UNIFORM APPROXIMATION OF ABHYANKAR VALUATION IDEALS IN FUNCTION FIELDS OF PRIME CHARACTERISTIC

RANKEYA DATTA

Abstract. We prove the prime characteristic analogue of a characteristic 0 result of Ein, Lazarsfeld and Smith [ELS03] on uniform approximation of valuation ideals associated to real-valued Abhyankar valuations centered on regular varieties.

Let X be a variety over a perfect field k of prime characteristic, with function field K. Suppose v is a real-valued valuation of K/k centered on X. Then for all $m \in \mathbb{R}$, we have the *valuation ideals*

$$\mathfrak{a}_m(X) \subseteq \mathcal{O}_X,$$

consisting of local sections f such that $v(f) \geq m$. When $X = \operatorname{Spec}(A)$, we use $\mathfrak{a}_m(A)$ to denote the ideal $\{a \in A : v(a) \geq m\}$ of A.

The goal of this paper is to use the theory of asymptotic test ideals in positive characteristic to prove the following uniform approximation result for Abhyankar valuation ideals established in the characteristic 0 setting by Ein, Lazarsfeld and Smith [ELS03].

Theorem A. *Let X be a regular variety over a perfect field k of prime characteristic with function field K. For any non-trivial, real-valued Abhyankar valuation v of K/k centered on X, there exists $e \geq 0$, such that for all $m \in \mathbb{R}_{\geq 0}$ and $\ell \in \mathbb{N}$,*

$$\mathfrak{a}_m(X)^\ell \subseteq \mathfrak{a}_{\ell m}(X) \subseteq \mathfrak{a}_{m-e}(X)^\ell.$$

Thus, the theorem says that the valuation ideals $\mathfrak{a}_{\ell m}$ associated to a real-valued Abhyankar valuation are uniformly approximated by powers of $\mathfrak{a}_m$. One can think of an Abhyankar valuation (Section 2.2) as a generalization of the notion of order of vanishing along a prime divisor on a normal model. As such, these are the valuations that are the most geometrically tractable (see [Spi90, ELS03, FJ04, JM12, Tem13, Tei14, RS14, Blu16, DS16] for some applications). For example, a key point in our proof of Theorem A is that Abhyankar valuations over perfect fields are locally uniformizable [KK05, Theorem 1], which implies that any real-valued Abhyankar valuation admits a center on a regular local ring such that the corresponding valuation ideals become monomial with respect to a suitable choice of a regular system of parameters (Proposition 2.3.3). In other words, real-valued Abhyankar valuations over perfect fields of prime characteristic are *quasi-monomial*, a result which in characteristic 0 follows from resolution of singularities [ELS03, Proposition 2.8].

In [ELS03] (see also [Blu16]), Theorem A is proved over a ground field of characteristic 0 using the machinery of *asymptotic multiplier ideals*, first defined in [ELS01] in order to prove a uniformity statement about symbolic powers of ideals on regular varieties. It has since become clear that in prime characteristic a *test ideal* is an analogue of a multiplier ideal. Introduced by Hochster and Huneke in their work on tight closure [HH90], the first link between test and multiplier ideals was forged by Smith [Smi00] and Hara [Har01], following which Hara and Yoshida introduced the notion of test ideals of pairs [HY03]. Even in the absence of vanishing theorems in positive characteristic, test ideals of pairs were shown to satisfy many of the usual properties of multiplier ideals of pairs that make the latter such an effective tool in birational geometry [HY03, HT04, Tak06] (see also Theorem 4.1.3).

In this paper, we employ an asymptotic version of the test ideal of a pair to prove Theorem A, drawing inspiration from the asymptotic multiplier ideal techniques in [ELS03]. However, instead of utilizing tight closure machinery, our approach to asymptotic test ideals is based on Schwede's dual and simpler reformulation of test ideals using p^{-e}*-linear maps*, which are like maps inverse to Frobenius [Sch10, Sch11] (see also [Smi95, LS01]).

Asymptotic test ideals are associated to graded families of ideals (Definition 4.2.1), an example of the latter being the family of valuation ideals $\mathfrak{a}_\bullet := \{\mathfrak{a}_m(A)\}_{m\in\mathbb{R}_{\geq 0}}$. For each $m \geq 0$, one constructs the *m-th asymptotic test ideal* $\tau_m(A, \mathfrak{a}_\bullet)$ of the family $\mathfrak{a}_\bullet$, and then Theorem A is deduced using

Theorem B. *Let v be a non-trivial real-valued Abhyankar valuation of K/k, centered on a regular local ring $(A, \mathfrak{m})$, where A is essentially of finite type over the perfect field k of prime characteristic with fraction field K. Then there exists $r \in A - \{0\}$ such that for all $m \in \mathbb{R}_{\geq 0}$,*

$$r \cdot \tau_m(A, \mathfrak{a}_\bullet) \subseteq \mathfrak{a}_m(A).$$

In other words, $\bigcap_{m\in\mathbb{R}_{\geq 0}}(\mathfrak{a}_m : \tau_m(A, \mathfrak{a}_\bullet)) \neq (0)$.

Finally, as in [ELS03], Theorem B also gives a new proof of a prime characteristic version of Izumi's theorem for arbitrary real-valued Abhyankar valuations with a common regular center (see also the more general work of [RS14]).

Corollary C (**Izumi's Theorem for Abhyankar valuations in prime characteristic**). *Let v and w be non-trivial real-valued Abhyankar valuations of K/k, centered on a regular local ring $(A, \mathfrak{m})$, as in Theorem B. Then there exists a real number $C > 0$ such that for all $x \in A - \{0\}$,*

$$v(x) \leq Cw(x).$$

Thus, Corollary C implies that the valuation topologies on A induced by two non-trivial real-valued Abhyankar valuations are equivalent.

Hara defined and used asymptotic test ideals to give a prime characteristic proof of uniform bounds on symbolic power ideals [Har05], which is independent of Hochster and Huneke's earlier proof [HH02] and similar to the multiplier ideal approach of [ELS01]. This paper continues the efforts of Hara and other researchers to use test ideals to prove prime characteristic analogues of statements in characteristic 0 that were established using multiplier ideals.

Structure of the paper: Section 1 establishes notation, and Section 2 is a brief survey of Abhyankar valuations, including a proof of local monomialization of real-valued Abhyankar valuations over perfect fields of arbitrary characteristic (Proposition 2.3.3). In Section 3, we collect some basic facts about the Frobenius endomorphism needed in our discussion of asymptotic test ideals. Section 4 is the technical heart of the paper, and after summarizing the construction and basic properties of test ideals of pairs, we embark on a description of asymptotic test ideals. Included are discussions on the behavior of asymptotic test ideals under étale (Subsection 4.3) and birational maps (Subsection 4.4), which are needed in order to reduce the proof of Theorem B to the case of monomial ideals in a polynomial ring. We prove Theorem B in Section 5, and in Section 6 deduce Theorem A and Corollary C using Theorem B.

1. Conventions

All rings are commutative with unity, and ring homomorphisms preserve the multiplicative identities. We use $\mathbb{N}$ to denote the positive integers $\{1, 2, 3, \dots\}$. A *local ring* for us is a ring with a unique maximal ideal, which is *not necessarily* Noetherian. A ring S is *essentially of finite type* over a ring R if S is the localization of a finitely generated R-algebra at some multiplicative set. Given a domain R, a *fractional ideal* of R is an R-submodule of the fraction field $\mathrm{Frac}(R)$.

Given a field k, a *variety* X over k will be an integral, separated scheme of finite type over k. We will sometimes write "X is a variety of K/k" to mean X is a variety over k with function field K. In this paper, the field k is usually perfect of characteristic $p > 0$, and X is usually *regular*, that is, all the local rings $\mathcal{O}_{X,x}$ are regular. Since regular local rings are unique factorization domains, Weil and Cartier divisors coincide on any regular variety.

Regular varieties X of dimension n over a perfect field are smooth of relative dimension n [BLR90, §2.2, Proposition 15]. Then $\omega_X \coloneqq \bigwedge^n \Omega_{X/k}$ is a line bundle on X, known as the *canonical bundle*, and any Cartier divisor D on X such that $\mathcal{O}_X(D) \cong \bigwedge^n \Omega_{X/k}$ is called a *canonical divisor*. The linear equivalence class of D is called the *canonical class*, and denoted K_X. By abuse of notation, we often denote a choice of a canonical divisor by K_X. Note ω_X is a dualizing sheaf for X. More precisely, if $f : X \to \mathrm{Spec}(k)$ is the smooth structure map, $f^!(k[0]) = \omega_X[\dim X]$ is the normalized dualizing complex of X [Har66, Chap. V, Theorem 8.3].

2. Abhyankar Valuations

In this section, we fix a ground field k of arbitrary characteristic, and a finitely generated field extension K of k. Additional restrictions on k will be imposed as needed.

2.1. **Background on valuations.** A *valuation v of K/k* with *value group* Γ_v (where Γ_v is a totally ordered abelian group written additively) is a surjective group homomorphism

$$v\colon K^\times \twoheadrightarrow \Gamma_v,$$

such that $v(k^\times) = \{0\}$, and for all $x, y \in K^\times$, if $x + y \neq 0$, then $v(x + y) \geq \min\{v(x), v(y)\}$. The *valuation ring* R_v of v is the ring $\{x \in K^\times : v(x) \geq 0\} \cup \{0\}$, which is local with maximal ideal $\mathfrak{m}_v \coloneqq \{x \in K^\times : v(x) > 0\} \cup \{0\}$ and residue field $\kappa_v \coloneqq R_v/\mathfrak{m}_v$. Both R_v and κ_v are k-algebras.

Given a local, integral k-algebra $(R, \mathfrak{m})$ with fraction field K, we say v is *centered* on R if $(R_v, \mathfrak{m}_v)$ dominates $(R, \mathfrak{m})$, that is, $R \subseteq R_v$ and $\mathfrak{m} = \mathfrak{m}_v \cap R$. Globally, given a variety X over k with function field K, we say v is *centered* on X if the canonical map $\mathrm{Spec}(K) \to X$ extends to a morphism $\mathrm{Spec}(R_v) \to X$. The image in X of the closed point of $\mathrm{Spec}(R_v)$ is called the *center* of v on X. Note when $X = \mathrm{Spec}(A)$, v has a center on X if and only if $A \subseteq R_v$.

We are primarily interested in valuations whose value groups are ordered subgroups of $\mathbb{R}$, a condition that is equivalent to the valuation rings having Krull dimension 1 [Mat89, Theorem 10.7]. For any such real-valued valuation v with center x on X and any $m \in \mathbb{R}$, one has the *m-th valuation ideal* $\mathfrak{a}_m(X) \subseteq \mathcal{O}_X$, where locally

$$\Gamma(U, \mathfrak{a}_m(X)) = \begin{cases} \{f \in \mathcal{O}_X(U) : v(f) \geq m\}, & \text{if } x \in U, \\ \mathcal{O}_X(U), & \text{if } x \notin U. \end{cases}$$

Note $\mathfrak{a}_m(X) = \mathcal{O}_X$ when $m \leq 0$. If $X = \mathrm{Spec}(A)$, we use $\mathfrak{a}_m(A)$ to denote the ideal $\{a \in A : v(a) \geq m\}$ of A, and when X or A is clear from context, we just write $\mathfrak{a}_m$.

