

FROBENIUS AND VALUATION RINGS

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ABSTRACT. The behavior of the Frobenius map is investigated for valuation rings of prime characteristic. We show that valuation rings are always F-pure. We introduce a generalization of the notion of strong F-regularity, which we call F-pure regularity, and show that a valuation ring is F-pure regular if and only if it is Noetherian. For valuations on function fields, we show that the Frobenius map is finite if and only if the valuation is Abhyankar; in this case the valuation ring is Frobenius split. For Noetherian valuation rings in function fields, we show that the valuation ring is Frobenius split if and only if Frobenius is finite, or equivalently, if and only if the valuation ring is excellent.

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1. INTRODUCTION

Classes of singularities defined using Frobenius—F-purity, Frobenius splitting, and the various variants of F-regularity—have played a central role in commutative algebra and algebraic geometry over the past forty years. The goal of this paper is a systematic study of these F-singularities in the novel, but increasingly important *non-Noetherian* setting of **valuation rings**.

Let R be a commutative ring of prime characteristic p . The Frobenius map is the ring homomorphism $R \xrightarrow{F} R$ sending each element to its p -th power. While simple enough, the Frobenius map reveals deep structural properties of a Noetherian ring of prime characteristic, and is a powerful tool for proving theorems for rings containing an arbitrary field (or varieties, say, over \mathbb{C}) by standard reduction to characteristic p techniques. Theories such as Frobenius splitting [33] and tight closure [22] are well-developed in the Noetherian setting, often under the additional assumption that the Frobenius map is finite. Since classically most motivating problems were inspired by algebraic geometry and representation theory, these assumptions seemed natural and not very restrictive. Now, however, good reasons are emerging to study F-singularities in certain non-Noetherian settings as well.

One such setting is cluster algebras [14]. An upper cluster algebra over \mathbb{F}_p need not be Noetherian, but recently it was shown that it is always Frobenius split, and indeed, admits a “cluster canonical” Frobenius splitting [4]. Likewise valuation rings are enjoying a resurgence of popularity despite rarely being Noetherian, with renewed interest in non-Archimedean geometry [9], the development of tropical geometry [17], and the valuative tree [13], to name just a few examples, as well as fresh uses in higher dimensional birational geometry (e.g. [10, 11, 6]).

For a Noetherian ring R , the Frobenius map is flat if and only if R is regular, by a famous theorem of Kunz [28]. As we observe in Theorem 3.1, the Frobenius map is always flat for a valuation ring. So in some sense, a valuation ring of characteristic p might be interpreted as a “non-Noetherian regular ring.”

On the other hand, some valuation rings are decidedly more like the local rings of smooth points on varieties than others. For example, for a variety X (over, say, an algebraically closed field of characteristic p), the Frobenius map is always finite. For valuation rings of the function field of X , however, we show that the Frobenius is finite if and only if the valuation is Abhyankar; see Theorem 5.1. In particular, for discrete valuations, finiteness of Frobenius is equivalent to the valuation being *divisorial*—that is, given by the order of vanishing along

a prime divisor on some birational model. Abhyankar valuations might be considered the geometrically most interesting ones (C.f. [12]), so it is fitting that their valuation rings behave the most like the rings of smooth points on a variety. Indeed, recently, the local uniformization problem for Abhyankar valuations was settled in positive characteristic [27].

One can weaken the demand that Frobenius is flat and instead require only that the Frobenius map is *pure* (see §2.5). Hochster and Roberts observed that this condition, which they dubbed *F-purity*, is often sufficient for controlling singularities of a Noetherian local ring, an observation at the heart of their famous theorem on the Cohen-Macaulayness of invariant rings [26, 25]. We show in Corollary 3.3 that any valuation ring of characteristic p is F-pure. Purity of a map is equivalent to its splitting under suitable finiteness hypotheses, but at least for valuation rings (which rarely satisfy said hypotheses), the purity of Frobenius seems to be better behaved and more straightforward than its splitting. Example 4.5.1 shows that not all valuation rings are Frobenius split, even in the Noetherian case.

Frobenius splitting has well known deep local and global consequences for algebraic varieties. In the local case, Frobenius splitting has been said to be a “characteristic p analog” of log canonical singularities for complex varieties, whereas related properties correspond to other singularities in the minimal model program [19, 37, 40, 46]. For projective varieties, Frobenius splitting is related to positivity of the anticanonical bundle; see [8, 33, 41, 42]. Although valuation rings are always F-pure, the question of their Frobenius splitting is subtle. Abhyankar valuations in function fields are Frobenius split (Theorem 5.1), but a discrete valuation ring is Frobenius split if and only if it is *excellent* in the sense of Grothendieck (Theorem 4.2.2). Along the way, we prove a simple characterization of the finiteness of Frobenius for a Noetherian domain in terms of excellence, which gives a large class of Noetherian domains in which Frobenius splitting implies excellence; see §2.6 for details.

Closely related to F-purity and Frobenius splitting are the various variants of F-regularity. Strong F-regularity was introduced by Hochster and Huneke [21] as a proxy for weak F-regularity—the property that all ideals are tightly closed—because it is easily shown to pass to localizations. Whether or not a weakly F-regular ring remains so after localization is a long standing open question in tight closure theory, as is the equivalence of weak F-regularity and strong F-regularity. Strong F-regularity has found many applications beyond tight closure, and is closely related to Ramanathan’s notion of “Frobenius split along a divisor” [35, 41]. A smattering of applications might include [2, 4, 5, 8, 15, 18, 34, 39, 36, 42, 43, 44, 41].

