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Uniform symbolic topologies in normal toric rings

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Given a normal toric algebra R, we compute a uniform integer $D = D(R) > 0$ such that the symbolic power $P^{(DN)} \subseteq P^N$ for all $N > 0$ and all monomial primes P. We compute the multiplier D explicitly in terms of the polyhedral cone data defining R, illustrating the output for Segre–Veronese algebras.

1. Introduction and conventions for the paper

Given any prime ideal P in a Noetherian ring R, its a**-th** ($a \in \mathbb{Z}_{>0}$) **symbolic power** ideal is the smallest P-primary ideal containing P^a, $P^{(a)} = P^a R_P \cap R := \{f \in R \colon uf \in P^a \text{ for some } u \in R - P\}$. Given a Noetherian commutative ring R, when is there an integer D, depending only on R, such that the symbolic power $P^{(Dr)} \subseteq P^r$ for all prime ideals $P \subseteq R$ and all positive integers r? In short, when does R have **uniform**

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symbolic topologies on primes [8]? Moreover, can we effectively compute the multiplier D in terms of simple data about R?

One celebrated affirmative answer is the *improved* Ein–Lazarsfeld–Smith Theorem: if R is a d-dimensional excellent regular ring, and $D = \max\{1, d-1\}$, then $Q^{(Dr)} \subseteq Q^r$ for all radical ideals $Q \subseteq R$ and all $r > 0$ [3,6,9]. Under mild stipulations in the non-regular setting, a local ring R regular on the punctured spectrum has uniform symbolic topologies on primes [7, Cor. 3.10], although explicit values for D remain elusive. See also our papers [10, Table 3.3] and [12, Thm. 1.2] for ADE rational surface singularities and for select domains with non-isolated singularities, respectively; see also the survey article [2, Thm. 3.29, Cor. 3.30] for module-finite direct summands of affine polynomial rings.

In this paper, we answer the above questions in the setting of torus-invariant primes in a normal toric (or monomial, or semigroup) algebra—the coordinate rings of normal affine toric varieties, hence also Cohen-Macaulay and combinatorially-defined. We state our two main results now for those readers already accustomed to the notation and terminology in Cox–Little–Schenck [1] and Fulton [4].

Theorem 1.1. *Let $C \subseteq N_{\mathbb{R}}$ be a full pointed rational polyhedral cone. Let $R_{\mathbb{F}} = \mathbb{F}[C^\vee \cap M]$ be the associated toric algebra over a field $\mathbb{F}$. Set $D := \max_{m \in \mathcal{B}} \langle m, v_C \rangle$, where $\mathcal{B}$ is the minimal generating set for $C^\vee \cap M$ and $v_C \in N$ is the sum of the primitive generators for C. Then*

$$P^{(D(r-1)+1)} \subseteq P^r$$

for all $r > 0$, and all monomial primes P in $R_{\mathbb{F}}$.

Corollary 1.2. *With notation as in Theorem 1.1, we assume further that C is simplicial. Define $T := \max\{\max_{m \in \mathcal{B}} \langle m, v_C \rangle, D\}$, where D is any positive integer such that $D \cdot \mathrm{Cl}(R_{\mathbb{F}}) = 0$. Then*

$$P^{(T(r-1)+1)} \subseteq P^r$$

for all $r > 0$, all monomial primes, and all height one primes in $R_{\mathbb{F}}$.

We quickly prove the latter now. To clarify, the **divisor class group** of a Noetherian normal domain R, $\mathrm{Cl}(R) = \mathrm{Cl}(\mathrm{Spec}(R))$, is the free abelian group on the set of height one prime ideals of R modulo relations of the form $a_1 P_1 + \ldots + a_r P_r = 0$ whenever the ideal $P_1^{(a_1)} \cap \ldots \cap P_r^{(a_r)}$ is principal.

