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Regularity analysis of metamaterial Maxwell's equations with random coefficients and initial conditions☆

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Abstract

In this paper we develop and analyze the stochastic collocation method for solving the time-dependent metamaterial Maxwell's equations subject to random coefficients and random initial conditions. We provide a rigorous regularity analysis of the solution with respect to the random variables. To our best knowledge, this is the first theoretical results derived for the stochastic metamaterial Maxwell equations. The rate of convergence is proved depending on the regularity of the solution. Numerical results are presented to confirm the theoretical analysis. We also demonstrate that the backward wave propagation phenomenon still exists in random metamaterial.

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1. Introduction

In recent years, solving stochastic partial differential equations (SPDEs) has been a very hot topic research in scientific computing. One of the main reasons is that model uncertainty is ubiquitous in many complex physical systems. Examples include uncertainties appearing in the mechanical properties of many biomaterials, polymeric fluids or composite materials, initial data for weather forecasting, and wave and fluid propagate through heterogeneous random media, etc. For PDE models, input uncertainties appear in coefficients, forcing terms, boundary and initial condition data, geometry, etc (cf. [1–4]). In electromagnetics, the fluctuations in the producing process (such as the lithography) of electromagnetic materials allow us to treat the permittivity and permeability as uncertain parameters (e.g., [5,6]). In simulating signal propagation in corrugated coaxial cables [7], the physical domain has some uncertainty.

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Due to the high dimensionality of stochastic solutions, it is very challenging to efficiently solve SPDEs with uncertain inputs. Through the great efforts of many researchers, two numerical methods become quite popular in solving SPDEs. One is the stochastic Galerkin method [8–11], which shows fast convergence rates with increasing order of expansions, provided that the solution of the underlying differential equation is sufficiently smooth in the random space. However, the system of equations resulting from the stochastic Galerkin methods is coupled and quite expensive to solve especially for problems requiring high-dimensional random spaces. Another popular method is the so-called stochastic collocation method (cf. [12,13]) by taking advantage of the strength of Monte Carlo methods and the stochastic Galerkin methods. The stochastic collocation method achieves fast convergence when the solutions are sufficiently smooth in the random space. More importantly, the stochastic collocation method is simple in implementation and the system of resulting equations is decoupled and hence is efficient to solve. So far, both the stochastic Galerkin method and the stochastic collocation method have been widely used to solve various problems, such as elliptic problems (e.g., [14–17]), parabolic equations (e.g., [18]), hyperbolic equations (e.g., [19,20]), random Helmholtz problem [21], and conservation laws (e.g., [22,23]), etc. More details can be found in recent review articles [11,24,25] and monographs [26–28].

Compared to many papers published for other problems, there are not many existing works on numerical methods for solving stochastic Maxwell's equations. In 2006, Chauviére et al. [6] solved the time-dependent Maxwell's equations by using both the stochastic Galerkin method and stochastic collocation method. Detailed comparisons of both methods are made for uncertainties caused by physical materials, by the source wave and by the physical domain. In 2015, Benner and Schneider [5] described several techniques for uncertainty quantification for the time-harmonic Maxwell's equations by using stochastic collocation method. The existence and uniqueness of stochastic Maxwell equations with additive noise were investigated in [29], and multi-symplectic difference method was proposed and analyzed for solving them [30]. However, how uncertainty propagates through the stochastic Maxwell's equations and what regularity we can expect is not investigated yet as Chauviére et al. pointed out [6, p. 774]. One of the main purposes of this paper is to fill this gap.

The rest of the paper is organized as follows. In Section 2, we first present detailed regularity analysis of the metamaterial Maxwell's equations with respect to random variables. Then we establish the convergence analysis for the stochastic collocation method developed to solving this model. Numerical results are presented in Section 3 to support our theoretical analysis. We conclude the paper in Section 4.

2. Maxwell's equations with random coefficients

Let $\boldsymbol{x} \in D \subset R^3$ be the spatial coordinate, t be the time variable from set $[0, T]$, and $(\Omega, \mathcal{A}, \mathcal{P})$ be a complete probability space, whose event space is Ω ($\omega \in \Omega$ is the event) and is equipped with σ-algebra $\mathcal{A}$, and $\mathcal{P}$ is a probability measure. Furthermore, we let $\rho(\boldsymbol{y}) : \Gamma \to R^+$ be a bounded joint probability density function of an R^d-valued random variable $\boldsymbol{y} = [y_1(\omega), \ldots, y_d(\omega)]$, $\omega \in \Omega$, whose image $\Gamma := \Pi_{n=1}^d \Gamma_n$, $\Gamma_n = y_n(\Omega) \in R$.

Consider the stochastic Maxwell's equations in metamaterials [31]: Find the random electric field $\boldsymbol{E}(\boldsymbol{x}, t, \boldsymbol{y})$, magnetic field $\boldsymbol{H}(\boldsymbol{x}, t, \boldsymbol{y})$, induced electric field $\boldsymbol{J}(\boldsymbol{x}, t, \boldsymbol{y})$ and magnetic fields $\boldsymbol{K}(\boldsymbol{x}, t, \boldsymbol{y}) : D \times (0, T) \times \Omega \to R^3$ such that P-almost everywhere in Ω, i.e., almost surely (a.s.) satisfy the following equations:

$$\epsilon(\boldsymbol{x}, \boldsymbol{y}(\omega))\partial_t \boldsymbol{E} = \nabla \times \boldsymbol{H} - \boldsymbol{J}, \tag{1}$$

$$\mu(\boldsymbol{x}, \boldsymbol{y}(\omega))\partial_t \boldsymbol{H} = -\nabla \times \boldsymbol{E} - \boldsymbol{K}, \tag{2}$$

$$\frac{1}{\epsilon(\boldsymbol{x}, \boldsymbol{y}(\omega))\omega_{pe}^2(\boldsymbol{x}, \boldsymbol{y}(\omega))}\partial_t \boldsymbol{J} + \frac{\Gamma_e(\boldsymbol{x}, \boldsymbol{y}(\omega))}{\epsilon(\boldsymbol{x}, \boldsymbol{y}(\omega))\omega_{pe}^2(\boldsymbol{x}, \boldsymbol{y}(\omega))}\boldsymbol{J} = \boldsymbol{E}, \tag{3}$$

$$\frac{1}{\mu(\boldsymbol{x}, \boldsymbol{y}(\omega))\omega_{pm}^2(\boldsymbol{x}, \boldsymbol{y}(\omega))}\partial_t \boldsymbol{K} + \frac{\Gamma_m(\boldsymbol{x}, \boldsymbol{y}(\omega))}{\mu(\boldsymbol{x}, \boldsymbol{y}(\omega))\omega_{pm}^2(\boldsymbol{x}, \boldsymbol{y}(\omega))}\boldsymbol{K} = \boldsymbol{H}, \tag{4}$$

subject to random initial conditions

$$\boldsymbol{E}(\boldsymbol{x}, t = 0, \boldsymbol{y}(\omega)) = \boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{y}(\omega)), \quad \boldsymbol{H}(\boldsymbol{x}, t = 0, \boldsymbol{y}(\omega)) = \boldsymbol{H}_0(\boldsymbol{x}, \boldsymbol{y}(\omega)), \tag{5}$$

$$\boldsymbol{J}(\boldsymbol{x}, t = 0, \boldsymbol{y}(\omega)) = \boldsymbol{J}_0(\boldsymbol{x}, \boldsymbol{y}(\omega)), \quad \boldsymbol{K}(\boldsymbol{x}, t = 0, \boldsymbol{y}(\omega)) = \boldsymbol{K}_0(\boldsymbol{x}, \boldsymbol{y}(\omega)), \tag{6}$$

and the perfect conducting (PEC) boundary condition:

$$\boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{0}, \quad \text{on } \partial D, \tag{7}$$

where $\boldsymbol{E}_0, \boldsymbol{H}_0, \boldsymbol{J}_0$ and $\boldsymbol{K}_0$ are some given functions. To accommodate the uncertainty or randomness of the material, we assume that the permittivity ϵ, permeability μ, electric plasma frequency ω_{pe}, magnetic plasma frequency ω_{pm}, electric damping frequency Γ_{pe} and magnetic damping frequency Γ_{pm} are all random. Here and below, $\boldsymbol{n}$ denotes the unit outward normal vector on the boundary ∂D, where $D \subset R^3$ is a bounded polyhedral domain with a Lipschitz boundary. For simplicity, we denote ∂_{sj} the jth derivative with respect to variable s, e.g., $s = t$ and y. We like to emphasize that here and below ∇ is only for spatial variable $\boldsymbol{x}$.

To solve problem (1)–(7), we use the Lagrange interpolation approach by following [32,33,12]. We first choose a set of Gauss–Lobatto collocation points $\{\boldsymbol{y}_k\}_{k=1}^{(N+1)^d} \in \Gamma$, where $N+1$ denotes the number of collocation points in each random variable space. We then solve the following system of equations at each collocation point $\boldsymbol{y}_j, j = 1, \ldots, (N+1)^d$:

$$\epsilon(\boldsymbol{x}, \boldsymbol{y}_j)\partial_t\hat{\boldsymbol{E}}(\boldsymbol{x}, t; \boldsymbol{y}_j) = \nabla \times \hat{\boldsymbol{H}}(\boldsymbol{x}, t; \boldsymbol{y}_j) - \hat{\boldsymbol{J}}(\boldsymbol{x}, t; \boldsymbol{y}_j), \tag{8}$$

$$\mu(\boldsymbol{x}, \boldsymbol{y}_j)\partial_t\hat{\boldsymbol{H}}(\boldsymbol{x}, t; \boldsymbol{y}_j) = -\nabla \times \hat{\boldsymbol{E}}(\boldsymbol{x}, t; \boldsymbol{y}_j) - \hat{\boldsymbol{K}}(\boldsymbol{x}, t; \boldsymbol{y}_j), \tag{9}$$

$$\frac{1}{\epsilon(\boldsymbol{x}, \boldsymbol{y}_j)\omega_{pe}^2(\boldsymbol{x}, \boldsymbol{y}_j)}\partial_t\hat{\boldsymbol{J}}(\boldsymbol{x}, t; \boldsymbol{y}_j) + \frac{\Gamma_e(\boldsymbol{x}, \boldsymbol{y}_j)}{\epsilon(\boldsymbol{x}, \boldsymbol{y}_j)\omega_{pe}^2(\boldsymbol{x}, \boldsymbol{y}_j)}\hat{\boldsymbol{J}}(\boldsymbol{x}, t; \boldsymbol{y}_j) = \hat{\boldsymbol{E}}(\boldsymbol{x}, t; \boldsymbol{y}_j), \tag{10}$$

$$\frac{1}{\mu(\boldsymbol{x}, \boldsymbol{y}_j)\omega_{pm}^2(\boldsymbol{x}, \boldsymbol{y}_j)}\partial_t\hat{\boldsymbol{K}}(\boldsymbol{x}, t; \boldsymbol{y}_j) + \frac{\Gamma_m(\boldsymbol{x}, \boldsymbol{y}_j)}{\mu(\boldsymbol{x}, \boldsymbol{y}_j)\omega_{pm}^2(\boldsymbol{x}, \boldsymbol{y}_j)}\hat{\boldsymbol{K}}(\boldsymbol{x}, t; \boldsymbol{y}_j) = \hat{\boldsymbol{H}}(\boldsymbol{x}, t; \boldsymbol{y}_j), \tag{11}$$

subject to the initial conditions

$$\hat{\boldsymbol{E}}(\boldsymbol{x}, t=0, \boldsymbol{y}_j) = \boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{y}_j), \quad \hat{\boldsymbol{H}}(\boldsymbol{x}, t=0, \boldsymbol{y}_j) = \boldsymbol{H}_0(\boldsymbol{x}, \boldsymbol{y}_j), \tag{12}$$

$$\hat{\boldsymbol{J}}(\boldsymbol{x}, t=0, \boldsymbol{y}_j) = \boldsymbol{J}_0(\boldsymbol{x}, \boldsymbol{y}_j), \quad \hat{\boldsymbol{K}}(\boldsymbol{x}, t=0, \boldsymbol{y}_j) = \boldsymbol{K}_0(\boldsymbol{x}, \boldsymbol{y}_j), \tag{13}$$

and the PEC boundary condition:

$$\boldsymbol{n} \times \hat{\boldsymbol{E}}(\boldsymbol{x}, t; \boldsymbol{y}_j) = \boldsymbol{0}, \quad \text{on } \partial D, \tag{14}$$

i.e., we can simply denote the approximate solution as

$$\boldsymbol{E}^N(\boldsymbol{x}, t; \boldsymbol{y}) = \sum_{k=1}^{(N+1)^d} \hat{\boldsymbol{E}}(\boldsymbol{x}, t; \boldsymbol{y}_k)\mathcal{L}_k(\boldsymbol{y}), \boldsymbol{H}^N(\boldsymbol{x}, t; \boldsymbol{y}) = \sum_{k=1}^{(N+1)^d} \hat{\boldsymbol{H}}(\boldsymbol{x}, t; \boldsymbol{y}_k)\mathcal{L}_k(\boldsymbol{y}), \tag{15}$$

$$\boldsymbol{J}^N(\boldsymbol{x}, t; \boldsymbol{y}) = \sum_{k=1}^{(N+1)^d} \hat{\boldsymbol{J}}(\boldsymbol{x}, t; \boldsymbol{y}_k)\mathcal{L}_k(\boldsymbol{y}), \boldsymbol{K}^N(\boldsymbol{x}, t; \boldsymbol{y}) = \sum_{k=1}^{(N+1)^d} \hat{\boldsymbol{K}}(\boldsymbol{x}, t; \boldsymbol{y}_k)\mathcal{L}_k(\boldsymbol{y}), \tag{16}$$

where $\mathcal{L}_k(\boldsymbol{y})$ are the tensor-product Lagrange interpolation polynomials. In Remark 2.1, we show that $u^N(\boldsymbol{x}, t; \boldsymbol{y})$ is just the interpolation of u, denoted as $I_N^y u = \sum_{k=1}^{(N+1)^d} u(\boldsymbol{x}, t; \boldsymbol{y}_k)\mathcal{L}_k(\boldsymbol{y})$, where $u = \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{J}, \boldsymbol{K}$.

To prove the convergence rate of this scheme, we first need to establish the regularity for the solution of our model problem (1)–(7). To simplify the notation and make the proof clear, sometimes we drop the explicit dependence of all physical parameters on $\boldsymbol{x}$ and $\boldsymbol{y}$.

Remark 2.1. To justify that $\boldsymbol{E}^N(\boldsymbol{x}, t; \boldsymbol{y}) = I_N^y\boldsymbol{E}$, $\boldsymbol{H}^N(\boldsymbol{x}, t; \boldsymbol{y}) = I_N^y\boldsymbol{H}$, $\boldsymbol{J}^N(\boldsymbol{x}, t; \boldsymbol{y}) = I_N^y\boldsymbol{J}$ and $\boldsymbol{K}^N(\boldsymbol{x}, t; \boldsymbol{y}) = I_N^y\boldsymbol{K}$, we denote the errors

$$\hat{\boldsymbol{E}}^N(\boldsymbol{x}, t; \boldsymbol{y}_j) = \boldsymbol{E}^N(\boldsymbol{x}, t; \boldsymbol{y}_j) - \boldsymbol{E}(\boldsymbol{x}, t; \boldsymbol{y}_j), \;\; \hat{\boldsymbol{H}}^N(\boldsymbol{x}, t; \boldsymbol{y}_j) = \boldsymbol{H}^N(\boldsymbol{x}, t; \boldsymbol{y}_j) - \boldsymbol{H}(\boldsymbol{x}, t; \boldsymbol{y}_j),$$
$$\hat{\boldsymbol{J}}^N(\boldsymbol{x}, t; \boldsymbol{y}_j) = \boldsymbol{J}^N(\boldsymbol{x}, t; \boldsymbol{y}_j) - \boldsymbol{J}(\boldsymbol{x}, t; \boldsymbol{y}_j), \;\; \hat{\boldsymbol{K}}^N(\boldsymbol{x}, t; \boldsymbol{y}_j) = \boldsymbol{K}^N(\boldsymbol{x}, t; \boldsymbol{y}_j) - \boldsymbol{K}(\boldsymbol{x}, t; \boldsymbol{y}_j).$$

Choosing $\boldsymbol{y} = \boldsymbol{y}_j$ in (1)–(7) and subtracting the resultants from the corresponding equations of (8)–(14), we can see that $\hat{\boldsymbol{E}}^N(\boldsymbol{x}, t; \boldsymbol{y}_j)$, $\hat{\boldsymbol{H}}^N(\boldsymbol{x}, t; \boldsymbol{y}_j)$, $\hat{\boldsymbol{J}}^N(\boldsymbol{x}, t; \boldsymbol{y}_j)$ and $\hat{\boldsymbol{K}}^N(\boldsymbol{x}, t; \boldsymbol{y}_j)$ satisfy the following equations:

$$\epsilon(\boldsymbol{x}, \boldsymbol{y}_j)\partial_t\hat{\boldsymbol{E}}^N(\boldsymbol{x}, t; \boldsymbol{y}_j) = \nabla \times \hat{\boldsymbol{H}}^N(\boldsymbol{x}, t; \boldsymbol{y}_j) - \hat{\boldsymbol{J}}^N(\boldsymbol{x}, t; \boldsymbol{y}_j), \tag{17}$$

$$\mu(\boldsymbol{x}, \boldsymbol{y}_j)\partial_t\hat{\boldsymbol{H}}^N(\boldsymbol{x}, t; \boldsymbol{y}_j) = -\nabla \times \hat{\boldsymbol{E}}^N(\boldsymbol{x}, t; \boldsymbol{y}_j) - \hat{\boldsymbol{K}}^N(\boldsymbol{x}, t; \boldsymbol{y}_j), \tag{18}$$

$$\frac{1}{\epsilon(\boldsymbol{x}, \boldsymbol{y}_j)\omega_{pe}^2(\boldsymbol{x}, \boldsymbol{y}_j)}\partial_t\hat{\boldsymbol{J}}^N(\boldsymbol{x}, t; \boldsymbol{y}_j) + \frac{\Gamma_e(\boldsymbol{x}, \boldsymbol{y}_j)}{\epsilon(\boldsymbol{x}, \boldsymbol{y}_j)\omega_{pe}^2(\boldsymbol{x}, \boldsymbol{y}_j)}\hat{\boldsymbol{J}}^N(\boldsymbol{x}, t; \boldsymbol{y}_j) = \hat{\boldsymbol{E}}^N(\boldsymbol{x}, t; \boldsymbol{y}_j), \tag{19}$$

$$\frac{1}{\mu(\boldsymbol{x},\boldsymbol{y}_j)\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y}_j)}\partial_t\hat{\boldsymbol{K}}^N(\boldsymbol{x},t;\boldsymbol{y}_j)+\frac{\Gamma_m(\boldsymbol{x},\boldsymbol{y}_j)}{\mu(\boldsymbol{x},\boldsymbol{y}_j)\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y}_j)}\hat{\boldsymbol{K}}^N(\boldsymbol{x},t;\boldsymbol{y}_j)=\hat{\boldsymbol{H}}^N(\boldsymbol{x},t;\boldsymbol{y}_j), \tag{20}$$

subject to the zero initial conditions

$$\hat{\boldsymbol{E}}^N(\boldsymbol{x},t=0,\boldsymbol{y}_j)=\hat{\boldsymbol{H}}^N(\boldsymbol{x},t=0,\boldsymbol{y}_j)=0, \tag{21}$$

$$\hat{\boldsymbol{J}}^N(\boldsymbol{x},t=0,\boldsymbol{y}_j)=\hat{\boldsymbol{K}}^N(\boldsymbol{x},t=0,\boldsymbol{y}_j)=0, \tag{22}$$

and the PEC boundary condition:

$$\boldsymbol{n}\times\hat{\boldsymbol{E}}^N(\boldsymbol{x},t;\boldsymbol{y}_j)=\boldsymbol{0},\quad \text{on } \partial D. \tag{23}$$

Multiplying (17)–(20) by $\hat{\boldsymbol{E}}^N(\boldsymbol{x},t=0,\boldsymbol{y}_j)$, $\hat{\boldsymbol{H}}^N(\boldsymbol{x},t;\boldsymbol{y}_j)$, $\hat{\boldsymbol{J}}^N(\boldsymbol{x},t;\boldsymbol{y}_j)$ and $\hat{\boldsymbol{K}}^N(\boldsymbol{x},t;\boldsymbol{y}_j)$, respectively, and integrating over D, we can easily see that (cf. proof of Lemma 3.12 in [31]):

$$\frac{1}{2}\frac{d}{dt}\left(\|\sqrt{\epsilon}\hat{\boldsymbol{E}}^N\|_{L^2(D)}^2+\|\sqrt{\mu}\hat{\boldsymbol{H}}^N\|_{L^2(D)}^2+\|\frac{\hat{\boldsymbol{J}}^N}{\sqrt{\epsilon}\omega_{pe}}\|_{L^2(D)}^2+\|\frac{\hat{\boldsymbol{K}}^N}{\sqrt{\mu}\omega_{pm}}\|_{L^2(D)}^2\right)(y_j)$$
$$+\int_D\left(\frac{\Gamma_e}{\epsilon\omega_{pe}^2}|\hat{\boldsymbol{J}}^N|^2+\frac{\Gamma_m}{\mu\omega_{pm}^2}|\hat{\boldsymbol{K}}^N|^2\right)=0. \tag{24}$$

Integrating (24) from $t=0$ to t and using the zero initial conditions, we easily have

$$\frac{1}{2}\left(\|\sqrt{\epsilon}\hat{\boldsymbol{E}}^N\|_{L^2(D)}^2+\|\sqrt{\mu}\hat{\boldsymbol{H}}^N\|_{L^2(D)}^2+\|\frac{\hat{\boldsymbol{J}}^N}{\sqrt{\epsilon}\omega_{pe}}\|_{L^2(D)}^2+\|\frac{\hat{\boldsymbol{K}}^N}{\sqrt{\mu}\omega_{pm}}\|_{L^2(D)}^2\right)(t,y_j)\le 0,$$

which leads to

$$\hat{\boldsymbol{E}}^N(\boldsymbol{x},t=0,\boldsymbol{y}_j)=\hat{\boldsymbol{H}}^N(\boldsymbol{x},t=0,\boldsymbol{y}_j)=\hat{\boldsymbol{J}}^N(\boldsymbol{x},t=0,\boldsymbol{y}_j)=\hat{\boldsymbol{K}}^N(\boldsymbol{x},t=0,\boldsymbol{y}_j)=0.$$

These justify that $\boldsymbol{E}^N(\boldsymbol{x},t;\boldsymbol{y})=I_N^y\boldsymbol{E}$, $\boldsymbol{H}^N(\boldsymbol{x},t;\boldsymbol{y})=I_N^y\boldsymbol{H}$, $\boldsymbol{J}^N(\boldsymbol{x},t;\boldsymbol{y})=I_N^y\boldsymbol{J}$ and $\boldsymbol{K}^N(\boldsymbol{x},t;\boldsymbol{y})=I_N^y\boldsymbol{K}$.