Traditionally, strong F-regularity has been defined only for Noetherian rings in which Frobenius is finite. To clarify the situation for valuation rings, we introduce a new definition which we call *F-pure regularity* (see Definition 6.1.1) requiring *purity* rather than *splitting* of certain maps. We show that F-pure regularity is better suited for arbitrary rings, but equivalent to strong F-regularity under the standard finiteness hypotheses; it also agrees with another generalization of strong F-regularity proposed by Hochster (using tight closure) in the local Noetherian case [20]. Likewise, we show that F-pure regularity is a natural and straightforward generalization of strong F-regularity, satisfying many expected properties—for example, regular rings are F-pure regular. Returning to valuation rings, in Theorem 6.5.1 we characterize F-pure regular valuation rings as precisely those that are Noetherian.

Finally, in §6.6, we compare our generalization of strong F-regularity with the obvious competing generalization, in which the standard definition in terms of splitting certain maps

is naively extended without assuming any finiteness conditions. To avoid confusion,¹ we call this *split F-regularity*. We characterize split F-regular valuation rings (at least in a certain large class of fields) as precisely those that are Frobenius split, or equivalently *excellent*; see Corollary 6.6.3. But we also point out that there are regular local rings that fail to be split F-regular, so perhaps split F-regularity is not a reasonable notion of “singularity.”

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2. PRELIMINARIES

Throughout this paper, all rings are assumed to be commutative, and of prime characteristic p unless explicitly stated otherwise. By a local ring, we mean a ring with a unique maximal ideal, *not necessarily Noetherian*.

2.1. Valuation Rings. We recall some basic facts and definitions about valuation rings (of arbitrary characteristic), while fixing notation. See [7, Chapter VI] or [32, Chapter 4] for proofs and details.

The symbol Γ denotes an ordered abelian group. Recall that such an abelian group is torsion free. The rational rank of Γ , denoted $\text{rat.rank } \Gamma$, is the dimension of the \mathbb{Q} -vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$.

Let K be a field. A **valuation on** K is a homomorphism

$$v : K^\times \rightarrow \Gamma$$

from the group of units of K , satisfying

$$v(x + y) \geq \min\{v(x), v(y)\}$$

for all $x, y \in K^\times$. We say that v is **defined over a subfield** k of K , or that v is a **valuation on** K/k , if v takes the value 0 on elements of k .

There is no loss of generality in assuming that v is surjective, in which case we say Γ (or Γ_v) is the **value group** of v . Two valuations v_1 and v_2 on K are said to be **equivalent** if there is an order preserving isomorphism of their value groups identifying $v_1(x)$ and $v_2(x)$ for all $x \in K^\times$. Throughout this paper, we identify equivalent valuations.

The **valuation ring of** v is the subring $R_v \subseteq K$ consisting of all elements $x \in K^\times$ such that $v(x) \geq 0$ (together with the zero element of K). Two valuations on a field K are

¹An earlier version of this paper used the terms *pure F-regularity* and *split F-regularity* for the two generalizations of classical strong F-regularity, depending upon whether maps were required to be pure or split. The names were changed at Karl Schwede’s urging to avoid confusion with terminology for pairs in [45].

equivalent if and only if they determine the same valuation ring. Hence a valuation ring of K is essentially the same thing as an equivalence class of valuations on K .

The valuation ring of v is local, with **maximal ideal** m_v consisting of elements of strictly positive values (and zero). The **residue field** R_v/m_v is denoted $\kappa(v)$. If v is a valuation over k , then both R_v and $\kappa(v)$ are k -algebras.

A valuation ring V of K can be characterized directly, without reference to a valuation, as a subring with the property that for every $x \in K$, either $x \in V$ or $x^{-1} \in V$. The valuation ring V uniquely determines a valuation v on K (up to equivalence), whose valuation ring in turn recovers V . Indeed, it is easy to see that the set of ideals of a valuation ring is totally ordered by inclusion, so the set of principal ideals Γ^+ forms a monoid under multiplication, ordered by $(f) \leq (g)$ whenever f divides g . Thus, Γ can be taken to be the ordered abelian group generated by the principal ideals, and the valuation $v : K^\times \rightarrow \Gamma$ is induced by the monoid map sending each non-zero $x \in V$ to the ideal generated by x . Clearly, the valuation ring of v is V . See [32, Chapter 4].

2.2. Extension of Valuations. Consider an extension of fields $K \subseteq L$. By definition, a valuation w on L is an **extension** of a valuation v on K if the restriction of w to the subfield K is v . Equivalently, w extends v if R_w **dominates** R_v , meaning that $R_v = R_w \cap K$ with $m_w \cap R_v = m_v$. In this case, there is an induced map of residue fields

$$\kappa(v) \hookrightarrow \kappa(w).$$

The **residue degree of w over v** , denoted by $f(w/v)$, is the degree of the residue field extension $\kappa(v) \hookrightarrow \kappa(w)$.

If w extends v , there is a natural injection of ordered groups $\Gamma_v \hookrightarrow \Gamma_w$, since Γ_v is the image of w restricted to the subset K . The **ramification index of w over v** , denoted by $e(w/v)$, is the index of Γ_v in Γ_w .

If $K \hookrightarrow L$ is a finite extension, then both the ramification index $e(w/v)$ and the residue degree $f(w/v)$ are *finite*. Indeed, if $K \subseteq L$ is a degree n extension, then

$$(2.2.0.1) \quad e(w/v)f(w/v) \leq n.$$

More precisely, we have

Proposition 2.2.1. [7, VI.8]. *Let $K \subseteq L$ be an extension of fields of finite degree n . For a valuation v on K , consider the set \mathcal{S} of all extensions (up to equivalence) w of v to L . Then*

$$\sum_{w_i \in \mathcal{S}} e(w_i/v)f(w_i/v) \leq n.$$

In particular, the set \mathcal{S} is finite. Furthermore, equality holds if and only if the integral closure of R_v in L is a finitely generated R_v -module.