Proof of Corollary 1.2. Since C is simplicial, $\#\,\mathrm{Cl}(R_{\mathbb{F}})$ is finite by [11, Thm. 3.6 and Lem. 3.8]. Now combine Theorem 1.1 with [11, Lem. 1.1], and take the maximum of the values. $\square$

Conventions: Throughout, $\mathbb{F}$ denotes an arbitrary ground field of arbitrary characteristic. All rings are commutative with identity—indeed, they are normal domains of finite type over $\mathbb{F}$. For conciseness of exposition, we deduce all results only in the case of full pointed cones. Results extend to non-full pointed cones verbatim, with full details recorded in the author's Ph.D. thesis [13, Ch. 3].

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2. Toric algebra notation and terminology preliminaries

As in Cox–Little–Schenck [1, Ch. 1, 3, 4] and Fulton [4, Ch. 1, 3], a lattice is a free abelian group of finite rank. We fix a perfect bilinear pairing $\langle\cdot,\cdot\rangle\colon M\times N\to\mathbb{Z}$ between two lattices M and N; this identifies M with $\mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$ and N with $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$. Our pairing extends to a perfect pairing of finite-dimensional vector spaces $\langle\cdot,\cdot\rangle\colon M_{\mathbb{R}}\times N_{\mathbb{R}}\to\mathbb{R}$, where $M_{\mathbb{R}}:=M\otimes_{\mathbb{Z}}\mathbb{R}$ and $N_{\mathbb{R}}:=N\otimes_{\mathbb{Z}}\mathbb{R}$.

Going forward, we fix a full pointed **N-rational** polyhedral cone $C\subseteq N_{\mathbb{R}}$ and its M-rational dual: respectively, for some **finite** subset $G\subseteq N-\{0\}$ these are closed, convex sets of the form

$$C=\mathrm{Cone}(G):=\left\{\sum_{v\in G}a_v\cdot v\colon\ \text{each } a_v\in\mathbb{R}_{\geq 0}\right\}\subseteq N_{\mathbb{R}},\ \text{and}$$

$$C^\vee:=\{w\in M_{\mathbb{R}}\colon\langle w,v\rangle\geq 0 \text{ for all } v\in C\}=\{w\in M_{\mathbb{R}}\colon\langle w,v\rangle\geq 0\text{ for all } v\in G\}.$$

By definition, the **dimension** of a cone in $M_{\mathbb{R}}$ or $N_{\mathbb{R}}$ is the dimension of the real vector subspace it spans; a cone is **full(-dimensional)** if it spans the full ambient space. A cone in $M_{\mathbb{R}}$ or $N_{\mathbb{R}}$ is **pointed (or strongly convex)** if it contains no line through the origin. A pointed full-dimensional cone C is said to be **simplicial** if it is generated by exactly $d=\dim_{\mathbb{R}}(N_{\mathbb{R}})$ elements in N.

There is a uniquely-determined *minimal* finite generating set $\mathcal{B}$ for the semigroup $C^\vee\cap M$, its **Hilbert basis**. This basis consists of the **irreducible** vectors $m\in C^\vee\cap M-\{0\}$, read, nonzero vectors that cannot be expressed as a sum of two nonzero vectors in $C^\vee\cap M$ [1, Prop. 1.2.17, Prop. 1.2.23].

Fix an arbitrary ground field $\mathbb{F}$. The semigroup ring $R_{\mathbb{F}}=\mathbb{F}[C^\vee\cap M]$ is the **toric $\mathbb{F}$-algebra associated to** C. This ring $R_{\mathbb{F}}$ is a normal domain of finite type over $\mathbb{F}$ [1, Thm. 1.3.5]. Note that $R_{\mathbb{F}}$ has an $\mathbb{F}$-basis $\{\chi^m\colon m\in C^\vee\cap M\}$ of monomials, giving $R_{\mathbb{F}}$ an M-grading, where $\deg(\chi^m):=m$. A **monomial ideal (also called an M-homogeneous or torus-invariant ideal)** in $R_{\mathbb{F}}$ is an ideal generated by a subset of these monomials.

