TENSOR-MULTINOMIAL SUMS OF IDEALS: PRIMARY DECOMPOSITIONS AND PERSISTENCE OF ASSOCIATED PRIMES

IRENA SWANSON AND ROBERT M. WALKER

ABSTRACT. Given a polynomial ring C over a field and proper ideals I and J whose generating sets involve disjoint variables, we determine how to embed the associated primes of each power of $I + J$ into a collection of primes described in terms of the associated primes of select powers of I and of J. We record two applications. First, in case the field is algebraically closed, we construct primary decompositions for powers of $I + J$ from primary decompositions for powers of I and J. Separately, we attack the persistence problem for associated primes of powers of an ideal in case one of I or J is a non-zero normal ideal.

1. INTRODUCTION

Throughout, $A = K[x_1, \ldots, x_a]$, $B = K[y_1, \ldots, y_b]$, and $C = A \otimes_K B = K[x_1, \ldots, x_a, y_1, \ldots, y_b]$, where $x_1, \ldots, x_a, y_1, \ldots, y_b$ are variables over a field K. We fix ideals $I \subsetneq A$ and $J \subsetneq B$. By abuse of notation, we use the same symbol to denote ideals in A or B and their expansions to C.

Our primary focus is on constructing primary decompositions for the powers of the ideal

$$I + J \subsetneq C,$$

which we call a **(two-term) tensor-multinomial sum of ideals**, clarifying our chosen title.

By a result of Brodmann [1] (or [2, Theorem 3.5]), the collection $\mathcal{A}_A(I) := \bigcup_{n=1}^{\infty} \mathrm{Ass}_A(A/I^n)$ of associated primes of powers of I is a finite set. By our abuse of notation, $\mathcal{A}_A(I) = \mathcal{A}_C(I)$.

Definition 1.1. Suppose $\mathcal{A}_A(I) = \mathcal{A}_C(I) = \{P_1, \ldots, P_r\}$ and $\mathcal{A}_B(J) = \mathcal{A}_C(J) = \{Q_1, \ldots, Q_s\}$. For each integer $m \geq 1$, fix primary decompositions

$$I^m = p_{m1} \cap \cdots \cap p_{mr} \quad \text{and} \quad J^m = q_{m1} \cap \cdots \cap q_{ms},$$

where for each $k = 1, \ldots, r$, the ideal p_{mk} is either P_k-primary or equals C, and for $\ell = 1, \ldots, s$, the ideal $q_{m\ell}$ is either Q_ℓ-primary or else C. We also set $p_{0k} = q_{0\ell} = C$ for all k, ℓ.

Definition 1.2. A list of ideals $\{L_c\}_{c \geq 0}$ in C is a **filtration** if $L_0 = C$ and $L_c \supseteq L_{c+1}$ for all $c \geq 0$.

Remark 1.3. This definition differs from Hà–Nguyen–Trung–Trung [4, Section 3] only in that we *do not* require L_1 to be non-zero and proper – upon inspection this assumption is inessential for the proofs of [4, Proposition 3.3, Theorem 3.4]. We briefly recall this point at the start of Section 3.

The technical heart of the paper lies in the following theorem, deduced in Section 3:

Theorem 1.4. *Fix a field K, and polynomial K-algebras $A = K[x_1, \ldots, x_a]$, $B = K[y_1, \ldots, y_b]$ and $C = A \otimes_K B = K[x_1, \ldots, x_a, y_1, \ldots, y_b]$. Let I be an ideal in A and J an ideal in B. Then for each integer $n > 0$,*

$$\mathrm{Ass}(C/(I+J)^n) \subseteq \bigcup \left\{ \mathcal{P} \in \mathrm{Min}(C/(P+Q)) : P \in \bigcup_{i=1}^{n} \mathrm{Ass}(C/I^i), Q \in \bigcup_{j=1}^{n} \mathrm{Ass}(C/J^j) \right\},$$

and moreover

$$\mathcal{A}_C(I+J) = \bigcup \{ \mathcal{P} \in \mathrm{Min}(C/(P+Q)) : P \in \mathcal{A}(I), Q \in \mathcal{A}(J) \}.$$

1

We record two applications of Theorem 1.4. First, we record the following theorem, our primary result in the paper – we construct primary decompositions for powers of $I + J$ from filtered primary components for powers of I and J in the case where K is algebraically closed.

Theorem 1.5 (Cf. Corollary 3.4). *Let K be an algebraically closed field and suppose that for each pair $1 \le k \le r$ and $1 \le \ell \le s$, the collections $\{p_{mk}\}_{m=0}^{\infty}$ and $\{q_{m\ell}\}_{m=0}^{\infty}$ from Definition 1.1 are filtrations. Then for each positive integer n,*

$$(I + J)^n = \bigcap_{k=1}^{r} \bigcap_{\ell=1}^{s} \left(\sum_{i=0}^{n} p_{ik} \cdot q_{n-i,\ell} \right),$$

where each $\sum_{i=0}^{n} p_{ik} \cdot q_{n-i,\ell}$ is either C or primary to the prime ideal $P_k + Q_\ell$. Furthermore,

$$\mathcal{A}_C(I + J) = \{P_k + Q_\ell \colon 1 \le k \le r, 1 \le \ell \le s\},$$

and so the cardinalities of these finite sets satisfy the relation

$$\#\mathcal{A}_C(I + J) = \#\mathcal{A}_A(I) \cdot \#\mathcal{A}_B(J).$$

Lemma 2.1 below indicates how to easily construct filtered collections – such as $\{p_{mk}\}_{m=0}^{\infty}$ and $\{q_{m\ell}\}_{m=0}^{\infty}$ in Theorem 1.5 – from arbitrary primary decompositions for powers of I, J, and $I + J$.