2.3. Abhyankar Valuations. Fix a field K finitely generated over a fixed ground field k , and let v be a valuation on K/k . By definition, the **transcendence degree** of v is the transcendence degree of the field extension

$$k \hookrightarrow \kappa(v).$$

The main result about the transcendence degree of valuations is due to Abhyakar [3]. See also [7, VI.10.3, Corollary 1].

Theorem 2.3.1 (Abhyankar's Inequality). *Let K be a finitely generated field extension of k , and let v be a valuation on K/k . Then*

$$(2.3.1.1) \quad \text{trans. deg } v + \text{rat. rank } \Gamma_v \leq \text{trans. deg } K/k.$$

Moreover if equality holds, then Γ_v is a finitely generated abelian group, and $\kappa(v)$ is a finitely generated extension of k .

We say v is an **Abhyankar valuation** if equality holds in Abhyankar's Inequality (2.3.1.1). Note that an Abhyankar valuation has a finitely generated value group, and its residue field is finitely generated over the ground field k .

Example 2.3.2. Let K/k be the function field of a normal algebraic variety X of dimension n over a ground field k . For a prime divisor Y of X , consider the local ring $\mathcal{O}_{X,Y}$ of rational functions on X regular at Y . The ring $\mathcal{O}_{X,Y}$ is a discrete valuation ring, corresponding to a valuation v (the order of vanishing along Y) on K/k ; this valuation is of rational rank one and transcendence degree $n - 1$ over k , hence Abhyankar. Such a valuation is called a **divisorial valuation**. Conversely, every rational rank one Abhyankar valuation is divisorial: for such a v , there exists some normal model X of K/k and a divisor Y such that v is the order of vanishing along Y [48, VI, §14, Thm 31].

Proposition 2.3.3. *Let $K \subseteq L$ be a finite extension of finitely generated field extensions of k , and suppose that w is valuation on L/k extending a valuation v on K/k . Then w is Abhyankar if and only if v is Abhyankar.*

Proof. Since L/K is finite, L and K have the same transcendence degree over k . On the other hand, the extension $\kappa(v) \subseteq \kappa(w)$ is also finite by (2.2.0.1), and so $\kappa(v)$ and $\kappa(w)$ also have the same transcendence degree over k . Again by (2.2.0.1), since Γ_w/Γ_v is a finite abelian group, $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_w/\Gamma_v = 0$. By exactness of

$$0 \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_v \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_w \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_w/\Gamma_v \rightarrow 0$$

we conclude that Γ_w and Γ_v have the same rational rank. The result is now clear from the definition of an Abhyankar valuation. \square

2.4. Frobenius. Let R be a ring of prime characteristic p . The Frobenius map $R \xrightarrow{F} R$ is defined by $F(x) = x^p$. We can denote the target copy of R by F_*R and view it as an R -module via restriction of scalars by F ; thus F_*R is both a ring (indeed, it is precisely R) and an R -module in which the action of $r \in R$ on $x \in F_*R$ produces $r^p x$. With this notation, the Frobenius map $F : R \rightarrow F_*R$ and its iterates $F^e : R \rightarrow F_*^e R$ are ring maps, as well as R -module maps. See [44, 1.0.1] for a further discussion of this notation.

We note that F_*^e gives us an exact covariant functor from the category of R -modules to itself. This is nothing but the usual restriction of scalars functor associated to the ring homomorphism $F^e : R \rightarrow R$.

For an ideal $I \subset R$, the notation $I^{[p^e]}$ denotes the ideal generated by the p^e -th powers of the elements of I . Equivalently, $I^{[p^e]}$ is the expansion of I under the Frobenius map, that is, $I^{[p^e]} = IF_*^e R$ as subsets of R .

The image of F^e is the subring $R^{p^e} \subset R$ of p^e -th powers. If R is reduced (which is equivalent to the injectivity of Frobenius), statements about the R -module $F_*^e R$ are equivalent to statements about the R^{p^e} -module R .

Definition 2.4.1. A ring R of characteristic p is **F-finite** if $F : R \rightarrow F_* R$ is a finite map of rings, or equivalently, if R is a finitely generated R^p -module. Note that $F : R \rightarrow F_* R$ is a finite map if and only if $F^e : R \rightarrow F_*^e R$ is a finite map for all $e > 0$.

F-finite rings are ubiquitous. For example, every perfect field is F-finite, and a finitely generated algebra over an F-finite ring is F-finite. Furthermore, F-finiteness is preserved under homomorphic images, localization and completion. This means that nearly every ring classically arising in algebraic geometry is F-finite. However, valuation rings even of F-finite fields are often not F-finite.

2.5. F-purity and Frobenius splitting. We first review purity and splitting for maps of modules over an arbitrary commutative ring A , not necessarily Noetherian or of prime characteristic. A map of A -modules $M \xrightarrow{\varphi} N$ is **pure** if for any A -module Q , the induced map

$$M \otimes_A Q \rightarrow N \otimes_A Q$$

is injective. The map $M \xrightarrow{\varphi} N$ is **split** if φ has a left inverse in the category of A -modules. Clearly, a split map is always pure. Although it is not obvious, the converse holds under a weak hypothesis:

Lemma 2.5.1. [26, Corollary 5.2] *Let $M \xrightarrow{\varphi} N$ be a pure map of A -modules where A is a commutative ring. Then φ is split if the cokernel $N/\varphi(M)$ is finitely presented.*

Definition 2.5.2. Let R be an arbitrary commutative ring of prime characteristic p .