3. Proof of main result and example computations

Proof of Theorem 1.1. We may fix a face $F \neq \{0\}$ of the full pointed rational cone C, and $P = P_F$ the corresponding monomial prime in $R = R_{\mathbb{F}}$. First, we note F has a uniquely-determined set G_F of primitive generators; recall that by definition, a vector $v \in N$ is **primitive** if $\frac{1}{k} \cdot v \notin N$ for all $k \in \mathbb{Z}_{>1}$. Fulton [4, p. 53] records a surjective M-graded ring map between integral domains

$$\phi_F : R_{\mathbb{F}} = \mathbb{F}[C^\vee \cap M] \twoheadrightarrow \mathbb{F}[F^* \cap M], \quad \phi_F(\chi^m) = \begin{cases} \chi^m & \text{if } \langle m, v \rangle = 0 \text{ for all } v \in F \\ 0 & \text{if } \langle m, v \rangle > 0 \text{ for some } v \in F. \end{cases}$$

The **monomial prime ideal of** F, $P_F := \ker(\phi_F)$, has height equal to $\dim(F)$. By [11, Lem. 3.1],

$$P_F = (\{\chi^m : m \in C^\vee \cap M \text{ and the integer } \langle m, v_F \rangle > 0\})R_{\mathbb{F}},$$

where G_F is the set of primitive generators of F and $v_F := \sum_{v \in G_F} v \in F \cap N$.

Lemma 1. *For each integer $E \geq 1$, we have $P_F^{(E)} \subseteq I_F(E) \subseteq P_F^{\lceil E/D' \rceil} \subseteq P_F^{\lceil E/D \rceil}$ where*

$$I_F(E) := (\chi^m : \langle m, v_F \rangle \geq E)\, R, \quad D := \max_{m \in \mathcal{B}} \langle m, v_C \rangle\,, \text{ and } D' := \max_{m \in \mathcal{B}} \langle m, v_F \rangle \leq D.$$

Proof. First, $I_F(E)$ is P_F-primary for all $E \geq 1$, i.e., if $sf \in I_F(E)$ for some $s \in R - P_F$, then $f \in I_F(E)$. As $I_F(E)$ is monomial, we may test this by fixing $\chi^m \in I_F(E)R_{P_F} \cap R$ and $\chi^q \in R - P_F$ such that $\chi^m \cdot \chi^q = \chi^{m+q} \in I_F(E)$: $\langle q, v_F \rangle = 0$, while $E \leq \langle m+q, v_F \rangle = \langle m, v_F \rangle + \langle q, v_F \rangle = \langle m, v_F \rangle$, so $\chi^m \in I_F(E)$. Thus all $I_F(E)$ are P_F-primary, and certainly $P_F^E \subseteq I_F(E)$. Thus $P_F^{(E)} \subseteq I_F(E)$, being the smallest P_F-primary ideal containing P_F^E.

Now fix any monomial $\chi^\ell \in I_F(E)$, say $\ell = \sum_{m \in \mathcal{B}} a_m \cdot m$ with $a_m \in \mathbb{Z}_{\geq 0}$. Let $S \subseteq \mathcal{B}$ consist of those $m \in \mathcal{B}$ such that the monomials χ^m form a minimal generating set for P. By linearity of $\langle \bullet, v_F \rangle$,

$$E \leq \langle \ell, v_F \rangle = \sum_{m \in \mathcal{B}} a_m \langle m, v_F \rangle = \sum_{m \in S} a_m \langle m, v_F \rangle \leq \sum_{m \in S} a_m \cdot D' \Longrightarrow \sum_{m \in S} a_m \geq \lceil E/D' \rceil.$$

Thus $\chi^\ell \in P_F^{\sum_{m \in S} a_m} \subseteq P_F^{\lceil E/D' \rceil}$, ergo $I_F(E) \subseteq P_F^{\lceil E/D' \rceil} \subseteq P_F^{\lceil E/D \rceil}$, proving the lemma. $\square$

To finish the proof of Theorem 1.1, set $E = D(r-1)+1$ in the lemma. Thus $P_F^{(D(r-1)+1)} \subseteq P_F^r$ for all $r > 0$, as desired. $\square$

