Special Issue Paper

Mathematical Methods in the Applied Sciences

Received 14 February 2016 Published online 14 February 2017 in Wiley Online Library

(wileyonlinelibrary.com) DOI: 10.1002/mma.4345
MOS subject classification: 35B27; 35J15

Two-scale cut-and-projection convergence; homogenization of quasiperiodic structures

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Communicated by G. Barbatis

We demonstrate how the problem of finding the effective property of quasiperiodic constitutive relations can be simplified to the periodic homogenization setting by transforming the original quasiperiodic material structure to a periodic heterogeneous material in a higher dimensional space. The characterization of two-scale cut-and-projection convergence limits of partial differential operators is presented. Copyright © 2017 John Wiley & Sons, Ltd.

Keywords: quasiperiodic composites; two-scale convergence; cut-and-projection; compactness; electrostatics

1. Introduction

The search for effective properties of material mixtures dates back to at least the second half of the 19th century with notably the works of Maxwell Garnett, Clausius-Mossotti, and Lord Rayleigh. The contributions in the field have been in the form of various mixing formulas based on physical insights and simplified models of the effect of dispersed phases in a matrix with landmark papers by condensed matter physicists such as Bruggeman and Landauer in the first half of the 20th century (see [1] for a review) that serve as an inspiration for mathematicians DeGiorgi and Spagnolo [2], Tartar [3], Bensoussan, Lions and Papanicolaou [4], Dal Maso [5].

A mathematical result typically states in which sense the solutions of sequences of partial differential equations (PDEs) with rapidly varying coefficients converge to the solution of PDEs with constant coefficients while the coefficient variation becomes more and more rapid. The PDEs with constant coefficients constitute models of processes taking place in homogeneous, and often anisotropic, materials, that is, the effective properties of the heterogeneous materials are given by the constant coefficients.

In [6], Nguetseng presented the concept of two-scale convergence, which was further developed in [7]. Two-scale convergence turned out to be a very useful concept in homogenizing periodic material mixtures. This is a generalization of the usual weak convergence, in which one uses oscillating test functions to capture the same scale oscillations in the sequence of functions that are investigated. As a consequence, one obtains limit functions that are defined on the product space $\mathbb{R}^n \times]0,1[^n$. A similar method is the periodic unfolding approach [8], in which one first maps the original sequence of functions to a sequence that is defined on $\mathbb{R}^n \times]0,1[^n$, and then takes the usual weak limit in suitable function spaces, using this extended domain. This is similar to the Fourier transform approach proposed in [9].

It is true that there are composites that have a periodic microstructure and many nonperiodic mixtures are very well modeled by periodic composites. However, there are many material systems that are more cumbersome to model, for example, stochastic high-contrast composites. There are also large period and quasiperiodic composites. For example, mixing two periodic materials may result in a mixture that has a very large periodicity (when there is a common periodicity that is large) or in a quasiperiodic material, in the case of a mixture of materials with rational and irrational periodicity, for example, see [10] for a setting in an almost periodic regime and [11, 12] for some recent work on one type of a stochastic and/or quasiperiodic multiscale homogenization setting.

Quasiperiodic materials can be described by periodic structures in higher spatial dimensions that are cut by hyperplanes and projected onto the lower dimensional space, typically $\mathbb{R}^3$, as proposed by the physicists Duneau and Katz 30 years ago [13]. This opens up the possibility to use standard periodic homogenization tools, for example, two-scale convergence, to homogenize quasiperiodic materials. To do that, one has to complement existing tools with the cut-and-projection operator; this was done in [14]. In this paper, we

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revisit this extension, *the two-scale cut-and-projection convergence* method presented in [14], to homogenize quasiperiodic structures. We characterize the limits of partial differential operators in this setting and demonstrate the method on an electrostatic problem.

