigcup_{u\in V} node\ u\ is\ not\ informed) \\
&\leq \sum_{u\in V} Pr(node\ u\ is\ not\ informed) \\
&= \sum_{u\in V} (1 - \mathbb{P}_{T_{spr}(\gamma)}(u)) \leq n \times O(n^{-\gamma-1}) = O(n^{-\gamma}),
\end{aligned} \tag{3}$$

the first inequality is by union bound.
Therefore, the probability that all nodes are informed is at least $1 - O(n^{-\gamma})$ after time $T_{spr}(G,\gamma)$, i.e., $T_{spr}(G,\gamma)$ is the stopping time required for all nodes to be informed with probability $1 - O(n^{-\gamma})$.

Then, the following definition is needed for the following analysis:

Definition 3. Given a multiplex network $G = \{G_m \triangleq (V,E_m)\}_{m=1}^M$, for each node $u \in V$, $d_m(u)$ and $Neg_m(u)$ are defined as the node degree and the neighbor set of u in layer m. The maximum total node degree is then defined as $\Delta_{max} = \max_{u\in V}(\sum_{m=1}^{M} d_m(u))$. The total neighbor set $Neg(u)$ of node u is defined as the set of all unique nodes connected to u in any layer, i.e., $v \in Neg(u)$ if $(u,v) \in \bigcup_{m=1}^{M} E_m$. If $v \in Neg(u)$, the link (u,v)'s existing layer set is defined as $L_{(u,v)} = \{\alpha; (u,v) \in E_\alpha\}$,[2] and the corresponding (u,v) link at layer α is denoted as $(u,v)_\alpha$.

For an information spreading process in a graph $G = (E,V)$ of size n with source $s \in V$, given the informed probability set $[\mathbb{P}_{t,s}(u)]_{u\in V}$ before round t, after gossiping at round t the informed probability set $[\mathbb{P}_{t+1,s}(u)]_{u\in V}$ is given by

$$\begin{aligned}
\mathbb{P}_{t+1,s}(u) =&(1 - (1 - \mathbb{P}_{t,pull}(u))(1 - \mathbb{P}_{t,push}(u))) \\
&\times (1 - \mathbb{P}_{t,s}(u)) + \mathbb{P}_{t,s}(u), \forall u \in V,
\end{aligned} \tag{4}$$

where $\mathbb{P}_{t,pull}(u)$ is the probability that node u pulls the information from its neighbor nodes in round t, and $\mathbb{P}_{t,push}(u)$ is the probability that node u gets the information by neighbors' push operation in round t. In the single network, node u successfully pulls the information from one of its neighbors when it contacts a node already informed, therefore the pull probability is given by

$$\mathbb{P}_{t,s,pull}(u) = \sum_{v\in Neg(u)} \frac{1}{d(u)} \mathbb{P}_{t,s}(v), \tag{5}$$

where $d(u)$ is the degree of u.

Also, node u successfully gets pushed the information by its neighbors when any of its neighbors already informed contacts u, so the push probability is given by

[2] Link (u,v) may exist in several layers simultaneously.

$$\mathbb{P}_{t,s,push}(u) = 1 - \prod_{v \in Neg(u)} \Big(1 - \frac{1}{d(v)}\mathbb{P}_{t,s}(v)\Big). \quad (6)$$

For a multiplex network G with M layers, node u successfully pulls the information when in any layers it successfully pulls the information. Therefore, the pull probability for an uninformed node u in network G is

$$\mathbb{P}_{t,pull}(u) = 1 - \prod_{m=1}^{M} \Big(1 - \sum_{v \in Neg_m(u)} \frac{1}{d_m(u)}\mathbb{P}_{t,s}(v)\Big), \quad (7)$$

For the push operation, in each round, node u gets pushed the information if any of its possible neighbor nodes already informed contacts it. For any $v \in Neg(u)$, the probability that v contacts node u is $1 - \prod_{l \in L_{(u,v)}} (1 - \frac{1}{d_l(v)})$. Therefore, the push probability for an uninformed node u in network G is

$$\mathbb{P}_{t,push}(u) = 1 - \prod_{v \in Neg(u)} \Bigg(1 - \Bigg(1 - \prod_{l \in L_{(u,v)}} \Bigg(1 - \frac{1}{d_l(v)}\Bigg)\Bigg)\mathbb{P}_{t,s}(v)\Bigg). \quad (8)$$

Remark 2. Due to the gossip nature, at each layer, an uninformed node only attempts to pull the information from one neighbor, but may get pushed the information from multiple neighbors. This accounts for the different expressions in Eq. (5) and Eq. (6) when considering the overlapping edges across the layers.

The informed probability model facilitates the demonstration of the beneficial effect of the multiplex network structure qualitatively, as shown in Theorem 1 below.

Theorem 1. Consider a multiplex network $G^{(1)} = \{G_1^{(1)}, ..., G_{M_1}^{(1)}\}$, and another multiplex network $G^{(2)}$ built upon $G^{(1)}$ by adding additional $M_2 - M_1$ layers, i.e., $G^{(2)} = \{G_1^{(1)}, ..., G_{M_1}^{(1)}, G_{M_1+1}^{(2)}, ..., G_{M_2}^{(2)}\}, M_2 > M_1$. Let $T_{spr}(G^{(1)}, \gamma)$ and $T_{spr}(G^{(2)}, \gamma)$ be the information spreading time for $G^{(1)}$ and $G^{(2)}$, respectively, then $T_{spr}(G^{(1)}, \gamma) \geq T_{spr}(G^{(2)}, \gamma)$.