An important feature of valuation ideals implicitly used throughout the paper is the following:

Lemma 2.1.1. *Given an affine variety* $\mathrm{Spec}(A)$*, if $\mathfrak{p}$ is the prime ideal of A corresponding to the center of a real-valued valuation v on* $\mathrm{Spec}(A)$*, then for all real numbers $m > 0$, the ideal $\mathfrak{a}_m(A)$ is $\mathfrak{p}$-primary. Moreover, $\mathfrak{a}_m(A_\mathfrak{p}) = \mathfrak{a}_m(A)A_\mathfrak{p}$.*

Proof. For $b \in A$, if $v(b) > 0$, then by the Archimedean property, $nv(b) = v(b^n) \geq m$, for some $n \in \mathbb{N}$. This shows that $\mathfrak{p}$ is the radical of $\mathfrak{a}_m(A)$. If $ab \in \mathfrak{a}_m(A)$ and $a \notin \mathfrak{a}_m(A)$, then $v(b) > 0$, so that for some n, $b^n \in \mathfrak{a}_m(A)$, as we just showed. Hence $\mathfrak{a}_m(A)$ is $\mathfrak{p}$-primary.

Note if $s \notin A - \mathfrak{p}$, $v(s) = 0$. Thus, the inclusion $\mathfrak{a}_m(A)A_\mathfrak{p} \subseteq \mathfrak{a}_m(A_\mathfrak{p})$ is clear. Conversely, if $a/s \in \mathfrak{a}_m(A_\mathfrak{p})$, since $v(a/s) = v(a) - v(s) = v(a)$, we get $a \in \mathfrak{a}_m(A)$, proving $\mathfrak{a}_m(A_\mathfrak{p}) \subseteq \mathfrak{a}_m(A)A_\mathfrak{p}$. □

2.2. **Abhyankar valuations.** Associated to a valuation v of K/k with value group Γ_v and residue field κ_v are two important invariants.

Definition 2.2.1. The **rational rank** of v, abbreviated $\operatorname{rat.rk} v$, is by definition the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_\mathbb{Z} \Gamma_v$, and the **transcendence degree of** v, abbreviated $\operatorname{tr.deg} v$, is the transcendence degree of κ_v over k.

The following fundamental inequality relates the rational rank and transcendence degree of a valuation (see [Abh56] for a generalization):

Theorem 2.2.2. [Bou89, VI, §10.3, Corollary 1] *Let v be a valuation of K/k, with value group Γ_v and residue field κ_v. Then*

$$\operatorname{rat.rk} v + \operatorname{tr.deg} v \leq \operatorname{tr.deg} K/k.$$

If equality holds in the above inequality, then Γ_v is a finitely generated abelian group (hence $\Gamma_v \cong \mathbb{Z}^{\oplus r}$, for $r = \operatorname{rat.rk} v$), and κ_v is a finitely generated field extension of k.

An **Abhyankar valuation** v of K/k is a valuation for which $\operatorname{rat.rk} v + \operatorname{tr.deg} v = \operatorname{tr.deg} K/k$.

Examples 2.2.3.

(1) (*Prototype*) Let X be a normal variety over k of dimension n, and D a prime divisor on X. Then we get a valuation of $K(X)/k$, denoted ord_D, which is called the *order of vanishing along D*. The value group of ord_D is $\mathbb{Z}$, and the valuation ring $\mathcal{O}_{X,D}$ equals the stalk at the generic point of D. Then ord_D is Abhyankar since it is has rational rank 1 and transcendence degree $n-1$.

In fact, Zariski showed that if v is an Abhyankar valuation of K/k of rational rank 1, then there exists some normal model X of K/k and a prime divisor D on X such that v is given by order of vanishing along D [SZ60, VI, §14, Theorem 31].

(2) There are real-valued Abhyankar valuations on $\mathbb{F}_p(X,Y)/\mathbb{F}_p$ which are not discrete. For example, let α be any irrational number, and Γ the subgroup $\mathbb{Z} \oplus \mathbb{Z}\alpha$ of $\mathbb{R}$ with the induced order. There exists a unique valuation v of $\mathbb{F}_p(X,Y)/\mathbb{F}_p$ with value group Γ such that

$$v(X) = 1 \text{ and } v(Y) = \alpha.$$

Then v is Abhyankar with rational rank 2 and transcendence degree 0. Note v is not a discrete valuation because Γ is not isomorphic to $\mathbb{Z}$.

(3) (*Non real-valued Abhyankar valuation*) Consider again the extension $\mathbb{F}_p(X,Y)/\mathbb{F}_p$. Giving $\mathbb{Z} \oplus \mathbb{Z}$ the lexicographical order, there exists a unique valuation v_{lex} of $\mathbb{F}_p(X,Y)/\mathbb{F}_p$ with value group $\mathbb{Z} \oplus \mathbb{Z}$ such that

$$v_{lex}(X) = (1,0) \text{ and } v_{lex}(Y) = (0,1).$$

Since the rational rank of v_{lex} is 2, it is also Abhyankar. However, there is no order preserving embedding of $\mathbb{Z} \oplus \mathbb{Z}$ with lex order into $\mathbb{R}$.

(4) (*Non-example*) Take the formal Laurent series field $\mathbb{F}_p((t))$. This has the t-adic valuation v_t over $\mathbb{F}_p$, whose corresponding valuation ring is the formal power series ring $\mathbb{F}_p[[t]]$. Choose an embedding of fields

$$\mathbb{F}_p(X,Y) \hookrightarrow \mathbb{F}_p((t))$$

that maps $X \mapsto t$, and Y to $p(t) \in \mathbb{F}_p[[t]]$ such that $\{t, p(t)\}$ are algebraically independent over $\mathbb{F}_p$. Restricting v_t to $\mathbb{F}_p(X,Y)$ gives a discrete valuation w of $\mathbb{F}_p(X,Y)/\mathbb{F}_p$, such that

$$\kappa_w = \mathbb{F}_p.$$

This is because κ_w contains $\mathbb{F}_p$, and is also contained in the residue field of $\mathbb{F}_p[[t]]$, which equals $\mathbb{F}_p$. Then rat.rk $w = 1$, tr.deg $w = 0$, and so w is not an Abhyankar valuation.

2.3. **Local uniformization of Abhyankar valuations.** Local uniformization of a valuation v of K/k is a local analogue of resolution of singularities. In its simplest form, it asks if there exists a regular variety of K/k on which v admits a center. Zariski solved local uniformization when k has characteristic 0 [Zar40], long before Hironaka's seminal work on resolution of singularities, and used it to simplify the proof of resolution of singularities of surfaces [Zar42]. More recently, de Jong's work on alterations [dJ96] showed that local uniformization can always be achieved up to a finite extension of the function field, and that in characteristic $p > 0$, this extension can even be taken to be purely inseparable [Tem13]. However, local uniformization remains elusive in positive characteristic, although for Abhyankar valuations Knaf and Kuhlmann establish the following:

Theorem 2.3.1. [KK05, Theorem 1.1] *Let K be a finitely generated field extension of any field k, and v an Abhyankar valuation of K/k with valuation ring $(R_v, \mathfrak{m}_v, \kappa_v)$. Suppose that κ_v is separable over k. Then given any finite subset $Z \subset R_v$, there exists a variety X of K/k, and a center x of v on X satisfying the following properties:*

(1) *$\mathcal{O}_{X,x}$ is a regular local ring of dimension equal to the rational rank of v.*
(2) *$Z \subseteq \mathcal{O}_{X,x}$, and there exists a regular system of parameters $x_1, \dots, x_d$ of $\mathcal{O}_{X,x}$ such that every $z \in Z$ admits a factorization*

$$z = ux_1^{a_1} \dots x_d^{a_d},$$

for some $u \in \mathcal{O}_{X,x}^\times$, and $a_i \geq 0$.

Remark 2.3.2. Any Abhyankar valuation v over a perfect field k satisfies Theorem 2.3.1 since κ_v/k is then automatically separable. There are other related approaches to local uniformization of Abhyankar valuations. For instance, see [Tem13, Section 5] and [Tei14, Corollary 7.25].

The presence of the set Z in Theorem 2.3.1 allows us to deduce that real-valued Abhyankar valuations over perfect fields of arbitrary characteristic can be locally monomialized. This will be important in the proof of Theorem B.

Proposition 2.3.3 (**Local monomialization**). *Assume k is perfect, and v is a non-trivial Abhyankar valuation of K/k of rational rank d, centered on an affine variety* $\operatorname{Spec}(R)$ *of K/k. Then there exists a variety* $\operatorname{Spec}(S)$ *of K/k, along with an inclusion of rings $R \hookrightarrow S$ satisfying:*

(a) $\operatorname{Spec}(S)$ *is regular, and v is centered at $x \in \operatorname{Spec}(S)$ such that $\mathcal{O}_{\operatorname{Spec}(S),x}$ is a regular local ring of dimension d, and the induced map of residue fields $\kappa(x) \hookrightarrow \kappa_v$ is an isomorphism.*
(b) *There exists a regular system of parameters $\{x_1, \dots, x_d\}$ of $\mathcal{O}_{\operatorname{Spec}(S),x}$ such that $v(x_1), \dots,$ $v(x_d)$ freely generate the value group Γ_v.*

If in addition v is real-valued, then the valuation ideals of $\mathcal{O}_{\operatorname{Spec}(S),x}$ are generated by monomials in $x_1, \dots, x_d$.

Proof. Since the value group Γ_v is free of rank d (Theorem 2.2.2), one can choose $r_1, \dots, r_d \in R_v$ such that $v(r_1), \dots, v(r_d)$ freely generate Γ_v. Also, because κ_v is a finitely generated field extension of k, there exist $y_1, \dots, y_j \in R_v$ whose images in κ_v generate κ_v over k. Let $t_1, \dots, t_n \in K$ be generators for R over k. Then $t_1, \dots, t_n \in R_v$ because v is centered on $\operatorname{Spec}(R)$. Defining

$$Z := \{t_1, \dots, t_n, y_1, \dots, y_j, r_1, \dots, r_d\},$$

by Theorem 2.3.1 there exists a variety X over k with function field K such that v is centered at a regular point $x \in X$ of codimension d, $Z \subseteq \mathcal{O}_{X,x}$, and there exists a regular system of parameters $\{x_1, \dots, x_d\}$ of $\mathcal{O}_{X,x}$ with respect to which every $z \in Z$ can be factorized as

$$z = u x_1^{a_1} \dots x_d^{a_d},$$

for some $u \in \mathcal{O}_{Y,y}^{\times}$, and integers $a_i \geq 0$. In particular, each $v(r_i)$ is a $\mathbb{Z}$-linear combination of $v(x_1), \dots, v(x_d)$, which shows that $\{v(x_1), \dots, v(x_d)\}$ also freely generates Γ_v. Moreover, by our choice of Z, $\kappa(x) \hookrightarrow \kappa_v$ is an isomorphism.

Since $t_1, \dots, t_n \in \mathcal{O}_{X,x}$, we have an inclusion $R \subseteq \mathcal{O}_{X,x}$. Now restricting to an affine neighborhood of x, we may assume $X = \operatorname{Spec}(S)$, where $t_1, \dots, t_n \in S$ and S is regular. Then by construction, $R \subseteq S$, and parts (a) and (b) of the Proposition are satisfied.