Prior to obtaining Theorem 1.4, we originally pursued Theorem 1.5 with a view towards attacking the following question on *persistence* of associated primes.

Question 1.6. Suppose that $\mathrm{Ass}(A/I^{n-1}) \subseteq \mathrm{Ass}(A/I^n)$ and $\mathrm{Ass}(B/J^{n-1}) \subseteq \mathrm{Ass}(B/J^n)$ for all integers $n > 0$. When is it the case that $\mathrm{Ass}(C/(I + J)^{n-1}) \subseteq \mathrm{Ass}(C/(I + J)^n)$ for all n as well?

This persistence property seems to hold for prime ideals in polynomial rings that we have been able to find in the literature and study using Macaulay2 [3]. We know of no prime ideal that fails to satisfy persistence. That said, it seems unlikely that this persistence of associated primes holds for all prime ideals in any polynomial ring over an arbitrary ground field.

As a second application of Theorem 1.4, we answer Question 1.6 affirmatively when one of I or J is a nonzero **normal ideal**, i.e., when all the powers of I or of J are integrally closed. Said ideal satisfies the persistence property by a result of Katz and Ratliff [7, (1.3) Theorem], and even if the other ideal does not, the corollary to follow indicates that their sum does satisfy it, indicating that the persistence property is *remarkably* persistent and robust under extension of scalars.

Corollary 1.7. *Fix a field K, along with polynomial rings $A = K[x_1, \ldots, x_a]$, $B = K[y_1, \ldots, y_b]$ and $C = A \otimes_K B = K[x_1, \ldots, x_a, y_1, \ldots, y_b]$. Let I be an ideal in A and J a non-zero normal ideal in B. Then $\mathrm{Ass}(C/(I + J)^{n-1}) \subseteq \mathrm{Ass}(C/(I + J)^n)$ for all positive integers n.*

Remark 1.8. Theorems 1.4 and 1.5 and Corollary 1.7 extend in two directions. Our proofs extend to the setting considered by Hà–Nguyen–Trung–Trung [4]. Namely, they fix Noetherian commutative algebras A and B over a common field K, such that $C = A \otimes_K B$ is Noetherian as well, along with non-zero ideals $I \subseteq A$ and $J \subseteq B$. Moreover, both results can be rendered for finite tensor products and tensor-multinomial sums of ideals, and deduced via inductive arguments; see the second author's paper [9, Proof of Multinomial Theorem 2.8] which is instructive in this vein.

2. Preparatory Lemmas

We continue to work with the polynomial K-algebras A, B, and C as in the Introduction. Of the lemmas recorded here, the ones cited in proofs in Section 3 are Lemmas 2.1, 2.5, and 2.7 – 2.9.

Lemma 2.1. *Consider the ideal collection $\{p_{mk}\}_{m=0}^{\infty}$ from Definition 1.1 for a fixed index $1 \leq k \leq r$. If we set $p'_{mk} = \cap_{i=0}^{m} p_{ik}$, then $\{p'_{mk}\}_{m=0}^{\infty}$ is a filtration such that p'_{mk} is either C or P_k-primary for each $m \geq 0$, and $I^m = \cap_k p'_{mk}$ is a primary decomposition with possible redundancies.*

Proof. Each p'_{mk} is a finite intersection of ideals that are either C or P_k-primary, and hence p'_{mk} is either C of P_k-primary. Since $I^i \supseteq I^m$ for all $i \leq m$, it follows that any P_k-primary component of I^i contains I^m. Thus $I^m \subseteq \cap_k \cap_{i=0}^{m} p_{ik} = \cap_k p'_{mk} \subseteq \cap_k p_{mk} = I^m$, so equality holds throughout. $\square$

Lemma 2.2. *For any ideals $I \subseteq A$ and $J_1, J_2 \subseteq B$,*

$$(I + J_1) \cap J_2 = IJ_2 + J_1 \cap J_2.$$

Similarly, for any ideals $I_1, I_2 \subseteq A$ and $J \subseteq B$, $I_1 \cap (I_2 + J) = I_1 \cap I_2 + I_1 J$. In particular, $I \cap J = IJ$.

Proof. We only prove the displayed equality because the second statement follows by symmetry, and because the last statement follows trivially from it. We adapt the proof for [4, Lemma 3.1]. First, some notation: given sets U and V in A and B, respectively, their simple tensor set is

$$U \otimes V = \{u \otimes v \colon u \in U, v \in V\}.$$

Let U be a K-vector space basis for I, and V a K-vector space basis for $J_1 \cap J_2$. Extend V to a K-basis V_i for each J_i, and extend U to a K-basis U^* for A and V_2 to a K-basis V_2^* for B. Then $U^* \otimes V_2^*$ is a K-basis for $A \otimes_K B$. Notice $(I + J_1) \cap J_2$ is generated by

