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Uniform Harbourne–Huneke bounds via flat extensions

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Over an arbitrary field $\mathbb{F}$, Harbourne [3] conjectured that the symbolic power $I^{(N(r-1)+1)} \subseteq I^r$ for all $r > 0$ and all homogeneous ideals I in $S = \mathbb{F}[\mathbb{P}^N] = \mathbb{F}[x_0, \ldots, x_N]$. The conjecture has been disproven for select values of $N \geq 2$: first by Dumnicki, Szemberg, and Tutaj-Gasińska in characteristic zero [7], and then by Harbourne and Seceleanu in positive characteristic [13]. However, the ideal containments above do hold when, e.g., I is a monomial ideal in S [3, Ex. 8.4.5].
As a sequel to [21], we present criteria for containments of type $I^{(N(r-1)+1)} \subseteq I^r$ for all $r > 0$ and certain classes of ideals I in a prodigious class of normal rings. Of particular interest is a result for monomial primes in tensor products of affine semigroup rings. Indeed, we explain how to give effective multipliers N in several cases including: the D-th Veronese subring of any polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$ ($n \geq 1$); and the extension ring $\mathbb{F}[x_1, \ldots, x_n, z]/(z^D - x_1 \cdots x_n)$ of $\mathbb{F}[x_1, \ldots, x_n]$.

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1. Introduction and conventions for the paper

Over an arbitrary field $\mathbb{F}$, let $S = \mathbb{F}[\mathbb{P}^N] = \mathbb{F}[x_0, x_1, \dots, x_N]$ be the standard $\mathbb{N}$-graded polynomial algebra. The groundbreaking work of Ein–Lazarsfeld–Smith and Hochster–Huneke [8,16] implies that the symbolic power $I^{(Nr)} \subseteq I^r$ for all graded ideals $0 \subsetneq I \subsetneq S$ and all integers $r > 0$. Using graded ideals of *star configurations* in $\mathbb{P}^N$, Bocci and Harbourne [4] showed that in securing these containments one cannot replace N by some integer $0 < C < N$, even asymptotically. In particular, $I^{(4)} \subseteq I^2$ holds for all graded ideals in $\mathbb{F}[\mathbb{P}^2]$, and Huneke asked whether an improvement $I^{(3)} \subseteq I^2$ holds for any radical ideal I defining a finite set of points in $\mathbb{P}^2$. Building on this, Harbourne proposed dropping the symbolic power from Nr down to the **Harbourne–Huneke bound** $Nr - (N-1) = N(r-1) + 1$ when $N \geq 2$ [3, Conj. 8.4.2]: i.e.,

$$I^{(N(r-1)+1)} \subseteq I^r \text{ for any graded ideal } 0 \subsetneq I \subsetneq S, \text{ all } r > 0, \text{ and all } N \geq 2. \quad (1)$$

There are several scenarios where these improved containments hold: for instance, they hold for all monomial ideals in S over any field [3, Ex. 8.4.5]; see the recent ideal containment problem survey by Szemberg and Szpond [20, Thm. 3.8], as well as recent work of Grifo–Huneke [12] in 2017.

However, Dumnicki, Szemberg, and Tutaj-Gasińska showed in characteristic zero [7] that the containment $I^{(3)} \subseteq I^2$ can fail for a radical ideal defining a point configuration in $\mathbb{P}^2$. Harbourne–Seceleanu showed in odd positive characteristic [13] that (1) can fail for pairs $(N, r) \neq (2, 2)$ and ideals I defining a point configuration in $\mathbb{P}^N$. Akesseh [1] cooks up many new counterexamples to (1) from these original constructions via finite, flat morphisms $\varphi^{\#} \colon \mathbb{P}^N \to \mathbb{P}^N$. No prime ideal counterexample is known.

Lately, there has been better sustained success in showing that a containment in (1) fails – and perhaps, more fervor. However, we want to revisit the fact that in arbitrary characteristic (1) holds for all monomial ideals in S. In particular, our investigation of Harbourne–Huneke bounds *improves upon* the fact that $P^{(N(r-1)+1)} \subseteq P^{(r)} = P^r$ for all $r > 0$ and for all monomial prime ideals P in S (i.e., monomial ideals generated by subsets of the variables $x_0, \dots, x_N$). Indeed, $P^{(r)} = P^r$ for all r whenever P is a complete intersection ideal in S, and for all $r > 0$, $N(r-1) + 1 \geq (r-1) + 1 = r$.

The goal of this paper is to show that a variant of (1) holds for several familiar classes of ideals (e.g., combinatorial ideals such as monomial primes) in certain non-regular rings – even though it already fails for a large class of ideals defining point configurations in $\mathbb{P}^N$, and hence can fail for arbitrary graded ideals in $\mathbb{F}[\mathbb{P}^N]$. More precisely, we work in the setting of rational surface singularities and higher-dimensional normal toric rings. First, we demonstrate how one can strengthen Lemmas 1.1 and 2.6 of our IJM paper [21] to a version involving a Harbourne–Huneke bound:

Lemma 1.1. *Let R be a Noetherian normal domain of positive Krull dimension whose global divisor class group* $\mathrm{Cl}(R) := \mathrm{Cl}(\mathrm{Spec}(R))$ *is annihilated by an integer $D > 0$. Then*

$$\mathfrak{q}^{(D(r-1)+s)} = (\mathfrak{q}^{(D)})^{r-1}\mathfrak{q}^{(s)}, \textit{ and } \quad \mathfrak{q}^{(D(r-1)+1)} \subseteq \mathfrak{q}^r$$

for all ideals $\mathfrak{q} \subseteq R$ *of pure height one, all* $r > 0$*, and all* $0 \leq s < D$*.*

When the domain R in this lemma is two-dimensional, $P^{(r)} = P^r$ when the ideal P is zero or maximal, and so we infer that $P^{(D(r-1)+1)} \subseteq P^r$ for all prime ideals P in R and all $r > 0$, and that $P^{(3)} \subseteq P^2$ for all primes when $D = 2$ works. As discussed in [21], the above lemma already applies to any two-dimensional, local rational singularity (Lipman [17]) and the coordinate rings of simplicial toric varieties; see Theorem 3.6 below. The intro to [21] gives Lipman's definition of two-dimensional, normal local rational singularities; Section 3 therein gives remarks on class groups, both for these singularities and for toric varieties. We prove a result for Veronese rings (Theorem 5.4 below) from which one can infer that the ideal containment in the lemma can be tight by example.

However, it is the result to follow that inspires the chosen title for this paper. It allows us to give first examples of the Harbourne–Huneke bound for all monomial primes in certain normal algebras of dimension three or higher, subalgebras of a Laurent polynomial ring that are generated by monomials. These domains are the coordinate rings of normal affine toric varieties, called toric rings, monomial rings, or affine semigroup rings. In this setting, we adduce a result (Proposition 2.1) on ideal containment preservation along faithfully flat ring extensions, as part of deducing the following

Theorem 1.2. *Let* $R_1, \ldots, R_n$ *be normal affine semigroup rings over a field* $\mathbb{F}$*. For each* $1 \leq i \leq n$*, suppose there is an integer* $D_i > 0$ *such that* $P^{(D_i(r-1)+1)} \subseteq P^r$ *for all* $r > 0$ *and all monomial primes* $P \subseteq R_i$*. Set* $D := \max\{D_1, \ldots, D_n\}$*. Then* $Q^{(D(r-1)+1)} \subseteq Q^r$ *for all* $r > 0$ *and any monomial prime* Q *in the normal affine semigroup ring* $R = R_1 \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} R_n$*.*

To clarify, a normal affine semigroup $\mathbb{F}$-algebra A has an $\mathbb{F}$-basis of Laurent monomials and an ideal in A is **monomial** if it is generated by monomials. See Section 3 for more details.

All normal toric rings of dimension at most two have finite cyclic divisor class group, and thus satisfy the hypotheses on the R_i factors in the theorem; aside from these cases, the factors R_i may be taken from the following classes of rings (including those of Krull dimension three or higher):

Theorem 1.3. *Let* $S = \mathbb{F}[x_1, \ldots, x_n]$ $(n \geq 1)$ *be a polynomial ring over an arbitrary field* $\mathbb{F}$ *and consider the module-finite extensions of normal toric rings* $V_D \subseteq S \subseteq H_D$*, where*

1. $V_D \subseteq S$ *is the* D*-th Veronese subring with its standard* $\mathbb{N}$*-grading, and*
2. $H_D = \mathbb{F}[z, x_1, \ldots, x_n]/(z^D - x_1 \cdots x_n)$ *is a hypersurface ring.*

Then $P^{(D(r-1)+1)} \subseteq P^r$ *for all* $r > 0$*, where* P *is a monomial ideal in any of the three rings.*

Conventions: All our rings are Noetherian and commutative with identity. From Section 4 onwards, our rings will be *affine* $\mathbb{F}$-algebras, that is, of finite type over a fixed field $\mathbb{F}$ of arbitrary characteristic. By *algebraic variety*, we will mean an integral scheme of finite type over the field $\mathbb{F}$.

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2. Symbolic powers, faithful flatness, and the proof of Lemma 1.1

2.1. Symbolic powers and faithful flatness

If I is any proper ideal in a nonzero Noetherian ring R, and $\mathrm{Ass}_R(R/I)$ is the set of associated primes of I, we define its **a-th ($a \in \mathbb{Z}_{>0}$) symbolic power** ideal $I^{(a)}$ by the rule:

$$I^{(a)} := I^a W^{-1} R \cap R, \text{ where } W = R - \bigcup \{P \colon P \in \mathrm{Ass}_R(R/I)\}.$$

Equivalently, $I^{(a)} = \{f \in R \colon sf \in I^a \text{ for some } s \in W\}$. While $I^a \subseteq I^{(a)}$ for all a, the converse can fail for $a > 1$: $I^{(1)} = I$ since W is the set of nonzerodivisors modulo I.