Suppose v is also real-valued, and $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}_{\operatorname{Spec}(S),x}$. We want to show that the valuation ideals $\mathfrak{a}_m$ of $\mathcal{O}_{\operatorname{Spec}(S),x}$ are monomial in $\{x_1, \dots, x_d\}$. For $m > 0$, since $\mathfrak{a}_m$ is $\mathfrak{p}$-primary, we know that $\mathfrak{p}^n \subseteq \mathfrak{a}_m$ for some $n \in \mathbb{N}$. Note $\mathfrak{p}^n$ has a monomial generating set $\{x_1^{\alpha_1} \dots x_d^{\alpha_d} : \alpha_1 + \dots + \alpha_d = n\}$. Modulo $\mathfrak{p}^n$, any non-zero element $t \in \mathfrak{a}_m$ can be expressed as a finite sum s with unit coefficients of monomials of the form $x_1^{\beta_1} \dots x_d^{\beta_d}$, with $0 < \beta_1 + \dots + \beta_d \leq n - 1$. Then expressing $t = s + u$, for $u \in \mathfrak{p}^n$, we see that $v(s) \geq m$ because both $v(t), v(u) \geq m$. However, $v(s)$ equals the smallest valuation of the monomials $x_1^{\beta_1} \dots x_d^{\beta_d}$ appearing in the sum since monomials have distinct valuations. Thus, each such $x_1^{\beta_1} \dots x_d^{\beta_d} \in \mathfrak{a}_m$, completing the proof. $\square$

Example 2.3.4. Let v_π be the valuation on $\mathbb{F}_p(X, Y, Z)/\mathbb{F}_p$ with value group $\mathbb{Z} \oplus \mathbb{Z}\pi \subset \mathbb{R}$ such that $v_\pi(X) = 1 = v_\pi(Y)$, $v_\pi(Z) = \pi$, and for any polynomial $\sum b_{\alpha\beta\gamma} X^\alpha Y^\beta Z^\gamma \in \mathbb{F}_p[X, Y, Z]$,

$$v_\pi(\sum b_{\alpha\beta\gamma} X^\alpha Y^\beta Z^\gamma) = \inf\{\alpha + \beta + \pi\gamma : b_{\alpha\beta\gamma} \neq 0\}.$$

One can verify that v_π is Abhyankar with $\operatorname{rat.rk} v_{pi} = 2$ and $\operatorname{tr.deg} v_\pi = 1$. For example Y/X is a unit in the valuation ring R_{v_π} whose image in the residue field is transcendental over $\mathbb{F}_p$. Note v_π is centered on $\mathbb{A}^3_{\mathbb{F}_p} = \operatorname{Spec}(\mathbb{F}_p(X, Y, Z))$ at the origin. However, the system of parameters X, Y, Z of the local ring at the origin do not freely the generate the value group. On the other hand, blowing up the origin and considering the affine chart $\operatorname{Spec}(\mathbb{F}_p[X, \frac{Y}{X}, \frac{Z}{X}])$, we see that v_π is centered on $\mathbb{F}_p[X, \frac{Y}{X}, \frac{Z}{X}]$ with center $(X, Z/X)$, and now the regular system of parameters $X, Z/X$ of the local ring $\mathbb{F}_p[X, \frac{Y}{X}, \frac{Z}{X}]_{(X, \frac{Z}{X})}$ do indeed freely generate the value group.

3. Characteristic p preliminaries

3.1. **The Frobenius Endomorphism.** We fix a prime number $p > 0$. For any ring R of characteristic p, we have the *Frobenius map*

$$F : R \to R$$

that sends $r \mapsto r^p$. We denote the target copy of R with R-module structure induced by restriction of scalars via F as F_*R, and for $x \in R$, we also sometimes denote the corresponding element of F_*R as $F_*(x)$. Thus, R acts on F_*R by

$$r \cdot F_*(x) = F_*(r^p x).$$

In what follows, both F and its iterates $F^e : R \to F^e_* R$ will play an important role. Note that the operation F^e_* gives an exact functor from $\operatorname{Mod}_R \to \operatorname{Mod}_R$ since it is just restruction of scalars.

Globally, if X is a scheme over $\mathbb{F}_p$, then we have the *(absolute) Frobenius endomorphism*

$$F : X \to X,$$

which on the underlying topological spaces is the identity map, while inducing a map on structure sheaves

$$\mathcal{O}_X \to F_*\mathcal{O}_X$$

by raising local sections to their p-th powers. By a common abuse of notation, we also denote the map on structure sheaves induced by Frobenius by $F : \mathcal{O}_X \to F_*\mathcal{O}_X$.

We will primarily be interested in the class of schemes for which the Frobenius morphism is finite. These have a special name.

Definition 3.1.1. A scheme X over $\mathbb{F}_p$ is **F-finite** if Frobenius on X is a finite morphism. A ring R is **F-finite** if $\mathrm{Spec}(R)$ is F-finite, or equivalently, if F_*R is a finitely generated R-module.

F-finite rings and schemes are ubiquitous. Indeed, it is easy to show that any ring essentially of finite type over a perfect (or even F-finite) field is F-finite. This means that all varieties in this paper are F-finite. More generally, finiteness of Frobenius is preserved under finite type ring maps, localization, and completion of Noetherian local rings. F-finite Noetherian rings also satisfy good geometric properties. For example, Kunz showed that F-finite Noetherian rings are excellent [Kun76], the converse being true under mild restrictions [DS17].

Quite remarkably, Frobenius is able to detect regularity of a local ring.

Theorem 3.1.2. [Kun69, Theorem 2.1] *Let $(R, \mathfrak{m})$ be a Noetherian local ring of prime characteristic. Then R is regular if and only if the Frobenius map $F : R \to F_*R$ is flat.*

Globally, Kunz's theorem implies that if X is an F-finite, locally Noetherian scheme, then X is regular if and only if $F_*\mathcal{O}_X$ is a locally free sheaf on X. Theorem 3.1.2 was the starting point of systematically using the Frobenius map to study singularities in prime characteristic, and the various notions of singularities proposed and studied since Kunz's result (such as F-purity, Frobenius splitting, F-regularity, etc.) try to measure how far F_*R is from being a flat R-module.

3.2. **p^{-e}-linear maps.** Let X be a scheme over $\mathbb{F}_p$. For sheaves of $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$, $\mathcal{O}_X$-linear maps of the form

$$\eta : F^e_*\mathcal{G} \to \mathcal{F}$$

will play an important role in this paper, especially when we introduce test ideals. Following [BB11], we call such maps **p^{-e}-linear maps**. The name comes from the fact that if $r \in \mathcal{O}_X(U), g \in F^e_*\mathcal{G}(U) = \mathcal{G}(U)$ are local sections, then

$$\eta(r^{p^e} g) = r\eta(g).$$

For R-modules M, N we similarly have p^{-e}-linear maps of R-modules which are R-linear maps

$$F^e_* N \to M.$$

Perhaps the simplest example of such a map is a Frobenius splitting.

Definition 3.2.1. A scheme X over $\mathbb{F}_p$ is **Frobenius split** if there exists a p^{-1}-linear map $\eta : F_*\mathcal{O}_X \to \mathcal{O}_X$ such that the composition $\mathcal{O}_X \xrightarrow{F} F_*\mathcal{O}_X \xrightarrow{\eta} \mathcal{O}_X$ is the identity. The map η is then called a **Frobenius splitting of** X. A ring R is **Frobenius split** if $\mathrm{Spec}(R)$ is Frobenius split, that is, the Frobenius map $F : R \to F_*R$ admits a left inverse in the category of R-modules.

Example 3.2.2. Let $R = \mathbb{F}_p[x, y]$, where x, y are indeterminates. Then F_*R is a free R-module with basis

$$\{x^i y^j : 0 \le i, j \le p-1\}.$$

One obtains a Frobenius splitting $F_*R \to R$ by sending $1 \mapsto 1$ and the other basis elements to 0.

Schemes with non-trivial p^{-e}-linear maps often satisfy surprising cohomological properties. For example, Mehta and Ramanathan, who coined the term *Frobenius splitting*, showed that Frobenius split projective varieties satisfy Kodaira vanishing [MR85], even though Kodaira vanishing is known to fail in general in characteristic p [Ray78].

Along similar lines, the following observation of Mehta and Ramanathan, that associates geometric data to p^{-e}-linear maps, is at the basis of many local and global results in positive characteristic. We will later use it to examine the behavior of test ideals under birational maps (Proposition 4.4.2).

Proposition 3.2.3. [MR85, BS13] *Suppose X is a regular variety over a perfect field k of prime characteristic, and K_X is a canonical divisor on X. Then for any divisor D,*

$$\mathcal{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D), \mathcal{O}_X) \cong F^e_*\mathcal{O}_X((1-p^e)K_X - D).$$

Proof. By assumption F^e is finite, and we have

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D), \mathcal{O}_X) &\cong \mathcal{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D) \otimes \mathcal{O}_X(K_X), \mathcal{O}_X(K_X)) \\ &\cong \mathcal{H}om_{\mathcal{O}_X}\big(F^e_*\big(\mathcal{O}_X(D) \otimes (F^e)^*\mathcal{O}_X(K_X)\big), \mathcal{O}_X(K_X)\big) \\ &\cong \mathcal{H}om_{\mathcal{O}_X}\big(F^e_*\big(\mathcal{O}_X(D + p^e K_X)\big), \mathcal{O}_X(K_X)\big),\end{aligned}$$

where the first isomorphism follows from properties of locally free sheaves, the second isomorphism from the projection formula [Har77, II, Exer 5.1(d)], and the third isomorphism from the fact that the transition functions of $\mathcal{O}_X(K_X)$ are raised to their p^e-th powers under $(F^e)^*$.

Using duality for finite morphisms [Har77, III, Exer 6.10] we then get

$$\mathcal{H}om_{\mathcal{O}_X}\big(F^e_*\big(\mathcal{O}_X(D + p^e K_X)\big), \mathcal{O}_X(K_X)\big) \cong F^e_*\mathcal{H}om_{\mathcal{O}_X}\big(\big(\mathcal{O}_X(D + p^e K_X)\big), (F^e)^!\mathcal{O}_X(K_X)\big),$$

where $(F^e)^!\mathcal{O}_X(K_X)$ is the quasicoherent sheaf on X that satisfies

$$F^e_*(F^e)^!\mathcal{O}_X(K_X) = \mathcal{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X(K_X)).$$

Since X is smooth over the perfect field k and F^e is finite, one can now use the isomorphism $(F^e)^!\mathcal{O}_X(K_X) \cong \mathcal{O}_X(K_X)$ [BS13, Pg. 135, no. (6) & (8)] to conclude that $\mathcal{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D), \mathcal{O}_X) \cong F^e_*\mathcal{O}_X((1-p^e)K_X - D)$. □

Remark 3.2.4. Taking global sections under the isomorphism of Proposition 3.2.3, we see that non-zero p^{-e}-linear maps $F^e_*\mathcal{O}_X(D) \to \mathcal{O}_X$, upto pre-multiplication by a unit of $\Gamma(X, \mathcal{O}_X)$, are in bijection with effective divisors on X linearly equivalent to $(1-p^e)K_X - D$. More generally, if K is the function field of X regarded as a constant sheaf, then p^{-e}-linear maps $F^e_*\mathcal{O}_X(D) \to K$ correspond to *not necessarily* effective divisors linearly equivalent to $(1-p^e)K_X - D$. The image of $F^e_*\mathcal{O}_X(D) \to K$ lies in $\mathcal{O}_X$ precisely when the associated divisor is effective [BS13, Exercise 4.13].