$$\big((U \otimes V_2^*) \cup (U^* \otimes V_1)\big) \cap (U^* \otimes V_2) = (U \otimes V_2) \cup \big(U^* \otimes (V_1 \cap V_2)\big)$$

and the right-hand side generates $IJ_2 + (J_1 \cap J_2)$. $\square$

Lemma 2.3. *For any index $k \in \mathbb{Z}_{>0}$, let $\mathcal{I}_k = \{I_{ik}\}_{i=0}^{\infty}$ consist of ideals in A and $\mathcal{J} = \{J_i\}_{i=1}^{\infty}$ consist of ideals in B with $J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots$. Then for any pair of integers $r \geq 1$ and $n \geq 0$,*

$$\bigcap_{k=1}^{r} \left(\sum_{i=0}^{n} I_{ik} J_{n-i} \right) = \sum_{i=0}^{n} \left(\bigcap_{k=1}^{r} \widetilde{I}_{ik} \right) J_{n-i},$$

where $\widetilde{I}_{ik} = \sum_{j=i}^{n} I_{jk}$. When $\mathcal{I}_k$ is a filtration in C for a given index k, then in fact $\widetilde{I}_{ik} = I_{ik}$.
    An analogous identity holds when the roles of A and B are switched.

Proof. We induce on r and then on n. The case $r = 1$ is trivial. Observe that for all k,

$$\sum_{i=0}^{n} I_{ik} J_{n-i} = \sum_{i=0}^{n} \widetilde{I}_{ik} J_{n-i}.$$

Replacing the I_{ik} with $\tilde{I}_{ik}$ for all k, it suffices to prove the lemma assuming the $\mathcal{I}_k$ are filtrations.
    By induction it suffices to prove the case $r = 2$. Several times in the proof we will use Lemma 2.2. Since $I_{0k} = C$ for all k, the claim holds for $n = 0$ as $I_{01} J_0 \cap I_{02} J_0 = J_0 = (I_{01} \cap I_{02}) J_0$. Now assume that $n > 0$, assuming the identity for $r = 2$ and $n - 1$. The first three equalities below, along with the sixth, use the easy fact that for any ideals L_1, L_2, L_3 in a ring R, if $L_1 \subseteq L_3$ then $(L_1 + L_2) \cap L_3 = L_1 + (L_2 \cap L_3)$; if $n \geq 2$, the fifth holds by applying Lemma 2.2 to the first boxed intersection and applying the induction hypothesis to the latter, first replacing I_{ik} with $I'_{ik} := I_{i+1,k}$

for $1 \leq i \leq n-1$. Set $\mathcal{S}_{k,t,u} := \sum_{i=u}^{t} I_{ik} J_{t-i}$ for pairs $t \geq 1$ and $u \geq 1$. Note that $\mathcal{S}_{k,t,u} = C$ if $t < u$ and otherwise lies in I_{vk} for triples $t \geq u \geq v \geq 0$. The left-hand intersection is

$$(J_n + \mathcal{S}_{1,n,1}) \cap (J_n + \mathcal{S}_{2,n,1}) = J_n + ((J_n + \mathcal{S}_{1,n,1}) \cap \mathcal{S}_{2,n,1}) \qquad [L_1 = J_n \subseteq L_3 = J_n + \mathcal{S}_{1,n,1}]$$

$$= J_n + \left(\left(J_n + (I_{11} \cap (J_{n-1} + \mathcal{S}_{1,n,2})) \right) \cap (I_{12} \cap (J_{n-1} + \mathcal{S}_{2,n,2})) \right) \quad [L_1 = \mathcal{S}_{k,n,1} \subseteq L_3 = I_{1k}, \ k = 1,2]$$

$$= J_n + \left((J_n + I_{11}) \cap (J_{n-1} + \mathcal{S}_{1,n,2}) \cap (I_{12} \cap (J_{n-1} + \mathcal{S}_{2,n,2})) \right) \quad [L_1 = J_n \subseteq L_3 = J_{n-1} + \mathcal{S}_{1,n,2}]$$

$$= J_n + \left((J_n + I_{11}) \cap I_{12} \cap (J_{n-1} + \mathcal{S}_{1,n,2}) \cap (J_{n-1} + \mathcal{S}_{2,n,2}) \right) \quad \text{[by reordering]}.$$

If $n = 1$, we are now done with Lemma 2.2, otherwise we continue with equalities:

$$= J_n + \left((J_n I_{12} + I_{11} \cap I_{12}) \cap \left(J_{n-1} + \sum_{i=2}^{n} (I_{i1} \cap I_{i2}) J_{n-i} \right) \right)$$

$$= J_n + \left(((J_n I_{12} + I_{11} \cap I_{12}) \cap J_{n-1}) + \sum_{i=2}^{n} (I_{i1} \cap I_{i2}) J_{n-i} \right) \quad [L_1 = \sum_{i=2}^{n} (I_{i1} \cap I_{i2}) J_{n-i} \subseteq L_3 = I_{11} \cap I_{12}]$$

$$= J_n + \left(J_n I_{12} + \left(\sum_{i=1}^{n} (I_{i1} \cap I_{i2}) J_{n-i} \right) \right) \qquad [L_1 = J_n I_{12} \subseteq L_3 = J_{n-1}],$$

which certainly equals the desired right-hand sum. The lemma then follows in full. $\qquad \square$

Remark 2.4. The same argument proves that if the J_i form a chain of ideals, then

$$\bigcap_{k=1}^{r} \left(\sum_{i=0}^{n} I_{ik} J_{n-i} \right) = \sum_{i=0}^{n} \left(\bigcap_{k=1}^{r} \widehat{I}_{ik} \right) J_{n-i},$$

where $\widehat{I}_{ik}$ is the sum of those I_{jk} for which $J_{n-i} \subseteq J_{n-j}$. Indeed, this follows from the identity $I_{ik} J_{n-i} + I_{jk} J_{n-j} = I_{ik} J_{n-i} + I_{jk} J_{n-i} + I_{jk} J_{n-j}$.