Proof: The key point is to look into the informed probability $\mathbb{P}_{t,s}(u)$ for each node u. From Eq. (7), the pull probability of node u in network $G^{(1)}$ is

$$\mathbb{P}_{t,s,pull}^{(1)}(u) = 1 - \prod_{m=1}^{M_1} \Big(1 - \sum_{v \in Neg_m^{(1)}(u)} \frac{1}{d_m(u)}\mathbb{P}_{t,s}^{(1)}(v)\Big). \quad (9)$$

Similarly for multiplex network $G^{(2)}$, the pull probability of node u is

$$\begin{aligned}\mathbb{P}_{t,s,pull}^{(2)}(u) &= 1 - \prod_{m=1}^{M_2} \Big(1 - \sum_{v \in Neg_m^{(2)}(u)} \frac{1}{d_m(u)}\mathbb{P}_{t,s}^{(2)}(v)\Big) \\ &= 1 - \prod_{m=1}^{M_1} \Big(1 - \sum_{v \in Neg_m^{(1)}(u)} \frac{1}{d_m(u)}\mathbb{P}_{t,s}^{(2)}(v)\Big) \\ &\times \prod_{m=M_1+1}^{M_2} \Big(1 - \sum_{v \in Neg_m^{(2)}(u)} \frac{1}{d_m(u)}\mathbb{P}_{t,s}^{(2)}(v)\Big). \end{aligned} \quad (10)$$

By Eq. (8), the push probabilities of node u in network $G^{(1)}$ and $G^{(2)}$ are

$$\begin{aligned}&\mathbb{P}_{t,s,push}^{(1)}(u) \\ &= 1 - \prod_{v \in Neg^{(1)}(u)} \Bigg(1 - \Big(1 - \prod_{l \in L_{(u,v)}^{(1)}} (1 - \frac{1}{d_l(v)})\Big)\mathbb{P}_{t,s}^{(1)}(v)\Bigg), \\ &\mathbb{P}_{t,s,push}^{(2)}(u) \\ &= 1 - \prod_{v \in Neg^{(2)}(u)} \Bigg(1 - \Big(1 - \prod_{l \in L_{(u,v)}^{(2)}} (1 - \frac{1}{d_l(v)})\Big)\mathbb{P}_{t,s}^{(2)}(v)\Bigg). \end{aligned} \quad (11)$$

Next, we need to show $\mathbb{P}_{t,s}^{(1)}(u) \leq \mathbb{P}_{t,s}^{(2)}(u), \forall t \in \mathbb{Z}, u, s \in V$. By induction, first given the same starting node s, $\mathbb{P}_{1,s}^{(1)}(u) = \mathbb{P}_{1,s}^{(2)}(u), \forall u \in V$. Then if $\mathbb{P}_{t,s}^{(1)}(u) \leq \mathbb{P}_{t,s}^{(2)}(u)$, from Eq. (7), $\mathbb{P}_{t,pull}^{(1)}(u) \leq \mathbb{P}_{t,pull}^{(2)}(u)$ since $M_2 > M_1$. Since $G^{(2)}$ is built upon $G^{(1)}$ by adding additional layers $G_{M_1+1}^{(2)}, ..., G_{M_2}^{(2)}$, then $Neg^{(1)}(u) \subset Neg^{(2)}(u)$ and $L_{(u,v)}^{(1)} \subset L_{(u,v)}^{(2)}, \forall u, v \in V$. Therefore, from Eq. (8), $\mathbb{P}_{t,push}^{(1)}(u) \leq \mathbb{P}_{t,push}^{(2)}(u)$. Finally,

$$\begin{aligned}&\mathbb{P}_{t+1,s}^{(1)}(u) \\ &= (1 - (1 - \mathbb{P}_{t,s,pull}^{(1)}(u))(1 - \mathbb{P}_{t,s,push}^{(1)}(u)))(1 - \mathbb{P}_{t,s}^{(1)}(u)) \\ &\quad + \mathbb{P}_{t,s}^{(1)}(u) \\ &\leq (1 - (1 - \mathbb{P}_{t,s,pull}^{(1)}(u))(1 - \mathbb{P}_{t,s,push}^{(1)}(u)))(1 - \mathbb{P}_{t,s}^{(2)}(u)) \\ &\quad + \mathbb{P}_{t,s}^{(2)}(u) \\ &\leq (1 - (1 - \mathbb{P}_{t,s,pull}^{(2)}(u))(1 - \mathbb{P}_{t,s,push}^{(2)}(u)))(1 - \mathbb{P}_{t,s}^{(2)}(u)) \\ &\quad + \mathbb{P}_{t,s}^{(2)}(u) \\ &= \mathbb{P}_{t+1,s}^{(2)}(u), \forall u \in V. \end{aligned} \quad (12)$$

Therefore, $\mathbb{P}_{t,s}^{(1)}(u) \leq \mathbb{P}_{t,s}^{(2)}(u), \forall t \in \mathbb{Z}, u, s \in V$. Then the proof is finished up by contradiction: given source node s, assume the information spreading times for networks $G^{(1)}$ and $G^{(2)}$ are denoted by T_1 and T_2, respectively. Then by definition $\max_u\{(1 - \mathbb{P}_{T_1,s}^{(1)}(u))\} \leq O(n^{-\gamma-1})$ and $\max_u\{(1 - \mathbb{P}_{T_2,s}^{(2)}(u))\} \leq O(n^{-\gamma-1})$. Also,

$$\max_u\{(1 - \mathbb{P}_{T_1,s}^{(2)}(u))\} \leq \max_u\{(1 - \mathbb{P}_{T_1,s}^{(1)}(u))\} \leq O(n^{-\gamma-1}). \quad (13)$$

If $T_2 \geq T_1$, then the above conclusion $\max_u\{(1 - \mathbb{P}_{T_1,s}^{(2)}(u))\} \leq O(n^{-\beta-1})$ contradicts the definition $T_2 = \inf\{t : \max_u\{(1 - \mathbb{P}_{t,s}^{(2)}(u))\} \leq O(n^{-\beta-1})\}$.

Therefore, for any given source node $s \in V$, $T_{spr}(G^{(2)}, \gamma, s) \leq T_{spr}(G^{(1)}, \gamma, s)$. Finally, by Def. 2, $T_{spr}(G^{(2)}, \gamma) \leq T_{spr}(G^{(1)}, \gamma)$. □

3.2 Idealized Information Spreading Time

Compared to the qualitative analysis above, the quantitative analysis for information spreading in the multiplex network is much more complicated, which is mainly caused by two factors: overlapping edges among layers and heterogeneous contacting probabilities at different layers. They render the exact estimation of information spreading time in the multiplex network intractable. Therefore, an idealized setting is considered in the following so that the corresponding information spreading time can be analyzed, which serves as a good lower bound for the information spreading time of the original multiplex network. First, to handle the overlapping edges among layers, an aggregated multigraph representation for a multiplex network is constructed as follows.

Definition 4. Given a multiplex network $G = \{G_m \triangleq (V, E_m)\}_{m=1}^{M}$, the corresponding aggregated multigraph $\tilde{G} = (\tilde{V}, \tilde{E})$ is defined such that $\tilde{V} = V$ and $\tilde{E} = \uplus_{m=1}^{M} E_m \triangleq \{(u,v)_\alpha; u \in V, v \in Neg(u), \alpha \in L_{(u,v)}\}$, where $\uplus$ stands for the non-unique set union, i.e., the same links at different layers are all kept.