The existence of non-trivial p^{-e}-linear maps puts restrictions on the geometry of a variety. For example, since a Frobenius splitting is a global section of $\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$, taking $D = 0$ in Proposition 3.2.3 we see that any Frobenius split, regular projective variety over a perfect field cannot have ample K_X. For other similar consequences, we recommend the survey [BS13].

4. Test Ideals

Beginning with a review of test ideals for pairs, the goal is to construct an asymptotic version that plays a role similar to asymptotic multiplier ideals in characteristic 0. We also examine how asymptotic test ideals transform under étale and birational ring maps.

4.1. Summary of test ideal for pairs.

First defined in [HY03], test ideals of pairs are intimately related to multiplier ideals of pairs. For example, it follows from results in [Smi00, Har01, HY03] that on reducing multiplier ideals of pairs modulo primes, one gets test ideals of pairs. However, instead of the tight-closure approach of [HY03], in this paper we adopt a dual point of view due to Schwede [Sch10], and define test ideals using p^{-e}-linear maps. For an excellent synthesis of the various viewpoints on test ideals we suggest the survey [ST12] to the interested reader.

Definition 4.1.1. Let R be an F-finite Noetherian domain, $\mathfrak{a} \subseteq R$ a non-zero ideal, and $t > 0$ a real number. The test ideal of the **pair** $(R, \mathfrak{a}^t)$ is defined to be the smallest *non-zero* ideal J of R such that for all $e \in \mathbb{N}$, and $\phi \in \mathrm{Hom}_R(F^e_* R, R)$,

$$\phi(F^e_*(J\mathfrak{a}^{\lceil t(p^e-1)\rceil})) \subseteq J.^1$$

It is denoted $\tau(R, \mathfrak{a}^t)$, or $\tau(\mathfrak{a}^t)$ when R is clear from context. If $\mathfrak{a} = R$, we usually just write $\tau(R)$, and call it **the test ideal of** R.

Remark 4.1.2. The existence of $\tau(R, \mathfrak{a}^t)$, which is not at all obvious from its definition, is a consequence of a deep result of Hochster and Huneke on the existence of *completely stable test elements* for Noetherian, F-finite domains [HH94, Theorem 5.10]. If $c \in R$ is such an element, then Hara and Takagi show that

$$\tau(\mathfrak{a}^t) = \sum_{e \in \mathbb{N}} \sum_{\phi} \phi(F^e_*(c\mathfrak{a}^{\lceil t(p^e-1)\rceil})),$$

where ϕ ranges over all elements of $\mathrm{Hom}_R(F^e_* R, R)$ [HT04, Lemma 2.1].

Having addressed the issue of the existence of test ideals, we now collect most of their basic properties, in part to highlight their similarity with multiplier ideals.

Theorem 4.1.3. *Suppose R is a Noetherian, F-finite domain with non-zero ideals $\mathfrak{a}$, $\mathfrak{b}$. Let $t > 0$ be a real number.*

(1) *If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\tau(\mathfrak{a}^t) \subseteq \tau(\mathfrak{b}^t)$.*
(2) *If the integral closures of $\mathfrak{a}$ and $\mathfrak{b}$ coincide, then $\tau(\mathfrak{a}^t) = \tau(\mathfrak{b}^t)$.*
(3) *If $s > t$, then $\tau(\mathfrak{a}^s) \subseteq \tau(\mathfrak{a}^t)$.*
(4) *For any $m \in \mathbb{N}$, $\tau((\mathfrak{a}^m)^t) = \tau(\mathfrak{a}^{mt})$.*
(5) *There exists some $\epsilon > 0$ depending on t such that for all $s \in [t, t+\epsilon]$, $\tau(\mathfrak{a}^s) = \tau(\mathfrak{a}^t)$.*
(6) *$\tau(R)$ defines the closed locus of prime ideals $\mathfrak{p}$ where $R_\mathfrak{p}$ is not strongly F-regular [2]. Thus, $\tau(R) = R$ if and only if R is strongly F-regular.*
(7) *We have $\tau(R)\mathfrak{a} \subseteq \tau(\mathfrak{a})$. Hence, if R is strongly F-regular (in particular regular), $\mathfrak{a} \subseteq \tau(\mathfrak{a})$.*
(8) *If $S \subset R$ is a multiplicative set, then $\tau(S^{-1}R, (\mathfrak{a}S^{-1}R)^t) = \tau(R, \mathfrak{a}^t)S^{-1}R$.*
(9) *If $(R, \mathfrak{m})$ is local, and $\widehat{R}$ denotes the $\mathfrak{m}$-adic completion of R, then $\tau(\widehat{R}, (\mathfrak{a}\widehat{R})^t) = \tau(R, \mathfrak{a}^t)\widehat{R}$.*
(10) *If R is regular, $x \in R$ a regular parameter, and $\overline{R} := R/xR$, then $\tau(\overline{R}, (\mathfrak{a}\overline{R})^t) \subseteq \tau(R, \mathfrak{a}^t)\overline{R}$.*
(11) ***(Subadditivity)** If R is regular and essentially of finite type over a perfect field, then for all $n \in \mathbb{N}$, $\tau(\mathfrak{a}^{nt}) \subseteq \tau(\mathfrak{a}^t)^n$.*

[1] In tight closure literature, this is usually called the *big* or *non-finitistic* test ideal of the pair $(R, \mathfrak{a}^t)$.

[2] We refrain from defining strong F-regularity [HH89, pg. 128] since we do not need this notion in any essential way. Note that regular F-finite domains are automatically strongly F-regular.

Indication of proof. For proofs and precise references for all statements, please consult [ST12, Section 6], or [SZ15, Theorem 4.6] when the ring is regular (the setting of this paper). □

Example 4.1.4 (**Test ideals of monomial ideals**). Let $\mathfrak{a}$ be a non-zero monomial ideal of the polynomial ring $R = k[x_1, \dots, x_n]$, where k is an F-finite field characteristic $p > 0$. For any real number $t > 0$, we let $P(t\mathfrak{a})$ denote the convex hull in $\mathbb{R}^n$ of the set

$$\{(ta_1, \dots, ta_n) \colon x_1^{a_1} \dots x_n^{a_n} \in \mathfrak{a}\},$$

and let $\mathrm{Int}(P(t\mathfrak{a}))$ be the points in this convex hull with integer coordinates. Then Hara and Yoshida show [HY03, Theorem 4.8] using the existence of log resolutions in the toric category that test and multiplier ideals of monomial ideals coincide, and so by [How01]

$$\tau(\mathfrak{a}^t) = \langle x_1^{b_1} \dots x_n^{b_n} \colon b_i \in \mathbb{N} \cup \{0\}, (b_1 + 1, \dots, b_n + 1) \in \mathrm{Int}(P(t\mathfrak{a}))\rangle.$$

4.2. **Asymptotic test ideals.** Asymptotic test ideals are defined for graded families of ideals, which we introduce first.

Definition 4.2.1. Let Φ be an additive sub-semigroup of $\mathbb{R}$, and R be a ring. A **graded family of ideals of R indexed by** Φ is a family of ideals $\{\mathfrak{a}_s\}_{s \in \Phi}$ such that for all $s, t \in \Phi$,

$$\mathfrak{a}_s \cdot \mathfrak{a}_t \subseteq \mathfrak{a}_{s+t}.$$

We also assume $\mathfrak{a}_s \neq 0$, for all s.

Examples 4.2.2.

(1) If $\mathfrak{a}$ is a non-zero ideal of a domain R, then $\{\mathfrak{a}^n\}_{n \in \mathbb{N} \cup \{0\}}$ is a graded family of ideals.
(2) If R is a Noetherian domain, the symbolic powers $\{\mathfrak{a}^{(n)}\}_{n \in \mathbb{N} \cup \{0\}}$ of a fixed non-zero ideal $\mathfrak{a}$ is an example of a graded family that was studied extensively in [ELS01, HH02].
(3) Let v be a non-trivial real-valued valuation of K/k centered on a domain R over k with fraction field K. Then the collection of valuation ideals $\{\mathfrak{a}_m(R)\}_{m \in \mathbb{R}_{\geq 0}}$ is a graded family of ideals by properties of a valuation (since v is non-trivial, the ideals $\mathfrak{a}_m$ are all non-zero).

Now suppose R is an F-finite, Noetherian domain of characteristic $p > 0$, and $\{\mathfrak{a}_m\}_{m \in \Phi}$ is a graded family of ideals of R, indexed by some sub-semigroup Φ of $\mathbb{R}$. Then for any real number $t > 0$, $m \in \Phi$, and $\ell \in \mathbb{N}$, we have

$$\tau(\mathfrak{a}_m^t) = \tau((\mathfrak{a}_m^\ell)^{t/\ell}) \subseteq \tau(\mathfrak{a}_{\ell m}^{t/\ell}).$$

Here the first equality follows from Theorem 4.1.3(4), and the inclusion follows from Theorem 4.1.3(1) using the fact that $\mathfrak{a}_m^\ell \subseteq \mathfrak{a}_{\ell m}$.

Thus, for any $m \in \Phi$, the set $\{\tau(\mathfrak{a}_{\ell m}^{1/\ell})\}_{\ell \in \mathbb{N}}$ is filtered under inclusion (for instance, $\tau(\mathfrak{a}_{\ell_1 m}^{1/\ell_1})$ and $\tau(\mathfrak{a}_{\ell_2 m}^{1/\ell_2})$ are both contained in $\tau(\mathfrak{a}_{\ell_1 \ell_2 m}^{1/\ell_1 \ell_2})$). Since R is a Noetherian ring, this implies that $\{\tau(\mathfrak{a}_{\ell m}^{1/\ell})\}_{\ell \in \mathbb{N}}$ has a unique maximal element under inclusion, which will be the m-th asymptotic test ideal.

Definition 4.2.3. For a graded family of ideals $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m \in \Phi}$ of a Noetherian, F-finite domain R, and for any $m \in \Phi$, we define the m**-th asymptotic test ideal of the graded system**, denoted $\tau_m(R, \mathfrak{a}_\bullet)$ (or $\tau_m(\mathfrak{a}_\bullet)$ when R is clear from context), as follows:

$$\tau_m(R, \mathfrak{a}_\bullet) \coloneqq \sum_{\ell \in \mathbb{N}} \tau(\mathfrak{a}_{\ell m}^{1/\ell}).$$

By the above discussion, $\tau_m(R, \mathfrak{a}_\bullet)$ equals $\tau(\mathfrak{a}_{\ell m}^{1/\ell})$, for some, equivalently all, $\ell \gg 0$.

Asymptotic test ideals satisfy appropriate analogues of properties satisfied by test ideals of pairs (Theorem 4.1.3), since they equal test ideals of suitable pairs. We highlight a few properties that will be important for us in the sequel.

Proposition 4.2.4. [Har05, SZ15] *Suppose R is a regular domain, essentially of finite type over a perfect field of positive characteristic, with a graded family of ideals $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m\in\Phi}$. We have the following:*

(1) *For any $m \in \Phi$, $\mathfrak{a}_m \subseteq \tau(\mathfrak{a}_m) \subseteq \tau_m(\mathfrak{a}_\bullet)$.*
(2) *For any $m \in \Phi$, $\ell \in \mathbb{N}$, $\mathfrak{a}_{\ell m} \subseteq \tau_{\ell m}(\mathfrak{a}_\bullet) \subseteq \tau_m(\mathfrak{a}_\bullet)^\ell$.*

Proof. We get (1) using Theorem 4.1.3(7), and the definition of asymptotic test ideals.