Lemma 2.5. *Suppose the ideal collections $\{p_{nk}\}_{n=0}^{\infty}$ and $\{q_{n\ell}\}_{n=0}^{\infty}$ from Definition 1.1 are filtrations for each fixed pair $1 \leq k \leq r$ and $1 \leq \ell \leq s$. Then for each positive integer n,*

$$(I + J)^n = \bigcap_{\ell=1}^{s} \bigcap_{k=1}^{r} \left(\sum_{i=0}^{n} p_{ik} \cdot q_{n-i,\ell} \right).$$

Proof. We invoke Lemma 2.3 twice:

$$\bigcap_{\ell=1}^{s} \bigcap_{k=1}^{r} \left(\sum_{i=0}^{n} p_{ik} q_{n-i,\ell} \right) = \bigcap_{\ell} \left(\sum_{i=0}^{n} \left(\bigcap_k p_{ik} \right) q_{n-i,\ell} \right) = \bigcap_{\ell} \left(\sum_{i=0}^{n} I^i q_{n-i,\ell} \right) = \sum_{i=0}^{n} I^i \left(\bigcap_{\ell} q_{n-i,\ell} \right)$$

$$= \sum_{i=0}^{n} I^i J^{n-i} = (I + J)^n. \qquad \square$$

Remark 2.6. If P_k is not associated to $I, I^2, \ldots, I^n$, then no components involving the p_{ik} are needed in the decomposition of $(I + J)^n$ in the lemma above. However, if P_k is associated to some I^i for $i < n$, then $p_{1k}, \ldots, p_{nk}$ may or may not be needed, as shown in examples in Section 4.

Lemma 2.7. *Let f be a non-zero divisor in A. Let I be an ideal in A and J an ideal in B. Then $(I + J) : f = (I : f) + J$.*

Proof. Let $c \in (I + J) : f$. Then by Lemma 2.2,

$$cf \in (I + J) \cap (f) = I \cap (f) + J(f) = (I : f)f + J(f),$$

so that $c \in (I : f) + J$. The other inclusion is easy. $\qquad\square$

Lemma 2.8. *Let $L_1 \subseteq L_2$ be proper ideals in a Noetherian ring R and let P be a prime ideal associated to L_1 but not to L_2. Then $(L_1 : P) \cap L_2$ properly contains L_1 and there exists $f \in (L_1 : P) \cap L_2$ such that $L_1 : f = P$.*

Proof. Certainly L_1 is contained in $(L_1 : P) \cap L_2$. Suppose that $L_1 = (L_1 : P) \cap L_2$. Let p be a P-primary component of L_1. Then $p : P$ is the only P-primary component on the right-hand side of the ideal equality, and so by the mix-and-match theorem of primary decompositions due to Yao [10], $p : P$ is also a primary component of L_1 on the left-hand side. But then $p : P^n$ is a primary component of L_1 for all non-negative integers n, but this is a contradiction as for large n, $p : P^n = R$. Thus $(L_1 : P) \cap L_2$ properly contains L_1.

Let L' be the intersection of all primary components of L_1 whose radicals properly contain L_1. Then P is not associated to $L_2 \cap L'$ and $L_1 \subseteq L_2 \cap L'$. Then by the previous paragraph there exists $f \in (L_1 : P) \cap L_2 \cap L'$ such that $f \notin L_1$. Then $P \subseteq L_1 : f$, and the latter is a proper ideal whose associated primes are all associated to L_1 and none properly contain P. Thus $P = L_1 : f$. $\qquad\square$

Lemma 2.9. *Let L be an ideal in a Noetherian ring R containing a non-zerodivisor. Let n be a positive integer and suppose that $L, L^2, \ldots, L^{n-1}$ are integrally closed. Suppose that P is associated to L^n. Then there exists $f \in L^{n-1}$ such that $L^n : f = P$.*

Proof. Let q_n be the intersection of primary components of L^n whose radicals properly contain P. Certainly $L^n \subseteq (L^n : P) \cap L^{n-1} \cap q_n$. Suppose that equality holds. By Lemma 2.8, P is associated to $L^{n-1} \cap q_n$, and hence to L^{n-1}. Colon the equality by L:

$$((L^n : L) : P) \cap (L^{n-1} : L) \cap (q_n : L) = L^n : L.$$

By the determinantal trick due to Prüfer (see [6, Corollary 1.1.8]) for all positive integers k, $L^k : L$ is a subset of the integral closure of L^{k-1}. By assumption this equals L^{k-1} if $k \leq n$. Thus

$$(L^{n-1} : P) \cap L^{n-2} \cap (q_n : L) = L^{n-1}.$$

By repeating this step we get that P is associated to $L^{n-2}, L^{n-3}, \ldots, L$ and that

$$(L : P) \cap (q_n : L^{n-1}) = (L : P) \cap L^0 \cap (q_n : L^{n-1}) = L.$$

If the P-primary component on the right is p, then the only P-primary component on the left is $p : P$, and so by the mix-and-match theorem of primary decompositions due to Yao [10], in a primary decomposition of L we can replace p with $p : P$, and similarly that with $p : P^2$, et cetera. But for large m, $p : P^m = R$, which says that P is not associated to L after all, which is a contradiction. Thus L^n is properly contained in $(L^n : P) \cap L^{n-1} \cap q_n$. Let f be in the latter ideal and not in L^n. Then $L^n : f$ is a proper ideal which contains P and has no associated primes strictly larger than P, so that $L^n : f = P$. $\qquad\square$