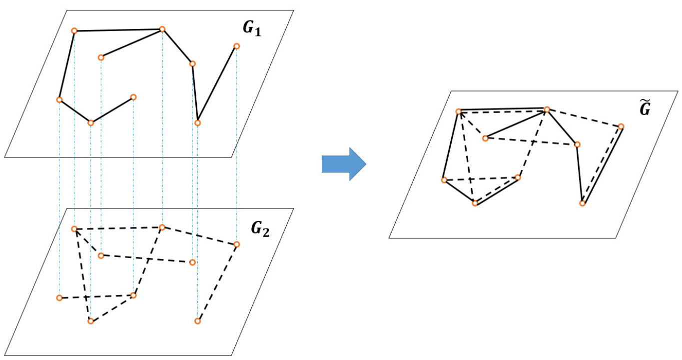

Fig. 1: From Multiplex Network to Aggregated Multigraph

Remark 3. As shown in Fig. 1, an aggregated multigraph is obtained from the corresponding multiplex network through a condensation process where the node set remains the same and all links in all layers from the original multiplex network are kept. As a result, multiple links can exist between the same pair of nodes u and v in the aggregated multigraph if they are connected in multiple layers of the original multiplex network (e.g., $(u,v) \in E_\alpha, (u,v) \in E_\beta, ...$). In the aggregated multigraph $\tilde{G}$, these links are considered different and marked as $(u,v)_\alpha, (u,v)_\beta, ...$, respectively.

To get around heterogeneous gossip at different layers, node u's contacting probability for each link $(u,v)_\alpha$ is unified as $\mathscr{P}(u) = \mathbb{P}((u,v)_\alpha) = \frac{1}{\min\limits_{m \in \{1,\ldots,M\}} d_m(u)}$ (denoted as link picking probability). This over-optimistic choice simplifies our analysis, while still providing a good lower bound as shown below.

The idealized setting is formed through a uniform gossip with link picking probability $\mathscr{P}(u), \forall u \in V$, on the aggregated multigraph $\tilde{G}$ constructed above. The information spreading time in this idealized setting for an arbitrary multiplex network is quantified below.

Definition 5. The *multiplex conductance* Φ_{mp} of a multiplex network $\{G_m \triangleq (V, E_m)\}_{m=1}^{M}$ is defined as

$$\Phi_{\text{mp}} = \min_{S \subset V, vol_T(S) \leq |E|_T} M \frac{|cut_T(S, V-S)|}{vol_T(S)}, \tag{14}$$

where $vol_T(S) = \sum\limits_{m=1}^{M} vol_m(S)$, $|E|_T = \sum\limits_{m=1}^{M} |E_m|$, and $|cut_T(S, V-S)| = \sum\limits_{m=1}^{M} |cut_m(S, V-S)|$. $vol_m(S)$ is the degree sum of all nodes in the node set S at layer m (volume), and $|cut_m(S, V-S)|$ is the number of edges between node set S and $V-S$ at layer m.

Remark 4. Multiplex conductance coincides with the original graph conductance [19] when $M = 1$. Note that the factor of M is deliberately introduced to reflect the multiple information spreading processes in the multiplex network setting.

Theorem 2. For an M-layer multiplex network with n nodes, the information spreading time in the idealized setting is at most $200(\gamma+2)\Phi_{\text{mp}}^{-1} \cdot (\log n + \frac{1}{2}\log M)$ rounds with probability $1 - O(n^{-\gamma})$, for some $\gamma > 0$, where Φ_{mp} is the multiplex conductance of this multiplex network.

First, the following sequence of random variables solely related to the pull operation is introduced to facilitate our analysis. Let $L_1, L_2, \ldots$ be a sequence of random variables with L_i $(i \geq 1)$ defined as follows. We distinguish two cases:

- If $vol_T(S_{i-1}) \leq |E|_T$, then by Definition 5, $|cut_T(S_{i-1}, U_{i-1})| \geq \frac{1}{M}\lceil \Phi_{\text{mp}} vol_T(S_{i-1}) \rceil \geq \frac{1}{M}\lceil \Phi_{\text{mp}} vol_T(S_0) \rceil$, where $U_{i-1} = V - S_{i-1}$ is the set of uninformed nodes at round $i-1$. Let $R = \lceil \Phi_{\text{mp}} vol_T(S_0) \rceil$ and E_i be an arbitrary subset of $\uplus_{m=1}^{M} cut_m(S_{i-1}, U_{i-1})$ consisting of $\frac{1}{M}R$ edges. Set E_i is (arbitrarily) fixed at the *beginning* of round i – before the round is executed. Define the *minimum volume* of a node u as $M \cdot \min\limits_m \{d_m(u)\}$. For each node $u \in U_{i-1}$, let $\delta_{i,u}$ be the $0/1$ random variable with $\delta_{i,u} = 1$ if and only if in round i node u pulls the information through some edge in E_i. Then, $L_i = \sum_{u \in U_{i-1}} (\delta_{i,u} M \cdot \min\limits_m \{d_m(u)\})$.
- If $vol_T(S_{i-1}) > |E|_T$, then $L_i = R$.

Then, the following lemma is needed to prove Theorem 2:

Lemma 1.

(a) $E[\sum_{k \leq i} L_k] = iR$ and $Var(\sum_{k \leq i} L_k) \leq iR\Delta_{max}$, where $R = \lceil \Phi_{\text{mp}} vol_T(S_0) \rceil$.

(b) If $vol_T(S_0) < \Delta_{max}$, then $Pr(vol_T(S_i) \geq \Delta_{max}) \geq 1/2$, for $i \geq 4\Delta_{max}/(\lceil \Phi_{\text{mp}} vol_T(S_0) \rceil)$.

(c) If $\Delta_{max} \leq vol_T(S_0) \leq |E|_T$ then $Pr(vol_T(S_i) \geq \min\{2vol_T(S_0), |E|_T + 1\}) \geq 1/2$, for $i \geq 4/\Phi_{\text{mp}}$.

(d) If $vol_T(S_0) > |E|_T$ then $Pr(vol_T(U_i) \leq vol_T(U_0)/2) \geq 1/2$, for $i \geq 6/\Phi_{\text{mp}}$.