Consider a flat map $\phi \colon A \to B$ of Noetherian rings. In what follows, the ideal $JB := \langle \phi(J) \rangle B$ for any ideal J in A, and $J^r B = (JB)^r$ for all $r \geq 0$ since the two ideals share a generating set. For any A-module E, the proof of Theorem 23.2 (ii) in Matsumura [18] shows that

$$\mathrm{Ass}_B(E \otimes_A B) = \bigcup_{P \in \mathrm{Ass}_A(E)} \mathrm{Ass}_B(B/PB). \tag{2}$$

We define a set $\mathcal{I}(A) = \{\text{proper ideals } I \subseteq A \colon \mathrm{Ass}_B(B/IB) = \{PB \colon P \in \mathrm{Ass}_A(A/I)\}\}$. Setting $E = A/I$ in (2), we observe that $I \in \mathcal{I}(A)$ if and only if the extended ideal PB is prime for all $P \in \mathrm{Ass}_A(A/I)$. Our paper [22] records a simple example to illustrate that in an arbitrary faithfully flat ring extension, $\mathcal{I}(A)$ need not contain all prime ideals in A, let alone all proper ideals.

Proposition 2.1. *Suppose $\phi \colon A \to B$ is a faithfully flat map of Noetherian rings. Then for each $I \in \mathcal{I}(A)$ and all integer pairs $(N, r) \in (\mathbb{Z}_{\geq 0})^2$, we have*

$$I^{(N)} B = (IB)^{(N)}, \tag{3}$$

and $I^{(N)} \subseteq I^r$ if and only if $(IB)^{(N)} = I^{(N)} B \subseteq I^r B = (IB)^r$.

Proof. First, $I^{(N)}B \subseteq (IB)^{(N)}$: indeed, if $f \in I^{(N)}$, then $sf \in I^N$ for some $s \in A$ such that

$$s \notin \bigcup_{P\in \mathrm{Ass}_A(A/I)} P \overset{(\star)}{=} \bigcup_{P\in \mathrm{Ass}_A(A/I)} (PB \cap A) = \left(\bigcup_{P\in \mathrm{Ass}_A(A/I)} PB \right) \cap A$$

where $(\star)$ holds by faithful flatness; it follows that $s \notin \bigcup_{P\in \mathrm{Ass}_A(A/I)} PB = \bigcup_{Q\in \mathrm{Ass}_B(B/IB)} Q$, where equality holds since $I \in \mathcal{I}(A)$ by hypothesis. We thus conclude that $f \in (IB)^{(N)}$.

By definition, $(IB)^{(N)}B_W = (IB)^N B_W = I^N B_W$ since all three ideals contract to $(IB)^{(N)}$, where $B_W = W^{-1}B$ denotes the ring obtained via localization of B at the multiplicative system

$$W = B - \left(\bigcup_{Q\in \mathrm{Ass}_B(B/IB)} Q \right) = B - \left(\bigcup_{P\in \mathrm{Ass}_A(A/I)} PB \right).$$

Notice that since $I^{(N)}B \subseteq (IB)^{(N)}$, the right-hand containment holds in

$$I^N B_W \subseteq I^{(N)} B_W = (I^{(N)}B)B_W \subseteq (IB)^{(N)}B_W = I^N B_W.$$

Thus $I^{(N)}B$ and $(IB)^{(N)}$ localize to the same ideal $I^N B_W$; contracting back to B, we conclude that (3) holds for all $N \geq 0$. Finally, (3) gives both implications of the second part of the proposition, adducing faithful flatness once more to contract an ideal containment to A. □

We adapt Proposition 2.1 later on (cf., Proposition 4.4) to prove Theorem 4.1, from which Theorem 1.2 follows as an immediate corollary.

2.2. Preliminaries on divisor class groups

Our main references are Fossum [9], Hartshorne [14, II.6], Hochster [15], and Matsumura [18, Ch. 11]. However, we opt to state mathematical definitions and results from these sources only for Noetherian normal domains, rather than for Krull domains in general as is done in [9].

Throughout, R will denote a Noetherian normal domain. Let $\mathcal{P}$ denote the set of height-one primes in R. As noted in Matsumura's chapter on Krull rings [18, Corollary of Thm. 12.3], when $f \in R$ is a nonzero nonunit, and ν_P is the discrete valuation on the DVR R_P (for $P \in \mathcal{P}$), we have a unique primary decomposition

$$(f)R = \bigcap_{P\in\mathcal{P}} P^{(N_P)}, \text{ where } N_P := \nu_P(f) = 0 \text{ for all but finitely many } P.$$

We define the **Weil divisor of** f to be $\operatorname{div}(f) := \sum_{P \in \mathcal{P}} N_P \cdot P$. Additionally, we define the *trivial* effective Weil divisor $\operatorname{div}(\langle 1 \rangle R) = \operatorname{div}(R) = [R] := 0$ of the unit ideal to have identically zero $\mathbb{Z}$-coefficients.

Definition 2.2. The **divisor class group** of a Noetherian normal domain R,

$$\operatorname{Cl}(R) = \operatorname{Cl}(\operatorname{Spec}(R)),$$

is the free abelian group on the set $\mathcal{P}$ of height one prime ideals of R modulo relations

$$a_1 P_1 + \ldots + a_r P_r = 0,$$

whenever the ideal $P_1^{(a_1)} \cap \ldots \cap P_r^{(a_r)}$ is principal.

In particular, $\operatorname{Cl}(R)$ is trivial if and only if R is a UFD [14, II.6]. Both conditions mean that every height one prime ideal in R is principal. We note that $P^{(a)} = P^a$ for all $a > 0$ and all height one primes P in a Noetherian UFD.

We now record three theorems without formal proof, consolidating some results from Ch. II, Sections 7, 8, and 10 of Fossum [9]. The first result consolidates some immediate consequences of a fact called Nagata's theorem [9, Thm. 7.1], on the behavior of class groups under localization.

Theorem 2.3 *(cf., Fossum [9, Cor. 7.2, Cor. 7.3]). Let S be a multiplicatively closed subset of a Noetherian normal domain A. Then:*

1. *The natural map $\operatorname{Cl}(A) \to \operatorname{Cl}(S^{-1}A)$ is a surjection of abelian groups. The kernel is generated by the classes of the height one prime ideals which meet S.*
2. *If S is generated by prime elements of A, then $\operatorname{Cl}(A) \to \operatorname{Cl}(S^{-1}A)$ is an isomorphism of abelian groups.*

The next two results will be especially useful in Section 3. They allow us to reduce class group computations to particularly nice cases where we end up enjoying a more incisive handle on computing class groups up to isomorphism.

Theorem 2.4 *(cf., Fossum [9, Thm. 8.1, Cor. 8.2]). Working with polynomial ring extensions of a Noetherian normal domain A, we have isomorphisms for any $n \in \mathbb{Z}_{>0}$:*

$$\operatorname{Cl}(A) \cong \operatorname{Cl}(A[X_1, \ldots, X_n]) \cong \operatorname{Cl}(A[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]).$$

Proof sketch. One can induce on n with base case $n = 1$. Assuming $n = 1$, the left-hand isomorphism is the content of Fossum [9, Thm. 8.1]. For the right-hand isomorphism, apply Theorem 2.3(2) to the polynomial ring $B = A[X]$ and the multiplicatively closed set $S \subseteq B$ generated by the prime element $X \in B$, so $S^{-1}B = A[X^{\pm 1}]$ is a Laurent polynomial ring in one variable over A. $\square$

Theorem 2.5 *(cf., Fossum [9, Cor. 10.3, Cor. 10.7]). Suppose that* $A = \oplus_{i=0}^{\infty} A_i$ *is an* $\mathbb{N}$*-graded Noetherian normal domain where* $A_0 = \mathbb{F}$ *is a field, with homogeneous maximal ideal* $\mathfrak{m} = \oplus_{i=1}^{\infty} A_i$*. Suppose that* $\mathbb{F}'$ *is any field extension of* $A_0 = \mathbb{F}$*, and that* $A' := A \otimes_{\mathbb{F}} \mathbb{F}'$ *is a Noetherian normal domain. Then* A' *is faithfully flat over* A *and the induced homomorphism* $\mathrm{Cl}(A) \to \mathrm{Cl}(A')$ *is injective.*

2.3. The proof of Lemma 1.1

We start by recalling the following proposition deduced in [21]. To clarify, an ideal has **pure height** h is every associated prime has height h. In particular, none are embedded.

Proposition 2.6 *(cf., [21, Prop. 2.5]). Let* R *be a Noetherian normal domain of positive Krull dimension, and* $\mathfrak{q}$ *any ideal of pure height one with associated primes* $P_1, \dots, P_c$*. Then:*

(a) There exist positive integers $b_1, \dots, b_c$*, uniquely determined by* $\mathfrak{q}$*, such that the symbolic power* $\mathfrak{q}^{(E)} = P_1^{(Eb_1)} \cap \dots \cap P_c^{(Eb_c)}$ *for all* $E \geq 0$*.*
(b) If either (1) $D \cdot \mathrm{Cl}(R) = 0$*, or* (2) *the class* $[\mathfrak{q}] \in \mathrm{Cl}(R)$ *has finite order* D*, then for all integers* $r \geq 0$*,* $\mathfrak{q}^{(Dr)} = (\mathfrak{q}^{(D)})^r$ *is principal and* $\mathfrak{q}^{(Dr)} \subseteq \mathfrak{q}^r$*.*

Per part (a) of this proposition, we may define Weil divisors

$$\mathrm{div}[\mathfrak{q}] := b_1 \cdot P_1 + \dots + b_c \cdot P_c, \quad \mathrm{div}[\mathfrak{q}^{(E)}] := E \cdot \mathrm{div}[\mathfrak{q}] = Eb_1 \cdot P_1 + \dots + Eb_c \cdot P_c,$$

where $E > 0$. In particular, $\mathrm{div}[\mathfrak{q}^{(A+B)}] = \mathrm{div}[\mathfrak{q}^{(A)}] + \mathrm{div}[\mathfrak{q}^{(B)}]$ for all nonnegative integers A and B.