- (a) The ring R is **Frobenius split** if the map $F : R \rightarrow F_* R$ splits as a map of R -modules, that is, there exists an R -module map $F_* R \rightarrow R$ such that the composition

$$R \xrightarrow{F} F_* R \rightarrow R$$

is the identity map.

- (b) The ring R is **F-pure** if $F : R \rightarrow F_* R$ is a pure map of R -modules.

A Frobenius split ring is always F-pure. The converse is also true under modest hypothesis:

Corollary 2.5.3. *A Noetherian F-finite ring of characteristic p is Frobenius split if and only if it is F-pure.*

Proof. The F-finiteness hypothesis implies that $F_* R$ is a finitely generated R -module. So a quotient of $F_* R$ is also finitely generated. Since a finitely generated module over a Noetherian ring is finitely presented, the result follows from Lemma 2.5.1. \square

2.6. F-finiteness and Excellence. Although we are mainly concerned with non-Noetherian rings in this paper, it is worth pointing out the following curiosity for readers familiar with Grothendieck's concept of an *excellent ring*, a particular kind of Noetherian ring expected to be the most general setting for many algebro-geometric statements [16, Définition 7.8.2].

Proposition 2.6.1. *A Noetherian domain is F-finite if and only if it is excellent and its fraction field is F-finite.*

Proof. If R is F-finite with fraction field K , then also $R \otimes_{R^p} K^p \cong K$ is finite over K^p , so the fraction field of R is F-finite. Furthermore, Kunz showed that F-finite Noetherian rings are excellent [29, Theorem 2.5].

We need to show that an excellent Noetherian domain with F-finite fraction field is F-finite. We make use of the following well-known property² of an excellent domain A : the integral closure of A in any finite extension of its fraction field is *finite* as a A -module [16, IV, 7.8.3 (vi)]. The ring R^p is excellent because it is isomorphic to R , and its fraction field is K^p . Since $K^p \hookrightarrow K$ is finite, the integral closure S of R^p in K is a finite R^p -module. But clearly $R \subset S$, so R is also a finitely generated R^p module, since submodules of a Noetherian module over a Noetherian ring are Noetherian. That is, R is F-finite. \square

Using this observation, we can clarify the relationship between F-purity and Frobenius splitting in an important class of rings.

Corollary 2.6.2. *For an excellent Noetherian domain whose fraction field is F-finite, Frobenius splitting is equivalent to F-purity.*

Proof of Corollary. Our hypothesis implies F-finiteness, so splitting and purity are equivalent by Lemma 2.5.1. \square

3. FLATNESS AND PURITY OF FROBENIUS IN VALUATION RINGS.

Kunz showed that for a Noetherian ring of characteristic p , the Frobenius map is flat if and only if the ring is regular [28, Theorem 2.1]. In this section, we show how standard results on valuations yield the following result:

Theorem 3.1. Let V be a valuation ring of characteristic p . Then the Frobenius map $F : V \rightarrow F_*V$ is faithfully flat.

This suggests that we can imagine a valuation ring to be “regular” in some sense. Of course, a Noetherian valuation ring is either a field or a one dimensional regular local ring— but because valuation rings are rarely Noetherian, Theorem 3.1 is not a consequence of Kunz’s theorem.

Theorem 3.1 follows from the following general result, whose proof we include for the sake of completeness.

Lemma 3.2. [7, VI.3.6, Lemma 1]. A finitely-generated, torsion-free module over a valuation ring is free. In particular, a torsion free module over a valuation ring is flat.

Proof. Let $M \neq 0$ be a finitely generated, torsion-free V -module. Choose a minimal set of generators $\{m_1, \dots, m_n\}$. If there is a non-trivial relation among these generators, then there exists $v_1, \dots, v_n \in V$ (not all zero) such that $v_1 m_1 + \dots + v_n m_n = 0$. Re-ordering if necessary, we may assume that v_1 is minimal among (non-zero) coefficients, that is, $(v_i) \subset (v_1)$ for all

²sometimes called the **Japanese** or **N2** property.

$i \in \{1, \dots, n\}$. Then for each $i > 1$, there exists $a_i \in V$ such that $v_i = a_i v_1$. This implies that

$$v_1(m_1 + a_2 m_2 + \dots + a_n m_n) = 0.$$

Since $v_1 \neq 0$ and M is torsion free, we get

$$m_1 + a_2 m_2 + \dots + a_n m_n = 0.$$

Then $m_1 = -(a_2 m_2 + \dots + a_n m_n)$. So M can be generated by the smaller set $\{m_2, \dots, m_n\}$ which contradicts the minimality of n . Hence $\{m_1, \dots, m_n\}$ must be a free generating set.

The second statement follows by considering a torsion-free module as a directed union of its finitely generated submodules, since a directed union of flat modules is flat [7, I.2.7 Prop 9] \square

Proof of Theorem 3.1. Observe that $F_* V$ is a torsion free V -module. So by Corollary 3.2, the module $F_* V$ is flat, which means the Frobenius map is flat. To see that Frobenius is faithfully flat, we need only check that $m F_* V \neq F_* V$ for m the maximal ideal of V [7, I.3.5 Prop 9(e)]. But this is clear: the element $1 \in F_* V$ is not in $m F_* V$, since $1 \in V$ is not in the ideal $m^{[p]}$. \square

Corollary 3.3. Every valuation ring of characteristic p is F -pure.