2.1. *Regularity analysis*

Lemma 2.1. *For problem (1)–(7) and any $t\in[0,T]$, we have*

$$\int_\Gamma\int_D\rho(\boldsymbol{y})\left(\epsilon(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{E}|^2+\mu(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{H}|^2+\frac{1}{\epsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})}|\boldsymbol{J}|^2+\frac{1}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})}|\boldsymbol{K}|^2\right)(t)dxdy$$
$$\le\int_\Gamma\int_D\rho(\boldsymbol{y})\left(\epsilon(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{E}_0|^2+\mu(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{H}_0|^2+\frac{1}{\epsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})}|\boldsymbol{J}_0|^2+\frac{1}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})}|\boldsymbol{K}_0|^2\right)dxdy.$$

Proof. Multiplying (1)–(4) by $2\rho(\boldsymbol{y})\boldsymbol{E}$, $2\rho(\boldsymbol{y})\boldsymbol{H}$, $2\rho(\boldsymbol{y})\boldsymbol{J}$ and $2\rho(\boldsymbol{y})\boldsymbol{K}$, respectively, then integrating over D and Γ, and adding the resultants, we have

$$\frac{d}{dt}\int_\Gamma\int_D\rho(\boldsymbol{y})\left(\epsilon(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{E}|^2+\mu(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{H}|^2+\frac{1}{\epsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})}|\boldsymbol{J}|^2+\frac{1}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})}|\boldsymbol{K}|^2\right)dxdy$$
$$+\int_\Gamma\int_D 2\rho(\boldsymbol{y})\left(\frac{\Gamma_e(\boldsymbol{x},\boldsymbol{y})}{\epsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})}|\boldsymbol{J}|^2+\frac{\Gamma_m(\boldsymbol{x},\boldsymbol{y})}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})}|\boldsymbol{K}|^2\right)dxdy=0, \tag{25}$$

where in the last step we used the PEC boundary condition (7) in the following identity

$$\int_\Gamma\int_D\rho(\boldsymbol{y})\boldsymbol{E}\cdot\nabla\times\boldsymbol{H}-\int_\Gamma\int_D\rho(\boldsymbol{y})\boldsymbol{H}\cdot\nabla\times\boldsymbol{E}=-\int_\Gamma\int_{\partial D}\rho(\boldsymbol{y})\boldsymbol{H}\cdot\boldsymbol{n}\times\boldsymbol{E}=0.$$

Integrating (25) with respect to t from $t=0$ to t concludes the proof. □

Lemma 2.2. *Denote*

$$C_{max1} := 2\max_{\overline{D}\times\overline{\Gamma}}\left(\frac{1}{\epsilon\mu}, \omega_{pe}^2 + \Gamma_e^2, \omega_{pm}^2 + \Gamma_m^2, \frac{1}{\epsilon^2\omega_{pe}^2}, \frac{1}{\mu^2\omega_{pm}^2}\right).$$

Then for problem (1)–(7) and any $t \in [0, T]$, we have

$$\int_\Gamma\int_D \rho(\boldsymbol{y})\left(\epsilon(\boldsymbol{x},\boldsymbol{y})|\partial_t\boldsymbol{E}|^2 + \mu(\boldsymbol{x},\boldsymbol{y})|\partial_t\boldsymbol{H}|^2 + \frac{1}{\epsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})}|\partial_t\boldsymbol{J}|^2 \right.$$
$$\left. + \frac{1}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})}|\partial_t\boldsymbol{K}|^2\right)(t)dxdy$$
$$\leq C_{max1}\int_\Gamma\int_D \rho(\boldsymbol{y})\left(\mu|\nabla\times\boldsymbol{H}_0|^2 + \epsilon|\nabla\times\boldsymbol{E}_0|^2 + \epsilon|\boldsymbol{E}_0|^2 + \mu|\boldsymbol{H}_0|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2\right)dxdy.$$

Proof. Taking the time derivative of (1)–(4), we obtain

$$\epsilon(\boldsymbol{x},\boldsymbol{y}(\omega))\partial_t(\partial_t\boldsymbol{E}) = \nabla\times(\partial_t\boldsymbol{H}) - \partial_t\boldsymbol{J}, \tag{26}$$

$$\mu(\boldsymbol{x},\boldsymbol{y}(\omega))\partial_t(\partial_t\boldsymbol{H}) = -\nabla\times(\partial_t\boldsymbol{E}) - \partial_t\boldsymbol{K}, \tag{27}$$

$$\frac{1}{\epsilon(\boldsymbol{x},\boldsymbol{y}(\omega))\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y}(\omega))}\partial_t(\partial_t\boldsymbol{J}) + \frac{\Gamma_e(\boldsymbol{x},\boldsymbol{y}(\omega))}{\epsilon(\boldsymbol{x},\boldsymbol{y}(\omega))\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y}(\omega))}\partial_t\boldsymbol{J} = \partial_t\boldsymbol{E}, \tag{28}$$

$$\frac{1}{\mu(\boldsymbol{x},\boldsymbol{y}(\omega))\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y}(\omega))}\partial_t(\partial_t\boldsymbol{K}) + \frac{\Gamma_m(\boldsymbol{x},\boldsymbol{y}(\omega))}{\mu(\boldsymbol{x},\boldsymbol{y}(\omega))\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y}(\omega))}\partial_t\boldsymbol{K} = \partial_t\boldsymbol{H}. \tag{29}$$

Multiplying (26)–(29) by $2\rho(\boldsymbol{y})\partial_t\boldsymbol{E}$, $2\rho(\boldsymbol{y})\partial_t\boldsymbol{H}$, $2\rho(\boldsymbol{y})\partial_t\boldsymbol{J}$ and $2\rho(\boldsymbol{y})\partial_t\boldsymbol{K}$, respectively, then integrating over D and Γ, and adding the resultants, we have

$$\frac{d}{dt}\int_\Gamma\int_D \rho(\boldsymbol{y})\left(\epsilon(\boldsymbol{x},\boldsymbol{y})|\partial_t\boldsymbol{E}|^2 + \mu(\boldsymbol{x},\boldsymbol{y})|\partial_t\boldsymbol{H}|^2 + \frac{1}{\epsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2}|\partial_t\boldsymbol{J}|^2 + \frac{1}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2}|\partial_t\boldsymbol{K}|^2\right)dxdy$$
$$+\int_\Gamma\int_D 2\rho(\boldsymbol{y})\left(\frac{\Gamma_e(\boldsymbol{x},\boldsymbol{y})}{\epsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})}|\partial_t\boldsymbol{J}|^2 + \frac{\Gamma_m(\boldsymbol{x},\boldsymbol{y})}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})}|\partial_t\boldsymbol{K}|^2\right)dxdy$$
$$= \int_\Gamma\int_D 2\rho(\boldsymbol{y})\left[\nabla\times(\partial_t\boldsymbol{H})\cdot\partial_t\boldsymbol{E} - \nabla\times(\partial_t\boldsymbol{E})\cdot\partial_t\boldsymbol{H}\right] = -\int_\Gamma\int_{\partial D} 2\rho(\boldsymbol{y})\boldsymbol{n}\times(\partial_t\boldsymbol{E})\cdot\partial_t\boldsymbol{H} = 0, \tag{30}$$

where in the last step we used integration by parts and the PEC boundary condition (7).

Integrating (30) with respect to t from $t = 0$ to t, then using the governing equations (1)–(4) and the Cauchy–Schwarz inequality, we have

$$\int_\Gamma\int_D \rho(\boldsymbol{y})\left(\epsilon(\boldsymbol{x},\boldsymbol{y})|\partial_t\boldsymbol{E}|^2 + \mu(\boldsymbol{x},\boldsymbol{y})|\partial_t\boldsymbol{H}|^2 + \frac{1}{\epsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2}|\partial_t\boldsymbol{J}|^2 + \frac{1}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2}|\partial_t\boldsymbol{K}|^2\right)(t)dxdy$$
$$\leq \int_\Gamma\int_D \rho(\boldsymbol{y})\left(\epsilon(\boldsymbol{x},\boldsymbol{y})|\partial_t\boldsymbol{E}_0|^2 + \mu(\boldsymbol{x},\boldsymbol{y})|\partial_t\boldsymbol{H}_0|^2 + \frac{1}{\epsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2}|\partial_t\boldsymbol{J}_0|^2 + \frac{1}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2}|\partial_t\boldsymbol{K}_0|^2\right)dxdy$$
$$\leq \int_\Gamma\int_D \rho\left[\frac{2}{\epsilon}(|\nabla\times\boldsymbol{H}_0|^2 + |\boldsymbol{J}_0|^2) + \frac{2}{\mu}(|\nabla\times\boldsymbol{E}_0|^2 + |\boldsymbol{K}_0|^2)\right.$$
$$\left. + \frac{2}{\epsilon\omega_{pe}^2}(\Gamma_e^2|\boldsymbol{J}_0|^2 + |\boldsymbol{E}_0|^2) + \frac{2}{\mu\omega_{pm}^2}(\Gamma_m^2|\boldsymbol{K}_0|^2 + |\boldsymbol{H}_0|^2)\right]dxdy$$
$$= \int_\Gamma\int_D \rho\left[\frac{2}{\epsilon\mu}(\mu|\nabla\times\boldsymbol{H}_0|^2 + \epsilon|\nabla\times\boldsymbol{E}_0|^2) + 2(\omega_{pe}^2 + \Gamma_e^2)\cdot\frac{1}{\epsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2\right.$$

$$+2(\omega_{pm}^2+\Gamma_m^2)\cdot\frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2+\frac{2}{\epsilon^2\omega_{pe}^2}\cdot\epsilon|\boldsymbol{E}_0|^2+\frac{2}{\mu^2\omega_{pm}^2}\cdot\mu|\boldsymbol{H}_0|^2\Bigg]dxdy$$

$$\leq C_{max1}\int_\Gamma\int_D\rho\left(\mu|\nabla\times\boldsymbol{H}_0|^2+\epsilon|\nabla\times\boldsymbol{E}_0|^2+\frac{1}{\epsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2+\frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2+\epsilon|\boldsymbol{E}_0|^2+\mu|\boldsymbol{H}_0|^2\right), \tag{31}$$

which concludes the proof. □

In the rest of the paper, we will use the following Gronwall inequality a lot.

Lemma 2.3. *If $Q(t)$ satisfies $\frac{dQ}{dt}\leq c_0Q+d_0$ for some constant $c_0\neq 0$ and d_0, then we have*

$$Q(t)\leq e^{c_0t}(Q(0)+\frac{d_0}{c_0}),\quad \forall\, t\geq 0.$$

Theorem 2.1. *Denote constant C_1:*

$$C_1=\max_{\overline{D}\times\overline{\Gamma}}\left(|\frac{\partial_{y_i}(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}|+|\partial_{y_i}\Gamma_e|,|\frac{\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}|+|\partial_{y_i}\Gamma_m|,|\frac{\partial_{y_i}\epsilon}{\epsilon}|,|\frac{\partial_{y_i}\mu}{\mu}|\right).$$

Then for any $t\in[0,T]$ and $i=1,\ldots,d$, we have

$$\int_\Gamma\int_D\rho(\boldsymbol{y})\left(\epsilon|\partial_{y_i}\boldsymbol{E}|^2+\mu|\partial_{y_i}\boldsymbol{H}|^2+\frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2+\frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2\right)(t)dxdy$$
$$\leq e^{C_1t}\int_\Gamma\int_D\rho(\boldsymbol{y})\left(\epsilon|\partial_{y_i}\boldsymbol{E}|^2+\mu|\partial_{y_i}\boldsymbol{H}|^2+\frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2+\frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2\right)(0)dxdy$$
$$+e^{C_1t}(1+C_{max1})\int_\Gamma\int_D\rho\left(\epsilon|\nabla\times\boldsymbol{E}_0|^2+\mu|\nabla\times\boldsymbol{H}_0|^2+\epsilon|\boldsymbol{E}_0|^2+\mu|\boldsymbol{H}_0|^2+\frac{1}{\epsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2\right.$$
$$\left.+\frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2\right).$$

Proof. Differentiating (1)–(4) with respect to any y_i $(i=1,\ldots,d)$, we obtain

$$\epsilon\partial_t(\partial_{y_i}\boldsymbol{E})-\nabla\times(\partial_{y_i}\boldsymbol{H})+\partial_{y_i}\boldsymbol{J}=-\partial_{y_i}\epsilon\cdot\partial_t\boldsymbol{E}, \tag{32}$$
$$\mu\partial_t(\partial_{y_i}\boldsymbol{H})+\nabla\times(\partial_{y_i}\boldsymbol{E})+\partial_{y_i}\boldsymbol{K}=-\partial_{y_i}\mu\cdot\partial_t\boldsymbol{H}, \tag{33}$$
$$\partial_t(\partial_{y_i}\boldsymbol{J})+\Gamma_e\cdot\partial_{y_i}\boldsymbol{J}-\epsilon\omega_{pe}^2\partial_{y_i}\boldsymbol{E}=\partial_{y_i}(\epsilon\omega_{pe}^2)\boldsymbol{E}-\partial_{y_i}\Gamma_e\cdot\boldsymbol{J}, \tag{34}$$
$$\partial_t(\partial_{y_i}\boldsymbol{K})+\Gamma_m\cdot\partial_{y_i}\boldsymbol{K}-\mu\omega_{pm}^2\partial_{y_i}\boldsymbol{H}=\partial_{y_i}(\mu\omega_{pm}^2)\boldsymbol{H}-\partial_{y_i}\Gamma_m\cdot\boldsymbol{K}. \tag{35}$$

Multiplying (32)–(35) by $2\rho(\boldsymbol{y})\partial_{y_i}\boldsymbol{E}$, $2\rho(\boldsymbol{y})\partial_{y_i}\boldsymbol{H}$, $\frac{2\rho(\boldsymbol{y})}{\epsilon\omega_{pe}^2}\partial_{y_i}\boldsymbol{J}$ and $\frac{2\rho(\boldsymbol{y})}{\mu\omega_{pm}^2}\partial_{y_i}\boldsymbol{K}$, respectively, then integrating over D and Γ, and adding the resultants, we have

$$\frac{d}{dt}\int_\Gamma\int_D\rho(\boldsymbol{y})\left(\epsilon|\partial_{y_i}\boldsymbol{E}|^2+\mu|\partial_{y_i}\boldsymbol{H}|^2+\frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2+\frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2\right)dxdy$$
$$+\int_\Gamma\int_D2\rho(\boldsymbol{y})\left(\frac{\Gamma_e(\boldsymbol{x},\boldsymbol{y})}{\epsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})}|\partial_{y_i}\boldsymbol{J}|^2+\frac{\Gamma_m(\boldsymbol{x},\boldsymbol{y})}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})}|\partial_{y_i}\boldsymbol{K}|^2\right)dxdy$$
$$=\int_\Gamma\int_D2\rho(\boldsymbol{y})\Bigg[-\partial_{y_i}\epsilon\cdot\partial_t\boldsymbol{E}\cdot\partial_{y_i}\boldsymbol{E}-\partial_{y_i}\mu\cdot\partial_t\boldsymbol{H}\cdot\partial_{y_i}\boldsymbol{H}+\frac{\partial_{y_i}(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}^2}\boldsymbol{E}\cdot\partial_{y_i}\boldsymbol{J}-\frac{\partial_{y_i}\Gamma_e}{\epsilon\omega_{pe}^2}\boldsymbol{J}\cdot\partial_{y_i}\boldsymbol{J}$$
$$+\frac{\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}^2}\boldsymbol{H}\cdot\partial_{y_i}\boldsymbol{K}-\frac{\partial_{y_i}\Gamma_m}{\mu\omega_{pm}^2}\boldsymbol{K}\cdot\partial_{y_i}\boldsymbol{K}\Bigg], \tag{36}$$

where in the last step we used the following identity:

$$\int_\Gamma\int_D \rho(\boldsymbol{y})(\nabla\times\partial_{y_i}\boldsymbol{H}\cdot\partial_{y_i}\boldsymbol{E}-\nabla\times\partial_{y_i}\boldsymbol{E}\cdot\partial_{y_i}\boldsymbol{H}) = -\int_\Gamma\int_{\partial D}\rho(\boldsymbol{y})(\boldsymbol{n}\times\partial_{y_i}\boldsymbol{E}\cdot\partial_{y_i}\boldsymbol{H}) = 0.$$

By the Cauchy–Schwarz inequality, it is easy to see that

$$\begin{aligned}&\int_\Gamma\int_D 2\rho(\boldsymbol{y})\partial_{y_i}\epsilon\partial_t\boldsymbol{E}\cdot\partial_{y_i}\boldsymbol{E} = \int_\Gamma\int_D 2\rho(\boldsymbol{y})\cdot\frac{\partial_{y_i}\epsilon}{\epsilon}\cdot\sqrt{\epsilon}\partial_t\boldsymbol{E}\cdot\sqrt{\epsilon}\partial_{y_i}\boldsymbol{E}\\ &\le \max_{\overline{\Gamma}\times\overline{D}}(|\frac{\partial_{y_i}\epsilon}{\epsilon}|)\int_\Gamma\int_D\rho(\boldsymbol{y})\left(\epsilon|\partial_t\boldsymbol{E}|^2+\epsilon|\partial_{y_i}\boldsymbol{E}|^2\right),\end{aligned}\tag{37}$$

$$\begin{aligned}&\int_\Gamma\int_D 2\rho(\boldsymbol{y})\partial_{y_i}\mu\partial_t\boldsymbol{H}\cdot\partial_{y_i}\boldsymbol{H} = \int_\Gamma\int_D 2\rho(\boldsymbol{y})\cdot\frac{\partial_{y_i}\mu}{\mu}\cdot\sqrt{\mu}\partial_t\boldsymbol{H}\cdot\sqrt{\mu}\partial_{y_i}\boldsymbol{H}\\ &\le \max_{\overline{\Gamma}\times\overline{D}}(|\frac{\partial_{y_i}\mu}{\mu}|)\int_\Gamma\int_D\rho(\boldsymbol{y})\left(\mu|\partial_t\boldsymbol{H}|^2+\mu|\partial_{y_i}\boldsymbol{H}|^2\right).\end{aligned}\tag{38}$$