3. Proofs of the Key Results

We continue to work with the polynomial K-algebras A, B, and C as in the Introduction. Of the lemmas recorded above, we require Lemmas 2.1, 2.5 and 2.7 –2.9 going forward.

Our proofs rely on Hà–Nguyen–Trung–Trung [4, Lemma 2.4, Theorem 2.5, Proposition 3.3, Proof of Theorem 3.4]. The proofs of these results in [4] work for filtrations as defined in Definition 1.2. By [4, Theorem 2.5], given nonzero finitely-generated modules M and N over A and B, respectively,

$$\mathrm{Ass}_C(M \otimes_K N) = \bigcup \{P \in \mathrm{Min}_C(C/\mathfrak{p} + \mathfrak{q}) \colon \mathfrak{p} \in \mathrm{Ass}_A(M), \ \mathfrak{q} \in \mathrm{Ass}_B(N)\}, \qquad (3.1)$$

5

in terms of sets of associated primes and minimal associated primes. By [4, Proposition 3.3], for any filtrations $\{I_i\}_{i \geq 0}$ and $\{J_j\}_{j \geq 0}$ in A and B, respectively, we have for any integer $n \geq 0$ an isomorphism of C-modules deduced at the level of K-vector spaces:

$$\frac{\sum_{i+j=n} I_i J_j}{\sum_{i+j=n+1} I_i J_j} \cong \bigoplus_{i=0}^{n} \left(I_i/I_{i+1} \otimes_K J_{n-i}/J_{n-i+1} \right). \tag{3.2}$$

Lemma 3.1. *Suppose that for each pair $1 \leq k \leq r$ and $1 \leq \ell \leq s$, the collections $\{p_{mk}\}_{m=0}^{\infty}$ and $\{q_{m\ell}\}_{m=0}^{\infty}$ from Definition 1.1 are filtrations. Then for each triple of integers $n \geq 1$, $k \in \{1, \ldots, r\}$, $\ell \in \{1, \ldots, s\}$,*

$$\mathrm{Ass}_C \left(C/(\sum_{i+j=n} p_{ik} \cdot q_{j\ell}) \right) \subseteq \mathrm{Min}_C(C/(P_k + Q_\ell)).$$

In particular, in case $\sum_{i+j=n} p_{ik} \cdot q_{j\ell}$ is proper and $P_k + Q_\ell$ is a prime ideal (e.g., this holds when K is algebraically closed), $\sum_{i+j=n} p_{ik} \cdot q_{j\ell}$ is primary to $P_k + Q_\ell$.

Proof. We adapt from the proof given for the Symbolic Power Binomial Theorem [4, Theorem 3.4]. Define $L_{t,k,\ell} := \sum_{i+j=t} p_{ik} \cdot q_{j\ell}$ for any integer $t \geq 1$. From the short exact sequences

$$0 \to L_{t-1,k,\ell}/L_{t,k,\ell} \to C/L_{t,k,\ell} \to C/L_{t-1,k,\ell} \to 0 \qquad (1 \leq t \leq n),$$

we may infer that

$$\mathrm{Ass}_C(C/L_{n,k,\ell}) \subseteq \bigcup_{t=1}^{n} \mathrm{Ass}_C(L_{t-1,k,\ell}/L_{t,k,\ell}).$$

By Display (3.2), we have

$$\mathrm{Ass}_C(L_{t-1,k,\ell}/L_{t,k,\ell}) = \bigcup_{i+j=t-1} \mathrm{Ass}_C \left(p_{i,k}/p_{i+1,k} \otimes_K q_{j,\ell}/q_{j+1,\ell} \right).$$

When the ideal $p_{i,k}/p_{i+1,k} \subseteq A/p_{i+1,k}$ is non-zero, its only associated prime ideal is P_k, and similarly the only associated prime ideal of $q_{j,\ell}/q_{j+1,\ell}$ is Q_ℓ. Thus by Display (3.1) we observe that

$$\mathrm{Ass}_C(L_{t-1,k,\ell}/L_{t,k,\ell}) \subseteq \mathrm{Min}_C(C/(P_k + Q_\ell)),$$

whence the lemma follows in full. $\qquad \square$

Lemma 3.2. *Let i and j be the least positive integers such that P is associated to I^i and Q is associated to J^j. Let $\mathcal{P} \in \mathrm{Min}(C/(P + Q))$. Then $\mathcal{P}$ is associated to $(I + J)^{i+j-1}$ and to no lower power of $I + J$.*