Proof: The proof of Lemma 1 is provided in the appendix, available in the online supplemental material. □

Remark 5. The random variables L_i are used to approximate the increment of $vol_T(S_i)$ over the time. With the expectation and variance of the sum of random variables L_i in Lemma 1(a), different increasing speeds of $vol_T(S_i)$ are shown in Lemma 1(b), (c), and (d), respectively, for three different stages ((d) actually shows the decreasing speed of its complement $vol_T(U_i)$).

Then, we can prove Theorem 2.

Proof for Theorem 2: When only the pull operation is considered, it can be seen from Lemma 1 that the total volume of the informed node set $vol_T(S_i)$ increases in different ways in the three different stages. The information spreading analysis is then divided into three stages. In the first stage, by Lemma 1(b), if given $vol_T(S_0) < \Delta_{max}$, after at most $\lceil 4\Delta_{max}/(\lceil \Phi_m vol_T(S_0)\rceil)\rceil \leq 5\Delta_{max}/(\lceil \Phi_m vol_T(S_0)\rceil)$ rounds, the total volume of the informed node set becomes at least Δ_{max} with probability at least $1/2$. Now, divide the information spreading process in this stage into phases each comprised of $5\Delta_{max}/(\lceil \Phi_m vol_T(S_0)\rceil)$ rounds. A phase is considered successful if the total volume of the informed node set at the end of that phase is at least Δ_{max}. Therefore, in the first $\lceil 2\beta \ln n\rceil$ phases, the probability that none of them is successful is at most $(1-1/2)^{\lceil 2\beta \ln n\rceil} \leq e^{-\lceil \beta \ln n\rceil} = O(n^{-\beta})$. Therefore, with at most $\lceil 2\beta \ln n\rceil \cdot 5\Delta_{max}/(\lceil \Phi_m vol_T(S_0)\rceil) \leq 3\beta \ln n \cdot 5\Delta_{max}/(\lceil \Phi_m vol_T(S_0)\rceil)$ rounds, the total volume of the informed node set has $vol_T(S_t) \geq \Delta_{max}$ with probability at least $1 - O(n^{-\beta})$.

In the second stage, by Lemma 1(c), if $\Delta_{max} \leq vol_T(S_0) \leq |E|_T$, then with probability at least $1/2$, it takes at most $\lceil 4/\Phi_m\rceil$ rounds until the total volume of the informed node set is increased to at least $\min\{2vol_T(S_0), |E|_T + 1\}$. Similarly, divide the information spreading process in this stage into phases of $\lceil 4/\Phi_m\rceil$ rounds each. A phase is successful if the total volume of the informed node set at the end of the phase is at least $\min\{2vol_T(S_i), |E|_T + 1\}$, where S_i is the set of informed nodes at the beginning of that phase. Then, for any k, the probability that the k-th phase is successful is at least $1/2$, regardless of the outcome of the previous $k-1$ phases. Let $B(k, 1/2)$ denote the binomial random variable for k trials each with success probability of $1/2$. Then, by the Chernoff bound, the probability that fewer than $\gamma = \log |E|_T$ of the first $k = (2\beta + 4)\gamma$ phases are successful is at most

$$Pr(B(k,1/2) \leq \gamma) \leq e^{-2(\gamma - k/2)^2/k} \leq e^{-\beta\gamma} = O(n^{-\beta}), \tag{15}$$

since $|E|_T \geq n-1$. And since at most γ successful phases are required until the total volume of the informed node set exceeds $|E|_T$, it follows that with probability $1 - O(n^{-\beta})$ the number of rounds required is at most $k\lceil 4/\Phi_m\rceil \leq k \cdot 5/\Phi_m \leq (2\beta+4)(2\log n + \log M)(5/\Phi_m)$ as $|E|_T \leq M \cdot n^2$.

Finally, by Lemma 1(d), if $vol_T(S_i) > |E|_T$ then, with probability at least $1/2$, it takes at most $\lceil 6/\Phi_m\rceil$ rounds until the total volume of the uninformed node set is halved. By similar reasoning as before, we can show that once the total volume of informed nodes has exceeded $|E|_T$, then $(2\beta + 4)(2\log n + \log M)(7/\Phi_m)$ rounds suffice to inform all nodes with probability $1 - O(n^{-\beta})$.

Combining all the above three cases and applying the union bound, we obtain that, with probability $1 - O(n^{-\beta})$, all nodes get informed within $50(\beta + 2)(\log n + \frac{1}{2}\log M)(\Phi_m^{-1} + \Delta_{max}/(\lceil \Phi_m vol_T(S_0)\rceil)$ rounds given any initial informed node set S_0 when only the pull operation is considered.

Then let v_{max} be the node of maximum total degree Δ_{max} (see Definition 3). From above, the pull operation distributes the information from v_{max} to any other nodes in $50(\beta + 2)\Phi_m^{-1} \cdot (1 + \frac{\Delta_{max}}{\Delta_{max}})(\log n + \frac{1}{2}\log M) = 100(\beta + 2)\Phi_m^{-1} \cdot (\log n + \frac{1}{2}\log M)$ rounds w.h.p. Then by Lemma 13 from [19] which shows the symmetry between the pull and push operation, the push operation can also spread to v_{max} the information started at an arbitrary source node s in $100(\beta+2)\Phi_m^{-1} \cdot (\log n + \frac{1}{2}\log M)$ rounds with the same high probability. Therefore, the push-pull operation can spread the information started at s to all nodes in $200(\beta + 2)\Phi_m^{-1} \cdot (\log n + \frac{1}{2}\log M)$ rounds w.h.p. □

Remark 6. By Theorem 2, we have shown that $\Theta(\Phi_{\text{mp}}^{-1} \cdot \log n)$ is a good estimate for information spreading time in a general multiplex network in the idealized setting. It becomes a true lower bound for the actual information spreading time when the layer number of multiplex networks is sufficiently large, as shown below.

Theorem 3. Given an M-layer multiplex network $G = \{G_m\}_{m=1}^M$ with n nodes, there always exists a constant $c_n > 0$ such that, when $M > c_n$, $\Theta(\Phi_{\text{mp}}^{-1} \cdot \log n)$ is a lower bound of the actual information spreading time.