Similarly, we can obtain

$$\int_\Gamma\int_D 2\rho(\boldsymbol{y})\cdot\frac{\partial_{y_i}(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}^2}\boldsymbol{E}\cdot\partial_{y_i}\boldsymbol{J} \le \max_{\overline{\Gamma}\times\overline{D}}(|\frac{\partial_{y_i}(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}|)\int_\Gamma\int_D\rho(\boldsymbol{y})(\epsilon|\boldsymbol{E}|^2+\frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2),\tag{39}$$

$$\int_\Gamma\int_D 2\rho(\boldsymbol{y})\cdot\frac{\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}^2}\boldsymbol{H}\cdot\partial_{y_i}\boldsymbol{K} \le \max_{\overline{\Gamma}\times\overline{D}}(|\frac{\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}|)\int_\Gamma\int_D\rho(\boldsymbol{y})(\mu|\boldsymbol{H}|^2+\frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2),$$

$$\int_\Gamma\int_D 2\rho(\boldsymbol{y})\cdot\frac{\partial_{y_i}\Gamma_e}{\epsilon\omega_{pe}^2}\boldsymbol{J}\cdot\partial_{y_i}\boldsymbol{J} \le \max_{\overline{\Gamma}\times\overline{D}}(|\partial_{y_i}\Gamma_e|)\int_\Gamma\int_D\frac{\rho(\boldsymbol{y})}{\epsilon\omega_{pe}^2}(|\boldsymbol{J}|^2+|\partial_{y_i}\boldsymbol{J}|^2),\tag{40}$$

$$\int_\Gamma\int_D 2\rho(\boldsymbol{y})\cdot\frac{\partial_{y_i}\Gamma_m}{\mu\omega_{pm}^2}\boldsymbol{K}\cdot\partial_{y_i}\boldsymbol{K} \le \max_{\overline{\Gamma}\times\overline{D}}(|\partial_{y_i}\Gamma_m|)\int_\Gamma\int_D\frac{\rho(\boldsymbol{y})}{\mu\omega_{pm}^2}(|\boldsymbol{K}|^2+|\partial_{y_i}\boldsymbol{K}|^2).\tag{41}$$

Denote constants C_2 and C_3 as follows:

$$C_2=\max_{\overline{D}\times\overline{\Gamma}}\left(|\frac{\partial_{y_i}(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}|,|\frac{\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}|,|\partial_{y_i}\Gamma_e|,|\partial_{y_i}\Gamma_m|\right),\quad C_3=\max_{\overline{D}\times\overline{\Gamma}}\left(|\frac{\partial_{y_i}\epsilon}{\epsilon}|,|\frac{\partial_{y_i}\mu}{\mu}|\right),$$

Let us introduce the notations

$$ENG_0(t)=\int_\Gamma\int_D\rho(\boldsymbol{y})\left(\epsilon|\boldsymbol{E}|^2+\mu|\boldsymbol{H}|^2+\frac{1}{\epsilon\omega_{pe}^2}|\boldsymbol{J}|^2+\frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}|^2\right)(t)dxdy,$$

and

$$ENG_1(t)=\int_\Gamma\int_D\rho(\boldsymbol{y})\left(\epsilon|\partial_{y_i}\boldsymbol{E}|^2+\mu|\partial_{y_i}\boldsymbol{H}|^2+\frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2+\frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2\right)(t)dxdy.$$

Substituting (37)–(41) into (36), we have

$$\frac{d}{dt}ENG_1(t)\le C_2ENG_0(t)+C_1ENG_1(t)+C_3\int_\Gamma\int_D\rho(\boldsymbol{y})\left(\epsilon|\partial_t\boldsymbol{E}|^2+\mu|\partial_t\boldsymbol{H}|^2\right).\tag{42}$$

Applying the Gronwall inequality stated in Lemma 2.3 to (42) and Lemmas 2.1–2.2, we have

$$\begin{aligned}&ENG_1(t)\\ &\le e^{C_1t}\left\{ENG_1(0)+\frac{1}{C_1}\left[C_2ENG_0(0)+C_3C_{max1}\left(ENG_0(0)+\int_\Gamma\int_D\rho(\epsilon|\nabla\times\boldsymbol{E}_0|^2+\mu|\nabla\times\boldsymbol{H}_0|^2)\right)\right]\right\}\\ &\le e^{C_1t}\left[ENG_1(0)+(1+C_{max1})(ENG_0(0)+\int_\Gamma\int_D\rho(\epsilon|\nabla\times\boldsymbol{E}_0|^2+\mu|\nabla\times\boldsymbol{H}_0|^2))\right],\end{aligned}$$

which concludes the proof. In the last step we used the fact that $C_2\le C_1$ and $C_3\le C_1$. □

Remark 2.2. If the physical parameters $\epsilon, \mu, \Gamma_e, \Gamma_m, \omega_{pe}, \omega_{pm}$ are independent of y_i, then $C_1 = C_2 = C_3 = 0$. Hence from (42) we easily see that Theorem 2.1 becomes

$$\int_\Gamma \int_D \rho(\mathbf{y}) \left(\epsilon|\partial_{y_i}\mathbf{E}|^2 + \mu|\partial_{y_i}\mathbf{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i}\mathbf{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\mathbf{K}|^2 \right)(t)dxdy$$
$$\leq \int_\Gamma \int_D \rho(\mathbf{y}) \left(\epsilon|\partial_{y_i}\mathbf{E}|^2 + \mu|\partial_{y_i}\mathbf{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i}\mathbf{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\mathbf{K}|^2 \right)(0)dxdy.$$

In the more general case, Theorem 2.1 shows that if the following initial conditions are L^2 bounded:

$$\int_\Gamma \int_D \rho(\mathbf{y}) \left(\epsilon|\partial_{y_i}\mathbf{E}|^2 + \mu|\partial_{y_i}\mathbf{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i}\mathbf{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\mathbf{K}|^2 \right)(0)dxdy \leq C,$$
$$\int_\Gamma \int_D \rho \left(\epsilon|\nabla\times\mathbf{E}_0|^2 + \mu|\nabla\times\mathbf{H}_0|^2 + \epsilon|\mathbf{E}_0|^2 + \mu|\mathbf{H}_0|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\mathbf{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\mathbf{K}_0|^2 \right) \leq C,$$

then the solution $(\mathbf{E}, \mathbf{H}, \mathbf{J}, \mathbf{K})$ of (1)–(7) is also L^2 bounded:

$$\int_\Gamma \int_D \rho(\mathbf{y}) \left(\epsilon|\partial_{y_i}\mathbf{E}|^2 + \mu|\partial_{y_i}\mathbf{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i}\mathbf{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\mathbf{K}|^2 \right)(t)dxdy \leq Ce^{C_1 t}.$$

This boundness guarantees that the mean squared error is $O(N^{-1})$ when the stochastic collocation method is used to solve the model problem (1)–(7). For details see Theorem 2.4 proved later.

To prove higher order convergence, we need to show that higher derivatives with respect to the random variables are L^2 bounded. Below we just present the proofs of L^2 boundness for the second-order derivatives, which depend on the estimates of $\nabla\times\mathbf{u}$, $\nabla\times\partial_t\mathbf{u}$ and $\nabla\times\partial_{y_i}\mathbf{u}$ for $\mathbf{u} = \mathbf{E}, \mathbf{H}, \mathbf{J}, \mathbf{K}$. These estimates will be proved in the following three lemmas.

Lemma 2.4. *Denote the constant* $C_1^* = \max_{\overline{D}\times\overline{\Gamma}} \left(|\frac{\nabla(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}| + |\nabla\Gamma_e|, |\frac{\nabla(\mu\omega_{pm}^2)}{\mu\omega_{pm}}| + |\nabla\Gamma_m|, |\frac{\nabla\epsilon}{\epsilon}|, |\frac{\nabla\mu}{\mu}| \right)$. *Then for any* $t \in [0, T]$ *and* $i = 1, \ldots, d$, *we have*

$$\int_\Gamma \int_D \rho(\mathbf{y}) \left(\epsilon|\nabla\times\mathbf{E}|^2 + \mu|\nabla\times\mathbf{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\nabla\times\mathbf{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\mathbf{K}|^2 \right)(t)dxdy$$
$$\leq e^{C_1^* t}(2 + C_{max1}) \int_\Gamma \int_D \rho(\mathbf{y}) \left(\epsilon|\nabla\times\mathbf{E}_0|^2 + \mu|\nabla\times\mathbf{H}_0|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\nabla\times\mathbf{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\mathbf{K}_0|^2 \right)$$
$$+ e^{C_1^* t}(1 + C_{max1}) \int_\Gamma \int_D \rho(\mathbf{y}) \left(\epsilon|\mathbf{E}_0|^2 + \mu|\mathbf{H}_0|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\mathbf{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\mathbf{K}_0|^2 \right).$$

Proof. Taking $\nabla\times$ of (1)–(4), and using the identity $\nabla\times(\phi\mathbf{u}) = \phi\nabla\times\mathbf{u} + \nabla\phi\times\mathbf{u}$ for any scalar function ϕ and vector function $\mathbf{u}$, we obtain

$$\epsilon\partial_t(\nabla\times\mathbf{E}) - \nabla\times(\nabla\times\mathbf{H}) + \nabla\times\mathbf{J} = -\nabla\epsilon\times\partial_t\mathbf{E}, \tag{43}$$
$$\mu\partial_t(\nabla\times\mathbf{H}) + \nabla\times(\nabla\times\mathbf{E}) + \nabla\times\mathbf{K} = -\nabla\mu\times\partial_t\mathbf{H}, \tag{44}$$
$$\partial_t(\nabla\times\mathbf{J}) + \Gamma_e\nabla\times\mathbf{J} - \epsilon\omega_{pe}^2\nabla\times\mathbf{E} = \nabla(\epsilon\omega_{pe}^2)\times\mathbf{E} - \nabla\Gamma_e\times\mathbf{J}, \tag{45}$$
$$\partial_t(\nabla\times\mathbf{K}) + \Gamma_m\nabla\times\mathbf{K} - \mu\omega_{pm}^2\nabla\times\mathbf{H} = \nabla(\mu\omega_{pm}^2)\times\mathbf{H} - \nabla\Gamma_m\times\mathbf{K}. \tag{46}$$

Denote

$$ENG_3(t) = \int_\Gamma \int_D \rho(\mathbf{y}) \left(\epsilon|\nabla\times\mathbf{E}|^2 + \mu|\nabla\times\mathbf{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\nabla\times\mathbf{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\mathbf{K}|^2 \right)(t)dxdy.$$

Multiplying (43)–(46) by $2\rho(\boldsymbol{y})\nabla\times\boldsymbol{E}$, $2\rho(\boldsymbol{y})\nabla\times\boldsymbol{H}$, $\frac{2\rho(\boldsymbol{y})}{\epsilon\omega_{pe}^2}\nabla\times\boldsymbol{J}$ and $\frac{2\rho(\boldsymbol{y})}{\mu\omega_{pm}^2}\nabla\times\boldsymbol{K}$, respectively, then integrating over D and Γ, and adding the resultants, we have

$$
\begin{aligned}
&\frac{d}{dt}ENG_3(t)+\int_\Gamma\int_D 2\rho(\boldsymbol{y})\left(\frac{\Gamma_e(\boldsymbol{x},\boldsymbol{y})}{\epsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})}|\nabla\times\boldsymbol{J}|^2+\frac{\Gamma_m(\boldsymbol{x},\boldsymbol{y})}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})}|\nabla\times\boldsymbol{K}|^2\right)dxdy\\
&=\int_\Gamma\int_D(\nabla\times\nabla\times\boldsymbol{H}\cdot 2\rho\nabla\times\boldsymbol{E}-\nabla\times\nabla\times\boldsymbol{E}\cdot 2\rho\nabla\times\boldsymbol{H})\\
&\quad-\int_\Gamma\int_D(\nabla\epsilon\times\partial_t\boldsymbol{E})\cdot 2\rho\nabla\times\boldsymbol{E}-\int_\Gamma\int_D(\nabla\mu\times\partial_t\boldsymbol{H})\cdot 2\rho\nabla\times\boldsymbol{H}\\
&\quad-\int_\Gamma\int_D(\nabla\Gamma_e\times\boldsymbol{J})\cdot\frac{2\rho}{\epsilon\omega_{pe}^2}\nabla\times\boldsymbol{J}+\int_\Gamma\int_D\nabla(\epsilon\omega_{pe}^2)\times\boldsymbol{E}\cdot\frac{2\rho}{\epsilon\omega_{pe}^2}\nabla\times\boldsymbol{J}\\
&\quad-\int_\Gamma\int_D(\nabla\Gamma_m\times\boldsymbol{K})\cdot\frac{2\rho}{\mu\omega_{pm}^2}\nabla\times\boldsymbol{K}+\int_\Gamma\int_D\nabla(\mu\omega_{pm}^2)\times\boldsymbol{H}\cdot\frac{2\rho}{\mu\omega_{pm}^2}\nabla\times\boldsymbol{K}:=\sum_{i=1}^7 Err_i.
\end{aligned}
\tag{47}
$$

Using integration by parts, (1), and boundary conditions $\boldsymbol{n}\times\boldsymbol{E}=0$ and $\boldsymbol{n}\times\boldsymbol{J}=0$, we obtain

$$
\begin{aligned}
&\int_\Gamma\int_D\nabla\times\nabla\times\boldsymbol{H}\cdot 2\rho\nabla\times\boldsymbol{E}=\int_\Gamma\int_{\partial D}\boldsymbol{n}\times(\nabla\times\boldsymbol{H})\cdot 2\rho\nabla\times\boldsymbol{E}+\int_\Gamma\int_D\nabla\times\boldsymbol{H}\cdot 2\rho\nabla\times\nabla\times\boldsymbol{E}\\
&=\int_\Gamma\int_{\partial D}\boldsymbol{n}\times(\epsilon\partial_t\boldsymbol{E}+\boldsymbol{J})\cdot 2\rho\nabla\times\boldsymbol{E}+\int_\Gamma\int_D\nabla\times\boldsymbol{H}\cdot 2\rho\nabla\times\nabla\times\boldsymbol{E}=\int_\Gamma\int_D\nabla\times\boldsymbol{H}\cdot 2\rho\nabla\times\nabla\times\boldsymbol{E},
\end{aligned}
$$

which leads to $Err_1=0$.

By the Cauchy–Schwarz inequality and the identity

$$\boldsymbol{u}\times\boldsymbol{v}=|\boldsymbol{u}|\cdot|\boldsymbol{v}|\sin\theta,\quad \text{where } \theta \text{ is the angle between } \boldsymbol{u} \text{ and } \boldsymbol{v},$$

we have

$$
\begin{aligned}
Err_2&=-\int_\Gamma\int_D 2\frac{\nabla\epsilon}{\epsilon}\times\sqrt{\rho\epsilon}\partial_t\boldsymbol{E}\cdot\sqrt{\rho\epsilon}\nabla\times\boldsymbol{E}\\
&\le(\max_{\overline{D}\times\overline{\Gamma}}|\frac{\nabla\epsilon}{\epsilon}|)\left(\int_\Gamma\int_D\rho\epsilon|\partial_t\boldsymbol{E}|^2+\int_\Gamma\int_D\rho\epsilon|\nabla\times\boldsymbol{E}|^2\right).
\end{aligned}
$$

Similarly, we can obtain

$$
\begin{aligned}
Err_3&\le(\max_{\overline{D}\times\overline{\Gamma}}|\frac{\nabla\mu}{\mu}|)\left(\int_\Gamma\int_D\rho\mu|\partial_t\boldsymbol{H}|^2+\int_\Gamma\int_D\rho\mu|\nabla\times\boldsymbol{H}|^2\right),\\
Err_4&\le(\max_{\overline{D}\times\overline{\Gamma}}|\nabla\Gamma_e|)\int_\Gamma\int_D\frac{\rho}{\epsilon\omega_{pe}^2}\left(|\boldsymbol{J}|^2+|\nabla\times\boldsymbol{J}|^2\right),\\
Err_5&\le(\max_{\overline{D}\times\overline{\Gamma}}|\frac{\nabla(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}|)\left(\int_\Gamma\int_D\rho\epsilon|\boldsymbol{E}|^2+\int_\Gamma\int_D\frac{\rho}{\epsilon\omega_{pe}^2}|\nabla\times\boldsymbol{J}|^2\right),\\
Err_6&\le(\max_{\overline{D}\times\overline{\Gamma}}|\nabla\Gamma_m|)\int_\Gamma\int_D\frac{\rho}{\mu\omega_{pm}^2}\left(|\boldsymbol{K}|^2+|\nabla\times\boldsymbol{K}|^2\right),\\
Err_7&\le(\max_{\overline{D}\times\overline{\Gamma}}|\frac{\nabla(\mu\omega_{pm}^2)}{\mu\omega_{pm}}|)\left(\int_\Gamma\int_D\rho\mu|\boldsymbol{H}|^2+\int_\Gamma\int_D\frac{\rho}{\mu\omega_{pm}^2}|\nabla\times\boldsymbol{K}|^2\right).
\end{aligned}
$$

Denote constants C_2^* and C_3^* as follows:

$$C_2^*=\max_{\overline{D}\times\overline{\Gamma}}\left(|\frac{\nabla(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}|,|\frac{\nabla(\mu\omega_{pm}^2)}{\mu\omega_{pm}}|,|\nabla\Gamma_e|,|\nabla\Gamma_m|\right),\quad C_3^*=\max_{\overline{D}\times\overline{\Gamma}}\left(|\frac{\nabla\epsilon}{\epsilon}|,|\frac{\nabla\mu}{\mu}|\right).$$

Recall the notation

$$ENG_0(t)=\int_\Gamma\int_D\rho(\boldsymbol{y})\left(\epsilon|\boldsymbol{E}|^2+\mu|\boldsymbol{H}|^2+\frac{1}{\epsilon\omega_{pe}^2}|\boldsymbol{J}|^2+\frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}|^2\right)(t)dxdy,$$

and substitute the above estimates into (47), then we have

$$\frac{d}{dt}ENG_3(t) \le C_2^* \cdot ENG_0(t) + C_1^* \cdot ENG_3(t) + C_3^* \int_\Gamma \int_D \rho(\boldsymbol{y}) \left(\epsilon|\partial_t \boldsymbol{E}|^2 + \mu|\partial_t \boldsymbol{H}|^2\right)$$
$$\le C_1^* \cdot ENG_3(t) + C_2^* \cdot ENG_0(0) + C_3^* C_{max1}(ENG_0(0) + \int_\Gamma \int_D \rho(\epsilon|\nabla \times \boldsymbol{E}_0|^2 + \mu|\nabla \times \boldsymbol{H}_0|^2)), \tag{48}$$

where we used Lemmas 2.1–2.2 in the last step.