Property (2) is crucial, and is a consequence of the subadditivity property of test ideals (Theorem 4.1.3(11)). The first inclusion $\mathfrak{a}_{\ell m} \subseteq \tau_{\ell m}(\mathfrak{a}_\bullet)$ follows from (1). For the second inclusion, for all $n \gg 0$, we have

$$\tau_{\ell m}(\mathfrak{a}_\bullet) = \tau(\mathfrak{a}_{n\ell m}^{1/n}) = \tau(\mathfrak{a}_{n\ell m}^{\ell/n\ell}),$$

and by subadditivity, $\tau(\mathfrak{a}_{n\ell m}^{\ell/n\ell}) \subseteq \tau(\mathfrak{a}_{n\ell m}^{1/n\ell})^\ell = \tau_m(\mathfrak{a}_\bullet)^\ell$, completing the proof. ☐

4.3. **(Asymptotic) test ideals and étale maps.** We study a transformation law for test ideals under essentially étale maps. Recall that an *essentially étale map of rings $A \to B$* is a formally étale map [DG64, IV_0, Définition 19.10.2] such that B is a localization of a finitely presented A-algebra. Note formally étale maps of *Noetherian* rings are automatically flat [DG64, IV_0, Théorème 19.7.1]. The main example of essentially étale maps for us will be a local homomorphism of Noetherian local rings $\varphi : (A, \mathfrak{m}_A, \kappa_A) \to (B, \mathfrak{m}_B, \kappa_B)$ that is flat, unramified ($\mathfrak{m}_A B = \mathfrak{m}_B$, $\kappa_A \hookrightarrow \kappa_B$ is finite separable), and essentially of finite type. Then φ is essentially étale by [Sta17, Tag 025B].

Proposition 4.3.1. [Stä16] *Let R be a regular domain essentially of finite type over an F-finite field, and $R \to S$ an essentially étale map. Then for any non-zero ideal $\mathfrak{a}$ of R, and a real number $t > 0$,*

$$\tau(S, (\mathfrak{a}S)^t) = \tau(R, \mathfrak{a}^t)S.$$

Indication of proof. Note $R \to S$ is injective since R is a domain, and $R \to S$ is flat. Therefore, $\mathfrak{a}S$ is a non-zero ideal of S, and $\tau((\mathfrak{a}S)^t)$ makes sense. Now for a proof, see [Stä16, Corollary 6.19] where the result is established in the more general setting where R is Gorenstein. ☐

A key point in the proof of [Stä16, Corollary 6.19] is the fact that for an essentially étale map of rings $A \to B$ of characteristic $p > 0$, the functor F^e_* commutes with base change. Although this fact is well-known, in F-singularity literature it is often stated with restrictive hypotheses on A on B that are not needed. Thus, we include a proof here of the general version.

Lemma 4.3.2. *Let $A \to B$ be an essentially étale map of rings of characteristic $p > 0$ (A, B are not necessarily Noetherian). Then the relative Frobenius map*

$$F_{B/A} : F^e_* A \otimes_A B \to F^e_* B. \tag{4.3.2.1}$$

is an isomorphism.

Proof. The isomorphism (4.3.2.1) is well-known when $A \to B$ is étale [SGA5, XV, Proposition 2(c)(2)]. Since we know F^e_* commutes with localization, (4.3.2.1) follows when B is an essentially étale A-algebra if one can show that B is then a localization of an étale A-algebra. So let C be a finitely presented A-algebra, and $S \subset C$ a multiplicative set such that

$$B = S^{-1}C.$$

Since $0 = \Omega_{B/A} = S^{-1}\Omega_{C/A}$, and C is finitely presented, there exists $f \in S$ such that

$$\Omega_{C[1/f]/A} = f^{-1}\Omega_{C/A} = 0,$$

that is $C[1/f]$ is an unramified A-algebra.

For any prime ideal $\mathfrak{q}$ of C that does not intersect S, we know that $C_{\mathfrak{q}} = (S^{-1}C)_{S^{-1}\mathfrak{q}}$ is formally smooth over A. Then the Jacobian criterion of local smoothness shows that there exists

$$g_{\mathfrak{q}} \in C - \mathfrak{q}$$

such that $C[1/g_{\mathfrak{q}}]$ is a smooth A-algebra. Here the main point is that formal smoothness of $C_{\mathfrak{q}}$ ensures $\Omega_{C_{\mathfrak{q}}/A}$ is free of the 'correct' rank for a presentation of C (see for example [Hoc17, Theorem on pg. 33]). Since $\{g_{\mathfrak{q}} : \mathfrak{q} \cap S = \emptyset\}$ generates the unit ideal in $S^{-1}C$, there is some $h \in S$ such that

$$h \in \sum_{\mathfrak{q}\cap S=\emptyset} g_{\mathfrak{q}}C.$$

Then $D(h) \subset \operatorname{Spec}(C)$ is smooth on an open cover, and so $C[1/h]$ is a smooth A-algebra.

This shows $C[1/fh]$ is an étale A-algebra, and because B is a further localization of $C[1/fh]$, we are done □

Proposition 4.3.1 has the following consequence for asymptotic test ideals:

Corollary 4.3.3. *Let $R \xrightarrow{\varphi} S$ be an essentially étale map, where R is regular domain, essentially of finite type over an F-finite field. Suppose $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m\in\Phi}$ is a graded family of non-zero ideals of R, and consider the family $\mathfrak{a}_\bullet S = \{\mathfrak{a}_m S\}_{m\in\Phi}$.*

1. *For all $m \in \Phi$, $\tau_m(S, \mathfrak{a}_\bullet S) = \tau_m(R, \mathfrak{a}_\bullet)S$.*
2. *If $\bigcap_{m\in\Phi}(\mathfrak{a}_m : \tau_m(R, \mathfrak{a}_\bullet)) \neq (0)$, then $\bigcap_{m\in\Phi}(\mathfrak{a}_m S : \tau_m(S, \mathfrak{a}_\bullet S)) \neq (0)$.*

Proof. Again, by the injectivity of φ, $\mathfrak{a}_\bullet S$ is a graded family of non-zero ideals of S. Then

$$\tau_m(S, \mathfrak{a}_\bullet S) \coloneqq \sum_{\ell\in\mathbb{N}} \tau\big((\mathfrak{a}_{\ell m}S)^{1/\ell}\big) = \sum_{\ell\in\mathbb{N}} \tau(\mathfrak{a}_{\ell m}^{1/\ell})S = \left(\sum_{\ell\in\mathbb{N}} \tau(\mathfrak{a}_{\ell m}^{1/\ell})\right)S = \tau_m(R, \mathfrak{a}_\bullet)S,$$

where the second quality follows from Proposition 4.3.1. This proves (1).

For (2), if r is a non-zero element in $\bigcap_{m\in\Phi}(\mathfrak{a}_m : \tau_m(R, \mathfrak{a}_\bullet))$, then using (1), $\varphi(r)$ is a non-zero element in $\bigcap_{m\in\Phi}(\mathfrak{a}_m S : \tau_m(S, \mathfrak{a}_\bullet S))$. □

4.4. **(Asymptotic) test ideals and birational maps.** We now examine the behavior of test ideals under birational ring maps. The main result (Proposition 4.4.2) is probably known to experts, but we include a proof, drawing inspiration from [HY03, BS13, ST14].

Setup 4.4.1. Let k be a perfect field of characteristic $p > 0$. Fix an extension $R \hookrightarrow S$ of regular, integral, finitely generated k-algebras such that $\operatorname{Frac}(R) = \operatorname{Frac}(S) = K$. Let $Y = \operatorname{Spec}(S)$, $X = \operatorname{Spec}(R)$, and

$$\pi : Y \to X$$

denote the birational morphism induced by the extension $R \subseteq S$. Choose canonical divisors K_Y and K_X that agree on the locus where π is an isomorphism, and let $K_{Y/X} \coloneqq K_Y - \pi^*K_X$. Define $\omega_{S/R} \coloneqq \Gamma(Y, \mathcal{O}_Y(K_{Y/X}))$. Then $\omega_{S/R}$ is a locally principal invertible fractional ideal of S, with inverse $\omega_{S/R}^{-1} = \Gamma(Y, \mathcal{O}_Y(-K_{Y/X}))$.

We use the following fact implicitly in the results of this subsection: *In Setup 4.4.1, if* $\mathfrak{I}$ *is a non-zero fractional ideal of* S*, then* $R \cap \mathfrak{I}$ *is a non-zero ideal of* R.

This follows by clearing the denominator of a non-zero element of $\mathfrak{I}$.

Proposition 4.4.2. *In Setup 4.4.1, if* $\mathfrak{a}$ *is a non-zero ideal of* S*, and* $\tilde{\mathfrak{a}}$ *denotes its contraction* $\mathfrak{a} \cap R$*, then for any real* $t > 0$*,*

$$\tau(R, \tilde{\mathfrak{a}}^t) \subseteq \big(\omega_{S/R} \cdot \tau(S, (\tilde{\mathfrak{a}}S)^t)\big) \cap R \subseteq \big(\omega_{S/R} \cdot \tau(S, \mathfrak{a}^t)\big) \cap R.$$

Proof. The inclusion $\big(\omega_{S/R} \cdot \tau((\tilde{\mathfrak{a}}S)^t)\big) \cap R \subseteq \big(\omega_{S/R} \cdot \tau(\mathfrak{a}^t)\big) \cap R$ is a consequence of $\tau((\tilde{\mathfrak{a}}S)^t) \subseteq \tau(\mathfrak{a}^t)$ (Theorem 4.1.3(1)).

By definition, $\tau(R, \tilde{\mathfrak{a}}^t)$ is the smallest non-zero ideal J of R under inclusion such that for all $e \in \mathbb{N}$, $\phi \in \operatorname{Hom}_R(F^e_* R, R)$,

$$\phi\big(F^e_*(J\tilde{\mathfrak{a}}^{\lceil t(p^e-1)\rceil})\big) \subseteq J.$$

Thus to finish the proof, it suffices to show the above containment for $J = (\omega_{S/R} \cdot \tau((\tilde{\mathfrak{a}}S)^t)) \cap R$. In fact, extending ϕ to a K-linear map

$$\phi_K : F^e_* K \to K,$$

it is enough to show that

$$\phi_K\big(F^e_*(\omega_{S/R} \cdot \tau((\tilde{\mathfrak{a}}S)^t) \cdot \tilde{\mathfrak{a}}^{\lceil t(p^e-1)\rceil})\big) \subseteq \omega_{S/R} \cdot \tau((\tilde{\mathfrak{a}}S)^t). \tag{4.4.2.1}$$

Our strategy will be to obtain an S-linear map $F^e_* S \to S$ from ϕ_K, and then use the defining property of $\tau((\tilde{\mathfrak{a}}S)^t)$ to prove (4.4.2.1).