1.1. Electrostatics

Let Ω be a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary. We consider the electrostatic equation,

$$\begin{cases} -\text{div } \sigma_\eta(\boldsymbol{x}) \,\text{grad } u_\eta(\boldsymbol{x}) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega \\ u_\eta|_{\partial\Omega} = 0 \end{cases} \tag{1}$$

where $f \in L^2(\Omega)$ and η is a parameter that tends to zero when the fine-scale structure in the composite becomes finer and finer. We assume that σ_η is bounded and coercive, that is, $\sigma_\eta \in L^\infty\left(\Omega; \mathbb{R}^{3\times 3}\right)$, and there exists a constant $c > 0$ such that

$$\sigma_\eta(\boldsymbol{x})\xi \cdot \xi \geq c|\xi|^2, \qquad \text{a.e. } \boldsymbol{x} \in \Omega, \ \forall \xi \in \mathbb{R}^3 \tag{2}$$

Standard estimates yield solutions that are uniformly bounded in $W_0^{1,2}(\Omega)$ with respect to η. Our objective is to find the effective, homogenized, equation when $\eta \to 0$.

1.2. Quasi-crystals

It is common in homogenization papers to assume that the medium is periodic, that is, in the static case that the electric conductivity is a periodic function of the three space variables. Nonetheless, we shall slightly depart from this hypothesis by rather assuming that there is some higher dimensional space within which one can define a periodic conductivity function (of more than three variables). As it turns out, this mathematical game allows for the analysis of a class of materials that are neither periodic nor random: Quasi-crystalline phases discovered by Shechtman [15] in the early eighties can be modeled by taking the cut-and-projection of a periodic structure in an higher dimensional space (typically $\mathbb{R}^6$ or $\mathbb{R}^{12}$) onto a hyperplane (such as the Euclidean space $\mathbb{R}^3$). In the sequel, we will only require the knowledge of a matrix $\mathbf{R}$ defining this cut-and-projection. In practice, physicists have access to the opto-geometric properties of a quasi-crystal through analysis of the symmetries on X-ray diffraction patterns (the so-called reciprocal pseudo-arrays), which are encompassed in the entries of $\mathbf{R}$. For instance, the relative permittivity of the quasi-crystal $Al_{63.5}Fe_{12.5}Cu_{24}$ is given by $\mathbf{R} : \mathbb{R}^3 \to \mathbb{R}^6$,

$$\begin{aligned} \varepsilon_r\big(\mathbf{R}\boldsymbol{x}\big) = &\varepsilon_r\big(n_\tau(x_1 + \tau x_2), n_\tau(\tau x_1 + x_3), n_\tau(x_2 + \tau x_3), \\ &n_\tau(-x_1 + \tau x_2), n_\tau(\tau x_1 - x_3), n_\tau(-x_2 + \tau x_3)\big) \end{aligned} \tag{3}$$

where n_τ is the normalization constant $1/\sqrt{2(2+\tau)}$, with the Golden number τ, and $\varepsilon_r \in L^\infty_\sharp(Y^6)$, that is, it is bounded almost everywhere on the hypercube $Y^6 =]0,1[^6$ and is periodic.

We note that there is an ambiguity in the definition of this relative permittivity as we could have defined it via a cut-and-projection from a periodic array in $\mathbb{R}^{12}$ onto $\mathbb{R}^3$ [13], $\varepsilon_r(\mathbf{R}'\boldsymbol{x})$ where $\mathbf{R}' : \mathbb{R}^3 \to \mathbb{R}^{12}$, that is, $\mathbf{R}'$ is a matrix with 12 rows and three columns and $\varepsilon_r \in L^\infty_\sharp(Y^{12})$. The conductivity in (1) is defined in the same way; hence, there is a potential pitfall in the homogenization process.

However, we shall see in the next section that the homogenized result does not actually depend upon $\mathbf{R} : \mathbb{R}^n \to \mathbb{R}^m$, provided it fulfills the criterion

$$\mathbf{R}^T\boldsymbol{k} \neq \mathbf{0}, \ \forall \boldsymbol{k} \in \mathbb{Z}^m \setminus \{\mathbf{0}\} \tag{4}$$

1.3. Two-scale cut-and-projection convergence

In this section, we recall some properties of two-scale convergence [7] in the quasiperiodic setting [14]. Let us consider a real valued matrix $\mathbf{R}$ with m rows and n columns. Similarly to the periodic case, our goal is to approximate an oscillating sequence $\{u_\eta(\boldsymbol{x})\}_{\eta \in]0,1[}$ by a sequence of two-scale quasiperiodic functions $u_0\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)$ where $u_0(\boldsymbol{x}, \cdot)$ is periodic on $\mathbb{R}^m$.