Proof of Lemma 1.1. Our proof of the first claim replaces $r - 1$ with $r \geq 0$. Per Proposition 2.6(b), suppose $\mathfrak{q}^{(Dr)} = (\mathfrak{q}^{(D)})^r = (f^r)$ is principal for all $r \geq 0$ and some nonzero $f \in R$. Now set $I = \mathfrak{q}^{(s)}$. Following the first proof in Hochster's notes [15], we have a short exact sequence

$$0 \to \frac{(f^r)R}{(f^r)I} \to \frac{R}{(f^r)I} \to \frac{R}{(f^r)R} \to 0$$

and $\frac{(f^r)R}{(f^r)I} \cong R/I$ as R-modules via the R-linear map $\phi\colon R \twoheadrightarrow \frac{(f^r)R}{(f^r)I}$ with $\phi(g) = \overline{gf^r}$. Thus per our exact sequence (cf., Matsumura [18, Thm. 6.3]),

$$\varnothing \neq \mathrm{Ass}_R(R/(f^r)I) \subseteq \mathrm{Ass}_R(R/I) \cup \mathrm{Ass}_R(R/(f^r)R)$$

and so $\mathrm{Ass}_R(R/(f^r)I)$ contains only height one primes since the latter two sets do. Finally, comparing Weil divisors of pure height one ideals

$$\operatorname{div}[(f^r)I = (\mathfrak{q}^{(D)})^r \mathfrak{q}^{(s)}] \stackrel{(*)}{=} \operatorname{div}[(f^r)R] + \operatorname{div}[I]$$
$$= \operatorname{div}[\mathfrak{q}^{(Dr)}] + \operatorname{div}[\mathfrak{q}^{(s)}] = \operatorname{div}[\mathfrak{q}^{(Dr+s)}].$$

As Hochster notes, one can check identity (*) after first localizing at each height one prime Q; in this case, the identity is obvious in a DVR. Per (*), the two pure height one ideals $\mathfrak{q}^{(Dr+s)}, (\mathfrak{q}^{(D)})^r \mathfrak{q}^{(s)}$ have the exact same primary decomposition and hence are equal. To conclude: since $\mathfrak{q}^{(D)} \subseteq \mathfrak{q}^{(1)} = \mathfrak{q}$, setting $s = 1$ yields $\mathfrak{q}^{(D(r-1)+1)} = (\mathfrak{q}^{(D)})^{r-1} \mathfrak{q}^{(1)} \subseteq \mathfrak{q}^{r-1+1} = \mathfrak{q}^r$. $\square$

We close by remarking that after adapting the statement of [21, Lem. 2.6] to feature the Harbourne–Huneke bounds, the exact same proof we gave in [21] will suffice. Namely, we reduce to the local case, and then invoke Lemma 1.1 from the present paper.

3. Toric algebra preliminaries

We review notation and relevant facts from toric algebra, citing Cox–Little–Schenck [5, Ch. 1, 3, 4] and Fulton [10, Ch. 1, 3]. A lattice is a free abelian group of finite rank. We fix a perfect bilinear pairing $\langle \cdot, \cdot \rangle \colon M \times N \to \mathbb{Z}$ between two lattices M and N; this identifies M with $\operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and N with $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. Our pairing extends to a perfect pairing of finite-dimensional vector spaces $\langle \cdot, \cdot \rangle \colon M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$, where $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$.

Fix an N**-rational** polyhedral cone and its M-rational dual: respectively, for some **finite** subset $G \subseteq N - \{0\}$ these are closed, convex sets of the form

$$C = \operatorname{Cone}(G) := \left\{ \sum_{v \in G} a_v \cdot v \colon \text{ each } a_v \in \mathbb{R}_{\geq 0} \right\} \subseteq N_{\mathbb{R}}, \text{ and}$$
$$C^\vee := \{ w \in M_{\mathbb{R}} \colon \langle w, v \rangle \geq 0 \text{ for all } v \in C \} = \{ w \in M_{\mathbb{R}} \colon \langle w, v \rangle \geq 0 \text{ for all } v \in G \}.$$

By definition, the **dimension** of a cone in $M_{\mathbb{R}}$ or $N_{\mathbb{R}}$ is the dimension of the real vector subspace it spans; a cone is **full(-dimensional)** if it spans the full ambient space. A cone in $M_{\mathbb{R}}$ or $N_{\mathbb{R}}$ is **pointed (or strongly convex)** if it contains no line through the origin. A **face** of C is a convex polyhedral cone F in $N_{\mathbb{R}}$ obtained by intersecting C with a hyperplane which is the kernel of a linear functional $m \in C^\vee$; F is **proper** if $F \neq C$. When C is both N-rational and pointed, so is every face F. Each such face $F \neq \{0\}$ has a uniquely-determined set G_F of primitive generators. By definition, $v \in N$ is **primitive** if $\frac{1}{k} \cdot v \notin N$ for all $k \in \mathbb{Z}_{>1}$.

There is a bijective inclusion-reversing correspondence between faces F of C and faces F^* of $C^\vee$, where $F^* = \{ w \in C^\vee \colon \langle w, v \rangle = 0 \text{ for all } v \in F \}$ is the face of $C^\vee$ **dual to** F [10, Sec. 1.2]. Under this correspondence, either cone is pointed if and only if the other is full, and

$$\dim(F) + \dim(F^*) = \dim(N_{\mathbb{R}}) = \dim(M_{\mathbb{R}}). \tag{4}$$

Fix an arbitrary ground field $\mathbb{F}$ and a cone C as above in $N_{\mathbb{R}}$. The semigroup ring $R_{\mathbb{F}} = \mathbb{F}[C^\vee \cap M]$ is the **toric $\mathbb{F}$-algebra associated to** C. This ring $R_{\mathbb{F}}$ is a normal domain of finite type over $\mathbb{F}$ [5, Thm. 1.3.5]. Note that $R_{\mathbb{F}}$ has an $\mathbb{F}$-basis $\{\chi^m : m \in C^\vee \cap M\}$ of monomials, giving $R_{\mathbb{F}}$ an M-grading, where $\deg(\chi^m) := m$. A **monomial ideal (also called an M-homogeneous or torus-invariant ideal)** in $R_{\mathbb{F}}$ is an ideal generated by a subset of these monomials. When $C^\vee$ is pointed, $R_{\mathbb{F}}$ also has a non-canonical $\mathbb{N}$-grading obtained by fixing any group homomorphism $M \to \mathbb{Z}$ taking positive values $C^\vee \cap M - \{0\}$. The set $\{\chi^m : m \in C^\vee \cap M - \{0\}\}$ generates the unique homogeneous maximal ideal $\mathfrak{m}$ under this $\mathbb{N}$-grading.

Remark 1. In forming the toric algebra $\mathbb{F}[C^\vee \cap M]$, there is no loss of generality in assuming C is pointed in $N_{\mathbb{R}}$. Indeed, because $C^\vee \cap M = C^\vee \cap M'$ where $M' = M \cap \{\mathbb{R}\text{-span of } C^\vee \text{ in } M_{\mathbb{R}}\}$, we may replace M by M' to assume $C^\vee$ is full in $(M')_{\mathbb{R}}$. Now, replacing N and C by the duals of M' and $C^\vee$, we may assume that C is pointed in $N' = \mathrm{Hom}_{\mathbb{Z}}(M', \mathbb{Z})$. See [5, Thm. 1.3.5] for details.

Fix a face F of a pointed rational cone C: [10, p. 53] records a surjective M-graded ring map

$$\phi_F : R_{\mathbb{F}} = \mathbb{F}[C^\vee \cap M] \twoheadrightarrow \mathbb{F}[F^* \cap M], \quad \phi_F(\chi^m) = \begin{cases} \chi^m & \text{if } \langle m, v\rangle = 0 \text{ for all } v \in F \\ 0 & \text{if } \langle m, v\rangle > 0 \text{ for some } v \in F. \end{cases}$$

Both rings are domains. The **monomial prime ideal of** F, $P_F := \ker(\phi_F)$, has height equal to $\dim(F)$. Conversely, any monomial prime of $R_{\mathbb{F}}$ corresponds bijectively to a face of C.

Lemma 3.1. *Fix a face F of a pointed rational cone C, and the monomial prime $P_F \subseteq R_{\mathbb{F}}$ above. Let G_F be the set of primitive generators of F, and set $v_F := \sum_{v \in G_F} v \in F \cap N$. Then*

$$P_F = (\{\chi^m : m \in C^\vee \cap M \text{ and the integer } \langle m, v_F\rangle > 0\})R_{\mathbb{F}}. \tag{5}$$

Proof. First, in defining $\phi_F(\chi^m)$ above, notice we can work with $v \in G_F$ without loss of generality. Now, fix $m \in C^\vee \cap M$. Then $\langle m, v\rangle \in \mathbb{Z}_{\geq 0}$ for all $v \in C \cap N$. As $\langle \cdot, \cdot \rangle$ is bilinear, (5) follows since a sum of nonnegative integers is positive if and only if one of the summands is positive. □

Proposition 3.2 *(Minkowski sum-ideal sum). Suppose $C \subseteq N_{\mathbb{R}}$ is a pointed rational polyhedral cone, and $R_{\mathbb{F}} = \mathbb{F}[C^\vee \cap M]$ is the corresponding toric $\mathbb{F}$-algebra. When a face $F = Cone(G_F) = \rho_1 + \ldots + \rho_\ell$ as a Minkowski sum of rays,*

$$P_F = \sum_{j=1}^{\ell} P_{\rho_j} \tag{6}$$

as a sum of ideals.

Proof. Let $G_F = \{u_{\rho_j}\colon 1 \leq j \leq \ell\}$ consist of the primitive ray generators. Any $v \in F$ satisfies

$$v = \sum_{j=1}^{\ell} a_j u_{\rho_j}, \text{ for some } a_1, \dots, a_\ell \in \mathbb{R}_{\geq 0}.$$

Given any $w \in C^\vee$, $\langle w, v\rangle \geq 0$ for all $v \in C$. Thus for $v \in F$ as above,

$$0 \leq \langle w, v\rangle = \sum_{j=1}^{\ell} a_j \langle w, u_{\rho_j}\rangle, \text{ for some } a_1, \dots, a_\ell \in \mathbb{R}_{\geq 0},$$

and so $\langle w, v\rangle$ is positive if and only if $\langle w, u_{\rho_j}\rangle > 0$ for some $1 \leq j \leq \ell$. We infer from this that the monomial ideals P_F and $\sum_{j=1}^{\ell} P_{\rho_j}$ have a generating set in common, and hence are equal. $\square$

Definition 3.3. With notation as in Proposition 3.2, we call (6) a **Minkowski sum-ideal sum decomposition** for P_F.