Example 3.1. Fix an arbitrary ground field $\mathbb{F}$ and integers $n \geq 2$ and $E \geq 2$. Let

$$R = \frac{\mathbb{F}[x_1, \ldots, x_n, z]}{(z^E - x_1 \cdots x_n)}.$$

Then Theorem 1.1 and its corollary ensure that $P^{(T(r-1)+1)} \subseteq P^r$ for all $r > 0$, all monomial primes, and all height one primes in R, where $T = \max\{n, E\}$. Indeed, R is a toric algebra arising from the simplicial full pointed cone $C \subseteq \mathbb{R}^n$ spanned by $\{e_n,\ E \cdot e_i + e_n : i = 1, \ldots, n-1\} \subseteq \mathbb{Z}^n$, where $e_1, \ldots, e_n$ denote the standard basis vectors in $\mathbb{R}^n$. We can compute that $\mathrm{Cl}(R) \cong (\mathbb{Z}/E\mathbb{Z})^{n-1}$ so $E \cdot \mathrm{Cl}(R) = 0$ [11, Ex. 5.6]. Meanwhile, in the notation of Theorem 1.1, $\mathcal{B} = \{e_1, \ldots, e_{n-1}, e_n, E \cdot e_n - e_1 - \cdots - e_{n-1}\} \subseteq \mathbb{Z}^n$ and the vector $v_C = n \cdot e_n + E \cdot (e_1 + \cdots + e_{n-1}) \in \mathbb{Z}^n$, so we compute that $\max_{m \in \mathcal{B}} \langle m, v_C \rangle = n$.

3.1. Closing example computation: Segre–Veronese algebras

Segre–Veronese algebras are a well-known class of normal toric rings.

Definition 3.2. Fix a family $A_1, \ldots, A_k$ of k standard graded algebras of finite type over $\mathbb{F}$, with $A_i = \mathbb{F}[a_{i,1}, \ldots, a_{i,b_i}]$ in terms of algebra generators. Their **Segre product** over $\mathbb{F}$ is the ring $S = (\#_{\mathbb{F}})_{i=1}^k A_i$ generated up to isomorphism as an $\mathbb{F}$-algebra by all k-fold products of the $a_{i,j}$.

Definition 3.3. We fix integers $E \geq 1$ and $m \geq 2$. Suppose $A = \mathbb{F}[x_1, \ldots, x_m]$ is a standard graded polynomial ring in m variables over a field $\mathbb{F}$. Let $V_{E,m} \subseteq A$ denote the E-th **Veronese subring** of A, the standard graded $\mathbb{F}$-subalgebra generated by all monomials of degree E in the x_i. There are $\binom{m-1+E}{E}$ such monomials; this number is the **embedding dimension** of $V_{E,m}$.

Definition 3.4. Fix k-tuples $\overline{E} = (E_1, \ldots, E_k) \in (\mathbb{Z}_{\geq 1})^k$ and $\overline{m} = (m_1, \ldots, m_k) \in (\mathbb{Z}_{\geq 2})^k$ of integers, with $k \geq 1$. Furthermore, we set $d(j) = \left(\sum_{i=1}^j m_i\right) - (j-1)$ for each $1 \leq j \leq k$: $d(k)$ is the Krull dimension of the Segre product $SV\left(\overline{E}, \overline{m}\right) = (\#_{\mathbb{F}})_{i=1}^k V_{E_i,m_i}$ of k Veronese rings in $m_1, \ldots, m_k$ variables, respectively; this is a **Segre–Veronese algebra with degree sequence $\overline{E}$.**

Theorem 3.5. *Suppose $A = SV\left(\overline{E}, \overline{m}\right)$ is a Segre–Veronese algebra over $\mathbb{F}$ with $\overline{E} = (E_1, \ldots, E_k)$. Let $D := \sum_{i=1}^k E_i$. Then $P^{(D(r-1)+1)} \subseteq P^r$ for all $r > 0$ and all monomial primes P in A.*

Proof. Given a lattice $N \cong \mathbb{Z}^d$ we will use $e_1, \ldots, e_d \in N$ to denote a choice of basis for N will dual basis $e_1^*, \ldots, e_d^*$ for M. In the setup of Theorem 1.1, the cardinality of the minimal generating set $\mathcal{B}$ of $C^\vee \cap M$ is the **embedding dimension** of $R_{\mathbb{F}} = \mathbb{F}[C^\vee \cap M]$ [1, Sec. 1.0, Proof of Thm. 1.3.10].