Applying the Gronwall inequality (cf. Lemma 2.3) to (48) and the facts that $C_2^* \le C_1^*$ and $C_3^* \le C_1^*$, we have

$$ENG_3(t) \le e^{C_1^* t}\left[ENG_3(0) + (1 + C_{max1})(ENG_0(0) + \int_\Gamma \int_D \rho(\epsilon|\nabla \times \boldsymbol{E}_0|^2 + \mu|\nabla \times \boldsymbol{H}_0|^2))\right]$$
$$\le e^{C_1^* t}(1 + C_{max1}) \int_\Gamma \int_D \rho \left(\epsilon|\boldsymbol{E}_0|^2 + \mu|\boldsymbol{H}_0|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2\right)$$
$$+ e^{C_1^* t}(2 + C_{max1}) \int_\Gamma \int_D \rho \left(\epsilon|\nabla \times \boldsymbol{E}_0|^2 + \mu|\nabla \times \boldsymbol{H}_0|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\nabla \times \boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla \times \boldsymbol{K}_0|^2\right),$$

which concludes the proof. □

Remark 2.3. Similar remark as Remark 2.2 holds true. More specifically, if the physical parameters ϵ, μ, Γ_e, Γ_m, ω_{pe}, ω_{pm} are independent of the spatial variable $\boldsymbol{x}$, then $C_1^* = C_2^* = C_3^* = 0$. In this case, Lemma 2.4 just becomes

$$\int_\Gamma \int_D \rho(\boldsymbol{y}) \left(\epsilon|\nabla \times \boldsymbol{E}|^2 + \mu|\nabla \times \boldsymbol{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\nabla \times \boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla \times \boldsymbol{K}|^2\right)(t)dxdy$$
$$\le \int_\Gamma \int_D \rho(\boldsymbol{y}) \left(\epsilon|\nabla \times \boldsymbol{E}|^2 + \mu|\nabla \times \boldsymbol{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\nabla \times \boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla \times \boldsymbol{K}|^2\right)(0)dxdy.$$

Lemma 2.5. *Denote constants C_4 and C_5 as*

$$C_4 = \max_{\overline{D}\times\overline{\Gamma}} \left(|\frac{\nabla\epsilon}{\epsilon\sqrt{\epsilon\mu}}| + |\frac{\nabla\mu}{\mu\sqrt{\epsilon\mu}}| + |\frac{\omega_{pe}\nabla\epsilon}{\epsilon}| + |\frac{\omega_{pm}\nabla\mu}{\mu}|, |\frac{\nabla(\mu\omega_{pm}^2)}{\mu\omega_{pm}}| + |\nabla\Gamma_m|, |\frac{\nabla(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}| + |\nabla\Gamma_e| \right),$$
$$C_5 = \max_{\overline{D}\times\overline{\Gamma}} \left(|\frac{\omega_{pe}\nabla\epsilon}{\epsilon}| + |\nabla\Gamma_e|, |\frac{\omega_{pm}\nabla\mu}{\mu}| + |\nabla\Gamma_m|, |\frac{\nabla(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}|, |\frac{\nabla(\mu\omega_{pm}^2)}{\mu\omega_{pm}}| \right).$$

Then for any $t \in [0, T]$ and $i = 1, \ldots, d$, we have

$$\int_\Gamma \int_D \rho(\boldsymbol{y}) \left(\epsilon|\nabla \times \partial_t \boldsymbol{E}|^2 + \mu|\nabla \times \partial_t \boldsymbol{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\nabla \times \partial_t \boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla \times \partial_t \boldsymbol{K}|^2\right)(t)dxdy$$
$$\le e^{C_4 t}(1 + \frac{C_5 C_{max1}}{C_4})[\int_\Gamma \int_D \rho(\boldsymbol{y}) \left(\epsilon|\nabla \times \partial_t \boldsymbol{E}|^2 + \mu|\nabla \times \partial_t \boldsymbol{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\nabla \times \partial_t \boldsymbol{J}|^2\right.$$
$$\left. + \frac{1}{\mu\omega_{pm}^2}|\nabla \times \partial_t \boldsymbol{K}|^2\right)(0)$$
$$+ \int_\Gamma \int_D \rho(\boldsymbol{y}) \left(\epsilon|\nabla \times \boldsymbol{E}_0|^2 + \mu|\nabla \times \boldsymbol{H}_0|^2 + \epsilon|\boldsymbol{E}_0|^2 + \mu|\boldsymbol{H}_0|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2\right)].$$

Proof. Taking ∂_t of (43)–(46), we obtain

$$\epsilon\partial_t(\nabla \times \partial_t \boldsymbol{E}) - \nabla \times (\nabla \times \partial_t \boldsymbol{H}) + \nabla \times \partial_t \boldsymbol{J} = -\nabla\epsilon \times \partial_t(\partial_t \boldsymbol{E}), \tag{49}$$
$$\mu\partial_t(\nabla \times \partial_t \boldsymbol{H}) + \nabla \times (\nabla \times \partial_t \boldsymbol{E}) + \nabla \times \partial_t \boldsymbol{K} = -\nabla\mu \times \partial_t(\partial_t \boldsymbol{H}), \tag{50}$$
$$\partial_t(\nabla \times \partial_t \boldsymbol{J}) + \Gamma_e \nabla \times \partial_t \boldsymbol{J} - \epsilon\omega_{pe}^2 \nabla \times \partial_t \boldsymbol{E} = \nabla(\epsilon\omega_{pe}^2) \times \partial_t \boldsymbol{E} - \nabla\Gamma_e \times \partial_t \boldsymbol{J}, \tag{51}$$

$$\partial_t(\nabla \times \partial_t \boldsymbol{K}) + \Gamma_m \nabla \times \partial_t \boldsymbol{K} - \mu\omega_{pm}^2 \nabla \times \partial_t \boldsymbol{H} = \nabla(\mu\omega_{pm}^2) \times \partial_t \boldsymbol{H} - \nabla \Gamma_m \times \partial_t \boldsymbol{K}. \tag{52}$$

Denote

$$ENG_4(t) = \int_\Gamma \int_D \rho(\boldsymbol{y}) \left(\epsilon |\nabla \times \partial_t \boldsymbol{E}|^2 + \mu |\nabla \times \partial_t \boldsymbol{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2} |\nabla \times \partial_t \boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2} |\nabla \times \partial_t \boldsymbol{K}|^2 \right)(t) dxdy,$$

and

$$ENG_5(t) = \int_\Gamma \int_D \rho(\boldsymbol{y}) \left(\epsilon |\partial_t \boldsymbol{E}|^2 + \mu |\partial_t \boldsymbol{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2} |\partial_t \boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2} |\partial_t \boldsymbol{K}|^2 \right)(t) dxdy.$$

Multiplying (49)–(52) by $2\rho(\boldsymbol{y})\nabla\times\partial_t\boldsymbol{E}$, $2\rho(\boldsymbol{y})\nabla\times\partial_t\boldsymbol{H}$, $\frac{2\rho(\boldsymbol{y})}{\epsilon\omega_{pe}^2}\nabla\times\partial_t\boldsymbol{J}$ and $\frac{2\rho(\boldsymbol{y})}{\mu\omega_{pm}^2}\nabla\times\partial_t\boldsymbol{K}$, respectively, then integrating over D and Γ, and adding the resultants, we have

$$\begin{aligned}
&\frac{d}{dt} ENG_4(t) + \int_\Gamma \int_D 2\rho(\boldsymbol{y}) \left(\frac{\Gamma_e(\boldsymbol{x},\boldsymbol{y})}{\epsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})} |\nabla \times \partial_t \boldsymbol{J}|^2 + \frac{\Gamma_m(\boldsymbol{x},\boldsymbol{y})}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})} |\nabla \times \partial_t \boldsymbol{K}|^2 \right) dxdy \\
&= \int_\Gamma \int_D (\nabla \times \nabla \times \partial_t \boldsymbol{H} \cdot 2\rho \nabla \times \partial_t \boldsymbol{E} - \nabla \times \nabla \times \partial_t \boldsymbol{E} \cdot 2\rho \nabla \times \partial_t \boldsymbol{H}) \\
&\quad - \int_\Gamma \int_D \nabla\epsilon \times \partial_t(\partial_t \boldsymbol{E}) \cdot 2\rho \nabla \times \partial_t \boldsymbol{E} - \int_\Gamma \int_D \nabla\mu \times \partial_t(\partial_t \boldsymbol{H}) \cdot 2\rho \nabla \times \partial_t \boldsymbol{H} \\
&\quad - \int_\Gamma \int_D (\nabla \Gamma_e \times \partial_t \boldsymbol{J}) \cdot \frac{2\rho}{\epsilon\omega_{pe}^2} \nabla \times \partial_t \boldsymbol{J} + \int_\Gamma \int_D \nabla(\epsilon\omega_{pe}^2) \times \partial_t \boldsymbol{E} \cdot \frac{2\rho}{\epsilon\omega_{pe}^2} \nabla \times \partial_t \boldsymbol{J} \\
&\quad - \int_\Gamma \int_D (\nabla \Gamma_m \times \partial_t \boldsymbol{K}) \cdot \frac{2\rho}{\mu\omega_{pm}^2} \nabla \times \partial_t \boldsymbol{K} + \int_\Gamma \int_D \nabla(\mu\omega_{pm}^2) \times \partial_t \boldsymbol{H} \cdot \frac{2\rho}{\mu\omega_{pm}^2} \nabla \times \partial_t \boldsymbol{K} := \sum_{i=1}^7 Err_i.
\end{aligned} \tag{53}$$

Using integration by parts, (1), and boundary conditions $\boldsymbol{n} \times \boldsymbol{E} = 0$ and $\boldsymbol{n} \times \boldsymbol{J} = 0$, we obtain

$$\begin{aligned}
&\int_\Gamma \int_D \nabla \times \nabla \times \partial_t \boldsymbol{H} \cdot 2\rho \nabla \times \partial_t \boldsymbol{E} \\
&= \int_\Gamma \int_{\partial D} \boldsymbol{n} \times (\nabla \times \partial_t \boldsymbol{H}) \cdot 2\rho \nabla \times \partial_t \boldsymbol{E} + \int_\Gamma \int_D \nabla \times \partial_t \boldsymbol{H} \cdot 2\rho \nabla \times \nabla \times \partial_t \boldsymbol{E} \\
&= \int_\Gamma \int_{\partial D} \boldsymbol{n} \times (\epsilon \partial_{t^2} \boldsymbol{E} + \partial_t \boldsymbol{J}) \cdot 2\rho \nabla \times \partial_t \boldsymbol{E} + \int_\Gamma \int_D \nabla \times \partial_t \boldsymbol{H} \cdot 2\rho \nabla \times \nabla \times \partial_t \boldsymbol{E} \\
&= \int_\Gamma \int_D \nabla \times \partial_t \boldsymbol{H} \cdot 2\rho \nabla \times \nabla \times \partial_t \boldsymbol{E},
\end{aligned}$$

which leads to $Err_1 = 0$.

Using (1) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&Err_2 = -\int_\Gamma \int_D \frac{\nabla\epsilon}{\epsilon} \times (\nabla \times \partial_t \boldsymbol{H} - \partial_t \boldsymbol{J}) \cdot 2\rho \nabla \times \partial_t \boldsymbol{E} \\
&= -\int_\Gamma \int_D 2\frac{\nabla\epsilon}{\epsilon\sqrt{\epsilon\mu}} \times (\sqrt{\rho\mu} \nabla \times \partial_t \boldsymbol{H}) \cdot \sqrt{\rho\epsilon} \nabla \times \partial_t \boldsymbol{E} + \int_\Gamma \int_D 2\frac{\omega_{pe}\nabla\epsilon}{\epsilon} \times (\sqrt{\frac{\rho}{\epsilon\omega_{pe}^2}} \partial_t \boldsymbol{J}) \cdot \sqrt{\rho\epsilon} \nabla \times \partial_t \boldsymbol{E} \\
&\le (\max_{\overline{D}\times\overline{\Gamma}} |\frac{\nabla\epsilon}{\epsilon\sqrt{\epsilon\mu}}|) \int_\Gamma \int_D \rho \left(\epsilon |\nabla \times \partial_t \boldsymbol{E}|^2 + \mu |\nabla \times \partial_t \boldsymbol{H}|^2 \right) \\
&\quad + (\max_{\overline{D}\times\overline{\Gamma}} |\frac{\omega_{pe}\nabla\epsilon}{\epsilon}|) \int_\Gamma \int_D \rho \left(\frac{1}{\epsilon\omega_{pe}^2} |\partial_t \boldsymbol{J}|^2 + \epsilon |\nabla \times \partial_t \boldsymbol{E}|^2 \right).
\end{aligned}$$

Using (2) and the Cauchy–Schwarz inequality, we can obtain

$$\begin{aligned}
Err_3 &= \int_\Gamma \int_D \frac{\nabla\mu}{\mu} \times (\nabla \times \partial_t \boldsymbol{E} + \partial_t \boldsymbol{K}) \cdot 2\rho \nabla \times \partial_t \boldsymbol{H} \\
&\le (\max_{\overline{D}\times\overline{\Gamma}} |\frac{\nabla\mu}{\mu\sqrt{\epsilon\mu}}|) \int_\Gamma \int_D \rho \left(\epsilon |\nabla \times \partial_t \boldsymbol{E}|^2 + \mu |\nabla \times \partial_t \boldsymbol{H}|^2 \right)
\end{aligned}$$

$$+(\max_{\overline{D}\times\overline{\Gamma}}|\frac{\omega_{pm}\nabla\mu}{\mu}|)\int_\Gamma\int_D\rho\left(\frac{1}{\mu\omega_{pm}^2}|\partial_t\boldsymbol{K}|^2+\mu|\nabla\times\partial_t\boldsymbol{H}|^2\right).$$

Similarly, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
Err_4 &\le (\max_{\overline{D}\times\overline{\Gamma}}|\nabla\Gamma_e|)\int_\Gamma\int_D\frac{\rho}{\epsilon\omega_{pe}^2}\left(|\partial_t\boldsymbol{J}|^2+|\nabla\times\partial_t\boldsymbol{J}|^2\right),\\
Err_5 &\le (\max_{\overline{D}\times\overline{\Gamma}}|\frac{\nabla(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}|)\left(\int_\Gamma\int_D\rho\epsilon|\partial_t\boldsymbol{E}|^2+\int_\Gamma\int_D\frac{\rho}{\epsilon\omega_{pe}^2}|\nabla\times\partial_t\boldsymbol{J}|^2\right),\\
Err_6 &\le (\max_{\overline{D}\times\overline{\Gamma}}|\nabla\Gamma_m|)\int_\Gamma\int_D\frac{\rho}{\mu\omega_{pm}^2}\left(|\partial_t\boldsymbol{K}|^2+|\nabla\times\partial_t\boldsymbol{K}|^2\right),\\
Err_7 &\le (\max_{\overline{D}\times\overline{\Gamma}}|\frac{\nabla(\mu\omega_{pm}^2)}{\mu\omega_{pm}}|)\left(\int_\Gamma\int_D\rho\mu|\partial_t\boldsymbol{H}|^2+\int_\Gamma\int_D\frac{\rho}{\mu\omega_{pm}^2}|\nabla\times\partial_t\boldsymbol{K}|^2\right).
\end{aligned}$$

Substituting the above estimates into (53) and using the notations $ENG_4(t)$ and $ENG_5(t)$ and Lemma 2.2, we have

$$\begin{aligned}
&\frac{d}{dt}ENG_4(t)\le C_4\cdot ENG_4(t)+C_5\cdot ENG_5(t)\\
&\le C_4\cdot ENG_4(t)+C_5C_{max1}[ENG_0(0)+\int_\Gamma\int_D\rho(\epsilon|\nabla\times\boldsymbol{E}_0|^2+\mu|\nabla\times\boldsymbol{H}_0|^2)].
\end{aligned}\tag{54}$$

Applying the Gronwall inequality (cf. Lemma 2.3) to (54), we have

$$\begin{aligned}
&ENG_4(t)\le e^{C_4}\left[ENG_4(0)+\frac{C_5C_{max1}}{C_4}(ENG_0(0)+\int_\Gamma\int_D\rho(\epsilon|\nabla\times\boldsymbol{E}_0|^2+\mu|\nabla\times\boldsymbol{H}_0|^2))\right]\\
&\le e^{C_4}(1+\frac{C_5C_{max1}}{C_4})[(ENG_4(0)+ENG_0(0)+\int_\Gamma\int_D\rho(\epsilon|\nabla\times\boldsymbol{E}_0|^2+\mu|\nabla\times\boldsymbol{H}_0|^2))],
\end{aligned}$$

which concludes the proof. □

Remark 2.4. If the physical parameters $\epsilon,\mu,\Gamma_e,\Gamma_m,\omega_{pe},\omega_{pm}$ are independent of the spatial variable $\boldsymbol{x}$, then the constants $C_4=C_5=0$. In this case, Lemma 2.5 simply reduces to

$$\begin{aligned}
&\int_\Gamma\int_D\rho(\boldsymbol{y})\left(\epsilon|\nabla\times\partial_t\boldsymbol{E}|^2+\mu|\nabla\times\partial_t\boldsymbol{H}|^2+\frac{1}{\epsilon\omega_{pe}^2}|\nabla\times\partial_t\boldsymbol{J}|^2+\frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_t\boldsymbol{K}|^2\right)(t)dxdy\\
&\le\int_\Gamma\int_D\rho(\boldsymbol{y})\left(\epsilon|\nabla\times\partial_t\boldsymbol{E}|^2+\mu|\nabla\times\partial_t\boldsymbol{H}|^2+\frac{1}{\epsilon\omega_{pe}^2}|\nabla\times\partial_t\boldsymbol{J}|^2+\frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_t\boldsymbol{K}|^2\right)(0)dxdy.
\end{aligned}$$

Theorem 2.2. *Denote constants $C_{6,1},C_{6,2},C_{6,3},C_{6,4}$ and C_6 as*

$$\begin{aligned}
C_{6,1}&=\max_{\overline{D}\times\overline{\Gamma}}\left(|\frac{\nabla\epsilon}{\epsilon\sqrt{\epsilon\mu}}|+|\frac{\omega_{pe}\nabla\epsilon}{\epsilon}|+|\frac{(\partial_{y_i}\epsilon)\nabla\epsilon}{\epsilon^2}|+|\frac{\nabla(\partial_{y_i}\epsilon)}{\epsilon}|+|\frac{\partial_{y_i}\epsilon}{\epsilon}|+|\frac{\nabla\mu}{\mu\sqrt{\epsilon\mu}}|\right),\\
C_{6,2}&=\max_{\overline{D}\times\overline{\Gamma}}\left(|\frac{\nabla\mu}{\mu\sqrt{\epsilon\mu}}|+|\frac{\omega_{pm}\nabla\mu}{\mu}|+|\frac{(\partial_{y_i}\mu)\nabla\mu}{\mu^2}|+|\frac{\nabla(\partial_{y_i}\mu)}{\mu}|+|\frac{\partial_{y_i}\mu}{\mu}|+|\frac{\nabla\epsilon}{\epsilon\sqrt{\epsilon\mu}}|\right),\\
C_{6,3}&=\max_{\overline{D}\times\overline{\Gamma}}\left(|\nabla\Gamma_e|+|\frac{\nabla(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}|+|\frac{\nabla(\partial_{y_i}(\epsilon\omega_{pe}^2))}{\epsilon\omega_{pe}}|+|\frac{\partial_{y_i}(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}|+|\nabla(\partial_{y_i}\Gamma_e)|+|\partial_{y_i}\Gamma_e|\right),\\
C_{6,4}&=\max_{\overline{D}\times\overline{\Gamma}}\left(|\nabla\Gamma_m|+|\frac{\nabla(\mu\omega_{pm}^2)}{\mu\omega_{pm}}|+|\frac{\nabla(\partial_{y_i}(\mu\omega_{pm}^2))}{\mu\omega_{pm}}|+|\frac{\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}|+|\nabla(\partial_{y_i}\Gamma_m)|+|\partial_{y_i}\Gamma_m|\right),\\
C_6&=\max\left(C_{6,1},C_{6,2},C_{6,3},C_{6,4}\right),\quad C_{12}=C_6+C_4+C_1+C_1^*.
\end{aligned}$$

Then for any $t \in [0, T]$ and $i = 1, \dots, d$, we have

$$\int_\Gamma \int_D \rho(\boldsymbol{y}) \left(\epsilon |\nabla \times \partial_{y_i} \boldsymbol{E}|^2 + \mu |\nabla \times \partial_{y_i} \boldsymbol{H}|^2 + \frac{1}{\epsilon \omega_{pe}^2} |\nabla \times \partial_{y_i} \boldsymbol{J}|^2 + \frac{1}{\mu \omega_{pm}^2} |\nabla \times \partial_{y_i} \boldsymbol{K}|^2 \right)(t) dx dy$$
$$\le e^{C_{12}t} C_{13} [\int_\Gamma \int_D \rho \left(\epsilon |\nabla \times \partial_{y_i} \boldsymbol{E}|^2 + \mu |\nabla \times \partial_{y_i} \boldsymbol{H}|^2 + \frac{1}{\epsilon \omega_{pe}^2} |\nabla \times \partial_{y_i} \boldsymbol{J}|^2 + \frac{1}{\mu \omega_{pm}^2} |\nabla \times \partial_{y_i} \boldsymbol{K}|^2 \right)(0) dx dy$$
$$+ \int_\Gamma \int_D \rho \left(\epsilon |\boldsymbol{E}_0|^2 + \mu |\boldsymbol{H}_0|^2 + \frac{1}{\epsilon \omega_{pe}^2} |\boldsymbol{J}_0|^2 + \frac{1}{\mu \omega_{pm}^2} |\boldsymbol{K}_0|^2 \right) dx dy$$
$$+ \int_\Gamma \int_D \rho \left(\epsilon |\nabla \times \boldsymbol{E}_0|^2 + \mu |\nabla \times \boldsymbol{H}_0|^2 + \frac{1}{\epsilon \omega_{pe}^2} |\nabla \times \boldsymbol{J}_0|^2 + \frac{1}{\mu \omega_{pm}^2} |\nabla \times \boldsymbol{K}_0|^2 \right) dx dy$$
$$+ \int_\Gamma \int_D \rho \left(\epsilon |\nabla \times \partial_t \boldsymbol{E}|^2 + \mu |\nabla \times \partial_t \boldsymbol{H}|^2 + \frac{1}{\epsilon \omega_{pe}^2} |\nabla \times \partial_t \boldsymbol{J}|^2 + \frac{1}{\mu \omega_{pm}^2} |\nabla \times \partial_t \boldsymbol{K}|^2 \right)(0) dx dy$$
$$+ \int_\Gamma \int_D \rho \left(\epsilon |\partial_{y_i} \boldsymbol{E}|^2 + \mu |\partial_{y_i} \boldsymbol{H}|^2 + \frac{1}{\epsilon \omega_{pe}^2} |\partial_{y_i} \boldsymbol{J}|^2 + \frac{1}{\mu \omega_{pm}^2} |\partial_{y_i} \boldsymbol{K}|^2 \right)(0) dx dy],$$

where constant $C_{13} > 0$ depends on parameters $\epsilon, \mu, \omega_{pe}, \omega_{pm}, \Gamma_e$ and Γ_m, but is independent of t.