As the matrix $\mathbf{R}$ is not uniquely defined, we first need to check that if g is a trigonometric polynomial, then the quasiperiodic function $f = g \circ \mathbf{R}$ admits the following (uniquely defined) ergodic mean (for $\mathbf{R} : \mathbb{R}^n \to \mathbb{R}^m$).

$$L(f) = \lim_{T \to +\infty} \frac{1}{(2T)^n} \int_{]-T,T[^n} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{Y^m} g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = [g] \tag{5}$$

where $[g]$ denotes the mean of g over the periodic cell Y^m in $\mathbb{R}^m$. As shown in [14], this is the case provided that $\mathbf{R}$ fulfills the criterion (4). We recall the statement in [14] of this elementary result as it underpins homogenization of quasi-crystals.

Lemma 1.1
Let $\mathbf{R} : \mathbb{R}^n \longmapsto \mathbb{R}^m$ ($m \geq n$) satisfy (4). Then, (5) holds true for any trigonometric polynomial g on $\mathbb{R}^m$.

This result suggests the following concept of two-scale convergence attached to a matrix $\mathbf{R}$.

Definition 1.1 (Distributional two-scale convergence)
Let Ω be an open bounded set in $\mathbb{R}^n$ and $Y^m =]0,1[^m$. We say that the sequence (u_η) two-scale converges in the distributional sense toward the function $u_0 \in L^2(\Omega \times Y^m)$ for a matrix $\mathbf{R}$, if for every $\varphi \in \mathcal{D}(\Omega; C^\infty_\sharp(Y^m))$:

$$\lim_{\eta \to 0} \int_\Omega u_\eta(\boldsymbol{x})\varphi\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\,\mathrm{d}\boldsymbol{x} = \iint_{\Omega \times Y^m} u_0(\boldsymbol{x},\boldsymbol{y})\phi(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \tag{6}$$

Definition 1.2 (Weak two-scale convergence)
Let Ω be an open bounded set in $\mathbb{R}^n$ and $Y^m =]0,1[^m$. We say that the sequence (u_η) two-scale converges weakly toward the function $u_0 \in L^2(\Omega \times Y^m)$ for a matrix $\mathbf{R}$, if for every $\varphi \in L^2(\Omega, C_\sharp(Y^m))$ (6) holds.

We denote weak two-scale convergence for a matrix $\mathbf{R}$ with $u_{\eta_k} \overset{\mathbf{R}}{\rightharpoonup} u_0$. The following result [14] ensures the existence of such two-scale limits when the sequence (u_η) is bounded in $L^2(\Omega)$ and $\mathbf{R}$ satisfies (4).

Proposition 1.1
Let Ω be an open bounded set in $\mathbb{R}^n$ and $Y^m =]0,1[^m$. If $\mathbf{R}$ is a matrix satisfying (4) and (u_η) is a bounded sequence in $L^2(\Omega)$, then there exist a vanishing subsequence η_k and a limit $u_0(\boldsymbol{x},\boldsymbol{y}) \in L^2(\Omega \times Y^m)$ (Y^m-periodic in $\boldsymbol{y}$) such that $u_{\eta_k} \overset{\mathbf{R}}{\rightharpoonup} u_0$ as $\eta_k \to 0$.

We will need to pass to the limit in integrals $\int_\Omega u_\eta\, v_\eta\, \mathrm{d}\boldsymbol{x}$ where $u_\eta \overset{\mathbf{R}}{\rightharpoonup} u_0$ and $v_\eta \overset{\mathbf{R}}{\rightharpoonup} v_0$. For this, we introduce the notion of strong two-scale (cut-and-projection) convergence for a matrix $\mathbf{R}$.