Remark 2. Adapting the proof of Proposition 3.2 accordingly, we could use any decomposition of F as a Minkowski sum of faces, the latter need not be rays.

Our next goalpost is Lemma 3.4 on decomposing monomial primes in tensor products of normal toric rings. Fix two pointed rational polyhedral cones $C_i = \mathrm{Cone}(S_i) \subset (N_i)_{\mathbb{R}}$ $(i = 1, 2)$, where each S_i consists of the primitive ray generators. Define lattices $N = N_1 \times N_2, M = M_1 \times M_2$ per the standing conventions. Let $\langle , \rangle_i \colon M_i \times N_i \to \mathbb{Z}$ and $\langle , \rangle \colon M \times N \to \mathbb{Z}$ indicate our three designated bilinear pairings.

Remark 3. While tedious, we could pedantically write down compatibility conditions to the effect that the output values of these pairings will agree relative to the obvious $\mathbb{Z}$-linear embeddings $N_i \hookrightarrow N$ and $M_i \hookrightarrow M$, e.g., $N_1 \cong N_1 \times \{0\}$. In particular, in a slight abuse of notation, going forward we identify

$$\langle , \rangle = \langle , \rangle_1 + \langle , \rangle_2.$$

This generalizes the usual dot product setup naturally, $\mathbb{Z}^E \subseteq \mathbb{R}^E$, where $E = m + n$ as a sum of positive integers.

The **product cone** $C = C_1 \times C_2$ in $N_{\mathbb{R}}$ is a pointed rational polyhedral cone. In terms of ray generators, C is generated as

$$C = (C_1 \times \{0\}) + (\{0\} \times C_2) = \mathrm{Cone}[(S_1 \times \{0\}) \cup (\{0\} \times S_2)] \subseteq N_{\mathbb{R}}.$$

Note that

$$C^\vee = (C_1 \times \{0\})^\vee \cap (\{0\} \times C_2)^\vee = C_1^\vee \times C_2^\vee.$$

For the right-hand equality, we defer to Remark 3.

Lemma 3.4. *For $n \geq 2$, let $R_1, \dots, R_n$ be normal toric rings over a field $\mathbb{F}$, built from pointed rational polyhedral cones $C_i \subseteq (N_i)_\mathbb{R}$, respectively. Consider the normal toric ring $R \cong R_1 \otimes_\mathbb{F} \cdots \otimes_\mathbb{F} R_n$. Every monomial prime ideal Q in R can be expressed as a sum $Q = \sum_{i=1}^n (P_i R)$ of expanded ideals, where each ideal $P_i \subseteq R_i$ is a monomial prime.*

Proof. Induce on n with base case $n = 2$; we focus on the base case for the remainder of the proof. Suppose $R_i = (R_i)_\mathbb{F} = \mathbb{F}[C_i^\vee \cap M_i]$, and

$$R = R_\mathbb{F} = \mathbb{F}[C^\vee \cap M] \cong R_1 \otimes_\mathbb{F} R_2.$$

Any monomial prime in R corresponds bijectively with a face of C. All faces of C are of the form $F = F_1 \times F_2$ where F_i is a face of C_i. Given F as stated, with $Q_F \subseteq R$ the corresponding monomial prime, the base case follows from proving that

(1) $Q_{F_1 \times F_2} = Q_{F_1 \times \{0\}} + Q_{\{0\} \times F_2}$; and
(2) As expansions of monomial ideals, $Q_{F_1 \times \{0\}} = P_{F_1} R$, $Q_{\{0\} \times F_2} = P_{F_2} R$.

The **Minkowski sum-ideal sum decomposition** (6) suffices to verify both claims. First, to see **(1)**, notice $F_1 \times F_2 = (F_1 \times \{0\}) + (\{0\} \times F_2)$ as a Minkowski sum of faces. As for **(2)**, (6) allows us to reduce verification to the case where the F_i are rays. We do so explicitly for $Q_{\rho \times \{0\}}$ where ρ is a ray of C_1. We will use notations χ^a, ϕ^b, ψ^c for characters in R, R_1, R_2 respectively. We express an arbitrary

$$w = (w_1, w_2) \in C^\vee \cap M = (C_1^\vee \cap M_1) \times (C_2^\vee \cap M_2), \tag{7}$$

where $w_i \in C_i^\vee \cap M_i$. For w as in (7), the three characters $\chi^w, \chi^{(w_1,0)} = \phi^{w_1}, \chi^{(0,w_2)} = \psi^{w_2}$ all lie in R. Indeed, given any $v = (v_1, v_2) \in C$ with $v_i \in C_i$, and w as in (7), all dot product terms below are *nonnegative*: deferring to Remark 3,

$$\langle w, v\rangle = \langle w_1, v_1\rangle + \langle w_2, v_2\rangle$$
$$\langle (w_1, 0), v\rangle = \langle w_1, v_1\rangle \geq 0, \quad \langle (0, w_2), v\rangle = \langle w_2, v_2\rangle \geq 0.$$

In particular, since $v \in C$ was arbitrary both $(w_1, 0)$ and $(0, w_2)$ lie in $C^\vee \cap M$.

Now suppose $\chi^w = \chi^{(w_1,0)}\chi^{(0,w_2)} = \phi^{w_1}\psi^{w_2} \in Q_{\rho\times\{0\}}$, i.e., $\langle w, v\rangle > 0$ for some vector $v = (v_1, v_2) \in \rho \times \{0\}$. Since $v_2 = 0$ here, equivalently $\langle w, v\rangle = \langle w_1, v_1\rangle > 0$ for some $v_1 \in \rho$, i.e., the character $\chi^{(w_1,0)} = \phi^{w_1} \in P_\rho R$. Since $\chi^{(0,w_2)} = \psi^{w_2} \in R$, $\chi^w = \phi^{w_1}\psi^{w_2} \in P_\rho R$. Thus $Q_{\rho\times\{0\}} \subseteq P_\rho R$. For the other inclusion: the characters

$\chi^{(w_1,0)} = \phi^{w_1}$ as above generate $P_\rho R$, and each such generator lies in $Q_{\rho\times\{0\}}$ since we already indicated above that $\chi^w \in Q_{\rho\times\{0\}}$ if and only if $\chi^{(w_1,0)} = \phi^{w_1} \in P_\rho R$. $\square$

3.1. Hilbert bases, non-full cones, and toric divisor theory

First, suppose the pointed cone C from Remark 1 is full. Then there is a uniquely-determined minimal generating set $\mathcal{B}$ for $C^\vee \cap M$, in the sense that any other generating set contains $\mathcal{B}$. The set $\mathcal{B}$ is called the **Hilbert basis** of the semigroup, and consists of the **irreducible** vectors $m \in C^\vee \cap M - \{0\}$; a vector $v \in C^\vee \cap M$ is irreducible if it cannot be expressed as a sum of two vectors $m \in C^\vee \cap M - \{0\}$. See [5, Prop. 1.2.17] and [5, Prop. 1.2.23] for details.

In case the pointed cone C is not full, the next proposition is handy.

Proposition 3.5. *Let $N'_{\mathbb{R}}$ be the $\mathbb{R}$-span of a pointed cone $C \subseteq N_{\mathbb{R}}$. Set $N' = N'_{\mathbb{R}} \cap N$, and consider C as a full-dimensional cone in $N'_{\mathbb{R}}$ (relabeled as C'). Let $M' = \operatorname{Hom}_{\mathbb{Z}}(N', \mathbb{Z})$ be the dual lattice. Then working over an arbitrary ground field $\mathbb{F}$, the toric ring $R_{\mathbb{F}} := \mathbb{F}[C^\vee \cap M]$ is isomorphic to $R'_{\mathbb{F}} \otimes_{\mathbb{F}} L$ where the toric ring $R'_{\mathbb{F}} := \mathbb{F}[(C')^\vee \cap M']$ and L is a Laurent polynomial ring over $\mathbb{F}$. In particular, there is a bijective correspondence between the monomial primes of $R'_{\mathbb{F}}$ and $R_{\mathbb{F}}$ given by expansion and contraction of ideals along the faithfully flat ring map $\varphi\colon R'_{\mathbb{F}} \hookrightarrow R'_{\mathbb{F}} \otimes L = R_{\mathbb{F}}$. Moreover, the divisor class groups of $R_{\mathbb{F}}$ and $R'_{\mathbb{F}}$ are isomorphic.*

Proof. While Cox–Little–Schenck [5, Proof of Prop. 3.3.9] yields the first assertion, Lemma 3.4 yields the second since a Laurent polynomial ring has no nonzero monomial primes. As for the class group assertion, $R_{\mathbb{F}}$ is a Laurent polynomial ring over $R'_{\mathbb{F}}$ after base change, so apply Theorem 2.4. $\square$

We now recall how to compute divisor class groups up to isomorphism when working over algebraically closed fields. Working over an algebraically closed field $\mathbb{F}$, fix a pointed cone C as in Remark 1 and the pair of rings $R_{\mathbb{F}}$ and $R'_{\mathbb{F}}$ as in Proposition 3.5. When $C \neq \{0\}$, each $\rho \in \Sigma(1)$, the collection of **rational rays (one-dimensional faces)** of C, yields a unique primitive generator $u_\rho \in \rho \cap N$ for C and a torus-invariant height one prime ideal P_ρ in $R'_{\mathbb{F}}$; cf., [5, Thm. 3.2.6]. The torus-invariant height one primes generate a free abelian group $\bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}P_\rho$ which maps surjectively onto the divisor class group of $R'_{\mathbb{F}}$. More precisely, we record the following well-known theorem; see [5, Ch. 4].