Proof. Taking $\nabla \times$ of (32)–(35), we obtain

$$\begin{aligned} &\epsilon \partial_t (\nabla \times \partial_{y_i} \boldsymbol{E}) - \nabla \times (\nabla \times \partial_{y_i} \boldsymbol{H}) + \nabla \times \partial_{y_i} \boldsymbol{J} \\ &\quad = -\nabla \epsilon \times \partial_{t y_i} \boldsymbol{E} - \nabla(\partial_{y_i} \epsilon) \times \partial_t \boldsymbol{E} - (\partial_{y_i} \epsilon) \partial_t (\nabla \times \boldsymbol{E}), \end{aligned} \tag{55}$$

$$\begin{aligned} &\mu \partial_t (\nabla \times \partial_{y_i} \boldsymbol{H}) + \nabla \times (\nabla \times \partial_{y_i} \boldsymbol{E}) + \nabla \times \partial_{y_i} \boldsymbol{K} \\ &\quad = -\nabla \mu \times \partial_{t y_i} \boldsymbol{H} - \nabla(\partial_{y_i} \mu) \times \partial_t \boldsymbol{H} - (\partial_{y_i} \mu) \partial_t (\nabla \times \boldsymbol{H}), \end{aligned} \tag{56}$$

$$\begin{aligned} &\partial_t (\nabla \times \partial_{y_i} \boldsymbol{J}) + \Gamma_e \nabla \times \partial_{y_i} \boldsymbol{J} - \epsilon \omega_{pe}^2 \nabla \times \partial_{y_i} \boldsymbol{E} = -\nabla \Gamma_e \times \partial_{y_i} \boldsymbol{J} + \nabla(\epsilon \omega_{pe}^2) \times \partial_{y_i} \boldsymbol{E} \\ &\quad + \nabla(\partial_{y_i}(\epsilon \omega_{pe}^2)) \times \boldsymbol{E} + (\partial_{y_i}(\epsilon \omega_{pe}^2)) \nabla \times \boldsymbol{E} - \nabla(\partial_{y_i} \Gamma_e) \times \boldsymbol{J} - (\partial_{y_i} \Gamma_e) \nabla \times \boldsymbol{J}, \end{aligned} \tag{57}$$

$$\begin{aligned} &\partial_t (\nabla \times \partial_{y_i} \boldsymbol{K}) + \Gamma_m \nabla \times \partial_{y_i} \boldsymbol{K} - \mu \omega_{pm}^2 \nabla \times \partial_{y_i} \boldsymbol{H} = -\nabla \Gamma_m \times \partial_{y_i} \boldsymbol{K} + \nabla(\mu \omega_{pm}^2) \times \partial_{y_i} \boldsymbol{H} \\ &\quad + \nabla(\partial_{y_i}(\mu \omega_{pm}^2)) \times \boldsymbol{H} + (\partial_{y_i}(\mu \omega_{pm}^2)) \nabla \times \boldsymbol{H} - \nabla(\partial_{y_i} \Gamma_m) \times \boldsymbol{K} - (\partial_{y_i} \Gamma_m) \nabla \times \boldsymbol{K}. \end{aligned} \tag{58}$$

Denote

$$\begin{aligned} ENG_6(t) = \int_\Gamma \int_D \rho(\boldsymbol{y}) \Bigg(&\epsilon |\nabla \times \partial_{y_i} \boldsymbol{E}|^2 + \mu |\nabla \times \partial_{y_i} \boldsymbol{H}|^2 + \frac{1}{\epsilon \omega_{pe}^2} |\nabla \times \partial_{y_i} \boldsymbol{J}|^2 \\ &+ \frac{1}{\mu \omega_{pm}^2} |\nabla \times \partial_{y_i} \boldsymbol{K}|^2 \Bigg)(t) dx dy. \end{aligned}$$

Multiplying (55)–(58) by $2\rho \nabla \times \partial_{y_i} \boldsymbol{E}$, $2\rho \nabla \times \partial_{y_i} \boldsymbol{H}$, $\frac{2\rho}{\epsilon \omega_{pe}^2} \nabla \times \partial_{y_i} \boldsymbol{J}$ and $\frac{2\rho}{\mu \omega_{pm}^2} \nabla \times \partial_{y_i} \boldsymbol{K}$, respectively, then integrating over D and Γ, and adding the resultants, we have

$$\frac{d}{dt} ENG_6(t) + \int_\Gamma \int_D 2\rho(\boldsymbol{y}) \left(\frac{\Gamma_e(\boldsymbol{x}, \boldsymbol{y})}{\epsilon(\boldsymbol{x}, \boldsymbol{y}) \omega_{pe}^2(\boldsymbol{x}, \boldsymbol{y})} |\nabla \times \partial_{y_i} \boldsymbol{J}|^2 + \frac{\Gamma_m(\boldsymbol{x}, \boldsymbol{y})}{\mu(\boldsymbol{x}, \boldsymbol{y}) \omega_{pm}^2(\boldsymbol{x}, \boldsymbol{y})} |\nabla \times \partial_{y_i} \boldsymbol{K}|^2 \right) dx dy$$
$$= \int_\Gamma \int_D (\nabla \times \nabla \times \partial_{y_i} \boldsymbol{H} \cdot 2\rho \nabla \times \partial_{y_i} \boldsymbol{E} - \nabla \times \nabla \times \partial_{y_i} \boldsymbol{E} \cdot 2\rho \nabla \times \partial_{y_i} \boldsymbol{H})$$
$$- \int_\Gamma \int_D \nabla \epsilon \times \partial_{t y_i} \boldsymbol{E} \cdot 2\rho \nabla \times \partial_{y_i} \boldsymbol{E} - \int_\Gamma \int_D [\nabla(\partial_{y_i} \epsilon) \times \partial_t \boldsymbol{E} + \partial_{y_i} \epsilon \cdot \partial_t (\nabla \times \boldsymbol{E})] \cdot 2\rho \nabla \times \partial_{y_i} \boldsymbol{E}$$
$$- \int_\Gamma \int_D \nabla \mu \times \partial_{t y_i} \boldsymbol{H} \cdot 2\rho \nabla \times \partial_{y_i} \boldsymbol{H} - \int_\Gamma \int_D [\nabla(\partial_{y_i} \mu) \times \partial_t \boldsymbol{H} + \partial_{y_i} \mu \cdot \partial_t (\nabla \times \boldsymbol{H})] \cdot 2\rho \nabla \times \partial_{y_i} \boldsymbol{H}$$

$$
\begin{aligned}
&+\int_\Gamma\int_D[-\nabla\Gamma_e\times\partial_{y_i}\boldsymbol{J}+\nabla(\epsilon\omega_{pe}^2)\times\partial_{y_i}\boldsymbol{E}+\nabla(\partial_{y_i}(\epsilon\omega_{pe}^2))\times\boldsymbol{E}+\partial_{y_i}(\epsilon\omega_{pe}^2)\cdot\nabla\times\boldsymbol{E}\\
&-\nabla(\partial_{y_i}\Gamma_e)\times\boldsymbol{J}-(\partial_{y_i}\Gamma_e)\nabla\times\boldsymbol{J}]\cdot\frac{2\rho}{\epsilon\omega_{pe}^2}\nabla\times\partial_{y_i}\boldsymbol{J}\\
&+\int_\Gamma\int_D[-\nabla\Gamma_m\times\partial_{y_i}\boldsymbol{K}+\nabla(\mu\omega_{pm}^2)\times\partial_{y_i}\boldsymbol{H}+\nabla(\partial_{y_i}(\mu\omega_{pm}^2))\times\boldsymbol{H}+\partial_{y_i}(\mu\omega_{pm}^2)\cdot\nabla\times\boldsymbol{H}\\
&-\nabla(\partial_{y_i}\Gamma_e)\times\boldsymbol{K}-(\partial_{y_i}\Gamma_m)\nabla\times\boldsymbol{K}]\cdot\frac{2\rho}{\mu\omega_{pm}^2}\nabla\times\partial_{y_i}\boldsymbol{K}:=\sum_{i=1}^{7}Err_i. \qquad (59)
\end{aligned}
$$

Below we will estimate each Err_i of (59). First, using integration by parts, (32), and boundary conditions $\boldsymbol{n}\times\boldsymbol{E}=0$ and $\boldsymbol{n}\times\boldsymbol{J}=0$, we obtain

$$
\begin{aligned}
&\int_\Gamma\int_D\nabla\times\nabla\times\partial_{y_i}\boldsymbol{H}\cdot2\rho\nabla\times\partial_{y_i}\boldsymbol{E}\\
=&\int_\Gamma\int_{\partial D}\boldsymbol{n}\times(\nabla\times\partial_{y_i}\boldsymbol{H})\cdot2\rho\nabla\times\partial_{y_i}\boldsymbol{E}+\int_\Gamma\int_D\nabla\times\partial_{y_i}\boldsymbol{H}\cdot2\rho\nabla\times\nabla\times\partial_{y_i}\boldsymbol{E}\\
=&\int_\Gamma\int_{\partial D}\boldsymbol{n}\times(\epsilon\partial_{ty_i}\boldsymbol{E}+\partial_{y_i}\boldsymbol{J}+\partial_{y_i}\epsilon\partial_t\boldsymbol{E})\cdot2\rho\nabla\times\partial_{y_i}\boldsymbol{E}+\int_\Gamma\int_D\nabla\times\partial_{y_i}\boldsymbol{H}\cdot2\rho\nabla\times\nabla\times\partial_{y_i}\boldsymbol{E}\\
=&\int_\Gamma\int_D\nabla\times\partial_{y_i}\boldsymbol{H}\cdot2\rho\nabla\times\nabla\times\partial_{y_i}\boldsymbol{E},
\end{aligned}
$$

which leads to $Err_1=0$.

Using (32) and the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
Err_2=&-\int_\Gamma\int_D\frac{\nabla\epsilon}{\epsilon}\times(\nabla\times\partial_{y_i}\boldsymbol{H}-\partial_{y_i}\boldsymbol{J}-\partial_{y_i}\epsilon\partial_t\boldsymbol{E})\cdot2\rho\nabla\times\partial_{y_i}\boldsymbol{E}\\
\le&(\max_{\overline{D}\times\overline{\Gamma}}|\frac{\nabla\epsilon}{\epsilon\sqrt{\epsilon\mu}}|)\int_\Gamma\int_D\rho\left(\epsilon|\nabla\times\partial_{y_i}\boldsymbol{E}|^2+\mu|\nabla\times\partial_{y_i}\boldsymbol{H}|^2\right)\\
&+(\max_{\overline{D}\times\overline{\Gamma}}|\frac{\omega_{pe}\nabla\epsilon}{\epsilon}|)\int_\Gamma\int_D\rho\left(\frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2+\epsilon|\nabla\times\partial_{y_i}\boldsymbol{E}|^2\right)\\
&+(\max_{\overline{D}\times\overline{\Gamma}}|\frac{(\partial_{y_i}\epsilon)\nabla\epsilon}{\epsilon^2}|)\int_\Gamma\int_D\rho\left(\epsilon|\partial_t\boldsymbol{E}|^2+\epsilon|\nabla\times\partial_{y_i}\boldsymbol{E}|^2\right).
\end{aligned}
$$

By the Cauchy–Schwarz inequality, we can obtain

$$
\begin{aligned}
Err_3=&-\int_\Gamma\int_D2\frac{\nabla(\partial_{y_i}\epsilon)}{\epsilon}\times\sqrt{\rho\epsilon}\partial_t\boldsymbol{E}\cdot\sqrt{\rho\epsilon}\nabla\times\partial_{y_i}\boldsymbol{E}-\int_\Gamma\int_D2\frac{\partial_{y_i}\epsilon}{\epsilon}\sqrt{\rho\epsilon}\partial_t(\nabla\times\boldsymbol{E})\cdot\sqrt{\rho\epsilon}\nabla\times\partial_{y_i}\boldsymbol{E}\\
\le&(\max_{\overline{D}\times\overline{\Gamma}}|\frac{\nabla(\partial_{y_i}\epsilon)}{\epsilon}|)\int_\Gamma\int_D\rho\left(\epsilon|\partial_t\boldsymbol{E}|^2+\epsilon|\nabla\times\partial_{y_i}\boldsymbol{E}|^2\right)\\
&+(\max_{\overline{D}\times\overline{\Gamma}}|\frac{\partial_{y_i}\epsilon}{\epsilon}|)\int_\Gamma\int_D\rho\left(\epsilon|\partial_t(\nabla\times\boldsymbol{E})|^2+\epsilon|\nabla\times\partial_{y_i}\boldsymbol{E}|^2\right).
\end{aligned}
$$

Similarly, by (33) and the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
Err_4=&\int_\Gamma\int_D\frac{\nabla\mu}{\mu}\times(\nabla\times\partial_{y_i}\boldsymbol{E}+\partial_{y_i}\boldsymbol{K}+\partial_{y_i}\mu\partial_t\boldsymbol{H})\cdot2\rho\nabla\times\partial_{y_i}\boldsymbol{H}\\
\le&(\max_{\overline{D}\times\overline{\Gamma}}|\frac{\nabla\mu}{\mu\sqrt{\epsilon\mu}}|)\int_\Gamma\int_D\rho\left(\epsilon|\nabla\times\partial_{y_i}\boldsymbol{E}|^2+\mu|\nabla\times\partial_{y_i}\boldsymbol{H}|^2\right)\\
&+(\max_{\overline{D}\times\overline{\Gamma}}|\frac{\omega_{pm}\nabla\mu}{\mu}|)\int_\Gamma\int_D\rho\left(\frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2+\mu|\nabla\times\partial_{y_i}\boldsymbol{H}|^2\right)\\
&+(\max_{\overline{D}\times\overline{\Gamma}}|\frac{(\partial_{y_i}\mu)\nabla\mu}{\mu^2}|)\int_\Gamma\int_D\rho\left(\mu|\partial_t\boldsymbol{H}|^2+\mu|\nabla\times\partial_{y_i}\boldsymbol{H}|^2\right).
\end{aligned}
$$

By similar arguments, we have

$$
\begin{aligned}
Err_5 \le\ & (\max_{\overline{D}\times\overline{\Gamma}} |\frac{\nabla(\partial_{y_i}\mu)}{\mu}|) \int_\Gamma \int_D \rho \left(\mu|\partial_t \boldsymbol{H}|^2 + \mu|\nabla\times\partial_{y_i}\boldsymbol{H}|^2\right) \\
& + (\max_{\overline{D}\times\overline{\Gamma}} |\frac{\partial_{y_i}\mu}{\mu}|) \int_\Gamma \int_D \rho \left(\mu|\partial_t(\nabla\times\boldsymbol{H})|^2 + \mu|\nabla\times\partial_{y_i}\boldsymbol{H}|^2\right), \\
Err_6 \le\ & (\max_{\overline{D}\times\overline{\Gamma}} |\nabla\Gamma_e|) \int_\Gamma \int_D \rho \left(\frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\nabla\times\partial_{y_i}\boldsymbol{J}|^2\right) \\
& + (\max_{\overline{D}\times\overline{\Gamma}} |\frac{\nabla(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}|) \int_\Gamma \int_D \rho \left(\epsilon|\partial_{y_i}\boldsymbol{E}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\nabla\times\partial_{y_i}\boldsymbol{J}|^2\right) \\
& + (\max_{\overline{D}\times\overline{\Gamma}} |\frac{\nabla(\partial_{y_i}(\epsilon\omega_{pe}^2))}{\epsilon\omega_{pe}}|) \int_\Gamma \int_D \rho \left(\epsilon|\boldsymbol{E}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\nabla\times\partial_{y_i}\boldsymbol{J}|^2\right) \\
& + (\max_{\overline{D}\times\overline{\Gamma}} |\frac{\partial_{y_i}(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}|) \int_\Gamma \int_D \rho \left(\epsilon|\nabla\times\boldsymbol{E}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\nabla\times\partial_{y_i}\boldsymbol{J}|^2\right) \\
& + (\max_{\overline{D}\times\overline{\Gamma}} |\nabla(\partial_{y_i}\Gamma_e)|) \int_\Gamma \int_D \rho \left(\frac{1}{\epsilon\omega_{pe}^2}|\boldsymbol{J}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\nabla\times\partial_{y_i}\boldsymbol{J}|^2\right) \\
& + (\max_{\overline{D}\times\overline{\Gamma}} |\partial_{y_i}\Gamma_e|) \int_\Gamma \int_D \rho \left(\frac{1}{\epsilon\omega_{pe}^2}|\nabla\times\boldsymbol{J}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\nabla\times\partial_{y_i}\boldsymbol{J}|^2\right),
\end{aligned}
$$

and

$$
\begin{aligned}
Err_7 \le\ & (\max_{\overline{D}\times\overline{\Gamma}} |\nabla\Gamma_m|) \int_\Gamma \int_D \rho \left(\frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_{y_i}\boldsymbol{K}|^2\right) \\
& + (\max_{\overline{D}\times\overline{\Gamma}} |\frac{\nabla(\mu\omega_{pm}^2)}{\mu\omega_{pm}}|) \int_\Gamma \int_D \rho \left(\mu|\partial_{y_i}\boldsymbol{H}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_{y_i}\boldsymbol{K}|^2\right) \\
& + (\max_{\overline{D}\times\overline{\Gamma}} |\frac{\nabla(\partial_{y_i}(\mu\omega_{pm}^2))}{\mu\omega_{pm}}|) \int_\Gamma \int_D \rho \left(\mu|\boldsymbol{H}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_{y_i}\boldsymbol{K}|^2\right) \\
& + (\max_{\overline{D}\times\overline{\Gamma}} |\frac{\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}|) \int_\Gamma \int_D \rho \left(\mu|\nabla\times\boldsymbol{H}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_{y_i}\boldsymbol{K}|^2\right) \\
& + (\max_{\overline{D}\times\overline{\Gamma}} |\nabla(\partial_{y_i}\Gamma_m)|) \int_\Gamma \int_D \rho \left(\frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_{y_i}\boldsymbol{K}|^2\right) \\
& + (\max_{\overline{D}\times\overline{\Gamma}} |\partial_{y_i}\Gamma_m|) \int_\Gamma \int_D \rho \left(\frac{1}{\mu\omega_{pm}^2}|\nabla\times\boldsymbol{K}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_{y_i}\boldsymbol{K}|^2\right).
\end{aligned}
$$

Let us introduce the notations:

$$
\begin{aligned}
& C_7 = \max_{\overline{D}\times\overline{\Gamma}} \left(|\frac{(\partial_{y_i}\epsilon)\nabla\epsilon}{\epsilon^2}| + |\frac{\nabla(\partial_{y_i}\epsilon)}{\epsilon}|, |\frac{(\partial_{y_i}\mu)\nabla\mu}{\mu^2}| + |\frac{\nabla(\partial_{y_i}\mu)}{\mu}| \right), \quad C_8 = \max_{\overline{D}\times\overline{\Gamma}} \left(|\frac{\partial_{y_i}\epsilon}{\epsilon}|, |\frac{\partial_{y_i}\mu}{\mu}| \right), \\
& C_9 = \max_{\overline{D}\times\overline{\Gamma}} \left(|\frac{\omega_{pe}\nabla\epsilon}{\epsilon}| + |\nabla\Gamma_e|, |\frac{\omega_{pm}\nabla\mu}{\mu}| + |\nabla\Gamma_m|, |\frac{\nabla(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}|, |\frac{\nabla(\mu\omega_{pm}^2)}{\mu\omega_{pm}}| \right), \\
& C_{10} = \max_{\overline{D}\times\overline{\Gamma}} \left(|\frac{\partial_{y_i}(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}|, |\frac{\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}|, |\partial_{y_i}\Gamma_e|, |\partial_{y_i}\Gamma_m| \right), \\
& C_{11} = \max_{\overline{D}\times\overline{\Gamma}} \left(|\frac{\nabla(\partial_{y_i}(\epsilon\omega_{pe}^2))}{\epsilon\omega_{pe}}|, |\frac{\nabla(\partial_{y_i}(\mu\omega_{pm}^2))}{\mu\omega_{pm}}|, |\nabla(\partial_{y_i}\Gamma_e)|, |\nabla(\partial_{y_i}\Gamma_m)| \right).
\end{aligned}
$$

Substituting the above estimates into (59) and using the notation $ENG_6(t)$, we have

$$\frac{d}{dt}ENG_6(t) \le C_6 \cdot ENG_6(t) + C_7 \int_\Gamma \int_D \rho \left(\epsilon|\partial_t \boldsymbol{E}|^2 + \mu|\partial_t \boldsymbol{H}|^2\right)$$
$$+ C_8 \int_\Gamma \int_D \rho \left(\epsilon|\partial_t(\nabla\times\boldsymbol{E})|^2 + \mu|\partial_t(\nabla\times\boldsymbol{H})|^2\right)$$
$$+ C_9 \int_\Gamma \int_D \rho \left(\epsilon|\partial_{y_i}\boldsymbol{E}|^2 + \mu|\partial_{y_i}\boldsymbol{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2\right)$$
$$+ C_{10} \int_\Gamma \int_D \rho \left(\epsilon|\nabla\times\boldsymbol{E}|^2 + \mu|\nabla\times\boldsymbol{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\nabla\times\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\boldsymbol{K}|^2\right)$$
$$+ C_{11} \int_\Gamma \int_D \rho \left(\epsilon|\boldsymbol{E}|^2 + \mu|\boldsymbol{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}|^2\right). \quad (60)$$