Definition 1.3 (Strong two-scale convergence)
A sequence u_η in $L^2(\Omega)$ is said to two-scale converge strongly, for a matrix $\mathbf{R}$, toward a limit u_0 in $L^2(\Omega \times Y^m)$, which we denote $u_\eta \overset{\mathbf{R}}{\twoheadrightarrow} u_0$, if and only if $u_{\eta_k} \overset{\mathbf{R}}{\rightharpoonup} u_0$ and

$$\|u_\eta(\boldsymbol{x})\|_{L^2(\Omega)} \to \|u_0(\boldsymbol{x},\boldsymbol{y})\|_{L^2(\Omega\times Y^m)} \tag{7}$$

This definition expresses that the effective oscillations of the sequence (u_η) are on the order of η. Moreover, these oscillations are fully identified by u_0. The following proposition provides us with a corrector type result for the sequence u_η when its limit u_0 is smooth enough:

Proposition 1.2
Let $\mathbf{R}$ be a linear map from $\mathbb{R}^n$ in R^m satisfying (4). Let u_η be a sequence bounded in $L^2(\Omega)$ such that $u_\eta \overset{\mathbf{R}}{\rightharpoonup} u_0(\boldsymbol{x},\boldsymbol{y})$ (weakly). Then

(i) u_η weakly converges in $L^2(\Omega)$ toward $u(\boldsymbol{x}) = \int_{Y^m} u_0(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$ and

$$\liminf_{\eta\to 0} \|u_\eta\|_{L^2(\Omega)} \geq \|u_0\|_{L^2(\Omega\times Y^m)} \geq \|u\|_{L^2(\Omega)} \tag{8}$$

(*ii*) Let v_η be another bounded sequence in $L^2(\Omega)$ such that $v_\eta \overset{\mathbf{R}}{\twoheadrightarrow} v_0$ (strongly), then

$$u_\eta v_\eta \to w(\boldsymbol{x}) \text{ in} \mathcal{D}'(\Omega) \text{ where } w(\boldsymbol{x}) = \int_{Y^m} u_0(\boldsymbol{x},\boldsymbol{y}) v_0(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \tag{9}$$

(*iii*) If $u_\eta \overset{\mathbf{R}}{\twoheadrightarrow} u_0(\boldsymbol{x},\boldsymbol{y})$ strongly and $\left\| u_0\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|_{L^2(\Omega)} \to \|u_0(\boldsymbol{x},\boldsymbol{y})\|_{L^2(\Omega\times Y^m)}$, then

$$\left\| u_\eta - u_0\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|_{L^2(\Omega)} \to 0 \tag{10}$$

Classes of functions such that $\left\| u_0\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|_{L^2(\Omega)} \to \|u_0(\boldsymbol{x},\boldsymbol{y})\|_{L^2(\Omega\times Y^m)}$ are said to be admissible for the two-scale cut-and-projection convergence. In particular, classes of functions in $L^2(\Omega, C_\sharp(Y^m))$ (dense subset in $L^2(\Omega\times Y^m)$) are admissible.

In order to homogenize PDEs, we need to identify the differential relationship between $\boldsymbol{\chi}$ and u_0, given a bounded sequence (u_η) in $W^{1,2}(\Omega)$ (such that $u_\eta \overset{\mathbf{R}}{\rightharpoonup} u_0$ and $\nabla u_\eta \overset{\mathbf{R}}{\rightharpoonup} \boldsymbol{\chi}$). This problem was solved by Allaire in the case of periodic functions [7] and Bouchitté *et al*. for quasiperiodic functions [14].

2. Definition of function spaces and associated cut-and-projection partial differential operators and some of their properties