Proof. By Lemma 2.8, there exist $f \in I^{i-1}$ and $g \in J^{j-1}$ such that $P = I^i : f$ and $J^j : g = Q$. Then

$$Pfg \subseteq I^i g \subseteq I^i J^{j-1} \subseteq (I + J)^{i+j-1},$$

and similarly $Qfg \subseteq (I + J)^{i+j-1}$. In particular,

$$P + Q \subseteq (I + J)^{i+j-1} : fg \subseteq \left(I^i + J^j \right) : fg,$$

and by Lemma 2.7, this is a subset of $(I^i : f) + (J^j : g) = P + Q$. Thus $P + Q = (I + J)^{i+j-1} : fg$. Since $\mathcal{P}$ is minimal over $P + Q$, there exists $c \in C$ such that $(P + Q) : c = \mathcal{P}$, so that $\mathcal{P} = (I + J)^{i+j-1} : cfg$, which means that $\mathcal{P}$ is associated to $(I + J)^{i+j-1}$.

Now suppose that $\mathcal{P}$ is associated to $(I + J)^n$. By Lemma 2.5, $\mathcal{P}$ is associated to some $L_{n,k,\ell} = \sum_{m=1}^{n} p_{mk} q_{n-m,\ell}$. By Lemma 3.1, $\mathcal{P}$ is minimal over $P_k + Q_\ell$. By Lemma [4, Lemma 2.4] we

conclude that $P = P_k$ and $Q = Q_\ell$. Since $p_{0k} = p_{1k} = \cdots = p_{i-1,k} = C = q_{0\ell} = q_{1\ell} = \cdots = q_{j-1,\ell}$,

$$L_{n,k,\ell} = p_{n-j+1,k} + q_{n-i+1,\ell} + \sum_{m=i}^{n-j} p_{mk}q_{n-m,\ell},$$

and for P_k-primary component to appear, $n - j + 1 \geq i$, i.e., $n \geq i + j - 1$. $\qquad\square$

At last, we are now set to deduce Theorems 1.4 and 1.5 and Corollary 1.7 from the Introduction.

Proof of Theorem 1.4. We want to show that for each integer $n \geq 1$,

$$\mathrm{Ass}(C/(I+J)^n) \subseteq \bigcup \left\{ \mathcal{P} \in \mathrm{Min}(C/(P+Q)) : P \in \bigcup_{i=1}^{n} \mathrm{Ass}(C/I^i), Q \in \bigcup_{j=1}^{n} \mathrm{Ass}(C/J^j) \right\},$$

and that

$$\mathcal{A}_C(I+J) = \bigcup\{\mathcal{P} \in \mathrm{Min}(C/(P+Q)) : P \in \mathcal{A}(I), Q \in \mathcal{A}(J)\}.$$

By Lemma 2.1, for each index pair k, l with $1 \leq k \leq r$ and $1 \leq \ell \leq s$ we can make monotone filtrations $\{p_{i,k}\}_{i \geq 0}$ and $\{q_{j,\ell}\}_{j \geq 0}$ behaving as stipulated in Definition 1.1. Set $L_{n,k,\ell} = \sum_{i=0}^{n} p_{i,k}q_{n-i,\ell}$. By Lemma 2.5, $(I+J)^n$ is the intersection of the $L_{n,k,\ell}$ as k and ℓ vary. Thus $\mathrm{Ass}(C/(I+J)^n)$ is a subset of $\bigcup_{k,\ell} \mathrm{Ass}(C/L_{n,k,\ell})$. Then Lemma 3.1 proves the inclusions $\subseteq$ in the two displays. The opposite inclusion in the latter display follows by Lemma 3.2. $\qquad\square$

Remark 3.3. Equality may fail in the first display involving $\mathrm{Ass}(C/(I+J)^n)$ – see Example 4.2.

Corollary 3.4. *Let K be an algebraically closed field, let $A = K[x_1, \ldots, x_a]$, $B = K[y_1, \ldots, y_b]$, and $C = A \otimes_K B = K[x_1, \ldots, x_a, y_1, \ldots, y_b]$ be polynomial K-algebras. Let I be an ideal in A and J an ideal in B, and suppose that for each pair $1 \leq k \leq r$ and $1 \leq \ell \leq s$, the collections $\{p_{mk}\}_{m=0}^{\infty}$ and $\{q_{m\ell}\}_{m=0}^{\infty}$ from Definition 1.1 are filtrations (perhaps manufactured via Lemma 2.1 first). Then for each positive integer n,*

$$(I+J)^n = \bigcap_{\ell=1}^{s} \bigcap_{k=1}^{r} \left(\sum_{i=0}^{n} p_{ik} \cdot q_{n-i,\ell} \right)$$

is a possibly redundant primary decomposition. Furthermore,

$$\mathcal{A}_C(I+J) = \bigcup\{P + Q : P \in \mathcal{A}(I), Q \in \mathcal{A}(J)\}$$

and so in terms of cardinality of sets, we have the relation

$$\#\mathcal{A}_C(I+J) = \#\mathcal{A}_C(I) \cdot \#\mathcal{A}_C(J).$$

Proof. When K is algebraically closed, the sum of expansions for a prime ideal in A and a prime ideal in B is a prime ideal in C – see Milne [8, Prop. 4.15]. Thus by Lemma 3.1, $\sum_{i=0}^{n} p_{ik} \cdot q_{n-i,\ell}$ is $(P_k + Q_\ell)$-primary if it is proper. The first display in the statement of the corollary is simply Lemma 2.5. The corollary then follows in full as a consequence of Theorem 1.4. $\qquad\square$