Applying Lemmas 2.2, 2.5, Theorem 2.1, Lemmas 2.4 and 2.1 to the C_7, C_8, C_9, C_{10} and C_{11} terms, respectively, we obtain

$$\frac{d}{dt}ENG_6(t) \le C_6 \cdot ENG_6(t) + C_7 C_{max1} \int_\Gamma \int_D \rho(\epsilon|\nabla\times\boldsymbol{E}_0|^2 + \mu|\nabla\times\boldsymbol{H}_0|^2 + \epsilon|\boldsymbol{E}_0|^2 + \mu|\boldsymbol{H}_0|^2$$
$$+ \frac{1}{\epsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2)dxdy + C_8 e^{C_4 t}(1 + \frac{C_5 C_{max1}}{C_4})[\int_\Gamma \int_D \rho(\epsilon|\nabla\times\partial_t\boldsymbol{E}|^2 + \mu|\nabla\times\partial_t\boldsymbol{H}|^2$$
$$+ \frac{1}{\epsilon\omega_{pe}^2}|\nabla\times\partial_t\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_t\boldsymbol{K}|^2)(0)dxdy$$
$$+ \int_\Gamma \int_D \rho\left(\epsilon|\nabla\times\boldsymbol{E}_0|^2 + \mu|\nabla\times\boldsymbol{H}_0|^2 + \epsilon|\boldsymbol{E}_0|^2 + \mu|\boldsymbol{H}_0|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2\right)]$$
$$+ C_9 e^{C_2 t}[\int_\Gamma \int_D \rho\left(\epsilon|\partial_{y_i}\boldsymbol{E}|^2 + \mu|\partial_{y_i}\boldsymbol{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2\right)(0)dxdy$$
$$+ (1 + C_{max1}) \int_\Gamma \int_D \rho\left(\epsilon|\nabla\times\boldsymbol{E}_0|^2 + \mu|\nabla\times\boldsymbol{H}_0|^2 + \epsilon|\boldsymbol{E}_0|^2 + \mu|\boldsymbol{H}_0|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2\right)]$$
$$+ C_{10} e^{C_1^* t}[(2 + C_{max1}) \int_\Gamma \int_D \rho\left(\epsilon|\nabla\times\boldsymbol{E}_0|^2 + \mu|\nabla\times\boldsymbol{H}_0|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\nabla\times\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\boldsymbol{K}_0|^2\right)$$
$$+ (1 + C_{max1}) \int_\Gamma \int_D \rho\left(\epsilon|\boldsymbol{E}_0|^2 + \mu|\boldsymbol{H}_0|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2\right)]$$
$$+ C_{11} \int_\Gamma \int_D \rho\left(\epsilon|\boldsymbol{E}_0|^2 + \mu|\boldsymbol{H}_0|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2\right). \quad (61)$$

Using Lemma 2.3 to (61) and absorbing those constants in (61), we conclude the proof. □

With Lemmas 2.4–2.5 and Theorem 2.2, we can prove the boundness of the second derivative with respect to the random variables.

Theorem 2.3. *Denote the following constants:*

$$C_{14,1} = \max_{\overline{D}\times\overline{\Gamma}}\left(|\frac{\partial_{y_i^2}\epsilon}{\epsilon}| + |\frac{2(\partial_{y_i}\epsilon)^2}{\epsilon^2}| + |\frac{2\partial_{y_i}\epsilon}{\epsilon\sqrt{\epsilon\mu}}| + |\frac{2\omega_{pe}\partial_{y_i}\epsilon}{\epsilon}|\right),$$
$$C_{14,2} = \max_{\overline{D}\times\overline{\Gamma}}\left(|\frac{\partial_{y_i^2}\mu}{\mu}| + |\frac{2(\partial_{y_i}\mu)^2}{\mu^2}| + |\frac{2\partial_{y_i}\mu}{\mu\sqrt{\epsilon\mu}}| + |\frac{2\omega_{pm}\partial_{y_i}\mu}{\mu}|\right),$$

$$C_{14,3} = \max_{\overline{D}\times\overline{\Gamma}} \left(|\frac{\epsilon \partial_{y_i^2}\Gamma_e}{\Gamma_e}| + |\frac{2\epsilon\partial_{y_i}\Gamma_e}{\Gamma_e}| + |\frac{\partial_{y_i^2}(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}| + |\frac{2\partial_{y_i}(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}| \right),$$

$$C_{14,4} = \max_{\overline{D}\times\overline{\Gamma}} \left(|\partial_{y_i^2}\Gamma_e| + |2\partial_{y_i}\Gamma_m| + |\frac{\partial_{y_i^2}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}| + |\frac{2\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}| \right),$$

$$C_{14} = \max\left(C_{14,1}, C_{14,2}, C_{14,3}, C_{14,4}\right), \quad C_{15} = C_{14} + C_{12} + C_2.$$

Then for any $t \in [0, T]$ and $i = 1, \ldots, d$, we have

$$\int_\Gamma \int_D \rho(\boldsymbol{y}) \left(\epsilon|\partial_{y_i^2}\boldsymbol{E}|^2 + \mu|\partial_{y_i^2}\boldsymbol{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i^2}\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i^2}\boldsymbol{K}|^2 \right)(t)dxdy$$

$$\le e^{C_{15}t} C_{20}[\int_\Gamma \int_D \rho \left(\epsilon|\nabla\times\partial_{y_i}\boldsymbol{E}|^2 + \mu|\nabla\times\partial_{y_i}\boldsymbol{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\nabla\times\partial_{y_i}\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_{y_i}\boldsymbol{K}|^2 \right)(0)dxdy$$

$$+ \int_\Gamma \int_D \rho \left(\epsilon|\boldsymbol{E}_0|^2 + \mu|\boldsymbol{H}_0|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2 \right)dxdy$$

$$+ \int_\Gamma \int_D \rho \left(\epsilon|\nabla\times\boldsymbol{E}_0|^2 + \mu|\nabla\times\boldsymbol{H}_0|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\nabla\times\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\boldsymbol{K}_0|^2 \right)dxdy$$

$$+ \int_\Gamma \int_D \rho \left(\epsilon|\nabla\times\partial_t\boldsymbol{E}|^2 + \mu|\nabla\times\partial_t\boldsymbol{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\nabla\times\partial_t\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_t\boldsymbol{K}|^2 \right)(0)dxdy$$

$$+ \int_\Gamma \int_D \rho \left(\epsilon|\partial_{y_i}\boldsymbol{E}|^2 + \mu|\partial_{y_i}\boldsymbol{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2 \right)(0)dxdy$$

$$+ \int_\Gamma \int_D \rho \left(\epsilon|\partial_{y_i^2}\boldsymbol{E}|^2 + \mu|\partial_{y_i^2}\boldsymbol{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i^2}\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i^2}\boldsymbol{K}|^2 \right)(0)dxdy],$$

where constant $C_{20} > 0$ depends on parameters $\epsilon, \mu, \omega_{pe}, \omega_{pm}, \Gamma_e$ and Γ_m, but is independent of t.

Proof. Differentiating (32)–(35) with respect to any y_i $(i = 1, \ldots, d)$, we obtain

$$\epsilon\partial_t(\partial_{y_i^2}\boldsymbol{E}) - \nabla\times(\partial_{y_i^2}\boldsymbol{H}) + \partial_{y_i^2}\boldsymbol{J} = -(\partial_{y_i^2}\epsilon)\partial_t\boldsymbol{E} - 2(\partial_{y_i}\epsilon)\partial_{ty_i}\boldsymbol{E}, \tag{62}$$

$$\mu\partial_t(\partial_{y_i^2}\boldsymbol{H}) + \nabla\times(\partial_{y_i^2}\boldsymbol{E}) + \partial_{y_i^2}\boldsymbol{K} = -(\partial_{y_i^2}\mu)\partial_t\boldsymbol{H} - 2(\partial_{y_i}\mu)\partial_{ty_i}\boldsymbol{H}, \tag{63}$$

$$\begin{aligned}&\partial_t(\partial_{y_i^2}\boldsymbol{J}) + \Gamma_e\partial_{y_i^2}\boldsymbol{J} - \epsilon\omega_{pe}^2\partial_{y_i^2}\boldsymbol{E}\\ &\quad = -(\partial_{y_i^2}\Gamma_e)\boldsymbol{J} - 2(\partial_{y_i}\Gamma_e)\partial_{y_i}\boldsymbol{J} + (\partial_{y_i^2}(\epsilon\omega_{pe}^2))\boldsymbol{E} + 2\partial_{y_i}(\epsilon\omega_{pe}^2)\partial_{y_i}\boldsymbol{E},\end{aligned} \tag{64}$$

$$\begin{aligned}&\partial_t(\partial_{y_i^2}\boldsymbol{K}) + \Gamma_m\partial_{y_i^2}\boldsymbol{K} - \mu\omega_{pm}^2\partial_{y_i^2}\boldsymbol{H}\\ &\quad = -(\partial_{y_i^2}\Gamma_m)\boldsymbol{K} - 2(\partial_{y_i}\Gamma_m)\partial_{y_i}\boldsymbol{K} + (\partial_{y_i^2}(\mu\omega_{pm}^2))\boldsymbol{H} + 2\partial_{y_i}(\mu\omega_{pm}^2)\partial_{y_i}\boldsymbol{H}.\end{aligned} \tag{65}$$

Denote

$$ENG_7(t) = \int_\Gamma \int_D \rho(\boldsymbol{y}) \left(\epsilon|\partial_{y_i^2}\boldsymbol{E}|^2 + \mu|\partial_{y_i^2}\boldsymbol{H}|^2 + \frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i^2}\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i^2}\boldsymbol{K}|^2 \right)(t)dxdy.$$

Multiplying (62)–(65) by $2\rho(\boldsymbol{y})\partial_{y_i^2}\boldsymbol{E}$, $2\rho(\boldsymbol{y})\partial_{y_i^2}\boldsymbol{H}$, $\frac{2\rho(\boldsymbol{y})}{\epsilon\omega_{pe}^2}\partial_{y_i^2}\boldsymbol{J}$ and $\frac{2\rho(\boldsymbol{y})}{\mu\omega_{pm}^2}\partial_{y_i^2}\boldsymbol{K}$, respectively, then integrating over D and Γ, and adding the resultants, we have

$$\frac{d}{dt}ENG_7(t) + \int_\Gamma \int_D 2\rho(\boldsymbol{y}) \left(\frac{\Gamma_e(\boldsymbol{x},\boldsymbol{y})}{\epsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})}|\partial_{y_i^2}\boldsymbol{J}|^2 + \frac{\Gamma_m(\boldsymbol{x},\boldsymbol{y})}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})}|\partial_{y_i^2}\boldsymbol{K}|^2 \right)dxdy$$

$$= -\int_\Gamma \int_D (\partial_{y_i^2}\epsilon)\partial_t\boldsymbol{E}\cdot 2\rho\partial_{y_i^2}\boldsymbol{E} - \int_\Gamma \int_D (2\partial_{y_i}\epsilon)\partial_{ty_i}\boldsymbol{E}\cdot 2\rho\partial_{y_i^2}\boldsymbol{E}$$

$$
\begin{aligned}
&-\int_\Gamma\int_D(\partial_{y_i^2}\mu)\partial_t\boldsymbol{H}\cdot 2\rho\partial_{y_i^2}\boldsymbol{H}-\int_\Gamma\int_D(2\partial_{y_i}\mu)\partial_{ty_i}\boldsymbol{H}\cdot 2\rho\partial_{y_i^2}\boldsymbol{H}\\
&-\int_\Gamma\int_D\frac{\partial_{y_i^2}\Gamma_e}{\epsilon\omega_{pe}^2}\boldsymbol{J}\cdot 2\rho\partial_{y_i^2}\boldsymbol{J}-\int_\Gamma\int_D\frac{2\partial_{y_i}\Gamma_e}{\epsilon\omega_{pe}^2}\partial_{y_i}\boldsymbol{J}\cdot 2\rho\partial_{y_i^2}\boldsymbol{J}\\
&+\int_\Gamma\int_D\frac{\partial_{y_i^2}(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}^2}\boldsymbol{E}\cdot 2\rho\partial_{y_i^2}\boldsymbol{J}+\int_\Gamma\int_D\frac{2\partial_{y_i}(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}^2}\partial_{y_i}\boldsymbol{E}\cdot 2\rho\partial_{y_i^2}\boldsymbol{J}\\
&-\int_\Gamma\int_D\frac{\partial_{y_i^2}\Gamma_m}{\mu\omega_{pm}^2}\boldsymbol{K}\cdot 2\rho\partial_{y_i^2}\boldsymbol{K}-\int_\Gamma\int_D\frac{2\partial_{y_i}\Gamma_m}{\mu\omega_{pm}^2}\partial_{y_i}\boldsymbol{K}\cdot 2\rho\partial_{y_i^2}\boldsymbol{K}\\
&+\int_\Gamma\int_D\frac{\partial_{y_i^2}(\mu\omega_{pm}^2)}{\mu\omega_{pm}^2}\boldsymbol{H}\cdot 2\rho\partial_{y_i^2}\boldsymbol{K}+\int_\Gamma\int_D\frac{2\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}^2}\partial_{y_i}\boldsymbol{H}\cdot 2\rho\partial_{y_i^2}\boldsymbol{K}.
\end{aligned}
\tag{66}
$$

By the Cauchy–Schwarz inequality, we have

$$
Err_1\le(\max_{\overline{D}\times\overline{\Gamma}}|\frac{\partial_{y_i^2}\epsilon}{\epsilon}|)\int_\Gamma\int_D\rho(\epsilon|\partial_t\boldsymbol{E}|^2+\epsilon|\partial_{y_i^2}\boldsymbol{E}|^2).
$$

Similarly, by (32) and the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
Err_2&=-\int_\Gamma\int_D 2\frac{\partial_{y_i}\epsilon}{\epsilon}(\nabla\times\partial_{y_i}\boldsymbol{H}-\partial_{y_i}\boldsymbol{J}-\partial_{y_i}\epsilon\partial_t\boldsymbol{E})\cdot 2\rho\partial_{y_i^2}\boldsymbol{E}\\
&\le(\max_{\overline{D}\times\overline{\Gamma}}|\frac{2\partial_{y_i}\epsilon}{\epsilon\sqrt{\epsilon\mu}}|)\int_\Gamma\int_D\rho(\mu|\nabla\times\partial_{y_i}\boldsymbol{H}|^2+\epsilon|\partial_{y_i^2}\boldsymbol{E}|^2)\\
&\quad+(\max_{\overline{D}\times\overline{\Gamma}}|\frac{2\omega_{pe}\partial_{y_i}\epsilon}{\epsilon}|)\int_\Gamma\int_D\rho(\frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2+\epsilon|\partial_{y_i^2}\boldsymbol{E}|^2)\\
&\quad+(\max_{\overline{D}\times\overline{\Gamma}}|\frac{2(\partial_{y_i}\epsilon)^2}{\epsilon^2}|)\int_\Gamma\int_D\rho(\epsilon|\partial_t\boldsymbol{E}|^2+\epsilon|\partial_{y_i^2}\boldsymbol{E}|^2).
\end{aligned}
$$

Similarly, by (33) and the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
Err_4&=\int_\Gamma\int_D 2\frac{\partial_{y_i}\mu}{\mu}(\nabla\times\partial_{y_i}\boldsymbol{E}+\partial_{y_i}\boldsymbol{K}+\partial_{y_i}\mu\partial_t\boldsymbol{H})\cdot 2\rho\partial_{y_i^2}\boldsymbol{H}\\
&\le(\max_{\overline{D}\times\overline{\Gamma}}|\frac{2\partial_{y_i}\mu}{\mu\sqrt{\epsilon\mu}}|)\int_\Gamma\int_D\rho(\epsilon|\nabla\times\partial_{y_i}\boldsymbol{E}|^2+\mu|\partial_{y_i^2}\boldsymbol{H}|^2)\\
&\quad+(\max_{\overline{D}\times\overline{\Gamma}}|\frac{2\omega_{pm}\partial_{y_i}\mu}{\mu}|)\int_\Gamma\int_D\rho(\frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2+\mu|\partial_{y_i^2}\boldsymbol{H}|^2)\\
&\quad+(\max_{\overline{D}\times\overline{\Gamma}}|\frac{2(\partial_{y_i}\mu)^2}{\mu^2}|)\int_\Gamma\int_D\rho(\mu|\partial_t\boldsymbol{H}|^2+\mu|\partial_{y_i^2}\boldsymbol{H}|^2).
\end{aligned}
$$

Similarly, by the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
Err_3&\le(\max_{\overline{D}\times\overline{\Gamma}}|\frac{\partial_{y_i^2}\mu}{\mu}|)\int_\Gamma\int_D\rho(\mu|\partial_t\boldsymbol{H}|^2+\mu|\partial_{y_i^2}\boldsymbol{H}|^2),\\
Err_5&\le(\max_{\overline{D}\times\overline{\Gamma}}|\frac{\epsilon\partial_{y_i^2}\Gamma_e}{\Gamma_e}|)\int_\Gamma\int_D\frac{\rho}{\epsilon\omega_{pe}^2}(|\boldsymbol{J}|^2+|\partial_{y_i^2}\boldsymbol{J}|^2),\\
Err_6&\le(\max_{\overline{D}\times\overline{\Gamma}}|\frac{2\epsilon\partial_{y_i}\Gamma_e}{\Gamma_e}|)\int_\Gamma\int_D\frac{\rho}{\epsilon\omega_{pe}^2}(|\partial_{y_i}\boldsymbol{J}|^2+|\partial_{y_i^2}\boldsymbol{J}|^2),\\
Err_7&\le(\max_{\overline{D}\times\overline{\Gamma}}|\frac{\partial_{y_i^2}(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}|)\int_\Gamma\int_D\rho(\epsilon|\boldsymbol{E}|^2+\frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i^2}\boldsymbol{J}|^2),\\
Err_8&\le(\max_{\overline{D}\times\overline{\Gamma}}|\frac{2\partial_{y_i}(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}|)\int_\Gamma\int_D\rho(\epsilon|\partial_{y_i}\boldsymbol{E}|^2+\frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i^2}\boldsymbol{J}|^2),
\end{aligned}
$$

$$Err_9 \le (\max_{\overline{D}\times\overline{\Gamma}}|\partial_{y_i^2}\Gamma_m|)\int_\Gamma\int_D \frac{\rho}{\mu\omega_{pm}^2}(|\boldsymbol{K}|^2+|\partial_{y_i^2}\boldsymbol{K}|^2),$$

$$Err_{10} \le (\max_{\overline{D}\times\overline{\Gamma}}|2\partial_{y_i}\Gamma_m|)\int_\Gamma\int_D \frac{\rho}{\mu\omega_{pm}^2}(|\partial_{y_i}\boldsymbol{K}|^2+|\partial_{y_i^2}\boldsymbol{K}|^2),$$

$$Err_{11} \le (\max_{\overline{D}\times\overline{\Gamma}}|\frac{\partial_{y_i^2}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}|)\int_\Gamma\int_D \rho(\mu|\boldsymbol{H}|^2+\frac{1}{\mu\omega_{pm}^2}|\partial_{y_i^2}\boldsymbol{K}|^2),$$

$$Err_{12} \le (\max_{\overline{D}\times\overline{\Gamma}}|\frac{2\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}|)\int_\Gamma\int_D \rho(\mu|\partial_{y_i}\boldsymbol{H}|^2+\frac{1}{\mu\omega_{pm}^2}|\partial_{y_i^2}\boldsymbol{K}|^2).$$

Denote the following constants:

$$C_{16} = \max_{\overline{D}\times\overline{\Gamma}}\left(|\frac{\partial_{y_i^2}\epsilon}{\epsilon}|+|\frac{2(\partial_{y_i}\epsilon)^2}{\epsilon^2}|,|\frac{\partial_{y_i^2}\mu}{\mu}|+|\frac{2(\partial_{y_i}\mu)^2}{\mu^2}|\right),$$

$$C_{17} = \max_{\overline{D}\times\overline{\Gamma}}\left(|\frac{\partial_{y_i^2}(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}|,|\frac{\partial_{y_i^2}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}|,|\frac{\epsilon\partial_{y_i^2}\Gamma_e}{\Gamma_e}|,|\partial_{y_i^2}\Gamma_m|\right),$$

$$C_{18} = \max_{\overline{D}\times\overline{\Gamma}}\left(|\frac{2\partial_{y_i}(\epsilon\omega_{pe}^2)}{\epsilon\omega_{pe}}|,|\frac{2\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}|,|\frac{2\epsilon\partial_{y_i}\Gamma_e}{\Gamma_e}|+|\frac{2\omega_{pe}\partial_{y_i}\epsilon}{\epsilon}|,|2\partial_{y_i}\Gamma_m|\right),$$

$$C_{19} = \max_{\overline{D}\times\overline{\Gamma}}\left(|\frac{2\partial_{y_i}\epsilon}{\epsilon\sqrt{\epsilon\mu}}|,|\frac{2\partial_{y_i}\mu}{\mu\sqrt{\epsilon\mu}}|\right).$$

Substituting the above estimates into (66), we obtain

$$\begin{aligned}\frac{d}{dt}ENG_7(t) \le{}& C_{14}\cdot ENG_7(t)+C_{16}\int_\Gamma\int_D\rho(\epsilon|\partial_t\boldsymbol{E}|^2+\mu|\partial_t\boldsymbol{H}|^2)\\ &+C_{17}\int_\Gamma\int_D\rho(\epsilon|\boldsymbol{E}|^2+\mu|\boldsymbol{H}|^2+\frac{1}{\epsilon\omega_{pe}^2}|\boldsymbol{J}|^2+\frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}|^2)\\ &+C_{18}\int_\Gamma\int_D\rho(\epsilon|\partial_{y_i}\boldsymbol{E}|^2+\mu|\partial_{y_i}\boldsymbol{H}|^2+\frac{1}{\epsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2+\frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2)\\ &+C_{19}\int_\Gamma\int_D\rho(\epsilon|\nabla\times\partial_{y_i}\boldsymbol{E}|^2+\mu|\nabla\times\partial_{y_i}\boldsymbol{H}|^2).\end{aligned} \quad (67)$$

Applying Lemmas 2.2, 2.1, Theorems 2.1, and 2.2 to the C_{16}, C_{17}, C_{18} and C_{19} terms, respectively, then using the Gronwall inequality (cf. Lemma 2.3) to the resultant, we conclude the proof. □

Remark 2.5. By similar techniques, we believe that if the random parameters are smooth enough, then higher derivatives with respect to the random vector $\boldsymbol{y}$ can be proved to be bounded similarly as stated in Theorems 2.1–2.3. Since the proofs will become quite technical and are similar, we skip the proofs for higher derivatives.