To carry out the homogenization analysis of PDEs defined on quasiperiodic domains, we need to pass to the limit when η goes to zero in gradient, divergence, and curl operators acting on solutions of PDEs. To do this, we introduce some suitable function spaces. We consider a matrix $\mathbf{R}$ satisfying (4) with m rows and n columns, that is, $\mathbf{R} : \mathbb{R}^n \to \mathbb{R}^m$, assuming we are considering PDEs defined on domains $\Omega \subset \mathbb{R}^n$. Any $u \in L^2(Y^m)$ can be represented by a Fourier series,

$$u(\boldsymbol{y}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^m} u_{\boldsymbol{k}} e^{2\pi i \boldsymbol{k}\cdot\boldsymbol{y}}, \; u_{\boldsymbol{k}} \in \mathbb{C}$$

which can be used to define a function $u_{\mathbf{R}} \in L^2(\Omega)$, by the cut-and-projection operation,

$$u_{\mathbf{R}}(\boldsymbol{x}) = u(\mathbf{R}\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^m} u_{\boldsymbol{k}} e^{2\pi i \boldsymbol{k}\cdot\mathbf{R}\boldsymbol{x}} = \sum_{\boldsymbol{k}\in\mathbb{Z}^m} u_{\boldsymbol{k}} e^{2\pi i (\mathbf{R}^T\boldsymbol{k})\cdot\boldsymbol{x}}, \; u_{\boldsymbol{k}} \in \mathbb{C}$$

The mapping $y = \mathbf{R}x$ and the chain rule yields that partial derivatives transform to

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{m} \mathbf{R}_{ji} \frac{\partial}{\partial y_j}$$

Hence, the gradient of $u_{\mathbf{R}}$ is given by

$$\text{grad } u_{\mathbf{R}}(x) = \sum_{k\in\mathbb{Z}^m\setminus\{0\}} 2\pi i \mathbf{R}^T k u_k e^{2\pi i(\mathbf{R}^T k)\cdot x} = \mathbf{R}^T \text{grad}_y\ u(\mathbf{R}x) = \mathbf{R}^T \nabla_y\ u(\mathbf{R}x)$$

where grad is the usual gradient in $\mathbb{R}^n$ and grad_y denotes the gradient in the range of $\mathbf{R}$ in $\mathbb{R}^m$. Vector valued functions $u \in L^2(Y^m;\mathbb{R}^3)$ are represented analogously, and the divergence is given by

$$\text{div } u_{\mathbf{R}}(x) = \sum_{k\in\mathbb{Z}^m\setminus\{0\}} 2\pi i \left(\mathbf{R}^T \nabla_y\right) \cdot u_k e^{2\pi i(\mathbf{R}^T k)\cdot x} = \left(\mathbf{R}^T \nabla_y\right) \cdot u(\mathbf{R}x)$$

where div is the divergence operator in $\mathbb{R}^n$ and ∇_y denotes the nabla operator in the range of $\mathbf{R}$ in $\mathbb{R}^m$. For $n = 3$, we get

$$\text{curl } u_{\mathbf{R}}(x) = \text{curl } u(\mathbf{R}x) = \sum_{k\in\mathbb{Z}^m\setminus\{0\}} 2\pi i \left(\mathbf{R}^T k\right) \times u_k e^{2\pi i(\mathbf{R}^T k)\cdot x} = \left(\mathbf{R}^T \text{grad}_y\right) \times u(\mathbf{R}x) = \left(\mathbf{R}^T \nabla_y\right) \times u(\mathbf{R}x)$$

We can use these representations to define $\mathbf{R}$-dependent gradient, divergence, and curl operators acting on functions defined on domains in $\mathbb{R}^m$. They are

$$\nabla_{\mathbf{R}}\ u(y) = \text{grad}_{\mathbf{R}}\ u(y) = \mathbf{R}^T \text{grad}_y\ u(y) = \mathbf{R}^T \nabla_y\ u(y)$$
$$\text{div}_{\mathbf{R}}\ u(y) = \left(\mathbf{R}^T \nabla_y\right) \cdot u(y)$$

and for $n = 3$,

$$\text{curl}_{\mathbf{R}}\ u(y) = \left(\mathbf{R}^T \text{grad}_y\right) \times u(y) = \left(\mathbf{R}^T \nabla_y\right) \times u(y)$$

Remark 2.1
The gradient operator $\text{grad}_{\mathbf{R}}$ is a directional derivative given by the projection on $\mathbb{R}^n$ of the usual gradient in $\mathbb{R}^m$. The operation to compute the divergence and curl uses the same projection of the nabla operator in combination with the usual nabla rules. The $\text{div}_{\mathbf{R}}$ operator can also be interpreted as the divergence in Y^m of the m-component vector $\mathbf{R}u$.