By Proposition 3.2.3, ϕ corresponds to a section $g \in \Gamma(X, \mathcal{O}_X((1-p^e)K_X))$, and then the pullback $g = \pi^* g$ is a global section of $\mathcal{O}_Y((1-p^e)\pi^* K_X) = \mathcal{O}_Y((1-p^e)(K_Y - K_{Y/X}))$. Using Proposition 3.2.3 again, $g = \pi^* g$ corresponds to a p^{-e}-linear map of $\mathcal{O}_Y$-modules $F^e_*\mathcal{O}_Y((1-p^e)K_{Y/X}) \to \mathcal{O}_Y$, which on taking global sections gives an S-linear map

$$\varphi_g : F^e_*(\omega_{S/R}^{1-p^e}) \to S.$$

The map φ_g can be constructed from ϕ in a natural way. For ease of notation, let

$$M := F^e_*(\omega_{S/R}^{1-p^e}).$$

We claim that algebraically, φ_g is obtained by restricting ϕ_K to the S-submodule M of $F^e_* K$, but this needs justification because $\phi_K|_M$ is *a priori* an S-linear map from $M \to K$, while φ_g maps into S. However, choosing a non-zero $f \in R$ such that $R_f \hookrightarrow S_f$ is an isomorphism, we see that on localizing at f, the extensions $\varphi_g[f^{-1}]$ of φ_g and $\phi_K|_M[f^{-1}]$ of $\phi_K|_M$ agree on the S-module $M_f = F^e_*(S_f) = F^e_*(R_f)$ with the map $\phi[f^{-1}]$. Thus, φ_g and $\phi_K|_M$ coincide on $M = F^e_*(\omega_{S/R}^{1-p^e})$, and so $\phi_K|_M$ maps into S because φ_g does.

Since the inclusion $\tau(R, \tilde{\mathfrak{a}}^t) \subseteq \omega_{S/R} \cdot \tau((\tilde{\mathfrak{a}}S)^t)$ can be checked locally on S, one may assume that $\omega_{S/R}^{-1}$ is principal, say $\omega_{S/R}^{-1} = cS$. Then left-mutiplication by $F^e_*(c^{p^e-1})$ induces an S-linear map $F^e_* S \to M$, yielding on composition an element

$$\widetilde{\phi} := F^e_* S \xrightarrow{F^e_*(c^{p^e-1})\cdot} M \xrightarrow{\phi_K|_M} S$$

of $\operatorname{Hom}_S(F^e_* S, S)$. Using $\omega_{S/R} = c^{-1}S$, we finally get

$$\begin{aligned} \phi_K\big(F^e_*(\omega_{S/R} \cdot \tau((\tilde{\mathfrak{a}}S)^t) \cdot \tilde{\mathfrak{a}}^{\lceil t(p^e-1)\rceil})\big) &= c^{-1} \cdot \phi_K\big(F^e_*(c^{p^e-1}\tau((\tilde{\mathfrak{a}}S)^t) \cdot \tilde{\mathfrak{a}}^{\lceil t(p^e-1)\rceil})\big) = \\ c^{-1} \cdot \widetilde{\phi}\big(F^e_*(\tau((\tilde{\mathfrak{a}}S)^t) \cdot \tilde{\mathfrak{a}}^{\lceil t(p^e-1)\rceil})\big) &\subseteq c^{-1}\tau((\tilde{\mathfrak{a}}S)^t) = \omega_{S/R} \cdot \tau((\tilde{\mathfrak{a}}S)^t), \end{aligned}$$

where the inclusion follows from the definition of $\tau((\tilde{\mathfrak{a}}S)^t)$, and the fact that $\widetilde{\phi} \in \operatorname{Hom}_S(F^e_* S, S)$. $\square$

Corollary 4.4.3. *Suppose in Setup 4.4.1, we are given a graded family $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m\in\Phi}$ of non-zero ideals of S. Denote by $\tilde{\mathfrak{a}}_\bullet$ the family $\{\mathfrak{a}_m \cap R\}_{m\in\Phi}$. Then*

(1) *For all $m \in \Phi$, $\tau_m(R, \tilde{\mathfrak{a}}_\bullet) \subseteq (\omega_{S/R} \cdot \tau_m(S, \mathfrak{a}_\bullet)) \cap R$.*
(2) *If $\bigcap_{m\in\Phi}(\mathfrak{a}_m : \tau_m(S, \mathfrak{a}_\bullet)) \neq (0)$, then $\bigcap_{m\in\Phi}(\mathfrak{a}_m \cap R : \tau_m(R, \tilde{\mathfrak{a}}_\bullet)) \neq (0)$.*

Proof. Clearly $\tilde{\mathfrak{a}}_\bullet$ is a graded family of non-zero ideals of R. Now (1) follows from Proposition 4.4.2 by choosing $\ell \gg 0$ such that $\tau_m(\tilde{\mathfrak{a}}_\bullet) = \tau((\mathfrak{a}_{\ell m} \cap R)^{1/\ell})$, and $\tau_m(\mathfrak{a}_\bullet) = \tau(S, \mathfrak{a}_{\ell m}^{1/\ell})$.

For (2), let $\mathfrak{J}$ denote the non-zero ideal $\bigcap_{m\in\Phi}(\mathfrak{a}_m : \tau_m(\mathfrak{a}_\bullet))$ of S. Note $\mathfrak{J} \cdot \omega_{S/R}^{-1} \cap R$ is a non-zero ideal of R because $\mathfrak{J} \cdot \omega_{S/R}^{-1}$ is a non-zero fractional ideal of S, and R and S have the same fraction field. Then for all $m \in \Phi$,

$$\begin{aligned}(\mathfrak{J} \cdot \omega_{S/R}^{-1} \cap R) \cdot \tau_m(\tilde{\mathfrak{a}}_\bullet) &\subseteq (\mathfrak{J} \cdot \omega_{S/R}^{-1} \cap R)\big((\omega_{S/R} \cdot \tau_m(\mathfrak{a}_\bullet)) \cap R\big)\\ &\subseteq \big(\mathfrak{J} \cdot \omega_{S/R}^{-1} \cdot \omega_{S/R} \cdot \tau_m(\mathfrak{a}_\bullet)\big) \cap R = (\mathfrak{J} \cdot \tau_m(\mathfrak{a}_\bullet)) \cap R \subseteq \mathfrak{a}_m \cap R.\end{aligned}$$

Thus, $(0) \neq \mathfrak{J} \cdot \omega_{S/R}^{-1} \cap R \subseteq \bigcap_{m\in\Phi}(\mathfrak{a}_m \cap R : \tau_m(\tilde{\mathfrak{a}}_\bullet))$. □

5. Proof of Theorem B

For a ring A of K/k admitting a center of v, we will say A **satisfies Theorem B for** v if $\bigcap_{m\in\mathbb{R}_{\geq 0}}(\mathfrak{a}_m : \tau_m(\mathfrak{a}_\bullet)) \neq (0)$, where $\mathfrak{a}_m$ are the valuation ideals of A associated to v.

To prove Theorem B we need the following general fact about primary ideals in a Noetherian domain, which in particular implies that if Theorem B holds for the local ring at the center x of a variety X of K/k, then it also holds on any affine open neighborhood of x.

Lemma 5.0.1. *Let A be a Noetherian domain, and $\mathfrak{p}$ a prime ideal of A.*

(1) *For any $\mathfrak{p}$-primary ideal $\mathfrak{a}$ of A, $\mathfrak{a}A_\mathfrak{p} \cap A = \mathfrak{a}$.*
(2) *Let $\{\mathfrak{a}_i\}_{i\in I}, \{J_i\}_{i\in I}$ be collections ideals of A such that each $\mathfrak{a}_i$ is $\mathfrak{p}$-primary. Then*

$$\bigcap_{i\in I}(\mathfrak{a}_i A_\mathfrak{p} : J_i A_\mathfrak{p}) = \left(\bigcap_{i\in I}(\mathfrak{a}_i : J_i)\right) A_\mathfrak{p}.$$

Thus, $\bigcap_{i\in I}(\mathfrak{a}_i A_\mathfrak{p} : J_i A_\mathfrak{p}) \neq (0)$ if and only if $\bigcap_{i\in I}(\mathfrak{a}_i : J_i) \neq (0)$.

Proof of Lemma 5.0.1. (1) follows easily from the definition of a primary ideal. For (2), the containment $\big(\bigcap_{i\in I}(\mathfrak{a}_i : J_i)\big)A_\mathfrak{p} \subseteq \bigcap_{i\in I}(\mathfrak{a}_i A_\mathfrak{p} : J_i A_\mathfrak{p})$ is easy to verify. Now let

$$\tilde{s} \in \bigcap_{i\in I}(\mathfrak{a}_i A_\mathfrak{p} : J_i A_\mathfrak{p}),$$

and choose $t \in A - \mathfrak{p}$ such that $t\tilde{s} \in A$, noting that $t\tilde{s}$ is also in the ideal $\bigcap_{i\in I}(\mathfrak{a}_i A_\mathfrak{p} : J_i A_\mathfrak{p})$. Then for all $i \in I$,

$$(t\tilde{s}) \cdot J_i \subseteq (t\tilde{s}) \cdot (J_i A_\mathfrak{p} \cap A) \subseteq \mathfrak{a}_i A_\mathfrak{p} \cap A = \mathfrak{a}_i,$$

where the last equality comes from (1). Thus, $t\tilde{s} \in \bigcap_{i\in I}(\mathfrak{a}_i : J_i)$, and so $\tilde{s} \in \big(\bigcap_{i\in I}(\mathfrak{a}_i : J_i)\big)A_\mathfrak{p}$, establishing the other inclusion. Since $A \to A_\mathfrak{p}$ is injective, the final statement is clear. □

Using Lemma 5.0.1, Theorem B is proved as follows:

Proof of Theorem B. Let $(A, \mathfrak{m}, \kappa_A)$ be the regular local ring v is centered on, where A is essentially of finite type over the perfect field k with fraction field K. Suppose $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_v) = d$ and $\operatorname{tr.deg} K/k = n$. Let R be a finitely generated, regular k-subalgebra of K with a prime ideal $\mathfrak{p}$ such that $A = R_{\mathfrak{p}}$. Using local monomialization (Proposition 2.3.3), choose a finitely generated, regular k-subalgebra S of K along with an inclusion $R \hookrightarrow S$ such that v is centered on the prime $\mathfrak{q}$ of S, and $S_{\mathfrak{q}}$ has Krull dimension d, with a regular system of parameters $\{x_1, \dots, x_d\}$ such that $v(x_1), \dots, v(x_d)$ freely generate the value group Γ_v. Note that if $\{\mathfrak{b}_m\}_{m \in \mathbb{R}_{\geq 0}}$ is the set of valuation ideals of S, then $\{\mathfrak{b}_m \cap R\}_{m \in \mathbb{R}_{\geq 0}}$ is the set of valuation ideals of R. Now if $S_{\mathfrak{q}}$ satisfies Theorem B, then so does S (Lemma 2.1.1 and Lemma 5.0.1), hence R (Corollary 4.4.3), hence also $R_{\mathfrak{p}} = A$ because $\mathfrak{p}$ is the center of v on R (using Lemma 5.0.1 again). Thus, it suffices to prove Theorem B for $A = S_{\mathfrak{q}}$.

The valuation ideals $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m \in \mathbb{R}_{\geq 0}}$ of A are then monomial in the regular system of parameters $x_1, \dots, x_d$ (Proposition 2.3.3). As A has dimension d, its residue field κ_A has transcendence degree $n - d$ over k. Then using the fact that k is perfect, choose a separating transcendence basis $\{t_1, \dots, t_{n-d}\}$ of κ_A/k, and pick $y_1, \dots, y_{n-d} \in A$ such that

$$y_i \equiv t_i \mod \mathfrak{m}.$$

By [Bou89, VI, §10.3, Theorem 1], $\{x_1, \dots, x_d, y_1, \dots, y_{n-d}\}$ is algebraically independent over k, and we obtain a local extension

$$j : k[x_1, \dots, x_d, y_1, \dots, y_{n-d}]_{(x_1, \dots, x_d)} \hookrightarrow A,$$

of local rings of the same dimension that is unramified by construction. Moreover, j is also flat [Mat89, Theorem 23.1], essentially of finite type, hence essentially étale.