2.2. Convergence analysis

To prove the convergence estimate for the stochastic collocation method, let us first recall the following interpolation error estimates.

Lemma 2.6 (*[34, p. 289–290]*)**.** *Let $I_N^y u$ denote the polynomial of degree N that interpolates u at the $(N+1)$ Gauss, or Gauss–Radau, or Gauss–Lobatto points $\{y_j\}_{j=0}^N$, i.e., $I_N^y u(y)=\sum_{j=0}^N u(y_j)\mathcal{L}_j(y)$. Then we have the interpolation error in the L^2-norm:*

$$\|u-I_N^y u\|_{L^2(-1,1)} \le CN^{-m}|u|_{H^m(-1,1)}, \quad \forall\, u\in H^m(-1,1) \text{ with } m\ge 1, \quad (68)$$

and the interpolation error in the H^l-norm:

$$\|u-I_N^y u\|_{H^l(-1,1)} \le CN^{2l-\frac{1}{2}-m}|u|_{H^m(-1,1)}, \quad \forall\, u\in H^m(-1,1) \text{ with } m\ge l\ge 1. \quad (69)$$

For the Gauss–Lobatto interpolation, we have the following optimal error estimate:

$$\|(u - I_N^y u)'\|_{L^2(-1,1)} \le CN^{1-m}|u|_{H^m(-1,1)}, \quad \forall\, u \in H^m(-1,1) \text{ with } m \ge 1. \tag{70}$$

Below are the extension of the above interpolation results to tensor product interpolation.

Lemma 2.7. *Let $I_N u = I_N^{y_1} I_N^{y_2} \cdots I_N^{y_d} u$ denote the d-dimension tensor product polynomial of the 1-D interpolation polynomial of degree N that interpolates u at the $(N+1)$ Gauss, or Gauss–Radau, or Gauss–Lobatto points $\{y_j\}_{j=0}^N$. Then we have the interpolation error in the L^2-norm* [34, (5.8.20)]*:*

$$\|u - I_N u\|_{L^2(\Gamma)} \le CN^{-m}|u|_{H^m(\Gamma)}, \quad \forall\, u \in H^m(\Gamma) \text{ with } m > d/2. \tag{71}$$

For the Gauss–Lobatto interpolation, we have the following optimal error estimate [34, (5.8.21)]*:*

$$\|u - I_N u\|_{H^1(\Gamma)} \le CN^{1-m}|u|_{H^m(\Gamma)}, \quad \forall\, u \in H^m(\Gamma) \text{ with } m > (d+1)/2. \tag{72}$$

To present the error estimate, recall that the mean (or expectation) of a function u is defined by

$$\mathcal{E}[u] = \int_\Gamma \int_D \rho(y) u(x,t,y) d\boldsymbol{x} dy, \tag{73}$$

and its mean square is defined by

$$\mathcal{M}[u] = \left(\int_\Gamma \int_D \rho(y) |u(x,t,y)|^2 d\boldsymbol{x} dy \right)^{1/2}. \tag{74}$$

Theorem 2.4. *Let $(\boldsymbol{E}, \boldsymbol{H})$ be the solution of (1)–(7), and $(\boldsymbol{E}^N, \boldsymbol{H}^N)$ be the stochastic collocation solution of (15). If the assumptions of Theorems* 2.1 and 2.2 *are satisfied, then the following mean and mean square errors hold: For any $0 < t \le T$,*

$$\mathcal{M}[\boldsymbol{E} - \boldsymbol{E}^N] + \mathcal{M}[\boldsymbol{H} - \boldsymbol{H}^N] + \mathcal{M}[\nabla \times (\boldsymbol{E} - \boldsymbol{E}^N)] + \mathcal{M}[\nabla \times (\boldsymbol{H} - \boldsymbol{H}^N)] \le C_T N^{-1}, \tag{75}$$

$$\mathcal{E}[|\boldsymbol{E} - \boldsymbol{E}^N|] + \mathcal{E}[|\boldsymbol{H} - \boldsymbol{H}^N|] + \mathcal{E}[\nabla \times (\boldsymbol{E} - \boldsymbol{E}^N)] + \mathcal{E}[\nabla \times (\boldsymbol{H} - \boldsymbol{H}^N)] \le C_T N^{-1}. \tag{76}$$

Here and below C_T is a constant depending on T but independent of N. Furthermore, if the assumptions of Theorem 2.4 *are satisfied, then we have the following higher error estimates: For any $0 < t \le T$,*

$$\mathcal{M}[\boldsymbol{E} - \boldsymbol{E}^N] + \mathcal{M}[\boldsymbol{H} - \boldsymbol{H}^N] + \mathcal{E}[|\boldsymbol{E} - \boldsymbol{E}^N|] + \mathcal{E}[|\boldsymbol{H} - \boldsymbol{H}^N|] \le C_T N^{-2}. \tag{77}$$

Finally, if the assumptions of Theorem 2.4 *are satisfied, for the Gauss–Lobatto interpolation, we have the error estimate for the derivative of the solution with respect to the random variables: For any $0 < t \le T$, and $j = 1, \ldots, d$,*

$$\mathcal{M}[\partial_{y_j}(\boldsymbol{E} - \boldsymbol{E}^N)] + \mathcal{M}[\partial_{y_j}(\boldsymbol{H} - \boldsymbol{H}^N)] + \mathcal{E}[|\partial_{y_j}(\boldsymbol{E} - \boldsymbol{E}^N)|] + \mathcal{E}[|\partial_{y_j}(\boldsymbol{H} - \boldsymbol{H}^N)|] \le CN^{-1}. \tag{78}$$

Proof. For any fixed $\boldsymbol{x}$, using (68) of Lemma 2.5 for $u = \boldsymbol{E}$ and $u = \boldsymbol{H}$ with $m = 1$, respectively, we have

$$\begin{aligned} &\int_\Gamma \left(\rho(y)\epsilon(\boldsymbol{x}, y)|\boldsymbol{E}(\boldsymbol{x}, t; y) - \boldsymbol{E}^N(\boldsymbol{x}, t; y)|^2 + \rho(y)\mu(\boldsymbol{x}, y)|\boldsymbol{H}(\boldsymbol{x}, t; y) - \boldsymbol{H}^N(\boldsymbol{x}, t; y)|^2 \right) dy \\ &\le CN^{-2} \int_\Gamma \left(\rho(y)\epsilon(\boldsymbol{x}, y)|\partial_y \boldsymbol{E}|^2 + \rho(y)\mu(\boldsymbol{x}, y)|\partial_y \boldsymbol{H}|^2 \right) dy. \end{aligned} \tag{79}$$

Similarly, using (68) of Lemma 2.5 for $u = \nabla \times \boldsymbol{E}$ and $u = \nabla \times \boldsymbol{H}$ with $m = 1$, respectively, we have

$$\begin{aligned} &\int_\Gamma \left(\rho(y)\epsilon(\boldsymbol{x}, y)|\nabla \times (\boldsymbol{E}(\boldsymbol{x}, t; y) - \boldsymbol{E}^N(\boldsymbol{x}, t; y))|^2 + \rho(y)\mu(\boldsymbol{x}, y)|\nabla \times (\boldsymbol{H}(\boldsymbol{x}, t; y) - \boldsymbol{H}^N(\boldsymbol{x}, t; y))|^2 \right) dy \\ &\le CN^{-2} \int_\Gamma \left(\rho(y)\epsilon(\boldsymbol{x}, y)|\partial_y(\nabla \times \boldsymbol{E})|^2 + \rho(y)\mu(\boldsymbol{x}, y)|\partial_y(\nabla \times \boldsymbol{H})|^2 \right) dy. \end{aligned} \tag{80}$$

Adding (79) and (80) together, then integrating the resultant with respect to $\boldsymbol{x}$ over D and using Theorems 2.1 and 2.3, we complete the proof of (75).

The estimates (77) can be proved similarly by using (68) of Lemma 2.5 with $m = 2$ and the higher regularity obtained in Theorem 2.3.

Similarly, using (70) of Lemma 2.5 with $m = 2$, and the higher regularity proved in Theorem 2.3, we obtain the proof of (78).

Finally, the mean errors follow from the standard inequality $\|u\|_{L^1} \leq C\|u\|_{L^2}$ and the estimates (75), (77) and (78). $\square$

With the above interpolation estimate, we can show that the overall errors for solving the metamaterial Maxwell's equations by the classical Yee scheme (cf. [35]) are estimated as follows. Denote the electric field solution of Yee scheme for any fixed random vector $\boldsymbol{y}$ as $\boldsymbol{E}^N$, and $\boldsymbol{E}^N_{h,\Delta t}$ for the electric field solution of the fully-discrete solution with stochastic collocation method imposed. Denote the discrete L^2-norm over the physical space D as $|\cdot|_{l^2(D)}$ (cf. [35]). Then we can obtain the discrete mean square error as following:

$$\begin{aligned}
&(\int_\Gamma \rho\epsilon|\boldsymbol{E} - \boldsymbol{E}^N_{h,\Delta t}|^2_{l^2(D)}dy)^{1/2} \leq (\int_\Gamma 2\rho\epsilon(|\boldsymbol{E} - \boldsymbol{E}^N|^2_{l^2(D)} + |\boldsymbol{E}^N - \boldsymbol{E}^N_{h,\Delta t}|^2_{l^2(D)})dy)^{1/2}\\
&\leq C[N^{-m} + (h^2 + (\Delta t)^2)], \qquad (81)
\end{aligned}$$

where we used the error estimate of Yee scheme and Theorem 2.4. The error estimate for other variables can be bounded similarly.

3. Numerical results

To justify our theoretical analysis, here we present some numerical results carried out for the metamaterial model in TM_z mode, whose governing equations are:

$$\mu_0 \frac{\partial H_{x_1}}{\partial t} = -\frac{\partial E}{\partial x_2} - K_{x_1} + g_1, \qquad (82)$$

$$\mu_0 \frac{\partial H_{x_2}}{\partial t} = \frac{\partial E}{\partial x_2} - K_{x_2} + g_2, \qquad (83)$$

$$\epsilon_0 \frac{\partial E}{\partial t} = \frac{\partial H_{x_2}}{\partial x_1} - \frac{\partial H_{x_1}}{\partial x_2} - J + g_3, \qquad (84)$$

$$\frac{\partial J}{\partial t} = \epsilon_0\omega_e^2 E - \Gamma_e J + g_4, \qquad (85)$$

$$\frac{\partial K_{x_1}}{\partial t} = \mu_0\omega_m^2 H_{x_1} - \Gamma_m K_{x_1} + g_5, \qquad (86)$$

$$\frac{\partial K_{x_2}}{\partial t} = \mu_0\omega_m^2 H_{x_2} - \Gamma_m K_{x_2} + g_6, \qquad (87)$$

where g_i $(1 \leq i \leq 6)$ are added source terms used to construct exact solutions for checking convergence rates. The parameters μ_0, ϵ_0, Γ_m, Γ_e, ω_m and ω_e are functions of spatial variable $\mathbf{x}$ and random vector ξ.

Example 1. In this test, we choose the following parameters:

$$\begin{aligned}
\epsilon_0(\mathbf{x},\xi) &= 1 + 0.01(\sin(\pi(\xi_1x_1 + \xi_2x_2 - 1)) + \cos(\pi(\xi_3x_1 + \xi_4x_2 - 1)) + \exp(-\xi_5x_1 - \xi_6x_2)),\\
\mu_0(\mathbf{x},\xi) &= 1 + 0.01(\sin(\pi(\xi_1x_1 + \xi_2x_2 - 1)) + \exp(-\xi_3x_1 - \xi_4x_2) + \cos(\pi(\xi_5x_1 + \xi_6x_2 - 1))),\\
\Gamma_e(\mathbf{x},\xi) &= \pi + 0.01(\cos(\pi(\xi_1x_1 + \xi_2x_2 - 1)) + \sin(\pi(\xi_3x_1 + \xi_4x_2 - 1)) + \exp(-\xi_5x_1 - \xi_6x_2)),\\
\Gamma_m(\mathbf{x},\xi) &= \pi + 0.01(\cos(\pi(\xi_1x_1 + \xi_2x_2 - 1)) + \exp(-\xi_3x_1 - \xi_4x_2) + \sin(\pi(\xi_5x_1 + \xi_6x_2 - 1))),\\
\omega_e(\mathbf{x},\xi) &= \pi + 0.01(\exp(-\xi_1x_1 - \xi_2x_2) + \cos(\pi(\xi_3x_1 + \xi_4x_2 - 1)) + \sin(\pi(\xi_5x_1 + \xi_6x_2 - 1))),\\
\omega_m(\mathbf{x},\xi) &= \pi + 0.01(\exp(-\xi_1x_1 - \xi_2x_2) + \sin(\pi(\xi_3x_1 + \xi_4x_2 - 1)) + \cos(\pi(\xi_5x_1 + \xi_6x_2 - 1))),
\end{aligned}$$

where ξ_i $(1 \leq i \leq 6)$ are uniform independent random variables on $[0, 1]$.

In our tests, we use Yee scheme (cf. [35]) to solve the TM_z model on physical domain $[0, 1]^2$ and time domain $[0, 1]$ with the exact solution given as

$$H_{x_1} = \sin(\pi x_1 + \mu_0)\cos(\pi x_2 + \mu_0)\exp(-\pi t),$$

Table 1
Errors of the solutions when the analytic solutions are infinitely smooth in both random and spatial variables.

Mesh	1/5	1/10	Rate	1/20	Rate	1/40	Rate
$\mathcal{E}[\lvert H_{x_1} - H_{x_1}^h \rvert]$	9.51281E−03	2.25052E−03	2.0796	4.90362E−04	2.1390	1.16331E−04	2.1259
$\mathcal{M}[\lvert H_{x_1} - H_{x_1}^h \rvert]$	9.51281E−03	2.25090E−03	2.0794	4.90442E−04	2.1389	1.16353E−04	2.1258
$\mathcal{E}[\lvert H_{x_2} - H_{x_2}^h \rvert]$	9.51281E−03	2.25462E−03	2.0770	4.91154E−04	2.1378	1.16533E−04	2.1252
$\mathcal{M}[\lvert H_{x_2} - H_{x_2}^h \rvert]$	9.51281E−03	2.25498E−03	2.0768	4.91231E−04	2.1377	1.16554E−04	2.1251
$\mathcal{E}[\lvert E - E_{x_1}^h \rvert]$	1.14777E−02	2.33418E−03	2.2978	5.27242E−04	2.2221	1.26109E−04	2.1670
$\mathcal{M}[\lvert E - E_{x_1}^h \rvert]$	1.14777E−02	2.33587E−03	2.2968	5.27710E−04	2.2215	1.26237E−04	2.1666
$\mathcal{E}[\lvert K_{x_1} - K_{x_1}^h \rvert]$	8.77292E−03	1.71711E−03	2.3531	3.72208E−04	2.2794	8.73870E−05	2.2154
$\mathcal{M}[\lvert K_{x_1} - K_{x_1}^h \rvert]$	8.77292E−03	1.71767E−03	2.3526	3.72326E−04	2.2792	8.74154E−05	2.2153
$\mathcal{E}[\lvert K_{x_2} - K_{x_2}^h \rvert]$	8.77292E−03	1.71711E−03	2.3531	3.72208E−04	2.2794	8.73870E−05	2.2154
$\mathcal{M}[\lvert K_{x_2} - K_{x_2}^h \rvert]$	8.77292E−03	1.71767E−03	2.3526	3.72326E−04	2.2792	8.74154E−05	2.2153
$\mathcal{E}[\lvert J - J_{x_1}^h \rvert]$	1.71215E−02	3.95199E−03	2.1152	9.11318E−04	2.1159	2.18246E−04	2.0998
$\mathcal{M}[\lvert J - J_{x_1}^h \rvert]$	1.71215E−02	3.95296E−03	2.1148	9.11584E−04	2.1156	2.18302E−04	2.0997

$$
\begin{aligned}
H_{x_2} &= -\cos(\pi x_1 + \mu_0)\sin(\pi x_2 + \mu_0)\exp(-\pi t),\\
E &= \sin(\pi x_1 + \epsilon_0)\sin(\pi x_2 + \epsilon_0)\exp(-\pi t),\\
K_{x_1} &= \pi^2 t\sin(\pi x_1)\cos(\pi x_2)\exp(-\pi t),\\
K_{x_2} &= -\pi^2 t\cos(\pi x_1)\sin(\pi x_2)\exp(-\pi t),\\
J &= \pi^2 t\sin(\pi x_1)\sin(\pi x_2)\exp(-\pi t).
\end{aligned}
$$

To test the convergence rate, we vary the partition size in the x_1 and x_2 directions $h_{x_1} = h_{x_2} = h$ from $1/5$ to $1/40$, and time step size from $1/10$ to $1/80$. We set time partition equals two times of spatial to guarantee the stability. In the same time, the partition numbers in random space vary from 1 to 8. We present the errors of all six components $(H_{x_1}, H_{x_2}, E, K_{x_1}, K_{x_2}, J)$ in the discrete $\mathcal{E}[\cdot]$ and $\mathcal{M}[\cdot]$ in Table 1. We can see clearly that all solutions show second order convergence which agrees with our theoretical result, since in this case the exact solution is infinitely smooth in both random and spatial variables and the overall error is dominated by the numerical scheme error.