Proof. Fix a valuation ring V of characteristic p . We have already seen that the Frobenius map $V \rightarrow F_* V$ is faithfully flat (Theorem 3.1). But any faithfully flat map of rings $A \rightarrow B$ is pure as a map of A -modules [7, I.3.5 Prop 9(c)]. \square

4. F-FINITE VALUATION RINGS

In this section, we investigate F -finiteness in valuation rings. We first prove Theorem 4.1.1 characterizing F -finite valuation rings as those V for which $F_* V$ is a *free* V -module. We then prove a numerical characterization of F -finiteness in terms of ramification index and residue degree for extensions of valuations under Frobenius in Theorem 4.3.1. This characterization is useful for constructing interesting examples, and later for showing that F -finite valuations are Abhyankar.

4.1. Finiteness and Freeness of Frobenius. For any domain R of characteristic p , we have already observed (see the proof of Proposition 2.6.1) that a necessary condition for F -finiteness is the F -finiteness of its fraction field. For this reason, we investigate F -finiteness of valuation rings only in F -finite ambient fields.

Theorem 4.1.1. *Let K be an F -finite field. A valuation ring V of K is F -finite if and only if $F_* V$ is a free V -module.*

Proof. First assume $F_* V$ is free over V . Since $K \otimes_R F_* V \cong F_* K$ as K -vector spaces, the rank of $F_* V$ over V must be the same as the rank of $F_* K$ over K , namely the degree $[F_* K : K] = [K : K^p]$. Since K is F -finite, this degree is finite, and so $F_* V$ is a free V -module of finite rank. In particular, V is F -finite.

Conversely, suppose that V is F -finite. Then F_*V is a finitely generated, torsion-free V -module. So it is free by Lemma 3.2. \square

Corollary 4.1.2. *An F -finite valuation ring is Frobenius split.*

Proof. One of the rank one free summands of F_*V is the copy of V under F , so this copy of V splits off F_*V . Alternatively, since $V \rightarrow F_*V$ is pure, we can use Lemma 2.5.1: the cokernel of $V \rightarrow F_*V$ is finitely presented because it is finitely generated (being a quotient of the finitely generated V -module F_*V) and the module of relations is finitely generated (by $1 \in F_*V$). \square

Remark 4.1.3. The same argument shows that any module finite extension $V \hookrightarrow S$ splits—in other words, every valuation ring is a *splinter* in the sense of [31]; see also [31, Lemma 1.2].

4.2. Frobenius Splitting in the Noetherian case. We can say more for Noetherian valuation rings. First we make a general observation about F -finiteness in Noetherian rings:

Theorem 4.2.1. *For a Noetherian domain whose fraction field is F -finite, Frobenius splitting implies F -finiteness (and hence excellence).*

Before embarking on the proof, we point out a consequence for valuation rings:

Corollary 4.2.2. *For a discrete valuation ring V whose fraction field is F -finite, the following are equivalent:*

- (i) V is Frobenius split;
- (ii) V is F -finite;
- (iii) V is excellent.

Proof of Corollary. A DVR is Noetherian, so equivalence of (i) and (ii) follows from combining Theorem 4.2.1 and Corollary 4.1.2. The equivalence with excellence follows from Proposition 2.6.1. \square

Remark 4.2.3. We have proved that all valuation rings are F -pure. However, not all valuation rings, even discrete ones on $\mathbb{F}_p(x, y)$, are Frobenius split, as Example 4.5.1 below shows.

The proof of Theorem 4.2.1 relies on the following lemma:

Lemma 4.2.4. *A Noetherian domain with F -finite fraction field is F -finite if and only there exists $\phi \in \text{Hom}_{R^p}(R, R^p)$ such that $\phi(1) \neq 0$.*

Proof of Lemma. Assuming such ϕ exists, we first observe that the canonical map to the double dual

$$R \rightarrow R^{\vee\vee} := \text{Hom}_{R^p}(\text{Hom}_{R^p}(R, R^p), R^p)$$

is injective. Indeed, let $x \in R$ be a non-zero element. It suffices to show that there exists $f \in R^{\vee} := \text{Hom}_{R^p}(R, R^p)$ such that $f(x) \neq 0$. Let $f = \phi \circ x^{p-1}$, where x^{p-1} is the R^p -linear map $R \rightarrow R$ given by multiplication by x^{p-1} . Then $f(x) = \phi(x^p) = x^p \phi(1) \neq 0$. This shows that the double dual map is injective.

Now, to show that R is a finitely generated R^p -module, it suffices to show that the larger module $R^{\vee\vee}$ is finitely generated. For this it suffices to show that R^\vee is a finitely generated R^p -module, since the dual of a finitely generated module is finitely generated.

We now show that R^\vee is finitely generated. Let M be a maximal free R^p -submodule of R . Note that M has finite rank (equal to $[K : K^p]$, where K is the fraction field of R) and that R/M is a torsion R^p -module. Since the dual of a torsion module is zero, dualizing the exact sequence $0 \rightarrow M \rightarrow R \rightarrow R/M \rightarrow 0$ induces an injection

$$R^\vee := \operatorname{Hom}_{R^p}(R, R^p) \hookrightarrow \operatorname{Hom}_{R^p}(M, R) = M^\vee.$$

Since M is a finitely generated R^p -module, also M^\vee , and hence its submodule R^\vee is finitely generated (R is Noetherian). This completes the proof that R is F-finite.