Theorem 3.6. *With notation as in Proposition 3.5, let $C \subseteq N_{\mathbb{R}}$ be a pointed cone with primitive generators $\Sigma(1)$ as described above. Then there is a short exact sequence of abelian groups*

$$0 \to M' \xrightarrow{\phi} \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}P_\rho \to \operatorname{Cl}(R'_{\mathbb{F}}) \to 0, \tag{8}$$

where $\phi(m) = \operatorname{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle P_\rho$*. Furthermore,* $\mathrm{Cl}(R_{\mathbb{F}})$ *and* $\mathrm{Cl}(R'_{\mathbb{F}})$ *are isomorphic,* $\mathrm{Cl}(R_{\mathbb{F}})$ *is finite abelian if and only if* C *is simplicial, and trivial if and only if* C *is smooth.*

Remark 4. This above result follows from [5, Prop. 3.3.9, Prop. 4.1.1–4.1.2, Thm. 4.1.3, Exer. 4.1.1–4.1.2, Prop. 4.2.2, Prop. 4.2.6, and Prop. 4.2.7], essentially consolidating what facts we need to bear in mind going forward in the manuscript.

Definition 3.7. The cone $C \subseteq N_{\mathbb{R}}$ is **simplicial** (**respectively, smooth**) if $C = \{0\}$ or the primitive ray generators form part of an $\mathbb{R}$-basis for $N_{\mathbb{R}}$ (resp., a $\mathbb{Z}$-basis for N). We also apply the adjectives simplicial and smooth to the corresponding toric algebra $R_{\mathbb{F}}$ and the toric $\mathbb{F}$-variety $\mathrm{Spec}(R_{\mathbb{F}})$.

Remark 5. In algebro-geometric language, if C as in Theorem 3.6 is simplicial, then all Weil divisors on $\mathrm{Spec}(R_{\mathbb{F}})$ are $\mathbb{Q}$-Cartier of index at most the order of $\mathrm{Cl}(R_{\mathbb{F}})$.

The next lemma says we can reduce all toric divisor class group computations to the case where $\mathbb{F}$ is algebraically closed, to leverage Theorem 3.6.

Lemma 3.8. *With notation as in Proposition 3.5, the divisor class groups* $\mathrm{Cl}(R_{\mathbb{F}}) \cong \mathrm{Cl}(R_{\overline{\mathbb{F}}})$ *are isomorphic.*

Proof. By now it is clear we can reduce to the case where C is a full pointed cone in $N_{\mathbb{R}}$. The algebra $R_{\mathbb{F}}$ admits an $\mathbb{N}$-grading with its zeroth graded piece being $\mathbb{F}$; see the passage above Remark 1. We may then cite Theorem 2.5 to conclude that up to isomorphism, $\mathrm{Cl}(R_{\mathbb{F}}) \subseteq \mathrm{Cl}(R_{\overline{\mathbb{F}}})$ as a subgroup. This improves to an equality for normal toric rings because the divisor classes of height one monomial primes belong to both groups and generate the latter by Theorem 3.6. □

Remark 6. Citing Lemma 3.8, we observe in passing that the two toric algebra results deduced in the IJM paper [21, Thm. 1.3, Cor. 3.2] extend to the case where we are working over arbitrary fields which need not be algebraically closed.

4. The proof of Theorem 1.2: new examples from old

Theorem 1.2 is an immediate corollary, indeed a uniform bound analogue, of the following

Theorem 4.1. *For* $n \geq 2$*, let* $R_1, \ldots, R_n$ *be normal toric rings over a field* $\mathbb{F}$*,* $P_i \subseteq R_i$ *monomial primes with* $1 \leq i \leq n$*. For each* $1 \leq i \leq n$*, suppose there is an integer* $D_i > 0$ *such that* $P_i^{(D_i(r-1)+1)} \subseteq P_i^r$ *for all* $r > 0$*. Set* $D = \max\{D_1, \ldots, D_n\}$*. Then* $Q^{(D(r-1)+1)} \subseteq Q^r$ *for all* $r > 0$*, where the monomial prime* $Q = \sum_{i=1}^n (P_i R) \subseteq R \cong R_1 \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} R_n$*.*

Prior to giving the proof, we will state two preliminary lemmas, proving the latter lemma.

Lemma 4.2 *([6, Ch. 3]). For any prime ideal P in a Noetherian ring S, and $N \in \mathbb{Z}_{\geq 0}$,*

$$P^{(N)} = P^N :_S (s)^\infty = \bigcup_{j \geq 0} (P^N :_S (s^j)) = P^N :_S (s^T)$$

for all $T \gg 0$ and any $s \notin P$ belonging to all embedded primes of P^N.

Lemma 4.3. *Given any proper ideal I in a Noetherian ring S, and $E \in \mathbb{Z}_{\geq 0}$,*

$$(1)\ I^{(N)} \subseteq I^{\lceil N/E \rceil} \text{ for all } N \geq 0 \iff (2)\ I^{(E(r-1)+1)} \subseteq I^r \text{ for all } r > 0.$$

Proof. The case $N = 0$ is trivial (the unit ideal is contained in itself), so we show equivalence when $N > 0$. Given $r > 0$, setting $N = E(r-1)+1$ in (1) gives (2). That (2) implies (1) follows from noticing that for any two positive integers N, r, we have $r = \lceil N/E \rceil$ if and only if $N = E(r-1)+j$ for some $1 \leq j \leq E$, and $I^{(m)} \subseteq I^{(n)}$ when $m \geq n$. □

Finally, we adapt Proposition 2.1 to a specialized form suited to the proof. The backdrop will be as follows. Fix a field $\mathbb{F}$. For $n \geq 2$, fix nonzero $\mathbb{F}$-algebras $R_1, \ldots, R_n$. Since $R_2 \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} R_n \neq 0$ is free and hence faithfully flat over $\mathbb{F}$, the tensor product $R = R_1 \otimes_{\mathbb{F}} R_2 \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} R_n$ is faithfully flat over R_1; indeed, R is faithfully flat over each R_i by permuting the tensor factor under consideration (cf., Exercise 9.11 in Altman–Kleiman [2]). Thus we can view the factors R_i as subrings of R.

Proposition 4.4. *Given the rings R_i and R as above, suppose that R and each factor R_i is Noetherian. Then for each $1 \leq i \leq n$, we have $I^{(N)}R = (IR)^{(N)}$ for all integers $N \geq 0$ where*

$$I \in \mathcal{I}(R_i) = \{\textit{proper ideals } I \subseteq R_i \colon \operatorname{Ass}_R(R/IR) = \{PR \colon P \in \operatorname{Ass}_{R_i}(R_i/I)\}\}.$$

Moreover, given $(N, r) \in (\mathbb{Z}_{\geq 0})^2$, $I^{(N)} \subseteq I^r$ if and only if $(IR)^{(N)} \subseteq (IR)^r$.

Proof of Theorem 4.1. By Propositions 3.5 and 4.4, we may assume that all the toric rings R_i and R are built from full-dimensional pointed rational polyhedral cones. For each $1 \leq i \leq n$, let $x_{i,1}, \ldots, x_{i,t_i}$ be the monomial algebra generators of R_i over $\mathbb{F}$ corresponding to the Hilbert basis. By the isomorphism $R \cong R_1 \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} R_n$, R is the $\mathbb{F}$-algebra generated by $\{x_{i,1}, \ldots, x_{i,t_i} \colon 1 \leq i \leq n\}$. We define $S(N) := \{(A_1, \ldots, A_n) \in (\mathbb{Z}_{\geq 0})^n \colon \sum_{i=1}^n A_i = N\}$ for each $N \geq 0$, so

$$Q^N = \left(\sum_{i=1}^n (P_i R)\right)^N = \sum_{(A_1,\ldots,A_n) \in S(N)} \prod_{i=1}^n (P_i R)^{A_i}.$$

We will in fact show that the monomial ideal

$$Q^{(N)} \subseteq \sum_{(A_1,\ldots,A_n)\in S(N)} \prod_{i=1}^{n} (P_iR)^{(A_i)}. \tag{9}$$

We may assume without loss of generality that all of the primes P_i are nonzero.

Take an arbitrary monomial $g = \prod_{i=1}^n m_i \in R$ where m_i is a monomial in the $x_{i,\ell}$ for $1 \le i \le n$ and $1 \le \ell \le t_i$. Re-indexing if necessary, we may assume that $P_iR = (x_{i,1},\ldots,x_{i,s_i})R$ for $1 \le i \le n$ where $1 \le s_i \le t_i$, and this is a minimal generating set in the sense of Nakayama's Lemma since R is $\mathbb{N}$-graded by the discussion preceding Remark 1. Define a "complement" monomial $\mathcal{M} = \prod_{i=1}^n \prod_{\ell=s_i+1}^{t_i} x_{i,\ell}$ consisting of all algebra generators **not** among the generators of the P_iR. Any embedded prime of a power of Q is graded (read, monomial), so that $Q^{(N)} = Q^N :_R (\mathcal{M})^\infty$ as a saturation per Lemma 4.2. If $g \in Q^{(N)}$, then for all $T \gg 0$, the monomial

$$g\mathcal{M}^T \in Q^N = \sum_{(A_1,\ldots,A_n)\in S(N)} \prod_{i=1}^{n} (P_iR)^{A_i},$$

whence for some $(A_1,\ldots,A_n) \in S(N)$ and each $1 \le j \le n$ we have

$$g\mathcal{M}^T = \prod_{i=1}^n m_i \left(\prod_{\ell=s_i+1}^{t_i} x_{i,\ell}\right)^T = m_j \prod_{i=1}^n m_i^{1-\delta_{ij}} \left(\prod_{\ell=s_i+1}^{t_i} x_{i,\ell}\right)^T \in \prod_{j=1}^n (P_jR)^{A_j},$$

where δ_{ij} is the Kronecker delta. We can express $(P_jR)^{(A_j)} = P_jR^{A_j} :_R (\mathcal{N})^\infty$ where $\mathcal{N}$ is any "complement" monomial built from powers of all algebra generators **not** among the generators of P_jR. Therefore, setting $\mathcal{N} = \mathcal{N}(j) = \prod_{i=1}^n m_i^{1-\delta_{ij}} \left(\prod_{\ell=s_i+1}^{t_i} x_{i,\ell}\right)^T$, we see $m_j \in (P_jR)^{(A_j)}$ for each $1 \le j \le n$. Thus $g = \prod_{j=1}^n m_j \in \prod_{j=1}^n (P_jR)^{(A_j)}$. Since $g \in Q^{(N)}$ was arbitrary, (9) is immediate.