Proof: First, note that the calculated information spreading time $\Theta(\Phi_{\text{mp}}^{-1} \cdot \log n)$ is a decreasing function of M ($\frac{|cut_T(S,V-S)|}{vol_T(S)} = \Omega(\frac{1}{n^2}) = \Theta(1)$ when n is fixed), and it will go to 0 as M goes to the infinity. Further note that the information spreading time of any scheme for any underlying network topology is lower bounded by some constant. Thus, there must exists a constant $c_n > 0$ such that when $M > c_n$, $\Theta(\Phi_{\text{mp}}^{-1} \cdot \log n)$ is a lower bound of the actual information spreading time. □

While Theorem 2 is interesting in theory, in practice, M is often a modest number. In this case, if it can be further shown that the information spreading capability of each layer is accurately measured (in the order sense) by the corresponding conductance, $\Theta(\Phi_{\text{mp}}^{-1} \cdot \log n)$ can still be a good lower bound for the actual information spreading time of a multiplex network. Two special cases of special interest are further discussed below.

Remark 7. If the single-layer bound is tight for information spreading time in every layer of the given multiplex network (i.e., $T_{spr,i} = \Theta(\Phi_i^{-1} \cdot \log n)$, where Φ_i is the conductance of the ith layer), then our bound is also tight in the corresponding aggregated multigraph $\tilde{G}$ (i.e., $\tilde{T}_{spr} = \Theta(\Phi_{\text{mp}}^{-1} \cdot \log n)$).
This is intuitively correct by the construction of the aggregated multigraph and the derivation of Theorem 2. In single networks, the above bound is known tight for many graph models of interest. Prominent examples include complete graphs, random geometric graphs (RGGs), and expander graphs (which include Preferential Attachment (PA) graphs as a special case); their information spreading times are $T_C = \Theta(\log n)$, $T_{RGG} = \Theta(\frac{\log n}{r})$, and $T_{EX} = \Theta(\log n)$, respectively, with the corresponding conductance given by $\Phi_C = \frac{1}{2}$, $\Phi_{RGG} = \Theta(r)$ (r is the radius of RGGs), and $\Phi_{EX} = \Theta(1)$.

Remark 8. If the single-layer bound is essentially tight for information spreading time in every layer of the given multiplex network (i.e., $\Phi^{-1} = \Theta(D(G))$, where $D(G)$ is the diameter of the graph G, and $\Phi^{-1} = \omega(\log n)$), then our bound is also essentially tight in the corresponding aggregated multigraph $\tilde{G}$.

The information spreading time $T_{spr} = O(\Phi^{-1} \cdot \log n)$ in the single network is determined by two factors: the inverse of graph conductance Φ^{-1} and the logarithm of network size $\log n$. It is also known that a spreading algorithm in any graph needs at least $\Omega(D(G))$ rounds to finish. If $\Phi^{-1} = \Theta(D(G))$, it indicates that the inverse of graph conductance is on the same order as the graph diameter. The extra $\log n$ in the original bound is due to the distributed nature of the gossip algorithm, therefore it is essentially tight or gossip is essentially effective in such graph structures. Ring graph is such an example, for which the information spreading bound is essentially tight, since its conductance $\Phi_{ring} = \frac{2}{n}$ and diameter $D_{ring} = \frac{n}{2}$.

Many types of graphs as pointed out belong to the above two categories, except for star-like graphs, for which the conductance $\Phi_{star} = 1$ and the information spreading time $T_{star} = 1$. Therefore, we expect that a multiplex network comprised of component layers constructed (or well modeled) by the above graphs will admit $\Theta(\Phi_{\text{mp}}^{-1} \cdot \log n)$ as a good lower bound for information spreading time[3]. In this case, the improvement in information spreading efficiency from a single network to a multiplex network can be orderly determined by $P_I = \frac{\Phi_{\text{mp}}}{\Phi}$.

3.3 Multiplex Conductance Evaluation and Performance Improvement

Our main result above, as a generalization and advance of the current art of the single network study, nicely encapsulates the gossip-based information spreading performance in a multiplex network into a metric, its multiplex conductance. In this part, the multiplex conductance is evaluated for some interesting multiplex networks to facilitate understanding.

As can be seen from Definition 5, the multiplex conductance is calculated through jointly minimizing the overall cut-volume ratio of the multiplex network. Evaluation of the single network conductance is in general a hard problem already, and existing results are mainly obtained for graphs with certain nice properties and usually in the order form [20, 22]. The existence of multiple layers and interactions among layers further complicates the evaluation. Therefore, as a first work in this area, we will focus on evaluating some multiplex networks with special structures like ring graphs, which allow us to obtain some inspiring results. The study of the multiplex conductance for multiplex networks with other types of network topologies will be investigated in our future work. Based on the definition of multiplex conductance, two key factors contribute to higher information spreading efficiency in multiplex networks as compared to the single network: 1) availability of multiple channels M, and 2) larger contacting opportunities $\frac{|cut_T(S,V-S)|}{vol_T(S)}$. There may be a common misconception that the best achievable improvement for information spreading in a multiplex network formed with similar-topology layers would be on the order of M. Our first example below dispels this misconception through an innovative design exploiting the second factor above.

3.3.1 Proposed Ring-Ring Multiplex Network

In this example, a novel ring-ring multiplex network structure is proposed to demonstrate the impact of increased contacting opportunities in a multiplex network, for which an improvement as large as $\Theta(\sqrt{n})$ can be achieved. In this proposed multiplex network structure (as shown in Fig. 2), nodes are identified with the integers $1, 2, ..., n$. In the first layer, each node with ID i connects with two nodes with IDs $((i+1) \mod n)$ and $((i-1) \mod n)$, which forms a normal ring structure. Next, assuming without loss of generality that $n+1$ is not a prime number, find out its factor r that is closet to $\sqrt{n}$. Then in the second layer, node with ID i is connected with two nodes with IDs $((i+r) \mod n)$ and $((i-r) \mod n)$ instead. Consider the right connection of each node, then in the second layer starts from node with ID 1, after $\frac{n+1}{r}$ such connections, it will reach the node with ID $(1+(n+1)) \mod n$ which is 2. The same process repeats for n times, the right connections will reach back to node 1, therefore the second layer is guaranteed with a ring structure.

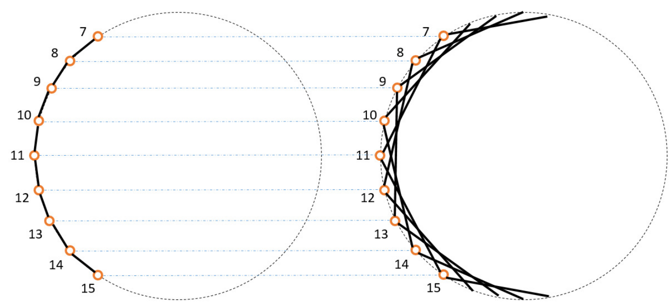

Fig. 2: Proposed Ring-Ring Structure $r = 4$

Then the multiplex conductance of this network structure is given by the following corollary:

Corollary 1. The multiplex conductance of the above proposed ring-ring multiplex network is $\Phi_{\text{mp}} = \frac{2\min\{r+1,(n+1)/r+1\}}{n}$.