For the converse, fix any K^p -linear splitting $\psi : K \rightarrow K^p$. Restricting to R produces an R^p -linear map to K^p . Since R is finitely generated over R^p , we can multiply by some non-zero element c of R^p to produce a non-zero map $\phi : R \rightarrow R^p$ such that $\phi(1) = c \neq 0$. The proof of Lemma 4.2.4 is complete. \square

Proof of Theorem 4.2.1. Let R be a domain with F-finite fraction field. A Frobenius splitting is a map $\phi \in \operatorname{Hom}_{R^p}(R, R^p)$ such that $\phi(1) = 1$. Theorem 4.2.1 then follows immediately from Lemma 4.2.4. \square

4.3. A numerical Criterion for F-finiteness. Consider the extension

$$K^p \subseteq K$$

where K any field of characteristic p . For any valuation v on K , let v^p denote the restriction to K^p . We next characterize F-finite valuations in terms of the ramification index and residue degree of v over v^p :

Theorem 4.3.1. *A valuation ring V of an F-finite field K of prime characteristic p is F-finite if and only if*

$$e(v/v^p)f(v/v^p) = [K : K^p],$$

where v is the corresponding valuation on K and v^p is its restriction to K^p .

Proof. First note that v is the only valuation of K extending v^p . Indeed, v is uniquely determined by its values on elements of K^p , since $v(x^p) = pv(x)$ and the value group of v is torsion-free. Furthermore, the valuation ring of v^p is easily checked to be V^p .

Observe that V is the integral closure of V^p in K . Indeed, since V is a valuation ring, it is integrally closed in K , but it is also obviously integral over V^p . We now apply Lemma 2.2.1. Since there is only one valuation extending v^p , the inequality

$$e(v/v^p)f(v/v^p) \leq [K : K^p]$$

will be an equality if and only if the integral closure of V^p in K , namely V , is finite over V^p . \square

The following simple consequence has useful applications to the construction of interesting examples of F-finite and non F-finite valuations:

Corollary 4.3.2. *Let V be a valuation ring of an F-finite field K of characteristic p . If $e(v/v^p) = [K : K^p]$ or $f(v/v^p) = [K : K^p]$, then V is F-finite.*

Remark 4.3.3. Theorem 4.3.1 and its corollary are easy to apply, because the ramification index and residue degree for the extension $V^p \hookrightarrow V$ can be computed in practice. Indeed, since Γ_{v^p} is clearly the subgroup $p\Gamma_v$ of Γ_v , we see that

$$(4.3.3.1) \quad e(v/v^p) = [\Gamma_v : p\Gamma_v].$$

Also, the local map $V^p \hookrightarrow V$ induces the residue field extension $\kappa(v^p) \hookrightarrow \kappa(v)$, which identifies the field $\kappa(v^p)$ with the subfield $(\kappa(v))^p$. This means that

$$(4.3.3.2) \quad f(v/v^p) = [\kappa(v) : \kappa(v)^p].$$

4.4. Examples of Frobenius Split Valuations. We can use our characterization of F-finite valuations to easily give examples of valuations on $\mathbb{F}_p(x, y)$ that are non-discrete but Frobenius split.

Example 4.4.1. Consider the rational function field $K = k(x, y)$ over a perfect field k of characteristic p . For an irrational number $\alpha \in \mathbb{R}$, let Γ be the ordered additive subgroup of \mathbb{R} generated by 1 and α . Consider the unique valuation $v : K^\times \rightarrow \Gamma$ determined by

$$v(x^i y^j) = i + j\alpha,$$

and let V be the corresponding valuation ring. Since $\Gamma \cong \mathbb{Z} \oplus \mathbb{Z}$ via the map which sends $a + b\alpha \mapsto (a, b)$, we see that the value group of v^p is $p\Gamma \cong p(\mathbb{Z} \oplus \mathbb{Z})$. Hence

$$e(v/v^p) = [\Gamma : p\Gamma] = p^2 = [K : K^p].$$

So V is F-finite by Corollary 4.3.2. Thus V is also Frobenius split by Corollary 4.1.2.

Example 4.4.2. Consider the lex valuation on the rational function field $K = k(x_1, x_2, \dots, x_n)$ over a perfect field k of characteristic p . This is the valuation $v : K^\times \rightarrow \mathbb{Z}^n$ on K/k defined by sending a monomial $x_1^{a_1} \dots x_n^{a_n}$ to $(a_1, \dots, a_n) \in \mathbb{Z}^{\oplus n}$, where $\Gamma = \mathbb{Z}^{\oplus n}$ is ordered lexicographically. Let V be the corresponding valuation ring. The value group of V^p is $p\Gamma$, so $e(v/v^p) = [\Gamma : p\Gamma] = p^n = [K : K^p]$. As in the previous example, Corollary 4.3.2 implies that V is F-finite, and so again F-split.

4.5. Example of a non-Frobenius Split Valuation. Our next example shows that discrete valuation rings are not always F-finite, even in the rational function field $\mathbb{F}_p(x, y)$. This is adapted from [48, Example on pg 62], where it is credited to F. K. Schmidt.

Example 4.5.1. Let $\mathbb{F}_p((t))$ be the fraction field of the discrete valuation ring $\mathbb{F}_p[[t]]$ of power series in one variable. Since the field of rational functions $\mathbb{F}_p(t)$ is countable, the uncountable field $\mathbb{F}_p((t))$ can not be algebraic over $\mathbb{F}_p(t)$. So we can find some power series

$$f(t) = \sum_{n=1}^{\infty} a_n t^n$$

in $\mathbb{F}_p[[t]]$ transcendental over $\mathbb{F}_p(t)$.