Finally, we show that **(!)** $Q^{(N)} \subseteq Q^{\lceil N/D\rceil}$ for all $N \ge 0$ where the integer $D = \max\{D_1,\ldots,D_n\}$. Using (9): where $\mathcal{A} = (A_1,\ldots,A_n) \in (\mathbb{Z}_{\ge 0})^n$,

$$Q^{(N)} \overset{(9)}{\subseteq} \sum_{\mathcal{A}\in S(N)} \prod_{i=1}^n (P_iR)^{(A_i)} \overset{\textbf{(A)}}{\subseteq} \sum_{\mathcal{A}\in S(N)} \prod_{i=1}^n (P_iR)^{\lceil A_i/D_i\rceil} \overset{\textbf{(B)}}{\subseteq} Q^{\lceil N/D\rceil}.$$

To see **(A)**, by hypothesis, for all $1 \le i \le n$, $P_i^{(D_i(r-1)+1)} \subseteq P_i^r$ for all $r > 0$, so $(P_iR)^{(D_i(r-1)+1)} \subseteq (P_iR)^r$ for all $r > 0$ by Proposition 4.4 since $P_i \in \mathcal{I}(R_i)$ by the proof of Lemma 3.4. Thus by Lemma 4.3, $(P_iR)^{(A_i)} \subseteq (P_iR)^{\lceil A_i/D_i\rceil}$ for all $A_i \ge 0$ and all $1 \le i \le n$. As for **(B)**: for each $(A_1,\ldots,A_n) \in S(N)$, we have $\prod_{i=1}^n (P_iR)^{\lceil A_i/D_i\rceil} \subseteq Q^{\lceil N/D\rceil}$; indeed, $\lceil A_i/D_i\rceil \ge \lceil A_i/D\rceil$ for each $1 \le i \le n$, and the integer $\sum_{i=1}^n \lceil A_i/D\rceil \ge \lceil(\sum_{i=1}^n A_i)/D\rceil = \lceil N/D\rceil$ for each $(A_1,\ldots,A_n) \in S(N)$. To finish, since $Q^{(N)} \subseteq Q^{\lceil N/D\rceil}$ for all $N \ge 0$, we invoke Lemma 4.3 again. □

5. Proving Theorem 1.3 in a refined form, class group computations

Theorem 1.3 is easy if $n = 1$ or $D = 1$: all rings in sight are polynomial rings and monomial primes are complete intersections. Thus for the remainder of this section, **we will assume that** $n \geq 2$ **and** $D \geq 2$. We will give presentations of our rings as subrings of the domain of Laurent polynomials $L = \mathbb{F}[s_1^{\pm 1}, \dots, s_{n-1}^{\pm 1}, u^{\pm 1}]$ in n indeterminates over the field $\mathbb{F}$. The proof will proceed in cases, starting with the ring $H_D = \frac{\mathbb{F}[x_1,\dots,x_n,z]}{(z^D - x_1 \cdots x_n)}$.

Remark 7. Maintaining all notation conventions from Section 3, in practice going forward we pick a basis $e_1, \dots, e_n$ of our lattice N with dual basis $e_1^*, \dots, e_n^*$ for the dual lattice M, so that both N and M are isomorphic to $\mathbb{Z}^n$. Then the pairing $\langle \cdot, \cdot \rangle$ becomes the usual dot product.

5.1. The hypersurface case

We first observe that H_D is a toric ring, up to isomorphism:

Lemma 5.1. *Consider the full-dimensional simplicial pointed rational polyhedral cone $\sigma_D^{(n)} \subseteq N_{\mathbb{R}} \cong \mathbb{R}^n$ whose ray generators are $De_i + e_n \in N$ for $1 \leq i < n$ and $e_n \in N$ in terms of the selected basis for N.*

1. *The Hilbert basis of the semigroup $(\sigma_D^{(n)})^\vee \cap M$ consists of $n+1$ vectors: the n dual basis vectors $e_1^*, \dots, e_n^*$, together with the vector $-e_1^* \cdots - e_{n-1}^* + De_n^* \in M$.*
2. *The toric ring $\mathbb{F}[(\sigma_D^{(n)})^\vee \cap M] \cong \frac{\mathbb{F}[x_1,\dots,x_n,z]}{(z^D - x_1 \cdots x_n)} = H_D$.*

Proof. The reader can use the hilbertBasis algorithm implemented in the `Polyhedra` package in Macaulay2 [11] to check (1). For (2), recall that to each $m = \sum_{i=1}^n m_i e_i^* \in (\sigma_D^{(n)})^\vee \cap M$ we assign a Laurent monomial $\chi^m = s_1^{m_1} \cdots s_{n-1}^{m_{n-1}} u^{m_n}$ in the semigroup ring $\mathbb{F}[(\sigma_D^{(n)})^\vee \cap M]$. Given (1), in terms of $\mathbb{F}$-algebra generators we have

$$\mathbb{F}[(\sigma_D^{(n)})^\vee \cap M] = \mathbb{F}\left[s_1, \dots, s_{n-1}, \frac{u^D}{(s_1 \cdots s_{n-1})}, u\right] \subseteq \mathbb{F}[s_1^{\pm 1}, \dots, s_{n-1}^{\pm 1}, u^{\pm 1}].$$

Given a polynomial ring $R = \mathbb{F}[x_1, \dots, x_{n-1}, x_n, z]$ in $n+1$ variables, consider the surjective algebra map $\phi\colon R = \mathbb{F}[x_1, \dots, x_{n-1}, x_n, z] \twoheadrightarrow \mathbb{F}[(\sigma_D^{(n)})^\vee \cap M]$ under which $x_i \mapsto s_i$ for each $1 \leq i \leq n-1$, $x_n \mapsto \frac{u^D}{(s_1 \cdots s_{n-1})}$, and $z \mapsto u$. Since $\dim(R) = \dim(\mathbb{F}[(\sigma_D^{(n)})^\vee \cap M]) + 1$, we conclude that the kernel of ϕ is a height one prime in the UFD R, and hence is principal. Now $F = z^D - x_1 \cdots x_n \in R$ is irreducible by Eisenstein's Criterion and belongs to the kernel of ϕ, so $\ker \phi = (F)$, and the isomorphism claim follows. $\square$

We now deduce the following refinement of Theorem 1.3 for H_D:

Theorem 5.2. *Take the ring* $H_D = \mathbb{F}[x_1, \ldots, x_n, z]/(z^D - x_1 \cdots x_n)$, *and* P *one of the monomial prime ideals of* H_D *(i.e.,* M*-graded /torus-invariant); assume* P *is nonzero and nonmaximal. When* $D \leq \mathrm{ht}(P)$ *(the* ***height*** *of* P*),* $P^{(E)} = P^E$ *for all* $E > 0$*. If* $D \geq \mathrm{ht}(P)$ *and* $E \equiv 1 \pmod{D}$*, then*

$$P^{(E)} \subseteq P^{\mathrm{ht}(P)\left(\frac{E-1}{D}\right)+1}.$$

In particular, $P^{(Dr)} \subseteq P^{(D(r-1)+1)} \subseteq P^{\mathrm{ht}(P)(r-1)+1} \subseteq P^r$ *for all* $r > 0$*.*

Proof. Citing the proof of Lemma 5.1(2), the height j prime ideal $P_j := (z, x_1, \ldots, x_j)H_D$, for $1 \leq j \leq n-1$, equals P_τ for the j-dimensional face τ of $\sigma_D^{(n)}$ generated by $De_i + e_n$ for $1 \leq i \leq j$. As a saturation, $P_j^{(E)} = P_j^E :_{H_D} (\prod_{i=j+1}^n x_i)^\infty$. Since $P_j^{(E)}$ is monomial, in chasing down inclusions below it suffices to discern which monomial classes $g = (z^\ell x_1^{a_1} \cdots x_j^{a_j})(x_{j+1}^{a_{j+1}} \cdots x_n^{a_n}) \in H_D$ multiply a power of $m = \prod_{i=j+1}^n x_i$ into P_j^E. For g as above, by definition $g \in P_j^{(E)}$ if and only if for all $T \gg 0$,

$$\begin{aligned} P_j^E \ni m^T g &= z^\ell \left(\prod_{i=j+1}^n x_i^{a_i+T}\right)\left(\prod_{i=1}^j x_i^{a_i}\right) \\ &= z^\ell \left(\prod_{i=1}^n x_i\right)^{T'} \left(\prod_{i=j+1}^n x_i^{a_i+T-T'}\right)\left(\prod_{i=1}^j x_i^{a_i-T'}\right) \\ &= \left(z^{D\cdot T'+\ell} \prod_{i=1}^j x_i^{a_i-T'}\right)\left(\prod_{i=j+1}^n x_i^{a_i+T-T'}\right), \end{aligned}$$

where $T' = T'(T) := \min(a_1, \ldots, a_j, a_{j+1}+T, \ldots, a_n+T) = \min(a_1, \ldots, a_j)$ for all $T \gg 0$. We conclude that $z^{D\cdot T'+\ell}\left(\prod_{i=1}^j x_i^{a_i-T'}\right) \in P_j^E$, and infer the inequality

$$(D-j)T' + \left(\sum_{i=1}^j a_i\right) + \ell \geq E. \tag{10}$$

Before proceeding, notice that since $T' \geq 0$, when $D \leq j$ so that the number $(D-j)T'$ is nonpositive, (10) implies that $\left(\sum_{i=1}^j a_i\right) + \ell \geq E$, so $(z^\ell x_1^{a_1} \cdots x_j^{a_j}) \in P_j^E$ and hence $g \in P_j^E$ already. Thus $P_j^{(E)} = P_j^E$ for all $E > 0$ when $D \leq j$, since both are generated by monomial classes. Thus in the remainder of the proof **we will assume that** $D \geq j = \mathrm{ht}(P_j)$, i.e., $D - j \geq 0$.