We now provide an explicit cone $C \subseteq N_{\mathbb{R}}$ and an explicit Hilbert basis $\mathcal{B}$ to feed into Theorem 1.1.

Fix k-tuples $\overline{E} \in (\mathbb{Z}_{\geq 1})^k$ and $\overline{m} \in (\mathbb{Z}_{\geq 2})^k$. Set $d(j) = \left(\sum_{i=1}^{j} m_i\right) - (j-1)$ for $1 \leq j \leq k$, while $d(0) = 0$. Given $SV\left(\overline{E}, \overline{m}\right) = (\#_{\mathbb{F}})_{i=1}^{k} V_{E_i, m_i}$, we fix a lattice $N \cong \mathbb{Z}^{d(k)}$ and record a cone $C = C\left(\overline{E}, \overline{m}\right) \subseteq N_{\mathbb{R}} \cong \mathbb{R}^{d(k)}$ as stipulated with $R_{\mathbb{F}} = \mathbb{F}[C^{\vee} \cap M] \cong SV\left(\overline{E}, \overline{m}\right)$. Specifically, consider the cone $C \subseteq N_{\mathbb{R}}$ generated by the following irredundant collection of primitive vectors:

$$\mathcal{A} = \bigcup_{1 \leq j \leq k} A_j, \text{ where } A_1 = \{e_1, \ldots, e_{m_1 - 1}, -e_1 - \cdots - e_{m_1 - 1} + E_1 \cdot e_{m_1}\},$$

and for each $2 \leq j \leq k$,

$$A_j = \left\{ e_h,\ E_j \cdot e_{m_1} - \sum_{h=d(j-1)+1}^{d(j)} e_h \colon d(j-1) + 1 \leq h \leq d(j) \right\}.$$

The semigroup $C^{\vee} \cap M$ is generated by the following set of irreducible vectors:

$$\mathcal{B} = \left\{ e^*_{m_1} + \sum_{j=1}^{k} \sum_{\ell=1}^{m_j - 1} a_{j,\ell} \cdot e^*_{d(j-1)+\ell} \colon 0 \leq \sum_{\ell=1}^{m_j - 1} a_{j,\ell} \leq E_j \text{ for } 1 \leq j \leq k \right\}.$$

Indeed, $\#\mathcal{B} = \prod_{j=1}^{k} \binom{m_j - 1 + E_j}{E_j}$, the embedding dimension of $SV(\overline{E}, \overline{m})$. Finally, one can record a bijection between the monomial generators of $R_{\mathbb{F}}$ and those typically used to present $SV(\overline{E}, \overline{m})$; cf., [11, Proof of Lem. 5.3] for how the bijection would look in the coordinates $a_{j,\ell}$ for each j.

Feeding $v_C = \sum_{u \in \mathcal{A}} u = (\sum_{j=1}^{k} E_j) \cdot e_{m_1}$ and $\mathcal{B}$ into Theorem 1.1 yields $D = \sum_{j=1}^{k} E_j$. $\square$

Over any perfect field $\mathbb{K}$, a Segre–Veronese algebra has uniform symbolic topologies on all primes, per [3, Thm. 2.2], [6, Thm. 1.1], and [7, Cor. 3.10]. However, no explicit multiplier is provided by these cited results; indeed, the case $k = 1$ was only addressed recently [2, Cor. 3.30]. By contrast, Theorem 3.5 gives an explicit multiplier for the torus-invariant primes in a Segre–Veronese algebra over an arbitrary field.

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