We define the following function spaces associated with the differential operators defined previously

$$\mathcal{H}_\sharp(\text{grad}_{\mathbf{R}}, Y^m) = \left\{u \in L^2_\sharp(Y^m) \mid \text{grad}_{\mathbf{R}}\ u \in L^2_\sharp(Y^m;\mathbb{R}^n)\right\} \tag{11}$$

$$\mathcal{H}_\sharp(\text{div}_{\mathbf{R}}, Y^m) = \left\{u \in L^2_\sharp(Y^m;\mathbb{R}^n) \mid \text{div}_{\mathbf{R}}\ u \in L^2_\sharp(Y^m)\right\} \tag{12}$$

$$\mathcal{H}_\sharp(\text{curl}_{\mathbf{R}}, Y^m) = \left\{u \in L^2_\sharp(Y^m;\mathbb{R}^3) \mid \text{curl}_{\mathbf{R}}\ u \in L^2_\sharp(Y^m;\mathbb{R}^3)\right\} \tag{13}$$

and

$$\mathcal{H}_\sharp(\text{div}_{\mathbf{R}_0}, Y^m) = \left\{u \in \mathcal{H}_\sharp(\text{div}_{\mathbf{R}}, Y^m) \mid \text{div}_{\mathbf{R}}\ u \equiv 0\right\} \tag{14}$$

$$\mathcal{H}_\sharp(\text{curl}_{\mathbf{R}_0}, Y^m) = \left\{u \in \mathcal{H}_\sharp(\text{curl}_{\mathbf{R}}, Y^m) \mid \text{curl}_{\mathbf{R}}\ u \equiv \mathbf{0}\right\} \tag{15}$$

We have the following lemma that states that any $\text{curl}_{\mathbf{R}}$-free vector field is given by the $\mathbf{R}$-gradient of a scalar potential.

Lemma 2.1
Any function $u \in \mathcal{H}_\sharp(\text{curl}_{\mathbf{R}_0}, Y^m)$ is given as $u = \text{grad}_{\mathbf{R}}\ \phi(y)$ for some scalar potential ϕ defined on Y^m.

We are now in the position to state that spaces $\mathcal{H}_\sharp(\text{curl}_{\mathbf{R}_0}, Y^m)$ and $\mathcal{H}_\sharp(\text{div}_{\mathbf{R}_0}, Y^m)$ are orthogonal to curls and gradients.

Lemma 2.2
The orthogonal space to $\mathcal{H}_\sharp(\text{curl}_{\mathbf{R}_0}, Y^m)$ is given by the space of $\mathbf{R}$-curls of functions defined as

$$\mathcal{M}_{\mathbf{R}_\sharp}(Y^m, \mathbb{R}^3) = \left\{w \in L^2_\sharp(Y^m;\mathbb{R}^3) \mid w(y) = \text{curl}_{\mathbf{R}} u(y)\ \right\} \tag{16}$$

for some vector valued function u in $\mathbb{R}^3$ defined on Y^m.

Lemma 2.3
The orthogonal space to $\mathcal{H}_\sharp(\text{div}_{\mathbf{R}_0}, Y^m)$ is given by the space of

$$\mathcal{L}_{\mathbf{R}_\sharp}(Y^m, \mathbb{R}^3) = \left\{w \in L^2_\sharp(Y^m;\mathbb{R}^3) \mid w(y) = \text{grad}_{\mathbf{R}}\phi(y)\right\} \tag{17}$$

for some scalar potential ϕ defined on Y^m.

3. Compactness results

In the following main compactness results, we let $\mathbf{R}$ be a matrix with m rows and three columns ($m > 3$) satisfying (4) and assume that Ω is an open bounded set of $\mathbb{R}^3$ with a Lipschitz boundary.