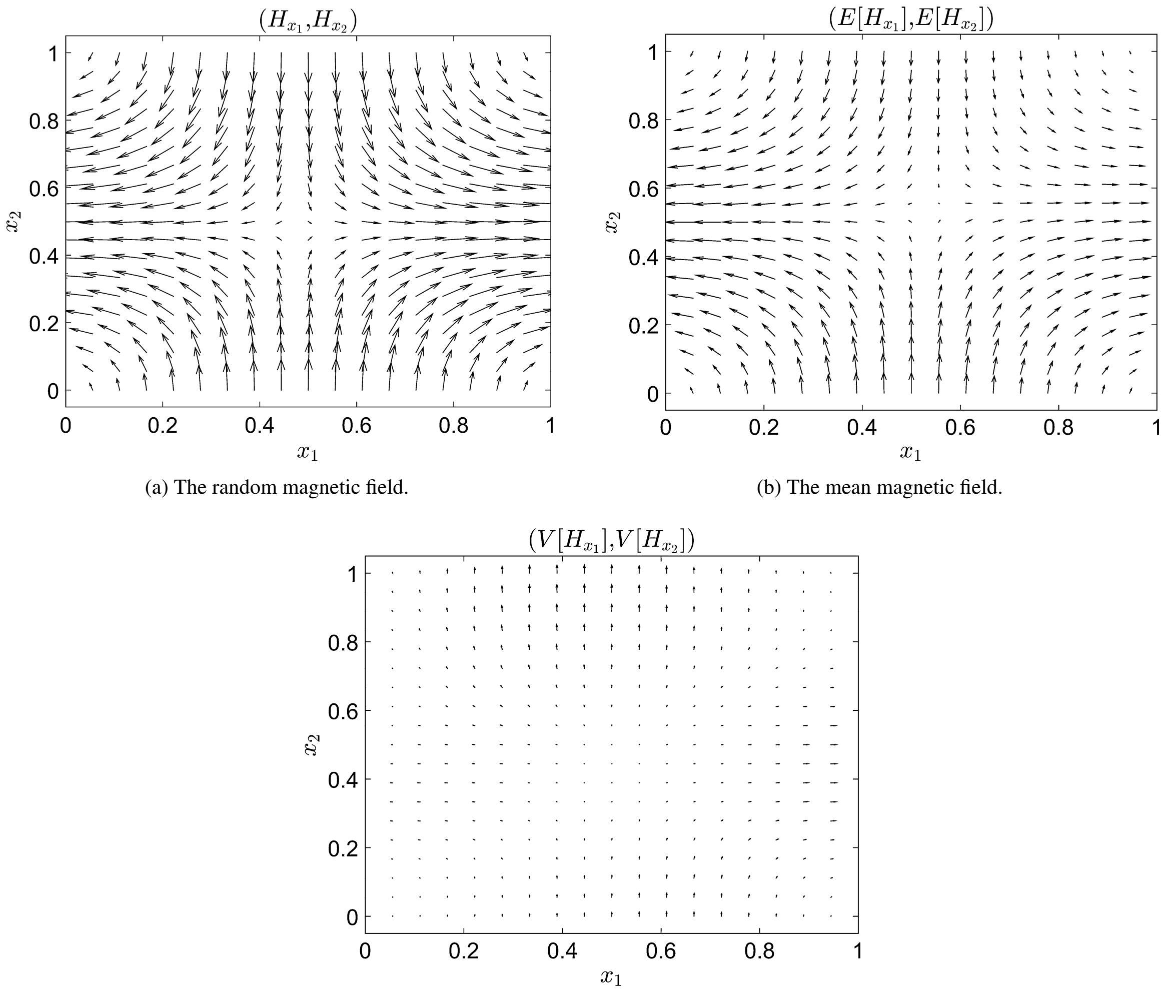
In Fig. 1, we present one sample magnetic field and its mean and variance obtained by solving the same problem by a 20×20 spatial uniform partition on $[0, 1]^2$. We set the initial conditions and boundary values using the above exact solution and no added source functions. Fig. 1 is obtained with the random vector $\xi = (0.8147, 0.9058, 0.1270, 0.9134, 0.6324, 0.0975)$, and shows that the mean magnetic field is similar to the sample field in this case.

Example 2. This example is used to test the convergence rate when the solution has limited regularity in the random variables. For simplicity, we use the same exact solution as Example 1 except H_{x_1} being given as:

$$
H_{x_1} = \sin(\pi x_1 + \mu_0)\cos(\pi x_2 + \mu_0)\exp(-\pi t) + (\xi_1 - \frac{\sqrt{2}}{2})^m \mathrm{sgn}\left(\xi_1 - \frac{\sqrt{2}}{2}\right), \quad m = 1, 2.
$$

We choose number $\sqrt{2}/2$ to avoid the case that some interpolation point falls at this cusp point. The corresponding source terms are obtained by plugging the exact solution into the governing equations. It is easy to check that the exact solutions are infinitely smooth except that $H_{x_1}(\Gamma) \in H^{m+1/2-\epsilon}(\Gamma)$ when $m = 1, 2$, respectively.

To investigate the convergence rate, we initialize the partition number for x_1, x_2, t and ξ as 10, 20, 40 and 2 respectively to make a uniform spatial and temporal partition and use a Gauss–Lobatto points for each random space. Then we double all partition numbers three times. The numerical results of original solutions are given in Tables 2 and 3 for $m = 1$ and $m = 2$, respectively. Table 2 shows that the error of H_{x_1} is about $O(N^{-1.3})$ in both mean and mean square norm defined earlier, and errors of other solutions are still $O(N^{-2})$ due to their infinite smoothness. This is consistent with our theoretical analysis. When $m = 2$, all the solutions have $O(N^{-2})$ convergence, which shows



(a) The random magnetic field.

(b) The mean magnetic field.

(c) The variance of magnetic field.

Fig. 1. Comparison of a random sample of magnetic field and its mean and variance obtained with $\xi = (0.8147, 0.9058, 0.1270, 0.9134, 0.6324, 0.0975)$.

clearly by the results stated in Table 3. Notice the rate of H_{x_1} is limited to 2 due to the 2nd convergent rate of the FDTD scheme. We also plotted the variances of the electric fields at variance times in Fig. 2.

Example 3. In this example, we solve a classic example showing the backward wave propagation in metamaterials (cf. [36,35]). This example assumes that a metamaterial slab of size $[0.024, 0.054]$ m $\times$ $[0.002, 0.062]$ m is located inside a vacuum of size $[0, 0.07]$ m $\times$ $[0, 0.064]$ m. An incident source wave is imposed as E field and excited at line $x = 0.004$ m ranging from $y = 0.025$ m to $y = 0.035$ m. The source wave varies in space as $\exp(-(x-0.03)^2/(50h)^2)$ where h is the partition size in space, and in time as:

$$f(t) = \begin{cases} 0 & t < 0 \text{ or } t > (2m+k)T_p \\ g_1(t)\sin(\omega_0 t) & 0 \le t < mT_p \\ \sin(\omega_0 t) & mT_p \le t < (m+k)T_p \\ g_2(t)\sin(\omega_0 t) & (m+k)T_p \le t < (2m+k)T_p \end{cases}$$

where

$$g_1(t) = 10x_1^3 - 15x_1^4 + 6x_1^5, \qquad x_1 = t/mT_p$$

Table 2
Errors of the solutions when $H_{x_1} \in H^{3/2-\epsilon}(\Gamma)$.

N	2	4	Rate	8	Rate	16	Rate
$\mathcal{E}[\lvert H_{x_1}-H^h_{x_1}\rvert]$	2.544284E−01	5.899390E−02	1.8103	1.560676E−02	1.6431	6.047332E−03	1.3678
$\mathcal{M}[\lvert H_{x_1}-H^h_{x_1}\rvert]$	2.972258E−01	7.969676E−02	1.6161	2.654922E−02	1.4797	1.024674E−02	1.3735
$\mathcal{E}[\lvert H_{x_2}-H^h_{x_2}\rvert]$	9.506088E−06	1.570312E−05	0.9623	5.537782E−06	1.7153	1.456298E−06	1.9270
$\mathcal{M}[\lvert H_{x_2}-H^h_{x_2}\rvert]$	1.576924E−05	1.125031E−05	1.1302	5.549302E−06	1.4701	1.465709E−06	1.9207
$\mathcal{E}[\lvert K_{x_1}-K^h_{x_1}\rvert]$	6.473112E−04	1.479130E−04	2.0982	3.443082E−05	2.0817	8.255164E−06	2.0603
$\mathcal{M}[\lvert K_{x_1}-K^h_{x_1}\rvert]$	6.473390E−04	1.479207E−04	2.0982	3.443912E−05	2.0816	8.255810E−06	2.0606
$\mathcal{E}[\lvert K_{x_2}-K^h_{x_2}\rvert]$	5.212914E−04	1.306984E−04	2.0072	3.209770E−05	2.0099	8.057718E−06	1.9940
$\mathcal{M}[\lvert K_{x_2}-K^h_{x_2}\rvert]$	5.213384E−04	1.307129E−04	2.0070	3.211124E−05	2.0095	9.062296E−06	1.9938
$\mathcal{E}[\lvert E-E^h\rvert]$	1.291961E−04	4.004254E−05	1.9222	9.690372E−06	2.0176	2.442342E−06	1.9883
$\mathcal{M}[\lvert E-E^h\rvert]$	1.297785E−04	4.008978E−05	1.9236	9.704608E−06	2.0176	2.445378E−06	1.9886
$\mathcal{E}[\lvert J-J^h\rvert]$	1.206363E−03	3.078828E−04	1.9984	7.689780E−05	2.0120	1.892578E−05	2.0226
$\mathcal{M}[\lvert J-J^h\rvert]$	1.206383E−03	3.078850E−04	1.9984	7.690346E−05	2.0119	1.892703E−05	2.0226

Table 3
Errors of the solutions when $H_{x_1} \in H^{5/2-\epsilon}(\Gamma)$.

N	2	4	Rate	8	Rate	16	Rate
$\mathcal{E}[\lvert H_{x_1}-H^h_{x_1}\rvert]$	4.707117E−02	6.770013E−03	2.7029	7.617172E−04	2.5807	1.891617E−04	2.0096
$\mathcal{M}[\lvert H_{x_1}-H^h_{x_1}\rvert]$	5.827503E−02	8.021087E−03	2.6621	1.028223E−03	2.5124	2.463826E−04	2.0612
$\mathcal{E}[\lvert H_{x_2}-H^h_{x_2}\rvert]$	2.053239E−05	3.153899E−06	1.9998	1.204564E−06	1.7503	2.786668E−07	2.1119
$\mathcal{M}[\lvert H_{x_2}-H^h_{x_2}\rvert]$	2.361496E−05	3.209744E−06	2.0270	1.213414E−06	1.7048	3.020466E−07	2.0062
$\mathcal{E}[\lvert K_{x_1}-K^h_{x_1}\rvert]$	3.563472E−04	8.236230E−05	2.0961	2.025432E−05	2.0996	4.483891E−06	2.1754
$\mathcal{M}[\lvert K_{x_1}-K^h_{x_1}\rvert]$	3.563567E−04	8.236546E−05	2.0961	2.025525E−05	2.0996	4.484184E−06	2.1754
$\mathcal{E}[\lvert K_{x_2}-K^h_{x_2}\rvert]$	3.563472E−04	8.236230E−05	2.0961	2.025432E−05	2.0996	4.483891E−06	2.1754
$\mathcal{M}[\lvert K_{x_2}-K^h_{x_2}\rvert]$	3.563567E−04	8.236546E−05	2.0961	2.025525E−05	2.0996	4.484184E−06	2.1754
$\mathcal{E}[\lvert E-E^h\rvert]$	8.567125E−05	1.922316E−05	2.0511	4.694926E−06	2.0016	1.198734E−06	1.9696
$\mathcal{M}[\lvert E-E^h\rvert]$	8.570607E−05	1.923237E−05	2.0509	4.697663E−06	2.0013	1.199810E−06	1.9691
$\mathcal{E}[\lvert J-J^h\rvert]$	4.566191E−04	1.075776E−04	2.0366	2.599359E−05	2.0100	6.630678E−06	1.9709
$\mathcal{M}[\lvert J-J^h\rvert]$	4.566465E−04	1.075871E−04	2.0366	2.599465E−05	2.0100	6.631424E−06	1.9708

$$g_2(t) = 1 - (10x_2^3 - 15x_2^4 + 6x_2^5), \qquad x_2 = (t-(m+k)T_p)/mT_p$$

here $T_p = 1/f_0$ and $\omega_0 = 2\pi f_0$. In this simulation, $m = 2$, $k = 100$, $f_0 = 30$ GHz.

This model is solved on a uniform mesh with time step size $\tau = 10^{-13}$ s $= 0.1$ ps and 12 perfectly matched layer (PML) imposed around the physical domain. For details can refer to our previous work [35]. We use the following random parameters for our simulation:

$$\begin{aligned}
&\epsilon_0(\mathbf{x},\xi) = 1.11\times 10^{-11}(1+\xi_1+\xi_2),\\
&\mu_0(\mathbf{x},\xi) = 10^{-6}/(1+\xi_1+\xi_2),\\
&\Gamma_m(\mathbf{x},\xi) = 10^{8}(1+10^{-4}(\xi_3-0.5)),\\
&\Gamma_e(\mathbf{x},\xi) = 10^{8}(1+10^{-4}(\xi_4-0.5)),\\
&\omega_m(\mathbf{x},\xi) = 2\pi\sqrt{2}\times 3\times 10^{10}(1+10^{-4}(\xi_5-0.5)),\\
&\omega_e(\mathbf{x},\xi) = 2\pi\sqrt{2}\times 3\times 10^{10}(1+10^{-4}(\xi_6-0.5)).
\end{aligned}$$

The obtained electric field at various time steps are plotted in Fig. 3, which shows that as the source wave enters the metamaterial slab, the wave propagates backward due to the negative refractive index of the metamaterial and

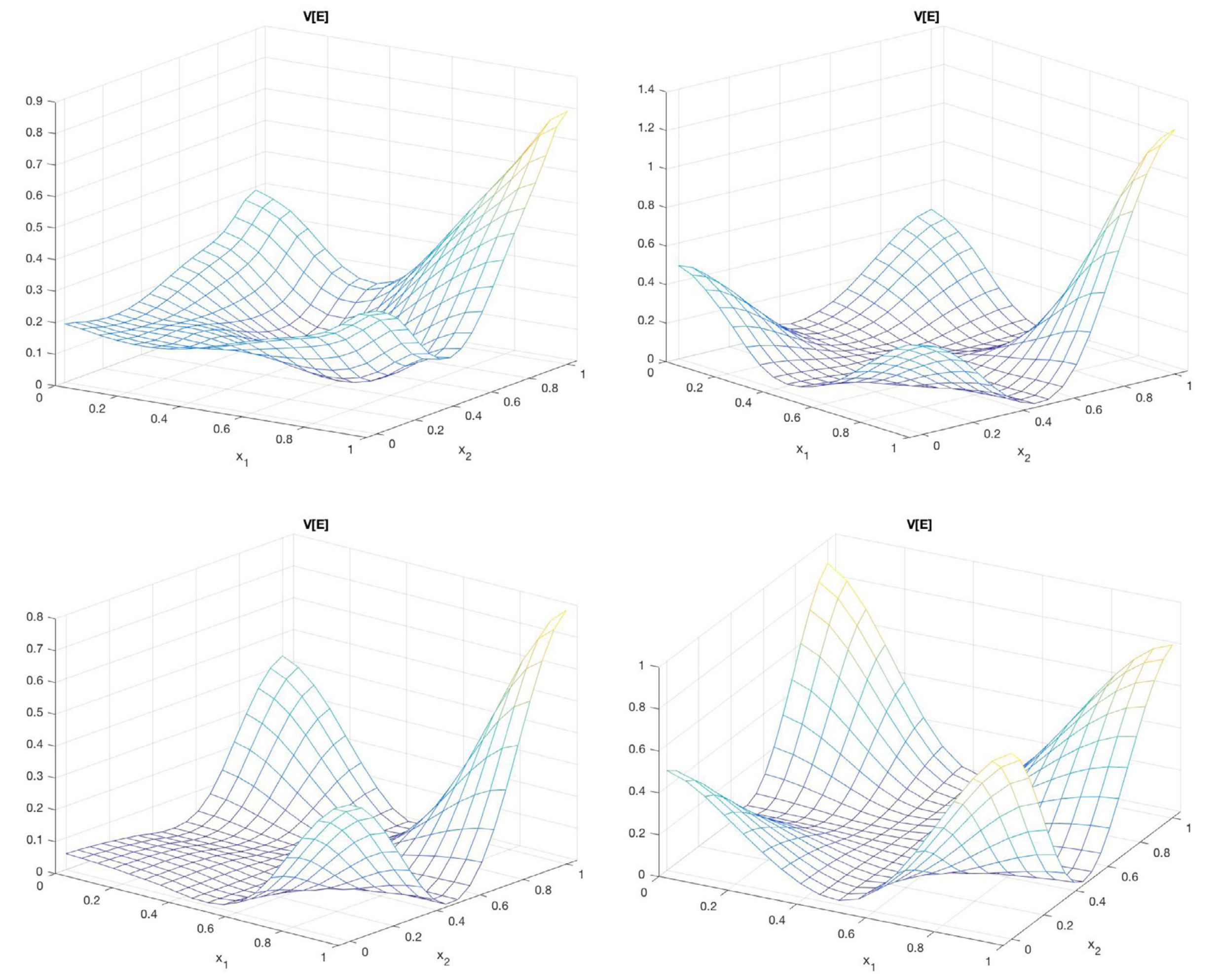


Fig. 2. Example 2. The variances of electronic fields at $t = 0.25$ (Top left), $t = 0.5$ (Top right), $t = 0.75$ (Bottom left) and $t = 1$ (Bottom right).

propagates forward after the wave moves out the metamaterial subdomain. This example shows that the backward wave propagation phenomenon still exists in the random metamaterial.

4. Conclusions

In this paper, we first establish the regularity analysis for the time-dependent Maxwell's equations in Drude metamaterial with respect to the random variables. Using the regularity result, we prove the error estimate for the stochastic collocation method developed to solve this model. Extensive numerical results are presented to justify the theoretical analysis. We also demonstrate the backward wave propagation phenomenon happened when the electromagnetic wave travels in the random metamaterial. We expect that similar results to the Drude model can be obtained for other metamaterial models too [31]. In the future, we plan to develop more efficient stochastic collocation method and even stochastic Galerkin method to Maxwell's equations. We will explore more practical wave propagation problems in random media and random inputs.

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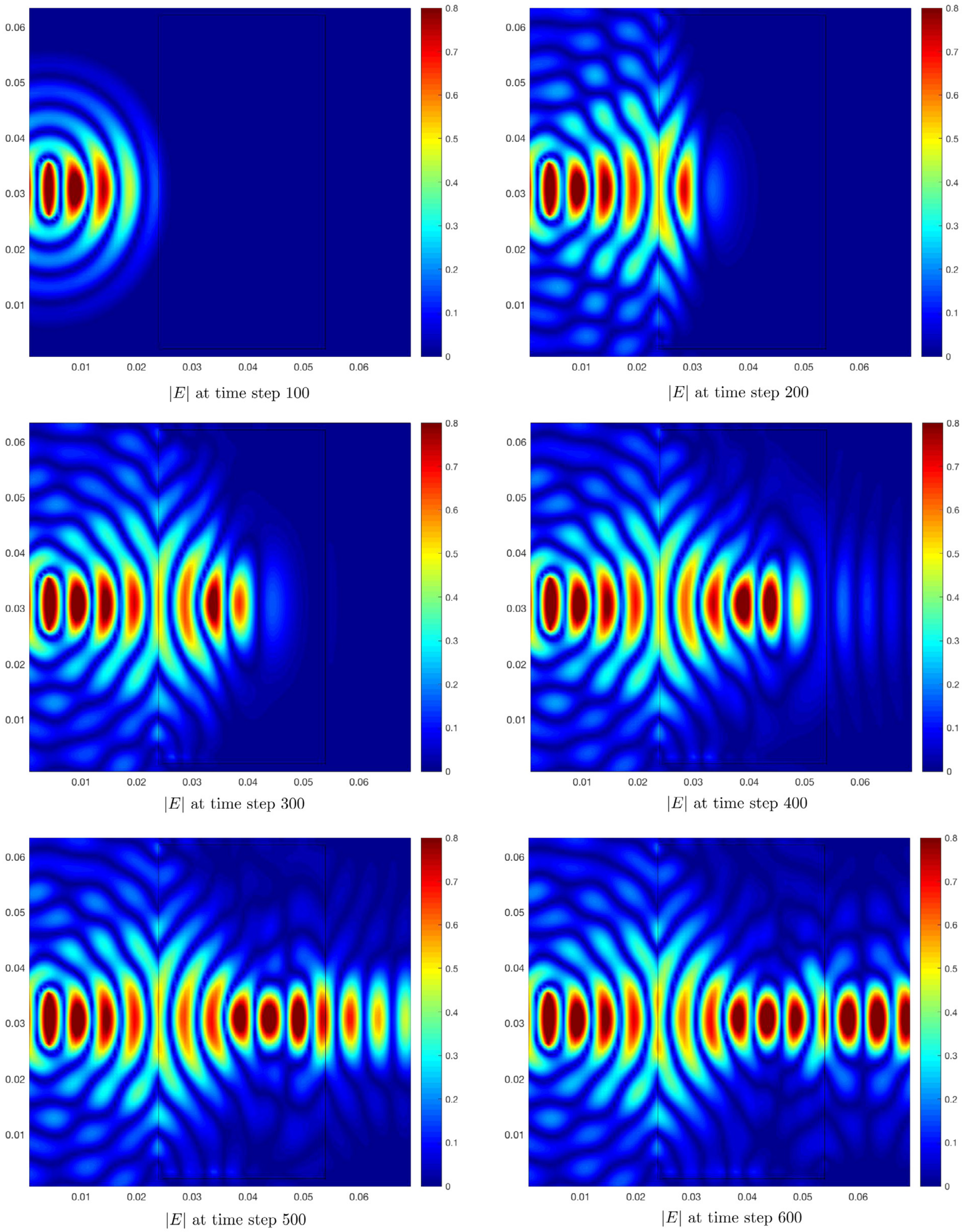

Fig. 3. The contour plot of electric field $|E|$ at various time steps.

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