Proof: The proof of Corollary 1 is provided in the appendix, available in the online supplemental material. □

In the proposed ring-ring structure, nodes in two layers maintain completely different connections, which is close to the idealized setting in the previous discussion. Therefore, as shown in Fig. 3, the predicted information spreading efficiency improvement $P_I = \frac{\Phi_{\text{mp}}}{\Phi} = \min\{r+1, (n+1)/r+1\}$ is quite tight in this scenario.

Remark 9. The second layer of the proposed ring-ring structure dramatically increases the size of cut (e.g., the size of the half-half cut is 2 in the original network but it becomes $2r+2$ in the proposed ring-ring structure) through the rewiring of nodes without increasing the

[3] Note that different layers may have different structures (such as one PA layer coupled with one RGG layer) so long as the single-layer bound is tight in each.

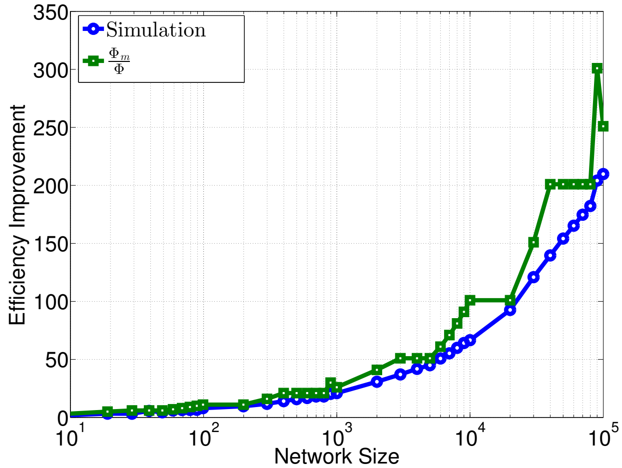

Fig. 3: Proposed Ring-Ring

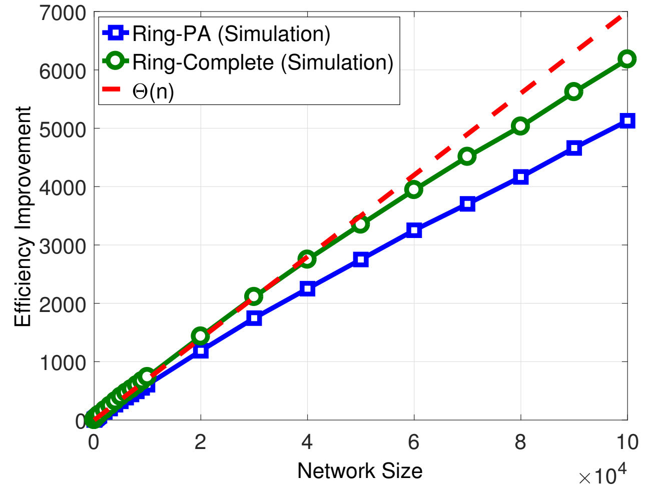

Fig. 4: Ring-Complete and Ring-PA

set volume. This smart design leads to an information spreading efficiency improvement on the order of $\min\{r+1, (n+1)/r+1\}$, which is close to $\sqrt{n}$ in most cases.

3.3.2 Different-Structure Multiplex Network

This example reveals increased contacting opportunities in a multiplex network from a new perspective. Note that multiplex networks don't need to be constrained by construction of similar-topology layers. Intuitively, a marriage of different structures can lead to a dramatic improvement in information spreading for the disadvantaged layer, as shown by the following two cases.

- *Ring-Complete Coupling:* If a ring graph is coupled with another complete graph to form a ring-complete multiplex network, the corresponding multiplex conductance may be computed as

$$\Phi_{\text{mp}} = \min_{|S|\le n/2} 2\frac{2+|S|(n-|S|)}{2|S|+|S|(n-1)} = \Theta(1). \tag{16}$$

  Then the performance improvement is $P_I = \Theta(\frac{\Phi_{\text{mp}}}{\Phi}) = \Theta(n)$.
- *Ring-PA Coupling:* Consider a PA graph with the parameter d, i.e., a new node connects to d existing nodes with probabilities proportional to their degrees, then the corresponding multiplex conductance of a ring-PA multiplex network is given by

$$\begin{aligned}
\Phi_{\text{mp}} = & \min_{\substack{S\in V,\\ vol_T(S)\le|E|_T}} 2\frac{|cut_r(S,V-S)|+|cut_{PA}(S,V-S)|}{vol_r(S)+vol_{PA}(S)} \\
\ge & \min_{\substack{S\in V,\\ vol_T(S)\le|E|_T}} 2\frac{2+|cut_{PA}(S,V-S)|}{2|S|+vol_{PA}(S)} \\
\approx & \min_{\substack{S\in V,\\ vol_T(S)\le|E|_T}} 2\frac{|cut_{PA}(S,V-S)|}{2|S|+vol_{PA}(S)} \\
\ge & \min_{\substack{S\in V,\\ vol_T(S)\le|E|_T}} 2\frac{|cut_{PA}(S,V-S)|}{2|S|+d|S|+|cut_{PA}(S,V-S)|} \\
\ge & \frac{2\alpha}{2+d+\alpha} = \Theta(1),
\end{aligned} \tag{17}$$

where α is the edge expansion of the PA graph. The last two inequalities are by the expansion properties of PA graphs [22]. Again the performance improvement is $\Theta(n)$. Both results are verified in Fig. 4.