Proposition 3.1
Let $\{u_\eta\}$ be a uniformly bounded sequence in $W^{1,2}(\Omega)$. Then there exist a subsequence $\{u_{\eta_k}\}$ and functions $u \in W^{1,2}(\Omega)$ and $\mathrm{grad}_{\mathbf{R}}\, u_1(\boldsymbol{x},\boldsymbol{y}) \in L^2(\Omega, L^2_\sharp(Y^m;\mathbb{R}^3))$, such that

$$u_{\eta_k} \overset{\mathbf{R}}{\twoheadrightarrow} u(\boldsymbol{x}), \qquad \mathrm{grad}\;u_{\eta_k} \overset{\mathbf{R}}{\rightharpoonup} \mathrm{grad}\;u(\boldsymbol{x}) + \mathrm{grad}_{\mathbf{R}}\, u_1(\boldsymbol{x},\boldsymbol{y}), \qquad \eta_k \to 0 \tag{18}$$

Proposition 3.2
Let $\{\boldsymbol{u}_\eta\}$ be a uniformly bounded sequence in $H(\mathrm{curl}\,,\Omega)$. Then there exist a subsequence $\{\boldsymbol{u}_{\eta_k}\}$ and functions $\boldsymbol{u}_0 \in H(\mathrm{curl}\,,\Omega)$, $\mathrm{grad}_{\mathbf{R}}\,\phi \in L^2(\Omega, L^2_\sharp(Y^m;\mathbb{R}^3))$, and $\mathrm{curl}_{\mathbf{R}}\,\boldsymbol{u}_1 \in L^2(\Omega, L^2_\sharp(Y^m;\mathbb{R}^3))$ such that

$$\boldsymbol{u}_{\eta_k} \overset{\mathbf{R}}{\rightharpoonup} \boldsymbol{u}_0(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{u}(\boldsymbol{x}) + \mathrm{grad}_{\mathbf{R}}\,\phi(\boldsymbol{x},\boldsymbol{y}), \qquad \mathrm{curl}\;\boldsymbol{u}_{\eta_k} \overset{\mathbf{R}}{\rightharpoonup} \mathrm{curl}\;\boldsymbol{u}(\boldsymbol{x}) + \mathrm{curl}_{\mathbf{R}}\,\boldsymbol{u}_1(\boldsymbol{x},\boldsymbol{y}), \qquad \eta_k \to 0 \tag{19}$$

where

$$\boldsymbol{u}(\boldsymbol{x}) = \int_{Y^m} \boldsymbol{u}_0(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \tag{20}$$

Proposition 3.3
Let $\{\boldsymbol{u}_\eta\}$ be a uniformly bounded sequence in $H(\mathrm{div}\,,\Omega)$. Then there exist a subsequence $\{\boldsymbol{u}_{\eta_k}\}$ and functions $\boldsymbol{u}_0 \in H(\mathrm{div}\,,\Omega)$ and $\boldsymbol{u}_1 \in L^2(\Omega, H_\sharp(\mathrm{div}\,,Y^m))$ such that

$$\boldsymbol{u}_{\eta_k} \overset{\mathbf{R}}{\rightharpoonup} \boldsymbol{u}_0(\boldsymbol{x},\boldsymbol{y}), \qquad \mathrm{div}\;\boldsymbol{u}_{\eta_k} \overset{\mathbf{R}}{\rightharpoonup} \mathrm{div}\;\boldsymbol{u}(\boldsymbol{x}) + \mathrm{div}_y\,\boldsymbol{u}_1(\boldsymbol{x},\boldsymbol{y}), \qquad \eta_k \to 0 \tag{21}$$

where $\boldsymbol{u}$ is given by (20).

4. Homogenization of the electrostatic problem

Consider the quasiperiodic heterogeneous electrostatic problem (1) with $\sigma_\eta = \sigma\left(\frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)$, that is,

$$\begin{cases} -\mathrm{div}\;\sigma\left(\frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\mathrm{grad}\;u_\eta(\boldsymbol{x}) = f(\boldsymbol{x})\,, & \boldsymbol{x}\in\Omega \\ u_\eta|_{\partial\Omega} = 0 \end{cases} \tag{22}$$