Since t and $f(t)$ are algebraically independent, there is an *injective* ring map

$$\mathbb{F}_p[x, y] \hookrightarrow \mathbb{F}_p[[t]] \quad \text{sending } x \mapsto t \text{ and } y \mapsto f(t)$$

which induces an extension of fields

$$\mathbb{F}_p(x, y) \hookrightarrow \mathbb{F}_p((t)).$$

Restricting the t -adic valuation on $\mathbb{F}_p((t))$ to the subfield $\mathbb{F}_p(x, y)$ produces a discrete valuation v of $\mathbb{F}_p(x, y)$. Let V denote its valuation ring.

We claim that V is not F-finite, a statement we can verify with Theorem 4.3.1. Note that $L = \mathbb{F}_p(x, y)$ is F-finite, with $[L : L^p] = p^2$. Since the value group Γ_v is \mathbb{Z} , we see that

$$e(v/v^p) = [\Gamma_v : p\Gamma_v] = p.$$

On the other hand, to compute the residue degree $f(v/v^p)$, we must understand the field extension $\kappa(v)^p \hookrightarrow \kappa(v)$. Observe that for an element $u \in \mathbb{F}_p(x, y)$ to be in V , its image in $\mathbb{F}_p((t))$ must be a power series of the form

$$\sum_{n=0}^{\infty} b_n t^n$$

where $b_n \in \mathbb{F}_p$. Clearly

$$v(u - b_0) > 0$$

which means that the class of $u = \sum_{n=0}^{\infty} b_n t^n$ in $\kappa(v)$ is equal to the class of b_0 in $\kappa(v)$. This implies that $\kappa(v) \cong \mathbb{F}_p$, so that $[\kappa(v) : \kappa(v)^p] = 1$. That is, $f(v/v^p) = 1$.

Finally, we then have that

$$e(v/v^p)f(v/v^p) = p \neq p^2 = [\mathbb{F}_p(x, y) : (\mathbb{F}_p(x, y))^p].$$

So V **cannot** be F-finite by Theorem 4.3.1. Thus this Noetherian ring is neither Frobenius split nor excellent by Corollary 4.2.2.

4.6. Finite Extensions. Frobenius properties of valuations are largely preserved under finite extension. First note that if $K \hookrightarrow L$ is a finite extension of F-finite fields, then $[L : L^p] = [K : K^p]$; this follows immediately from the commutative diagram of fields

$$\begin{array}{ccc} L & \longleftrightarrow & K \\ \uparrow & & \uparrow \\ L^p & \longleftrightarrow & K^p \end{array}$$

To wit, $[L : K^p] = [L : K][K : K^p] = [L : L^p][L^p : K^p]$ and $[L : K] = [L^p : K^p]$, so that $[L : L^p] = [K : K^p]$. Moreover,

Proposition 4.6.1. *Let $K \hookrightarrow L$ be a finite extension of F-finite fields of characteristic p . Let v be a valuation on K and w an extension of v to L . Then:*

- (i) *The ramification indices $e(v/v^p)$ and $e(w/w^p)$ are equal.*
- (ii) *The residue degrees $f(v/v^p)$ and $f(w/w^p)$ are equal.*
- (iii) *The valuation ring for v is F-finite if and only if the valuation ring for w is F-finite.*

Proof. By (2.2.0.1), we have

$$[\Gamma_w : \Gamma_v][\kappa(w) : \kappa(v)] \leq [L : K],$$

so both $[\Gamma_w : \Gamma_v]$ and $[\kappa(w) : \kappa(v)]$ are finite. Of course, we also know that the ramification indices $e(w/w^p) = [\Gamma_w : p\Gamma_w]$ and $e(v/v^p) = [\Gamma_v : p\Gamma_v]$ are finite, as are the residue degrees $f(w/w^p) = [\kappa(w) : \kappa(w)^p]$ and $f(v/v^p) = [\kappa(v) : \kappa(v)^p]$.

(i) In light of (4.3.3.1), we need to show that $[\Gamma_w : p\Gamma_w] = [\Gamma_v : p\Gamma_v]$. Since Γ_w is torsion-free, multiplication by p induces an isomorphism $\Gamma_w \cong p\Gamma_w$, under which the subgroup Γ_v corresponds to $p\Gamma_v$. Thus $[p\Gamma_w : p\Gamma_v] = [\Gamma_w : \Gamma_v]$. Using the commutative diagram of finite index abelian subgroups

$$\begin{array}{ccc} \Gamma_w & \longleftrightarrow & \Gamma_v \\ \uparrow & & \uparrow \\ p\Gamma_w & \longleftrightarrow & p\Gamma_v, \end{array}$$

we see that $[\Gamma_w : p\Gamma_w][p\Gamma_w : p\Gamma_v] = [\Gamma_w : \Gamma_v][\Gamma_v : p\Gamma_v]$. Whence $[\Gamma_w : p\Gamma_w] = [\Gamma_v : p\Gamma_v]$.

(ii) In light of (4.3.3.2), we need to show that $[\kappa(w) : \kappa(w)^p] = [\kappa(v) : \kappa(v)^p]$. We have $[\kappa(w)^p : \kappa(v)^p] = [\kappa(w) : \kappa(v)]$, so the result follows from computing the extension degrees in the commutative diagram of finite field extensions

$$\begin{array}{ccc} \kappa(w) & \longleftrightarrow & \kappa(v) \\ \uparrow & & \uparrow \\ \kappa(w)^p & \longleftrightarrow & \kappa(v)^p. \end{array}$$

(iii) By (i) and (ii) we get $e(w/w^p) = e(v/v^p)$ and $f(w/w^p) = f(v/v^p)$. Therefore

$$e(w/w^p)f(w/w^p) = e(v/v^p)f(v/v^p).$$

Since also $[L : L^p] = [K : K^p]$, we see using Theorem 4.3.1 that w is F -finite if and only if v is F -finite. \square

5. F-FINITENESS IN FUNCTION FIELDS

An important class of fields are *function fields over a ground field k* . By definition, a field K is a function field over k if it is a finitely generated field extension of k . These are the fields that arise as function fields of varieties over a (typically algebraically closed) ground field k . What more can be said about valuation rings in this important class of fields?

