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# Minimal Hypersurfaces with Arbitrarily Large Area

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For $3 \leq n \leq 7$, we prove that a bumpy closed Riemannian $n$-manifold contains a sequence of connected embedded closed minimal surfaces with unbounded area.

## 1 Introduction

The goal of this note is to prove the following result.

**Theorem 1.1.** For $3 \leq n \leq 7$, suppose that $(M^n, g)$ is a connected, closed Riemannian $n$-manifold with a bumpy metric. Then, there is a sequence $\Sigma_j \subset (M^n, g)$ of connected, embedded, closed minimal surfaces with $\text{area}(\Sigma_j) \to \infty$.

We recall here the following standard definition.

**Definition 1.2.** We say that a metric $g$ on a Riemannian manifold $M^n$ is *bumpy* if there is no immersed closed minimal hypersurface $\Sigma^{n-1}$ with a non-trivial Jacobi field.

By work of White [21, 22], bumpy metrics are generic in the sense of Baire category. Here, "generic" will always mean in the Baire category sense.

The key quantifier in Theorem 1.1 is *connected*. Indeed, thanks to the resolution of Marques–Neves's multiplicity-one conjecture by the authors [3] for $n = 3$ and recently by Zhou [23] for $3 \leq n \leq 7$, if one does not require that the $\Sigma_j$ are connected, then

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Theorem 1.1 would be an immediate consequence of the following statement concerning the existence of minimal surfaces $\Sigma_p$ realizing the $p$-widths (note that if $(M^n, g)$ does not satisfy the Frankel property, then it is not clear that the number of connected components of $\Sigma_p$ is uniformly bounded).

**Theorem 1.3** ($n = 3$ [3], $3 \leq n \leq 7$ [14, 23]). For $3 \leq n \leq 7$, suppose that $(M^n, g)$ is a closed Riemannian $n$-manifold with a bumpy metric. Then, there is a constant $C = C(M, g) > 0$ so that for each positive integer $p$, there is a smooth embedded closed minimal surface $\Sigma_p$ so that

- each component of $\Sigma_p$ is two sided,
- the area of $\Sigma_p$ satisfies $C^{-1}p^{\frac{1}{n}} \leq \mathrm{area}_g(\Sigma_p) \leq Cp^{\frac{1}{n}}$, and
- the index of $\Sigma_p$ is satisfies $\mathrm{ind}(\Sigma_p) = p$.

We emphasize that the $p$-widths were introduced by Gromov, Guth, and Marques–Neves [7, 10, 13] in the Almgren–Pitts setting (as considered in the work of Zhou [23]) and were understood in the Allen–Cahn setting (as considered in the work of the authors [3]) by Gaspar–Guaraco [6, 8].

Recently, there has been exciting progress on the existence and behavior of min-max minimal hypersurfaces in Riemannian manifolds; besides those works already mentioned, we point out that Liokumovich–Marques–Neves [12] have proven a Weyl law for the $p$-widths. This a key component of the proof by Irie–Marques–Neves that minimal surfaces are generically dense [11]. These results were extended to the Allen–Cahn setting by Gaspar–Guaraco [9]. Marques–Neves–Song have proven that, generically, there is an equidistributed set of minimal hypersurfaces [15]. Song has proven Yau's conjecture on infinitely many minimal surfaces in all cases [20]. Finally, we note that Zhou–Zhu's min-max theory for prescribed mean curvature [24] was instrumental in the proof of Zhou's multiplicity-one result.

Theorem 1.1 is a consequence of the following statement.

**Theorem 1.4.** For $3 \leq n \leq 7$, suppose that $(M^n, g)$ is a connected Riemannian $n$-manifold with a bumpy metric. Then, either:

1. there exists a sequence of connected closed embedded stable minimal hypersurfaces $\Sigma_j \subset (M^n, g)$ with $\mathrm{area}_g(\Sigma_j) \to \infty$ or
2. the hypersurfaces $\Sigma_p$ from Theorem 1.3 have at least one connected component $\Sigma_p'$ with $\mathrm{area}_g(\Sigma_p') \geq Cp^{\frac{1}{n}}$ for some $C = C(M, g) > 0$ independent of $p$.

We note that for $(M^3, g)$ with a bumpy metric of positive scalar curvature, the first case never occurs by [1, 2], and moreover, we can conclude that $\mathrm{ind}(\Sigma_p') \to \infty$. On the other hand, Colding–Minicozzi have shown that any 3-manifold admits a (bumpy) metric in which the first alternative occurs [4].