Theorem 4.1
There exists a subsequence of $\{u_\eta\}$ that converges weakly in $W_0^{1,2}(\Omega)$ to the solution u of the homogenized equation

$$\begin{cases} -\mathrm{div}\;\sigma^h\mathrm{grad}\,u(\boldsymbol{x}) = f(\boldsymbol{x})\,, & \boldsymbol{x}\in\Omega \\ u|_{\partial\Omega} = 0 \end{cases} \tag{23}$$

where

$$\sigma^h_{ik} = \int_{Y^m} \sigma_{ij}(\boldsymbol{y})\left(\delta_{jk} - \nabla_{\mathbf{R}_j}\,\chi^k(\boldsymbol{y})\right)\mathrm{d}\boldsymbol{y} \tag{24}$$

and $\nabla_{\mathbf{R}_j}\chi^k$ solves the local equation

$$\int_{Y^m} \sigma_{ij}(\boldsymbol{y})\left(\delta_{jk} - \nabla_{\mathbf{R}j}\,\chi^k(\boldsymbol{y})\right)\nabla_{\mathbf{R}i}\phi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = 0 \tag{25}$$

Proposition 4.1
The local equation (25) has a unique gradient solution $\nabla_{\mathbf{R}}\chi^k \in L^2(Y^m;\mathbb{R}^3)$.

Remark 4.1
The local problem in a strong formulation reads

$$-\mathrm{div}_{\boldsymbol{y}}\;\mathbf{R}\sigma(\boldsymbol{y})\mathbf{R}^T\mathrm{grad}_{\boldsymbol{y}}\;u_1(\boldsymbol{x},\boldsymbol{y}) = \mathrm{div}_{\boldsymbol{y}}\;\mathbf{R}\sigma(\boldsymbol{y})\,\mathrm{grad}\;u_0(\boldsymbol{x})$$

where $\mathrm{div}_{\boldsymbol{y}}\;\mathbf{R}\sigma(\boldsymbol{y})\mathbf{R}^T\mathrm{grad}_{\boldsymbol{y}}$ is a degenerated elliptic operator. This is pointed out in other papers, for example, in [12]. But in the weak formulation, Equation (25), we only force the local equation in three directions (i.e., if $n = 3$, $\mathbb{R}^n$), corresponding to the real space coordinate axes. The corresponding projected gradients are bounded, and the potentials are not needed for the effective properties in Equation (24). The gradient of potential, $\mathrm{grad}_{\boldsymbol{y}}\;u_1(\boldsymbol{x},\boldsymbol{y})$, may still be unbounded in $L^2(\Omega, L^2(Y^m;\mathbb{R}^m))$.

We have the following corrector result that resembles that of Theorem 2.1 in [10], in which a corrector result is given for quasiperiodic monotone operators. It follows from the fact that $\nabla u(\boldsymbol{x}) \cdot \nabla_{\mathbf{R}} \boldsymbol{\chi}$ with $\boldsymbol{\chi} = (\chi^1, \chi^2, \chi^3)$ are admissible test functions in the sense of Proposition 1.2 (see the proof of Theorem 2.6 in [7]).

Proposition 4.2 (Correctors)
We have

$$\lim_{\eta \to 0} \left\| \nabla u_\eta(\boldsymbol{x}) - \nabla u(\boldsymbol{x}) - \nabla u(\boldsymbol{x}) \cdot \nabla_{\mathbf{R}} \boldsymbol{\chi} \left(\frac{\mathbf{R}\boldsymbol{x}}{\eta} \right) \right\|_{L^2(\Omega;\mathbb{R}^3)} = 0$$

5. Conclusions

We have demonstrated how quasiperiodic structures can be homogenized by using the property that they are generated by a periodic geometry in a higher dimensional space. Homogenization of an electrostatic problem has been given as a first canonical example. Proofs will be provided elsewhere together with homogenization of other problems of interest to physicists, such as homogenization of the bianisotropic Maxwell system, as a quasiperiodic counterpart to the random case studied in [11], as well as homogenization of non-linear problems in L^p spaces.

Acknowledgments

We would like to thank Panagiotis Souganidis for informing us about Reference [12] and the anonymous referees for suggesting Reference [11]. S. G. is thankful for a European Research Council (ERC) Starting Grant (279673).

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