In this case, assuming $E \equiv 1 \pmod{D}$, we now show that $P_j^{(E)} \subseteq P_j^{1+j\left(\frac{E-1}{D}\right)}$. Fix a monomial

$$g = \left(z^\ell \prod_{i=1}^{j} x_i^{a_i} \right) \left(\prod_{i=j+1}^{n} x_i^{a_i} \right) \in P_j^{(E)},$$

and $T' = \min(a_1, \ldots, a_j)$ exactly as before. Now $g \in P_j^G$ where $G := \ell + \sum_{i=1}^{j} a_i$. The more involved case for us is when (**) $T' \leq (E-1)/D$: otherwise

$$G \geq a_1 + \cdots + a_j \geq jT' \geq j(E-1)/D + 1,$$

whence one easily infers that $g \in P_j^{j\left(\frac{E-1}{D}\right)+1}$. Assuming (**), we now show that $G \geq j\left(\frac{E-1}{D}\right) + 1$. Suppose to the contrary that $G \leq j\left(\frac{E-1}{D}\right)$. Since $g \in P_j^{(E)}$, inequality (10) above says

$$(D-j)T' + G = (D-j)T' + \left(\sum_{i=1}^{j} a_i \right) + \ell \geq E \Longrightarrow G \geq E - (D-j)T'.$$

Then since $E - 1 - DT' \geq 0$ by (**), and $D - j \geq 0$, we see that

$$\begin{aligned} j\,(E-1) = Dj\left(\frac{E-1}{D}\right) \geq DG &\geq DE - D(D-j)T' \\ &= D(E-1) + D - D(D-j)T' \\ &= j(E-1) + D + (D-j)(E-1-DT') \\ &\geq j(E-1) + D + (D-j)(0) \\ &= j(E-1) + D \end{aligned}$$

a contradiction. Thus $G \geq j\left(\frac{E-1}{D}\right) + 1$, so $g \in P_j^{1+j\left(\frac{E-1}{D}\right)}$. In particular, when $E = D(r-1)+1$, we have $P_j^{(D(r-1)+1)} \subseteq P_j^{1+j(r-1)}$. Finally, applying coordinate changes according to every permutation of $x_{[n]} := \{x_1, \ldots, x_n\}$, any (nonzero, nonmaximal) monomial prime ideal in H_D can be obtained from the P_j running through all indices $1 \leq j \leq n-1$, along with obtaining the desired containments. □

5.2. The Veronese case

Let $\mathbb{N} = \mathbb{Z}_{\geq 0}$ denote the set of **nonnegative integers**. To start,

Lemma 5.3. *Consider the full-dimensional simplicial pointed rational polyhedral cone $\eta_D^{(n)} \subseteq N_{\mathbb{R}} \cong \mathbb{R}^n$ whose ray generators are e_i for $1 \leq i < n$ along with the vector $-e_1 - \ldots - e_{n-1} + De_n$ in terms of the basis selected for N.*

1. *The Hilbert basis of the semigroup* $(\eta_D^{(n)})^\vee \cap M$ *is the set of vectors*

$$\left\{ e_n^* + \sum_{i=1}^{n-1} a_i e_i^* \in M \colon \textit{all } a_i \geq 0 \textit{ and } 0 \leq \sum_{i=1}^{n-1} a_i \leq D \right\}.$$

2. *The toric ring* $\mathbb{F}[(\eta_D^{(n)})^\vee \cap M] \cong V_D$*, the* D*-th Veronese subring of the polynomial ring* $\mathbb{F}[s_1, \ldots, s_{n-1}, u]$ *in the* n *indeterminates* $s_1, \ldots, s_{n-1}, u$*.*

Proof. The reader can use the hilbertBasis algorithm implemented in the `Polyhedra` package in Macaulay2 [11] to check (1). Given (1), as an algebra over $\mathbb{F}$, we have

$$\begin{aligned}\mathbb{F}[(\eta_D^{(n)})^\vee \cap M] &= \mathbb{F}\left[s_1^{a_1} \cdots s_{n-1}^{a_{n-1}} u \colon \text{each } a_i \geq 0,\, 0 \leq \sum_{i=1}^{n-1} a_i \leq D \right] \\ &\cong \frac{\mathbb{F}[x_{(a_1,\ldots,a_{n-1})} \colon \text{each } a_i \geq 0,\, 0 \leq \sum_{i=1}^{n-1} a_i \leq D]}{(x_e x_f - x_g x_h \colon e + f = g + h \in \mathbb{N}^{n-1})}.\end{aligned}$$

Within the polynomial ring $\mathbb{F}[s_1, \ldots, s_{n-1}, u]$, applying the correspondence

$$s_1^{a_1} \cdots s_{n-1}^{a_{n-1}} u \longleftrightarrow s_1^{a_1} \cdots s_{n-1}^{a_{n-1}} u^{D - a_1 - \cdots - a_{n-1}}$$

takes the generators in the presentation of $\mathbb{F}[(\eta_D^{(n)})^\vee \cap M]$ and recovers the usual presentation of V_D in terms of degree D monomials in n variables. Therefore, (2) holds: $\mathbb{F}[(\eta_D^{(n)})^\vee \cap M] \cong V_D$. $\square$

We use the toric presentation of V_D to deduce the following refinement of Theorem 1.3 for V_D:

Theorem 5.4. *Over an arbitrary field* $\mathbb{F}$*, take the* D*-th Veronese subring* $V_D \subseteq \mathbb{F}[s_1, \ldots, s_{n-1}, u]$ *and* P *one of the monomial prime ideals of* V_D*. When* P *is nonzero and nonmaximal,* $P^{(E)} \subseteq P^r$ *if and only if* $r \leq \lceil E/D \rceil$*. In particular,* $P^{(Dr)} \subseteq P^{(D(r-1)+1)} \subseteq P^r$ *for all* $r > 0$ *and the right-hand containment is sharp.*

Proof. Picking up from Lemma 5.3, for all $1 \leq j \leq n-1$, we define height one primes

$$P_j = P_{e_j} = \left(s_1^{a_1} \cdots s_{n-1}^{a_{n-1}} u \colon a_j > 0,\ \text{and } 1 \leq \sum_{b=1}^{n-1} a_b \leq D \right) V_D.$$

Then by the **Minkowski sum-ideal sum decomposition** (6) $P_{j_1 < \cdots < j_k} := P_{j_1} + \cdots + P_{j_k}$ is a prime of height $1 \leq k \leq n-1$ for each size-k subset $j_1 < \ldots < j_k$ of $[n-1] = \{1, \ldots, n-1\}$. In particular, we focus on $P_{1 < \cdots < k} = (s^{\bar{a}} u \colon \bar{a} \in T_k) V_D$, where

$$T_k := \left\{ \bar{a} = (a_1, \ldots, a_{n-1}) \in \mathbb{N}^{n-1} \colon 1 \leq \sum_{b=1}^{k} a_b \leq \sum_{b=1}^{n-1} a_b \leq D \right\}.$$

Any monomial g in $P^{(E)}_{1<\cdots<k} \subseteq P_{1<\cdots<k} \subseteq P_{1<\cdots<n-1}$ belongs to $P_{1<\cdots<k}$ and so decomposes (for some $B \geq 0$) as

$$g = u^B \prod_{\bar{a} \in T_{n-1}} (s^{\bar{a}} u)^{i_{\bar{a}}} = \prod_{\bar{a} \in T_k} (s^{\bar{a}} u)^{i_{\bar{a}}} \left(u^B \prod_{\bar{a} \in T_{n-1} - T_k} (s^{\bar{a}} u)^{i_{\bar{a}}} \right) \in P^{\sum_{\bar{a} \in T_k} i_{\bar{a}}}_{1<\cdots<k}.$$

Note that this factorization of g into two monomial pieces (T_k versus $T_{n-1} - T_k$) is unique up to applying the Veronese relations $s^{\bar{e}}u \cdot s^{\bar{f}}u = s^{\bar{g}}u \cdot s^{\bar{h}}u \quad (\bar{e} + \bar{f} = \bar{g} + \bar{h})$. Setting the monomial $m := u \cdot \prod_{\bar{a} \in T_{n-1} - T_k} s^{\bar{a}} u \in V_D$ to be the **product** of the monomials $s_1^{a_1} \cdots s_{n-1}^{a_{n-1}} u$ with $a_j = 0$ for all $1 \leq j \leq k (\leq n-1)$, we have $P^{(E)}_{1<\cdots<k} = P^E_{1<\cdots<k} :_{V_D} (m)^\infty$, and the monomial g is in $P^{(E)}_{1<\cdots<k}$ precisely when for all $T \gg 0$,

$$g \cdot m^T = \left(u^{B+T} \prod_{\bar{a} \in T_k} (s^{\bar{a}} u)^{i_{\bar{a}}} \right) \prod_{\bar{a} \in T_{n-1} - T_k} (s^{\bar{a}} u)^{i_{\bar{a}} + T} \in P^E_{1<\cdots<k}.$$

In particular, the monomial in parentheses is in $P^E_{1<\cdots<k}$ so it is a multiple of some E-fold product of generators of $P_{1<\cdots<k} = (s^{\bar{a}} u \colon \bar{a} \in T_k) V_D$. Thus we infer that two inequalities must hold, signifying we have enough u's and s_j's ($1 \leq j \leq k$) at our disposal, respectively, to feasibly form such a E-fold product. These inequalities are **(1)** $\sum_{\bar{a} \in T_k} i_{\bar{a}} + B + T \geq E$, and **(2)** the sum

$$\sum_{\bar{a} \in T_k} i_{\bar{a}} (a_1 + \cdots + a_k) = \sum_{j=1}^{D} \ell_j \cdot j \geq E,$$

where $\ell_j := \sum_{\bar{a} \in T_{k,j}} i_{\bar{a}}$, $T_{k,j} := \{\bar{a} \in T_k \colon \text{ the partition } a_1 + \cdots + a_k = j\}$. Indeed,

$$E \leq \sum_{j=1}^{D} \ell_j \cdot j \leq D \left(\sum_{j=1}^{D} \ell_j \right) \Longrightarrow \sum_{j=1}^{D} \ell_j \geq \lceil E/D \rceil,$$

so **(2)** implies that **(3)** $\sum_{\bar{a} \in T_k} i_{\bar{a}} = \sum_{j=1}^{D} \ell_j \geq \lceil E/D \rceil$.[1] For any monomial $g \in P^{(E)}_{1<\cdots<k}$, **(3)** implies that $g \in P^{\lceil E/D \rceil}_{1<\cdots<k}$. Thus $P^{(E)}_{1<\cdots<k} \subseteq P^{\lceil E/D \rceil}_{1<\cdots<k}$ for all $E > 0$.