3.3.3 Identical-Graph Multiplex Network

Finally, the impact of layer number M is explored, considering a special type of multiplex networks formed with M layers of identical topology. When M identical graphs having conductance Φ form a multiplex network, the overall multiplex conductance can be shown as

$$\begin{aligned}
\Phi_{\text{mp}} &= \min_{S\in V, vol_T(S)\le|E|_T} M\cdot\frac{M\cdot|cut(S,V-S)|}{M\cdot vol(S)} \\
&= M\Phi.
\end{aligned} \tag{18}$$

Therefore, an improvement of order M is expected for information spreading efficiency in this setting, which is generally an over-estimate, as the effect of link collision at different layers is ignored. This effect is most severe for the identical-layer structure, and can be partially corrected by considering the average meaningful contact for each node. For a node i with degree d_i in each layer, the average number of meaningful contacts in each time slot after eliminating duplicate contacts is $d_i(1-(1-\frac{1}{d_i})^M)$. Therefore, the information spreading efficiency improvement for the whole network is upper bounded by $\Delta(1-(1-\frac{1}{\Delta})^M)$, where Δ is the largest node degree across all layers in the network. By dividing the single layer information spreading time by M and $\Delta(1-(1-\frac{1}{\Delta})^M)$, respectively, the idealized and corrected information spreading time can be calculated as shown by the bottom three grouped curves (with diamond markers) and middle three grouped curves (with square markers) in Fig. 5. When Δ is small (like in the ring graph), this simple correction leads to an improvement as shown in Fig. 5. For multiplex networks constructed by independent layers, the over-estimation error is usually not a concern.

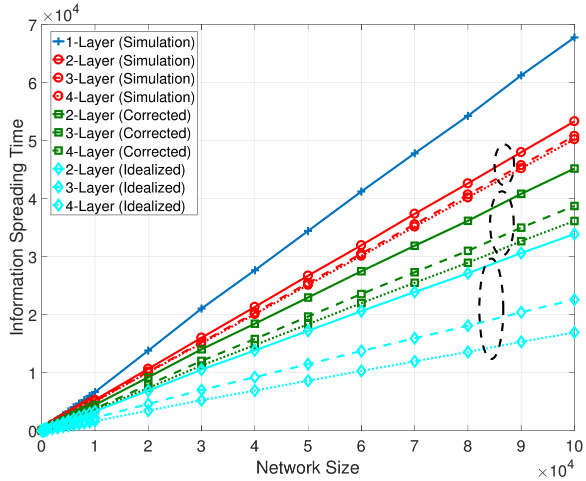

Fig. 5: Information Spreading in Multiplex Networks with Identical Ring Graphs

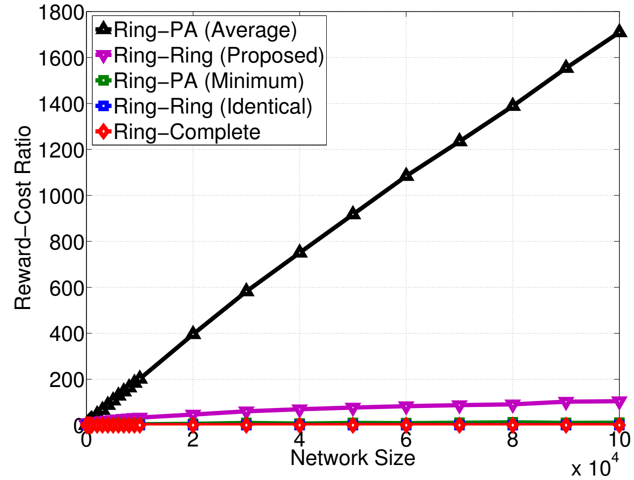

Fig. 6: User Aspect

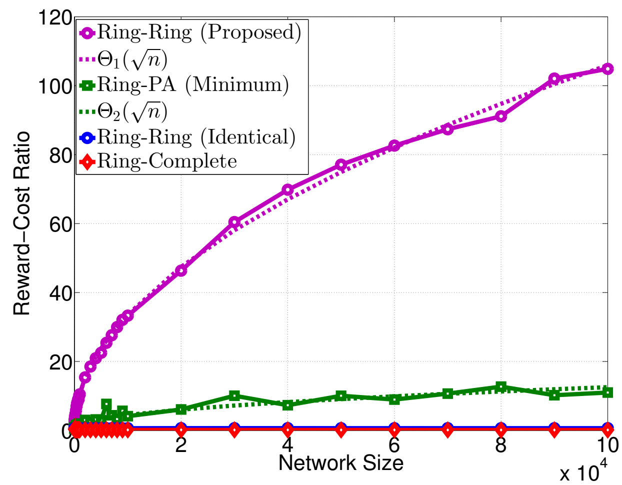

Fig. 7: Enlargement of Fig. 6

4 Layer Cost and Information Spreading Efficiency

In real life, the adoption of a new layer comes with an additional cost. In this section, the tradeoff between the improvement of information spreading efficiency and the additional layer cost is discussed. In particular, we will exam two types of cost below.[4]

- *Network Cost for a User:* The cost of additional layers from a user u's aspect is measured by the total degree of this node in all layers, i.e., $C_{ML}(u) = \sum_{m=1}^{M} d_m(u)$. Therefore, the corresponding cost increase for the user u is $C_{I,U}(u) = \frac{\sum_{m=1}^{M} d_m(u)}{d(u)}$, where $d(u)$ is the node degree of u in the initial single network.
- *Network Cost for the Network Designer:* Different from the user's aspect, for a network designer, the cost of a new layer is better measured by the number of total edges of it. Then the cost increase for the network designer is $C_{I,N} = \frac{\sum_{m=1}^{M} |E_m|}{|E|}$, where $|E|$ is the number of total edges of the initial single network.

With the cost increment C_I for each multiplex network, and the corresponding information spreading efficiency improvement P_I, the reward-cost ratio can be defined as $RC = \frac{P_I}{C_I}$. In the following, four related cases are examined to shed some light on the tradeoff between performance improvement and cost:

1) From a **ring graph** to a **two-identical-ring multiplex network**: In this case, the information spreading efficiency improvement is 2. The cost increments for both the user and the network designer are $C_{I,U} = C_{I,N} = 2$. Then the reward-cost ratios are $RC_U = RC_N = 1$.
2) From a **ring graph** to a **two-layer ring-complete multiplex network**: In this case, the performance improvement is $P_I = \Theta(\frac{\Phi_{\text{mp}}}{\Phi}) = \Theta(n)$ by Eq. (16). The cost increments for both the user and the network designer are $C_{I,U} = C_{I,N} = \frac{n+1}{2}$. Then the reward-cost ratios are $RC_U = RC_N = \Theta(1)$. Therefore, even though the ring-complete coupling brings dramatic improvement in information spreading efficiency, the associated cost somewhat offsets its effectiveness.
3) From a **ring graph** to a **two-layer ring-preferential-attachment (PA) multiplex network**: In this case, the performance improvement is $P_I = \Theta(n)$ by Eq. (17). But the cost increment is now different for the user and network designer, since the cost increments are different for different nodes due to network irregularity. Both the maximum cost increment $C_{I,U}^{max}$ and average cost increment $C_{I,U}^{ave}$ are considered for the user. Since the largest node degree in the PA network with n nodes grows as $\Theta(\sqrt{n})$ [23], the maximum cost increment is $C_{I,U}^{max} = \Theta(\sqrt{n})$. However, the average node degree for the PA network is $2d$. Then the average cost increment is $C_{I,U}^{ave} = d + 1$. Therefore, the minimum reward-cost ratio and average reward-cost ratio for the user are $RC_U^{min} = \Theta(\sqrt{n})$ and $RC_U^{ave} = \Theta(n)$, respectively. For the network designer, the cost increment is $C_{I,N} = d + 1$, and the reward-cost ratio is $RC_N = \Theta(n)$. Clearly, the ring-PA coupling is much more effective

[4]The choices of cost in this paper mainly serve to facilitate relevant discussion. In practice, other meaningful costs may also be considered.