We further note that in [3] we have shown how to use the monotonicity formula find a component $\Sigma_p''$ of $\Sigma_p$ with $\mathrm{genus}(\Sigma_p'') \geq C^{-1}\mathrm{ind}(\Sigma_p'') \geq p^{\frac{2}{3}}$ (the genus bound follows from the work of Ejiri–Micallef [5]). The same argument applies in higher dimensions to find $\Sigma_p'' \subset (M^n, g)$ with $\mathrm{ind}(\Sigma_p'') \geq Cp^{1-\frac{1}{n}}$. It is not clear to us, however, that $\Sigma_p'$ and $\Sigma_p''$ can be taken to be the same.

## 2 Proof of Theorem 1.4

**Lemma 2.1.** For $3 \leq n \leq 7$, consider $(M^n, g)$ a closed Riemannian $n$-manifold with a bumpy metric. Then for $A_0 > 0$, there are finitely many closed embedded stable minimal hypersurfaces with area at most $A_0$.

**Proof.** This follows from curvature estimates [16–18] and the proof of [4] (see also the proof of [19]). Namely, if there is a sequence $\Sigma_j$ of stable embedded minimal hypersurfaces with uniformly bounded area, then they have uniformly bounded curvature. Thus, after passing to a subsequence, they converge smoothly (possibly with multiplicity) to a stable minimal hypersuface $\Sigma_\infty$. Passing to a double cover if $\Sigma_\infty$ is one sided, we can thus find—for $j$ sufficiently large—graphs $u_j : \Sigma_\infty \to \mathbf{R}$ so that the exponential normal graph of $u_j$ is some component of $\Sigma_j$. We have that $u_j \to 0$ in $C^\infty(\Sigma_\infty)$. After normalizing by $\|u_j\|_{C^{2,\alpha}(\Sigma_\infty)}$, it is standard to pass to a further subsequence to find a non-trivial Jacobi field on (a cover of) $\Sigma_\infty$. This contradicts the bumpyness of $(M^n, g)$.

Alternatively, the implicit function theorem guarantees that any strictly stable minimal surface has a (strictly) mean convex tubular neighborhood (as before, passing to a double cover if the surface is one sided). No minimal surface (besides the original one) can be completely contained in this neighborhood by the maximum principle. However, if there were infinitely many embedded stable minimal surfaces with bounded area, then curvature estimates and area bounds would allow one to find a sequence of such surfaces converging (possibly with multiplicity) to a limiting such surface. This cannot occur by considering the aforementioned mean convex neighborhood. ■

**Lemma 2.2.** For $3 \leq n \leq 7$, consider $(M^n, g)$ a connected closed Riemannian $n$-manifold. Assume there are at most $N$ stable minimal surfaces in $(M^n, g)$. Then $M$ contains at most $\max\{2N, 1\}$ disjoint closed embedded unstable minimal hypersurfaces.

***Proof of Lemma 2.2.*** Suppose there are $Q \geq 2$ disjoint unstable embedded minimal hypersurfaces $\Sigma_1, \dots, \Sigma_Q$. Consider

$$\tilde{M} := \text{metric completion of } M \setminus \bigcup_{i=1}^{Q} \Sigma_i.$$

We can give $\tilde{M}$ the structure of a compact Riemannian manifold with boundary $\partial\tilde{M}$. Observe that each component of $\partial\tilde{M}$ is isometric to either one of the two-sided $\Sigma_i$ or the two-sided double cover of one of the one-sided $\Sigma_i$. Each $\Sigma_i$ (or its double cover) will correspond to at most two components of $\partial\tilde{M}$ in this way. Note that each component of $\partial\tilde{M}$ is unstable (for the components that correspond to two-sided $\Sigma_j$, this is clear; in the one-sided case, this follows by lifting an unstable variation to the double cover). Write $\tilde{M} = \tilde{M}_1 \cup \tilde{M}_2$, where $\tilde{M}_1$ is the union of the connected components of $\tilde{M}$ with exactly one boundary component and $\tilde{M}_2$ is the union of components of $\tilde{M}$ with at least two boundary components.