Let $\widetilde{A} \coloneqq k[x_1, \dots, x_d, y_1, \dots, y_{n-d}]_{(x_1, \dots, x_d)}$. It is easy to see that $\mathfrak{a}_\bullet \cap \widetilde{A} \coloneqq \{\mathfrak{a}_m \cap \widetilde{A}\}_{m \in \mathbb{R}_{\geq 0}}$ is the collection of valuation ideals of $\widetilde{A}$ with respect to the restriction of v to $\operatorname{Frac}(\widetilde{A})$. Moreover, if S is a set of monomials in $x_1, \dots, x_d$ generating $\mathfrak{a}_m$, and I_m is the ideal of $\widetilde{A}$ generated by S, then $I_m = I_m A \cap \widetilde{A} = \mathfrak{a}_m \cap \widetilde{A}$, where the first equality follows by faithful flatness of j. Thus, each $\mathfrak{a}_m \cap \widetilde{A}$ is generated by the same monomials in $x_1, \dots, x_d$ that generate $\mathfrak{a}_m$. Then to prove the theorem, it suffices to show by Corollary 4.3.3 that

$$\bigcap_{m \in \mathbb{R}_{\geq 0}} (\mathfrak{a}_m \cap \widetilde{A} : \tau_m(\mathfrak{a}_\bullet \cap \widetilde{A})) \neq (0).$$

But now we are in the setting of Example 4.1.4 since we are dealing with monomial ideals in the localization of a polynomial ring. We claim that

$$x_1 \dots x_d \in \bigcap_{m \in \mathbb{R}_{\geq 0}} (\mathfrak{a}_m \cap \widetilde{A} : \tau_m(\mathfrak{a}_\bullet \cap \widetilde{A})).$$

Choose $\ell \in \mathbb{N}$ such that $\tau_m(\mathfrak{a}_\bullet \cap \widetilde{A}) = \tau((\mathfrak{a}_{\ell m} \cap \widetilde{A})^{1/\ell})$. Since $\mathfrak{a}_{\ell m} \cap \widetilde{A}$ is generated by $\{x_1^{a_1} \dots x_d^{a_d} : \sum a_i v(x_i) \geq \ell m\}$, and test ideals commute with localization, we then know by Example 4.1.4 that $\tau_m(\mathfrak{a}_\bullet \cap \widetilde{A}) = \tau((\mathfrak{a}_{\ell m} \cap \widetilde{A})^{1/\ell})$ is generated by monomials $x_1^{b_1} \dots x_d^{b_d}$ such that $(b_1 + 1, \dots, b_d + 1)$ is in the convex hull of

$$\left\{ \left(\frac{a_1}{\ell}, \dots, \frac{a_d}{\ell} \right) : a_i \in \mathbb{N} \cup \{0\}, \sum \frac{a_i}{\ell} v(x_i) \geq m \right\}.$$

Then clearly $\sum (b_i + 1) v(x_i) \geq m$, that is, $(x_1 \dots x_n) \cdot x_1^{b_1} \dots x_d^{b_d} \in \mathfrak{a}_m \cap \widetilde{A}$. This shows $(x_1 \dots x_n) \cdot \tau_m(\mathfrak{a}_\bullet \cap \widetilde{A}) \subseteq \mathfrak{a}_m \cap \widetilde{A}$. $\square$

Remark 5.0.2. The transformation law for test ideals under essentially étale maps (Proposition 4.3.1) and its asymptotic version (Corollary 4.3.3) are results of independent interest. Thus we chose to illustrate one of their applications in our proof of Theorem B, although this can be avoided. Indeed, after reducing the proof of Theorem B to the case of a regular local center $(A, \mathfrak{m}_A, \kappa_A)$ with a regular system of parameters $\{r_1, \dots r_d\}$ whose valuations freely generate the value group, the behavior of test ideals under completion gives another way of proving Theorem B. Briefly, using the structure theory of complete local rings, identify $\widehat{A}$ with a power-series ring

$$\kappa_A[[x_1, \dots, x_d]],$$

where $r_i \mapsto x_i$ under this identification. Since the graded family of valuation ideals $\mathfrak{a}_\bullet$ of A are monomial in $\{r_1, \dots, r_d\}$ (Proposition 2.3.3)(9)), the graded family $\mathfrak{a}_\bullet \widehat{A}$ consists of ideals monomial in $x_1, \dots, x_d$. Explicitly, $\mathfrak{a}_m \widehat{A}$ is generated by

$$\{x_1^{\alpha_1} \dots x_d^{\alpha_d} : \alpha_1 v(r_1) + \dots + \alpha_d v(r_d) \geq m\}.$$

As the formation of test ideals commutes with completion (Theorem 4.1.3 (9)), for any $m \in \mathbb{R}_{\geq 0}$, $\tau_m(\widehat{A}, \mathfrak{a}_\bullet \widehat{A}) = \tau_m(A, \mathfrak{a}_\bullet)\widehat{A}$, and so by faithful flatness of the canonical map $A \to \widehat{A}$, to prove that Theorem B holds for A, it suffices to show that

$$x_1 \dots x_d \in \bigcap_{m \in \mathbb{R}_{\geq 0}} (\mathfrak{a}_m \widehat{A} : \tau_m(\widehat{A}, \mathfrak{a}_\bullet \widehat{A})). \tag{5.0.2.1}$$

However, $\kappa_A[[x_1, \dots, x_d]]$ is also the $(x_1, \dots, x_d)$-adic completion of $\kappa_A[x_1, \dots, x_d]_{(x_1, \dots, x_d)}$, and so we are reduced to analyzing test ideals of monomial ideals in a polynomial ring (Example 4.1.4). Then the argument in the final paragraph of the proof of Theorem B can be repeated verbatim to obtain (5.0.2.1).

6. Consequences of Theorem B

Throughout this section k is a perfect field of prime characteristic, X a regular variety over k with function field K, and v a non-trivial, real-valued Abhyankar valuation of K/k centered on $x \in X$.

6.1. **Proof of Theorem A.**

Our goal is to show that there exists $e \geq 0$ such that for all $m \in \mathbb{R}_{\geq 0}$, $\ell \in \mathbb{N}$,

$$\mathfrak{a}_m(X)^\ell \subseteq \mathfrak{a}_{\ell m}(X) \subseteq \mathfrak{a}_{m-e}(X)^\ell.$$

From now we also assume $m > 0$, as otherwise all the ideals equal $\mathcal{O}_X$.

Let $(\mathfrak{a}_\bullet)_x := \{\mathfrak{a}_m(\mathcal{O}_{X,x})\}_{m \in \mathbb{R}_{\geq 0}}$ denote the graded system of valuation ideals of the center $\mathcal{O}_{X,x}$, and using Theorem B, fix a nonzero $\tilde{s} \in \mathcal{O}_{X,x}$ such that

$$\tilde{s} \in \bigcap_{m \in \mathbb{R}_{\geq 0}} \big(\mathfrak{a}_m(\mathcal{O}_{X,x}) : \tau_m((\mathfrak{a}_\bullet)_x)\big).$$

Define $e := v(\tilde{s})$.

Since the inclusion $\mathfrak{a}_m^\ell \subseteq \mathfrak{a}_{\ell m}$ is clear, it suffices to show that for the above choice of e,

$$\Gamma(U, \mathfrak{a}_{\ell m}) \subseteq \Gamma(U, \mathfrak{a}_{m-e}^\ell), \tag{6.1.0.1}$$

for all $m \in \mathbb{R}_{\geq 0}$, $\ell \in \mathbb{N}$, and affine open $U \subseteq X$. Furthermore, we may assume U contains the center x of v, as otherwise $\Gamma(U, \mathfrak{a}_{\ell m})$ and $\Gamma(U, \mathfrak{a}_{m-e}^\ell)$ both equal $\mathcal{O}_X(U)$. We use $(\mathfrak{a}_\bullet)_U$ to denote the collection $\{\mathfrak{a}_m(U)\}_{m \in \mathbb{R}_{\geq 0}}$ of valuation ideals of $\mathcal{O}_X(U)$.

Utilizing Lemma 2.1.1 and Lemma 5.0.1(2), express $\tilde{s}$ as a fraction s_U/t, for some non-zero

$$s_U \in \bigcap_{m\in\mathbb{R}_{\geq 0}} \big(\mathfrak{a}_m(U) : \tau_m((\mathfrak{a}_\bullet)_U)\big),$$

and $t \in \mathcal{O}_X(U)$ such that $t_x \in \mathcal{O}_{X,x}^\times$. Then $v(s_U) = v(\tilde{s}) = e$, and it follows that for all $m \in \mathbb{R}_{\geq 0}$,

$$\tau_m\big((\mathfrak{a}_\bullet)_U\big) \subseteq \mathfrak{a}_{m-e}(U).$$

Proposition 4.2.4(2) implies that $\Gamma(U, \mathfrak{a}_{\ell m}) \subseteq \tau_m\big((\mathfrak{a}_\bullet)_U\big)^\ell$, and we obtain (6.1.0.1) by observing that

$$\Gamma(U, \mathfrak{a}_{\ell m}) \subseteq \tau_m\big((\mathfrak{a}_\bullet)_U\big)^\ell \subseteq \mathfrak{a}_{m-e}(U)^\ell = \Gamma(U, \mathfrak{a}_{m-e}^\ell).$$

6.2. **Proof of Corollary C.** We want to prove that if v, w are two non-trivial real-valued Abhyankar valuations of K/k, centered on a regular local ring $(A, \mathfrak{m})$ essentially of finite type over k with fraction field K, then there exists $C > 0$ such that for all $x \in A$,

$$v(x) \leq Cw(x).$$

Our argument is similar to [ELS03], and is provided for completeness.

We let $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m\in\mathbb{R}_{\geq 0}}$ denote the collection of valuation ideals of A associated to v, and $\mathfrak{b}_\bullet = \{\mathfrak{b}_m\}_{m\in\mathbb{R}_{\geq 0}}$ the collection associated to w. Since A is Noetherian, there exists a non-zero $x \in \mathfrak{m}$ such that for all non-zero y in $\mathfrak{m}$,

$$w(x) \leq w(y).$$

Otherwise, one can find a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathfrak{m}$ such that $w(x_1) > w(x_2) > w(x_3) > \dots$, giving us a strictly ascending chain of ideals $\mathfrak{b}_{w(x_1)} \subsetneq \mathfrak{b}_{w(x_2)} \subsetneq \mathfrak{b}_{w(x_3)} \subsetneq \dots$. For the rest of the proof, let

$$\delta \coloneqq \inf\{v(x) : x \in \mathfrak{m} - \{0\}\}.$$

Claim 6.2.1. *There exists $p > 0$ such that for all $\ell \in \mathbb{N}$, $\mathfrak{a}_{\ell p} \subseteq \mathfrak{b}_{\ell\delta}$.*

Assuming the claim, let $C \coloneqq 2p/\delta$, and suppose there exists $x_0 \in \mathfrak{m}$ such that $v(x_0) > Cw(x_0)$. Now choose $\ell \in \mathbb{N}$ such that

$$(\ell - 1)\delta \leq w(x_0) < \ell\delta. \tag{6.2.1.1}$$

Such an ℓ exists by the Archimedean property of $\mathbb{R}$, and moreover, $\ell \geq 2$ since $w(x_0) \geq \delta$. Clearly, $x_0 \notin \mathfrak{b}_{\ell\delta}$, and multiplying (6.2.1.1) by C, we get

$$2(\ell - 1)p \leq Cw(x_0) < 2\ell p.$$

But $\ell \geq 2$ implies $\ell p \leq 2(\ell - 1)p \leq Cw(x_0) < v(x_0)$. Then $x_0 \in \mathfrak{a}_{\ell p}$, contradicting $\mathfrak{a}_{\ell p} \subseteq \mathfrak{b}_{\ell\delta}$.