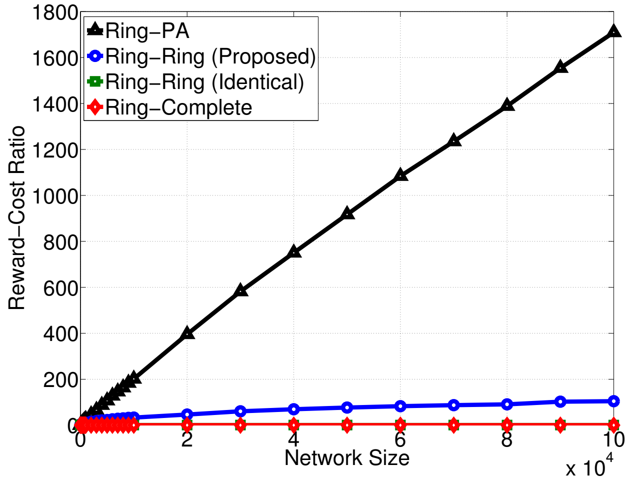

Fig. 8: Designer Aspect

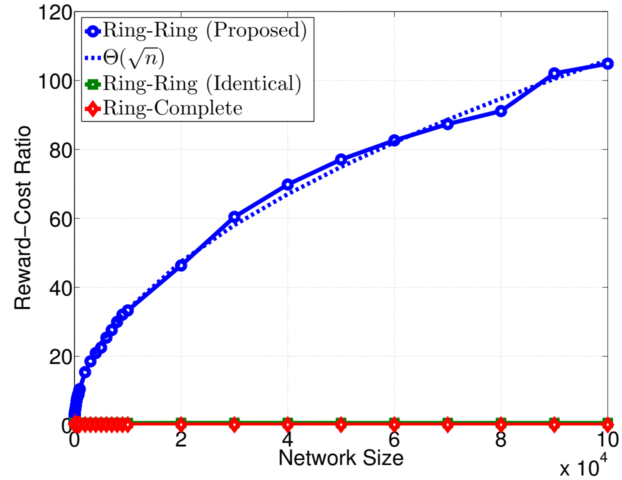

Fig. 9: Enlargement of Fig. 8

as compared to the ring-complete coupling when the layer cost is explicitly considered.

4) From a **ring graph** to the **ring-ring structure proposed in 3.3**: It has been shown that the efficiency improvement is $P_I = \min\{r+1, (n+1)/r+1\}$. The cost increments for both the user and the network designer are $C_{I,U} = C_{I,N} = 2$, then the reward-cost ratios are $RC_U = RC_N = \frac{\min\{r+1,(n+1)/r+1\}}{2}$, which is close to $\Theta(\sqrt{n})$ in most scenarios.

Remark 10. Figs. 6 and 7 simulate the actual reward-cost ratios from the user's perspective, while Figs. 8 and 9 present the results from the network designer's viewpoint. From both the discussion above and simulations, it can be seen that, overall speaking, the ring-PA coupling is the most effective multiplex structure as far as information spreading is concerned. Since PA graph is a popular model for social platforms, this result partially reveals the beneficial impact of utilizing social networks for information distribution.

On the other hand, the preferential attachment design is much more complicated as compared with our proposed ring-ring structure. Therefore, as far as the overall network design is concerned (instead of utilizing existing network structures), our proposed structure provides the insight that a careful planning of the new layer structure can greatly improve the overall efficiency for information spreading without incurring much additional cost.

The goal of this section is to study the benefit of facilitating the information spreading in multiplex networks through additional layers. We picked three different topologies, ring graph, complete graph, and PA graph, as candidates for each layer of multiplex networks. The complete graph represents the idealized single network structure which serves as a base line compared to any other topologies. The PA graph, which has a large amount of edges, a very small diameter, and is well-connected, is chosen as an extreme but realizable topology for a single network. In reality, it corresponds to the social network that greatly speeds up the information spreading process on top of the traditional communication networks. Despite of the effectiveness in spreading information, its existence requires a huge amount of management and computation resource. The ring network, which has a much smaller number of edges, a large diameter, and is very poor-connected, is chosen as another extreme but realizable case opposed to the PA network. And, in reality, it can correspond to the type of networks that a small group of people want to build to facilitate their own information spreading with a limited budget (small number of edges) and the real spatial constraint (poor-connected) on top of the existing social and communication networks.

5 Conclusion and Future Work

In this work, the gossip based information spreading is studied in multiplex networks. By defining the new metric multiplex conductance Φ_{mp}, $\Theta(\Phi_{\text{mp}}^{-1} \cdot \log n)$ is found to be a good estimate for information spreading in many multiplex networks of interest. The multiplex conductance is then evaluated for several interesting multiplex networks to help understand information spreading potentials of multiplex networks. By further taking the additional layer cost into consideration, the tradeoff between the cost and the information spreading efficiency improvement is discussed from both the user's and the network designer's perspective to facilitate the understanding in this burgeoning area.

Many interesting problems remain open in this research field. We are able to estimate the information spreading time only with the idealized setting. In future work, we plan to estimate the actual information spreading time for the gossip based model in multiplex networks. And we also plan to extend this study to more general network settings like multilayer networks. Meanwhile, the expected spreading time and extinction time has not been studied in the other information spreading model-SI/SIR like epidemic model, which will be our focus in the next step. Finally, we plan to look into other interesting networking problems like community detection and influence maximization in multilayer and multiplex network settings.

Acknowledgement

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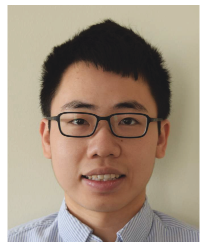

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