[1] Together, inequalities **(1)** and **(3)** are equivalent to

$$\sum_{\bar{a} \in T_k} i_{\bar{a}} = \sum_{j=1}^{D} \ell_j \geq \max\{\lceil E/D \rceil, E - (B+T)\} = \lceil E/D \rceil \text{ for all } T \geq E.$$

Additionally if we consider R with its **standard $\mathbb{N}$-grading**, then the minimal degree of a monomial (e.g., a monomial generator) in $P^r_{1<\cdots<k}$ is r. Noticing that for $1 \le j \le k$, the degree $\lceil E/D \rceil$ monomial $(s_j^D u)^{\lceil E/D \rceil} \in P^E_{1<\cdots<k} : (u^{(E+1)-\lceil E/D \rceil}) \subseteq P^E_{1<\cdots<k} : (m^{(E+1)-\lceil E/D \rceil}) \subseteq P^{(E)}_{1<\cdots<k}$, we obtain the only-if part of: for each $1 \le k \le n$, $P^{(E)}_{1<\cdots<k} \subseteq P^r_{1<\cdots<k}$ if and only if $r \le \lceil E/D \rceil$.

Setting $E = Dr - (D-1) = D(r-1)+1$, we have $\lceil E/D = (r-1) + 1/D \rceil = r$, so that $P^{(Dr-(D-1))}_{1<\cdots<k} \subseteq P^r_{1<\cdots<k}$ for all $r > 0$ and this containment is sharp.

In review, our argument **does not** depend crucially on which size-k index subset $j_1 < \ldots < j_k$ of $[n] = \{1, 2, \ldots, n\}$ we worked with; going with $1 < 2 < \ldots < k$ merely simplifies notation. In other words, in applying suitable permutations of the algebra generators for V_D, one obtains the above characterization of ideal containment for all of the monomial prime ideals in the ring **having one of the P_j as an ideal summand.** To handle monomial primes having the height one prime

$$P_{(-1,\ldots,-1,D)} = \left(s_1^{a_1} \cdots s_{n-1}^{a_{n-1}} u \colon 0 \le \sum_{i=1}^{n-1} a_i \le D-1 \right)$$

as a summand, we use the $\mathbb{F}$-algebra isomorphisms $\phi_j \colon V_D \to V_D$ $(1 \le j \le n-1)$ under which a monomial algebra generator $g = s_1^{a_1} \cdots s_j^{a_j} \cdots s_{n-1}^{a_{n-1}} u$ with $0 \le A := \sum_{i=1}^{n-1} a_i \le D$ is sent to

$$\phi_j(g) = \begin{cases} s_1^{a_1} \cdots s_j^{D-A} \cdots s_{n-1}^{a_{n-1}} u & \text{if } A \le D-1 \text{ and } a_j = 0 \\ s_1^{a_1} \cdots s_j^{0} \cdots s_{n-1}^{a_{n-1}} u & \text{if } A = D \text{ and } a_j > 0 \\ g & \text{if } A \le D-1 \text{ and } a_j > 0 \\ g & \text{if } A = D \text{ and } a_j = 0. \end{cases}$$

We note that $\phi_j^2 = \phi_j \circ \phi_j$ is the identity, and the height one prime $\phi_j(P_{(-1,\ldots,-1,D)}) = P_j$: indeed, when $h = s_1^{a_1} \cdots s_j^{a_j} \cdots s_{n-1}^{a_{n-1}} u$ is a generator of P_j, $a_j > 0$; when $A \le D-1$, $h = \phi_j(h)$, or else $D - A = 0$, $a_j = D - \left(\sum_{1 \le i \ne j \le n-1} a_i \right) > 0$, and $h = \phi_j(g)$ where $g = s_1^{a_1} \cdots s_j^{0} \cdots s_{n-1}^{a_{n-1}} u \in P_{(-1,\ldots,-1,D)}$. Moreover, we conclude that a (sharp) containment $Q^{(m)} \subset Q^r$ for any monomial prime Q with P_j as a summand translates under ϕ_j to a (sharp) containment $(Q')^{(m)} \subset (Q')^r$ for a monomial prime Q' of the same height as Q, with $P_{(-1,\ldots,-1,D)}$ replacing P_j as an ideal summand. Having analyzed ideals with one of the P_j as a summand quite thoroughly, this final observation completes the proof. $\square$

As advertised in the introduction, we want to close by drawing a connection between Lemma 1.1 and Theorems 5.2 and 5.4, e.g., to see that the containments in the lemma can be tight by example.

Remark 5.5. With notation as in Theorem 3.6, we note that if $C \subseteq N_{\mathbb{R}}$ is a full pointed rational polyhedral cone, then we have the following presentation for the divisor class group:

$$\mathrm{Cl}(\mathbb{F}[C^\vee \cap M]) \cong \frac{\bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}\cdot[D_\rho]}{\langle \sum_{\rho\in\Sigma(1)} \langle e_i^*, u_\rho\rangle [D_\rho] = 0 \colon 1 \le i \le n\rangle},$$

where the $e_i^* \in M$ form the dual basis to the basis $e_1, \ldots, e_n \in N$ chosen in N.

Example 5.6. We work with the polyhedral cones in the proof of Theorem 1.3, showing that $\mathrm{Cl}(H_D) \cong (\mathbb{Z}/D\mathbb{Z})^{n-1}$ and $\mathrm{Cl}(V_D) \cong \mathbb{Z}/D\mathbb{Z}$. Although these class group facts are well known in certain circles and can be deduced by other means (see e.g., [19]), for completeness of exposition we include succinct computations.

1. The cone $\sigma_D^{(n)} \subseteq N_\mathbb{R}$ has ray generators $f_i = De_i + e_n$ for $1 \le i < n$ and e_n, and

$$\begin{aligned}\mathrm{Cl}(\mathbb{F}[(\sigma_D^{(n)})^\vee \cap \mathbb{Z}^n]) &\cong \frac{\mathbb{Z}\cdot[\mathbf{D_{e_n}}] \oplus \bigoplus_{i=1}^{n-1} \mathbb{Z}\cdot[D_{f_i}]}{\langle D[D_{f_i}] = 0 \ (1 \le i < n), [\mathbf{D_{e_n}}] = -[\mathbf{D_{f_1}}] - \cdots - [\mathbf{D_{f_{n-1}}}]\rangle} \\ &\cong \frac{\mathbb{Z}\cdot -[\mathbf{D_{f_1}}] - \cdots - [\mathbf{D_{f_{n-1}}}] \oplus \bigoplus_{i=1}^{n-1} \mathbb{Z}\cdot[D_{f_i}]}{\langle D[D_{f_1}] = 0, \ldots, D[D_{f_{n-1}}] = 0\rangle} \\ &= \frac{\bigoplus_{i=1}^{n-1} \mathbb{Z}\cdot[D_{f_i}]}{\langle D[D_{f_1}] = 0, \ldots, D[D_{f_{n-1}}] = 0\rangle} \\ &\cong \boxed{(\mathbb{Z}/D\mathbb{Z})^{n-1}.}\end{aligned}$$

2. The cone $\eta_D^{(n)} \subseteq N_\mathbb{R}$ has ray generators e_i for $1 \le i < n$ and $f_n = De_n - \sum_{i=1}^{n-1} e_i$, and

$$\begin{aligned}\mathrm{Cl}(\mathbb{F}[(\eta_D^{(n)})^\vee \cap \mathbb{Z}^n]) &\cong \frac{\mathbb{Z}\cdot[\mathbf{D_{f_n}}] \oplus \bigoplus_{i=1}^{n-1} \mathbb{Z}\cdot[D_{e_i}]}{\langle [\mathbf{D_{e_i}}] - [\mathbf{D_{f_n}}] = \mathbf{0}\ (\mathbf{1 \le i < n}),\ D[D_{f_n}] = 0\rangle} \\ &\cong \frac{\mathbb{Z}\cdot[D_{f_n}]}{\langle D[D_{f_n}] = 0\rangle} \\ &\cong \boxed{(\mathbb{Z}/D\mathbb{Z}).}\end{aligned}$$

6. Lingering questions related to Theorem 1.2

To summarize, we have deduced two existence criteria for uniform Harbourne–Huneke bounds. Lemma 1.1 holds for ideals of pure height one in a Noetherian normal domain. And Theorem 1.2 holds for monomial primes in finite tensor products of normal toric rings; we deduced Theorem 1.3 to increase the range of examples that can be used as tensor factors. These criteria cover a prodigious class of normal toric rings. We include the following illustrative example:

Example 6.1. Let $p_i \in \mathbb{N}$ be the i-th prime number, R_i the p_i-th Veronese subring of $\mathbb{F}[X_{i,1}, \ldots, X_{i,14641}]$. Set

$$R(n) = (\bigotimes_{i=1}^{n})_{\mathbb{F}} R_i, \quad \sigma(n) = \prod_{i=1}^{n} p_i \text{ (the \textbf{primorial function})}.$$

One can compute that $\mathrm{Cl}(R(n)) \cong \mathbb{Z}/\sigma(n)$ via toric divisor theory, so Lemma 1.1 says that

$$\mathfrak{q}^{(\sigma(n)(r-1)+1)} \subseteq \mathfrak{q}^r$$

for all ideals $\mathfrak{q} \subseteq R(n)$ of pure height one, and all $r > 0$. Also, $D = p_n$ in Theorem 1.2, covering **all** 2^{14641n} **monomial primes in** $R(n)$. The multiplier for monomial primes climbs much slower than the multiplier in pure height one as n climbs to infinity.

We close with a few natural lines for further investigation.

1. Does the conclusion of Theorem 1.2 extend to monomial primes in any simplicial toric ring? Can we identify a candidate mechanism (e.g., group-theoretic) to help explain and verify these Harbourne–Huneke bounds in height two or higher for a larger class of ideals than monomial primes?
2. Given the role of tensor products in our manuscript, do analogues of Theorems 1.2 and 4.1 hold for other graded ring constructions in the toric setting, such as Segre products?

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