Proof of Corollary 1.7. Let $\mathcal{P}$ be associated to $(I+J)^{n-1}$. By Theorem 1.4, $\mathcal{P}$ is minimal over an ideal of the form $P_k + Q_\ell$, where P_k is associated to I^i and Q_ℓ is associated to J^j for some $i, j \in \{1, \ldots, n-1\}$. Indeed take i and j to be the smallest positive integers such that P_k is associated to I^i and Q_ℓ is associated to J^j. Lemma 3.2 then says that $i + j - 1 \leq n - 1$, i.e., that $j \leq n - i$. By a result of Katz and Ratliff [7, (1.3) Theorem], since powers of J are integrally closed, Q_ℓ is associated to J^{n-i} as well.

By Lemma 2.8, there exists $f \in I^{i-1}$ such that $P_k = I^i : f$, and by Lemma 2.9, there exists $g \in J^{n-i-1}$ such that $Q_\ell = J^{n-i} : g$. Then a proof similar to the beginning part of the proof of Lemma 3.2 shows that $(I+J)^n : fg = P_k + Q_\ell$, and so the prime ideal $\mathcal{P}$ minimal over $P_k + Q_\ell$ is associated to $(I+J)^n$. The corollary follows in full. $\qquad\square$

4. A Few Concluding Examples

We close with illustrative examples addressing Remark 3.3. We use notation as in Definition 1.1. Recall that given a prime ideal $\mathfrak{q}$ in a Noetherian ring R, its c-**th symbolic power** ($c \in \mathbb{Z}_{>0}$) is

$$\mathfrak{q}^{(c)} = \mathfrak{q}^c R_{\mathfrak{q}} \cap R = \{f \in R \colon uf \in \mathfrak{q}^c \text{ for some } u \in R - \mathfrak{q}\}.$$

Example 4.1. In this example, for all $n \geq 1$,

$$\mathrm{Ass}(C/(I+J)^n) = \left\{ P + Q : P \in \bigcup_{i=1}^n \mathrm{Ass}(C/I^i), Q \in \bigcup_{i=1}^n \mathrm{Ass}(C/J^i) \right\}.$$

Let $I = (x_1^4, x_1^3 x_2, x_1^2 x_2^2 x_3, x_1 x_2^3, x_2^4)$ and $J = (y_1^3 - y_2 y_3, y_2^2 - y_1 y_3, y_3^2 - y_1^2 y_2)$. Then $P_1 = (x_1, x_2)$ is a minimal prime over I and $P_2 = (x_1, x_2, x_3)$ is associated only to I and to no other power of I. For all $n \geq 1$ we have $p_{n1} = (x_1, x_2)^{4n}$, and for filtration sake we set $p_{n2} = p_{12} = (x_1^4, x_1^3 x_2, x_1 x_2^3, x_2^4, x_3)$. The ideal $Q_1 = J$ is the prime ideal defining the monomial curve (t^3, t^4, t^5), and by [5], $Q_2 = (y_1, y_2, y_3)$ is associated to all higher powers of J. Here $q_{n1} = J^{(n)}$, $q_{12} = C$ and other q_{n2} are proper ideals such that $\{q_{n2}\}_{n=0}^\infty$ is a filtration. Below we need the fact that for all positive n, $J^{n-1} \cap q_{n\ell}$ properly contains J^n. For $\ell = 2$ this holds by comparing the J-primary components and noting that the symbolic powers of J are distinct, and for $\ell = 1$ we first observe that for a large integer M, $q_{n-1,1} = J^{n-1} : y_1^M$ and $q_{n1} = J^n : y_1^M$, so that $q_{n-1,2} = J^{n-1} + (y_1^M)$ and $q_{n2} = J^n + (y_1^M)$. Then

$$J^{n-1} \cap q_{n2} = J^{(n-1)} \cap (J^n + (y_1^M)) = J^n + J^{(n-1)} \cap (y_1^M) = J^n + y_1^M J^{(n-1)},$$

and $y_1^M J^{(n-1)}$ is not a subset of J^n as it is not a subset after localizing at J. Thus $J^{n-1} \cap q_{n1}$ properly contains J^n. By Lemma 2.5,

$$I + J = \bigcap_{k=1}^2 (p_{0k} q_{11} + p_{1k} q_{01}) = \bigcap_{k=1}^2 (q_{11} + p_{1k}) = \bigcap_{k=1}^2 (J + p_{1k}),$$

and clearly both components are needed. For $n \geq 1$,

$$(I + J)^n = \bigcap_{\ell=1}^2 \bigcap_{k=1}^2 \left(\sum_{i=0}^n p_{ik} q_{n-i,\ell} \right),$$

and here all four components are needed as we prove next. By Lemma 2.3,

$$\bigcap_{\ell=1}^2 \left(\sum_{i=0}^n p_{i1} q_{n-i,\ell} \right) = \sum_{i=0}^n p_{i1} J^{n-i} = \sum_{i=0}^n (x_1, x_2)^{4i} J^{n-i} = \left((x_1, x_2)^4 + J \right)^n.$$