Proof of Claim 6.2.1: By our choice of δ, $\mathfrak{b}_\delta = \mathfrak{m}$. Thus, for all $\ell \in \mathbb{N}$, $\mathfrak{m}^\ell \subseteq \mathfrak{b}_{\ell\delta}$. Since by Theorem B

$$\bigcap_{m\in\mathbb{R}_{\geq 0}} \big(\mathfrak{a}_m : \tau_m(\mathfrak{a}_\bullet)\big) \neq (0), \tag{6.2.1.2}$$

there must exist some $p > 0$ such that $\tau_p(\mathfrak{a}_\bullet) \subseteq \mathfrak{m}$. Otherwise, for all $m \in \mathbb{R}_{\geq 0}$, $\tau_m(\mathfrak{a}_\bullet) = A$, which would imply that any $s \in \bigcap_{m\in\mathbb{R}} \big(\mathfrak{a}_m : \tau_m(\mathfrak{a}_\bullet)\big)$ is also an element of $\bigcap_{m\in\mathbb{R}_{\geq 0}} \mathfrak{a}_m = (0)$, contradicting (6.2.1.2). Then by Proposition 4.2.4(2), for all $\ell \in \mathbb{N}$,

$$\mathfrak{a}_{\ell p} \subseteq \tau_p(\mathfrak{a}_\bullet)^\ell \subseteq \mathfrak{m}^\ell \subseteq \mathfrak{b}_{\ell\delta}. \qquad \square$$

7. Acknowledgments

I thank my advisor Karen Smith for her prescient suggestion to pursue this line of research, for her support, for always generously sharing her ideas, and for her comments improving the exposition of this paper. Preliminary results were obtained during my stay at the University of Utah, and I am grateful to Karl Schwede for the invitation. I benefitted immensely from discussions with Karl and Linquan Ma in Utah and later at the 2017 Arizona Winter School. I thank Kevin Tucker, Manuel Blickle, Johan de Jong, Tiankai Liu, Takumi Murayama, Axel Stäbler and Hagen Knaf for useful conversations, Mel Hochster for pointing out a result in his notes useful for Lemma 4.3.2, and Harold Blum for explaining his proof of Theorem A in characteristic 0 using log-discrepancies of valuations [Blu16]. In addition, Harold and Takumi's meticulous reading of a draft has improved this paper's clarity. My work was supported by a department fellowship at University of Michigan and NSF grant DMS #1501625.

References

[Abh56] S. Abhyankar, *On the valuations centered in a local domain*, American Journal of Mathematics **78** (1956), no. 2, 321–348.

[BB11] M. Blickle and G. Böckle, *Cartier modules: finiteness results*, Journal für Reine und Angewandte Mathematik **661** (2011), 85–123.

[BLR90] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, A Series of Modern Surveys in Mathematics, vol. 21, Springer Berlin Heidelberg, 1990.

[Blu16] H. Blum, *Existence of valuations with smallest normalized volume*, arXiv:1606.08894, 2016.

[Bou89] N. Bourbaki, *Commutative algebra, Chapters 1-7*, Springer-Verlag Berlin Heidelberg, 1989.

[BS13] M. Blickle and K. Schwede, *p^{-1}-Linear maps in algebra and geometry*, vol. Commutative Algebra: expository papers dedicated to Eisenbud on his 65th birthday, pp. 123–205, Springer New York, 2013.

[DG64] J.d. Dieudonné and A. Grothendieck, *Éléments de géométrie algébrique: IV. étude local des schémas et des morphisms de schémas*, Publications mathématiques de l'I.H.É.S **20** (1964), 5–259.

[dJ96] A.J. de Jong, *Smoothness, semi-stability and alterations*, Publications Mathématiques de l'IHÉS **83** (1996), 51–93.

[DS16] R. Datta and K.E. Smith, *Frobenius and valuation rings*, Algebra and Number Theory **10** (2016), no. 5, 1057–1090.

[DS17] ———, *Excellence in prime characteristic*, arXiv:1704.03628, April 2017.

[ELS01] L. Ein, R. Lazarsfeld, and K.E. Smith, *Uniform bounds and symbolic powers on smooth varieties*, Inventiones Mathematicae **144** (2001), 241–252.

[ELS03] ———, *Uniform approximation of Abhyankar valuation ideals in smooth function fields*, American Journal of Mathematics **125** (2003), no. 2, 409–440.

[FJ04] C. Favre and M. Jonsson, *The Valuative Tree*, Lecture notes in Mathematics, vol. 1853, Springer-Verlag Berlin Heidelberg, 2004.

[Har66] R. Hartshorne, *Residues and duality*, Lecture notes in Mathematics, vol. 20, Springer Berlin Heidelberg, 1966.

[Har77] ———, *Algebraic Geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag New York, 1977.

[Har01] N. Hara, *Geometric interpretation of tight closure and test ideals*, Trans. Amer. Math. Soc. **353** (2001), no. 5, 1885–1906.

[Har05] ———, *A characteristic p analog of multiplier ideals and applications*, Communications in Algebra **33** (2005), no. 10, 3375–3388.

[HH89] M. Hochster and C. Huneke, *Tight closure and strong F-regularity*, Mémoires de la S. M. F. 2^e série **38** (1989), 119–133.

[HH90] ———, *Tight closure, invariant theory and the Briançon-Skoda theorem*, Journal of the Amer. Math. Soc. **3** (1990), no. 1, 31–116.

[HH94] ———, *F-regularity, test elements and smooth base change*, Trans. Amer. Math. Soc. **346** (1994), 1–62.

[HH02] ———, *Comparison of symbolic and ordinary powers of ideals*, Inventiones Mathematicae **147** (2002), no. 2, 349–369.

[Hoc17] M. Hochster, *Zariski's Main Theorem, structure of smooth, unramified end étale morphisms, henselian rings and henselization, Artin approximation, and reduction to characteristic p*, unpublished lecture notes available at `http://www.math.lsa.umich.edu/~hochster/615W17/615.pdf` (2017).

[How01] J.A. Howald, *Multiplier ideals of monomial ideals*, Trans. Amer. Math. Soc. **353** (2001), no. 7, 2665–2671.
[HT04] N. Hara and S. Takagi, *On a generalization of test ideals*, Nagoya Math Journal **175** (2004), 59–74.
[HY03] N. Hara and K-i. Yoshida, *A generalization of tight closure and multiplier ideals*, Trans. Amer. Math. Soc. **355** (2003), 3413–3174.
[JM12] M. Jonsson and M. Mustaţă, *Valuations and asymptotic invariants for sequences of ideals*, Annales de l'institut Fourier **62** (2012), no. 6, 2145–2209.
[KK05] H. Knaf and F-V. Kuhlmann, *Abhyankar places admit local uniformization in any characteristic*, Annales scientifiques de l'École Normale Supérieure, Série 4 **38** (2005), no. 6, 833–846.
[Kun69] E. Kunz, *Characterizations of regular local rings of characteristic p*, American Journal of Mathematics **91** (1969), no. 3, 772–784.
[Kun76] ———, *On Noetherian rings of characteristic p*, American Journal of Mathematics **98** (1976), no. 4, 999–1013.
[LS01] G. Lyubeznik and K.E. Smith, *On the commutation of the test ideal with localization and completion*, Trans. Amer. Math. Soc. **353** (2001), no. 8, 3149–3180.
[Mat89] H. Matsumura, *Commutative Ring Theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge, 1989.
[MR85] V.B. Mehta and A. Ramanathan, *Frobenius splitting and cohomology vanishing for Schubert varieties*, Annals of Math. **122** (1985), no. 2, 27–40.
[Ray78] M. Raynaud, *Contre-exemple au "vanishing theorem" en caractéristique* $p > 0$, vol. C.P. Ramanujam-a tribute, pp. 273–278, Tata Institute of Fudamental Research, Narosa Pub. House, 1978.
[RS14] G. Rond and M. Spivakovsky, *The analogue of Izumi's theorem for Abhyankar valuations*, Journal of London Math. Soc. **90** (2014), no. 3, 725–740.
[Sch10] K. Schwede, *Centers of F-purity*, Mathematische Zeitschrift **265** (2010), no. 3, 687–714.
[Sch11] ———, *Test ideals in non-*$\mathbb{Q}$*-Gorenstein rings*, Trans. Amer. Math. Soc. **363** (2011), no. 11, 5925–5941.
[SGA5] *Cohomologie l-adique et fonctions L*, Lecture Notes in Mathematics, Vol. 589, Springer-Verlag, Berlin-New York, 1977 (French). *Séminaire de Géometrie Algébrique du Bois-Marie* 1965-1966 (SGA 5); Edité par Luc Illusie.
[Smi95] K.E. Smith, *Test ideals in local rings*, Trans. Amer. Math. Soc. **347** (1995), no. 9, 3453–3472.
[Smi00] ———, *The multiplier ideal is a universal test ideal*, Comm. Algebra **28** (2000), no. 12, 5915–5929.
[Spi90] M. Spivakovsky, *Valuations in function fields of surfaces*, American Journal of Mathematics **112** (1990), no. 1, 107–156.
[ST12] K. Schwede and K. Tucker, *A survey of test ideals*, vol. Progress in Commutative Algebra 2, pp. 39–100, De Gruyter, 2012.
[ST14] ———, *On the behavior of test ideals under finite morphisms*, Journal of Algebraic Geometry **23** (2014), no. 3, 399–443.
[Stä16] A. Stäbler, *V-filtrations in positive characteristic and test modules*, Trans. Amer. Math. Soc. **368** (2016), no. 11, 7777–7808.
[Sta17] The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu`, 2017.
[SZ60] P. Samuel and O. Zariski, *Commutative Algebra, Vol II*, Springer-Verlag Berlin Heidelberg, 1960.
[SZ15] K.E. Smith and W. Zhang, *Frobenius splitting in commutative algebra*, Commutative algebra and noncommutative algebraic geometry, vol I, MSRI Publications, vol. 67, pp. 291–345, Cambridge, 2015.
[Tak06] S. Takagi, *Formulas for multiplier ideals on singular varieties*, American Journal of Mathematics **128** (2006), no. 6, 1345–1362.
[Tei14] B. Teissier, *Overweight deformations of affine toric varieties and local uniformization* , vol. Valuation Theory in Interaction, EMS Series of Congress Reports, pp. 474–565, EMS Publishing House, Zürich, 2014.
[Tem13] M. Temkin, *Inseparable local uniformization*, Journal of Algebra **373** (2013), no. 1, 65–119.
[Zar40] O. Zariski, *Local uniformization on algebraic varieties*, Annals of Math. **41** (1940), no. 4, 852–896.
[Zar42] ———, *A simplified proof for the resolution of singularities of an algebraic surface*, Annals of Math. **43** (1942), no. 3, 583–593.

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1043, USA

E-mail address: `rankeya@umich.edu`