We claim that $\partial\tilde{M}_2$ has at least $Q$ components. Indeed, consider some $\Sigma_i$. Suppose first that $\Sigma_i$ is two sided and separating. Consider $\Omega_\pm$ the components of $\tilde{M}$ containing $\Sigma_i$ in their boundary. Suppose that $\Omega_- \cup \Omega_+ \subset \tilde{M}_1$. In this case, it is clear that $Q = 1$, and $\Sigma_i$ was the only unstable surface in the family, contradicting our assumption that $Q \geq 2$. Thus, at least one of the $\Omega_\pm$ is contained in $\tilde{M}_2$. This determines at least one element of $\partial\tilde{M}_2$. Now, suppose that $\Sigma_i$ is two sided but not separating. It is clear that any component of $\partial\tilde{M}$ associated to $\Sigma_i$ cannot bound a component of $\tilde{M}_1$ (otherwise $\Sigma_i$ would be separating). Finally, if $\Sigma_i$ was one sided and some component of $\partial\tilde{M}$ bounded a component of $\tilde{M}_1$, then as before, we would have $Q = 1$, a contradiction.

Given this, Lemma 2.2 is a consequence of the following claim.

**Claim 2.3.** For $3 \leq n \leq 7$, consider $(\breve{M}^n, g)$ a connected compact Riemannian manifold with $J \geq 2$ boundary components that are all unstable minimal hypersurfaces. Then, there are at least $J/2$ connected closed embedded stable minimal surfaces in the interior of $\breve{M}$.

■

***Proof of the claim.*** Write the components of $\partial\breve{M}$ as $\Gamma_1, \dots, \Gamma_J$. Because $J \geq 2$,

$$[\Gamma_i] \neq 0 \in H_{n-1}(\breve{M}; \mathbf{Z}_2).$$

Find $\Gamma_1' \in [\Gamma_1]$ minimizing area in the homology class. Because each component of the boundary is an unstable minimal surface, $\Gamma_1'$ is contained in the interior of $\check{M}$. There is thus an open set $\check{\Omega}_1 \subset \check{M}$ so that $\partial\check{\Omega}_1 = \Gamma_1 \cup \Gamma_1'$. Set $\check{M}_1 := \check{M} \setminus \check{\Omega}_1$. This manifold has $J-1$ unstable boundary components and at least one stable boundary component. For $\Gamma_2$ one of the unstable boundary components minimizes area in homology to get $\Gamma_2'$. Each component of $\Gamma_2'$ is contained in the interior of $\check{M}_1$ or coincides with one of the stable components of $\partial\check{M}_1$. In either case, we can repeat this process $J$ times. We thus consider the stable minimal surfaces $\Gamma_1', \dots, \Gamma_J'$ in $\check{M}$. We note that, by construction, the sets $\check{\Omega}_1, \dots, \check{\Omega}_J$ are all disjoint, but that some of the connected components of $\cup_{j=1}^J \Gamma_j'$ might be repeated; the claim will follow once we show each connected component occurs at most twice in this union. Indeed, consider the set $\mathcal{B}$ of pairs $(\Gamma, \check{\Omega}_i)$, where $\check{\Omega}_i$ is as above for some $i \in \{1, \dots, J\}$ and $\Gamma$ is a connected component of $\cup_{j=1}^J \Gamma_j'$ with $\Gamma \subset \partial\check{\Omega}_j$. On one hand, for each $i \in J$, there is at least one $\Gamma$ with $(\Gamma, \check{\Omega}_i) \in \mathcal{B}$. On the other hand, for any $\Gamma$, there are at most two distinct indices $a, b$ so that $(\Gamma, \check{\Omega}_a), (\Gamma, \check{\Omega}_b) \in \mathcal{B}$. Putting these two facts together, there must be at least $J/2$ distinct components $\Gamma$. ■

This completes the proof of Lemma 2.2.

***Proof of Theorem* 1.4.** If the first possibility fails, then there is a uniform area bound for embedded stable minimal hypersurfaces. Thus, Lemma 2.1 implies that there are finitely many stable embedded minimal hypersurfaces in $(M^n, g)$. Call this number $N$. Applying Lemma 2.2, we find that there are at most $N + \max\{2N, 1\}$ pairwise disjoint embedded minimal surfaces in $(M, g)$. Applied to $\Sigma_p$, there is at least one component $\Sigma_p'$ with $\text{area}(\Sigma_p') \geq Cp^{\frac{1}{n}}$. ■

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