The intersection of this with $\sum_{i=0}^n p_{i2} q_{n-i,\ell} = q_{n\ell} + p_{12} \sum_{i=1}^n q_{n-i,\ell} = q_{n\ell} + (x_1^4, x_1^3 x_2, x_1 x_2^3, x_2^4, x_3)$, contains $x_1^2 x_2^2 (J^{n-1} \cap q_{n\ell})$ which is not in $(I + J)^n$. Thus for $n \geq 2$ in the intersection of $(I + J)^n$ as in Lemma 2.5 we cannot omit any component involving P_2. We certainly cannot omit the minimal component $P_1 + Q_1$, and we cannot omit the component for $P_1 + Q_2$ because

$$\left(\sum_{i=0}^n p_{i1} q_{n-i,1} \right) \cap \left(\sum_{i=0}^n p_{i2} q_{n-i,1} \right) \cap \left(\sum_{i=0}^n p_{i2} q_{n-i,2} \right) = \left(\sum_{i=0}^n (x_1, x_2)^{4i} q_{n-i,1} \right) \cap \left(\sum_{i=0}^n p_{i2} J^{n-i} \right)$$

$$= \left(\sum_{i=0}^n (x_1, x_2)^{4i} q_{n-i,1} \right) \cap \left(J^n + (x_1^4, x_1^3 x_2, x_1 x_2^3, x_2^4, x_3) \right)$$

contains $x_3 q_{n1}$ and is thus not a subset of $(I + J)^n$. This proves that for all $n \geq 2$, $(I + J)^n$ has four associated primes.

Example 4.2. In this example, for all $n \geq 2$,

$$\operatorname{Ass}(C/(I+J)^n) \subsetneq \left\{ P + Q : P \in \bigcup_{i=1}^n \operatorname{Ass}(C/I^i), Q \in \bigcup_{i=1}^n \operatorname{Ass}(C/J^i) \right\}.$$

Let I be as in the previous example, and let $J = (y_1^4, y_1^3 y_2, y_1^2 y_2^2 y_3, y_1 y_2^3, y_2^4)$, which is the ideal I when replacing $x_i \mapsto y_i$. Thus I and J each have two associated primes and higher powers have only one associated prime. It is straightforward to show that $I + J$ has four associated primes, namely all the combinations $P_i + Q_j$. We prove next that for all $n \geq 2$, $P_2 + Q_2$ is not associated to $(I+J)^n$, i.e., that the component $p_{12} + q_{12}$ is redundant in the intersection in the Lemma 2.3. Namely,

$$\left(\sum_{i=0}^n p_{i1} q_{n-i,1} \right) \cap \left(\sum_{i=0}^n p_{i1} q_{n-i,2} \right) \cap \left(\sum_{i=0}^n p_{i2} q_{n-i,1} \right) = \left(\sum_{i=0}^n p_{i1} J^{n-i} \right) \cap (q_{n1} + p_{12}),$$

and by the nature of monomial ideals and since $q_{n1} = J^n$, this intersection equals

$$\sum_{i=0}^n p_{i1} \left(J^{n-i} \cap q_{n1} \right) + \sum_{i=0}^n \left(p_{i1} \cap p_{12} \right) J^{n-i} = J^n + \sum_{i=0}^n I^i J^{n-i} = (I+J)^n.$$

References

[1] M. Brodmann, Asymptotic Stability of $\operatorname{Ass}(M/I^n M)$, Proc. of the AMS **74** (1979). no.1, 16-18. ↑1

[2] H. Dao, A. De Stefani, E. Grifo, C. Huneke, and L. Núñez-Betancourt, Symbolic Powers of Ideals, 2017. To appear in *Advances in Singularities and Foliations: Geometry, Topology and Applications*, Springer Proceedings in Mathematics & Statistics. arXiv/1708.03010. ↑1

[3] D.R. Grayson and M.E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, 1992. Available at http://www.uiuc.edu/Macaulay2/. ↑2

[4] H. T. Hà, H. D. Nguyen, N. V. Trung, and T. N. Trung, *Symbolic Powers of Sums of Ideals*, 2017. arXiv/1702.01766. ↑1, 2, 3, 5, 6

[5] C. Huneke, *The primary components of and integral closures of ideals in 3-dimensional regular local rings*, Mathematische Annalen **275** (1986), no. 4, 617–635. ↑8

[6] C. Huneke and I. Swanson, *Integral closure of ideals, rings, and modules*, London Mathematical Society Lecture Note Series, 336, Cambridge University Press, Cambridge, 2006. ↑5

[7] D. Katz and L. J. Ratliff, *On the prime divisors of IJ when I is integrally closed*, Archiv der Mathematik **51** (1988), 520–522. ↑2, 7

[8] J.S. Milne, *Algebraic Geometry*, 2013. Lecture Notes, Version 5.22. Online Link. ↑7

[9] R.M. Walker, Uniform Symbolic Topologies via Multinomial Expansions, 2017. To appear in *Proceedings of the AMS*. arXiv/1703.04530. ↑2

[10] Y. Yao, *Primary decomposition: compatibility, independence and linear growth*, Proceedings of the American Mathematical Society **130** (2002), no. 6, 1629–1637. ↑5

DEPARTMENT OF MATHEMATICS, REED COLLEGE, 3203 SE WOODSTOCK BLVD, PORTLAND, OR, 97202
E-mail address: `iswanson@reed.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI, 48109
E-mail address: `robmarsw@umich.edu`