We saw in Example 4.5.1 that not every valuation of an F -finite function field is F -finite. However, the following theorem gives a nice characterization of those that are.

Theorem 5.1. Let K be a finitely generated field extension of an F -finite ground field k . The following are equivalent for a valuation v on K/k :

- (i) The valuation v is Abhyankar.
- (ii) The valuation ring R_v is F -finite.
- (iii) The valuation ring R_v is a free R_v^p -module.

Furthermore, when these equivalent conditions hold, it is also true that R_v is Frobenius split.

Since Abhyankar valuations have finitely generated value groups and residue fields, the following corollary holds.

Corollary 5.2. An F -finite valuation of a function field over an F -finite field k has finitely generated value group and its residue field is a finitely generated field extension of k .

For example, valuations whose value groups are \mathbb{Q} can never be F-finite.

Remark 5.3. In light of Proposition 2.6.1, we could add a fourth item to the list of equivalent conditions in Theorem 5.1 in the *Noetherian* case: the valuation R_v is excellent. The theorem says that the only discrete valuation rings (of function fields) that are F-finite are the divisorial valuation rings or equivalently, the excellent DVRs.

To prove Theorem 5.1, first recall that the equivalence of (ii) and (iii) was already established in Theorem 4.1.1. The point is to connect these conditions with the Abyhankar property. Our strategy is to use Theorem 4.3.1, which tells us that a valuation v on K is F-finite if and only if

$$e(v/v^p)f(v/v^p) = [K : K^p].$$

We do this by proving two propositions—one comparing the rational rank of v to the ramification index $e(v/v^p)$, and the other comparing the transcendence degree of v to the residue degree $f(v/v^p)$.

Proposition 5.4. Let v be a valuation of rational rank s on an F-finite field K . Then

$$e(v/v^p) \leq p^s,$$

with equality when the value group Γ_v is finitely generated.

Proof. To see that equality holds when Γ_v is finitely generated, note that in this case, $\Gamma_v \cong \mathbb{Z}^{\oplus s}$. So $\Gamma_v/p\Gamma_v \cong (\mathbb{Z}/p\mathbb{Z})^{\oplus s}$, which has cardinality p^s . That is, $e(v/v^p) = p^s$.

It remains to consider the case where Γ may not be finitely generated. Nonetheless, since $e(v/v^p)$ is finite (C.f. 2.2.0.1), we do know that $[\Gamma_v : p\Gamma_v] = e(v/v^p)$ is finite. So the proof of Proposition 5.4 comes down to the following simple lemma about abelian groups:

Lemma 5.5. Let Γ be a torsion free abelian group of rational rank s . Then $[\Gamma : p\Gamma] \leq p^s$.

It suffices to show that $\Gamma/p\Gamma$ is a vector space over $\mathbb{Z}/p\mathbb{Z}$ of dimension $\leq s$. For if d is the dimension of $\Gamma/p\Gamma$, then $[\Gamma : p\Gamma] = p^d$. So let t_1, \dots, t_n be elements of Γ whose classes modulo $p\Gamma$ are linearly independent over $\mathbb{Z}/p\mathbb{Z}$. Then we claim that the t_i are \mathbb{Z} -independent elements of Γ . Assume to the contrary that there is some non-trivial relation $a_1 t_1 + \dots + a_n t_n = 0$, for some integers a_i . Since Γ is torsion-free, we can assume without loss of generality, that at least one a_j is not divisible by p . But now modulo $p\Gamma$, this relation produces a non-trivial relation on classes of the t_i in $\Gamma/p\Gamma$, contrary to the fact that these are linearly independent. This shows that any $\mathbb{Z}/p\mathbb{Z}$ -linearly independent subset of $\Gamma/p\Gamma$ must have cardinality at most s . Then the lemma, and hence Proposition 5.4, are proved. \square

Proposition 5.6. Let K be a finitely generated field extension of an F-finite ground field k . Let v be a valuation of transcendence degree t on K over k . Then

$$f(v/v^p) \leq p^t [k : k^p],$$

with equality when $\kappa(v)$ is finitely generated over k .

Proof. The second statement follows immediately from the following well-known fact, whose proof is an easy computation:

Lemma 5.7. A finitely generated field L of characteristic p and transcendence degree n over k satisfies $[L : L^p] = [k : k^p] p^n$.

It remains to consider the case where $\kappa(v)$ may not be finitely generated. Because K/k is a function field, Abhyankar's inequality (2.3.1.1) guarantees that the transcendence degree of $\kappa(v)$ over k is finite. Let x_1, \dots, x_t be a transcendence basis. There is a factorization

$$k \hookrightarrow k(x_1, \dots, x_t) \hookrightarrow \kappa(v)$$

where the second inclusion is algebraic. The proposition follows immediately from:

Lemma 5.8. If $L' \subseteq L$ is an algebraic extension of F-finite fields, then $[L : L^p] \leq [L' : L'^p]$.

To prove this lemma, recall that Proposition 4.6.1 ensures that $[L : L^p] = [L' : L'^p]$ when $L' \subseteq L$ is *finite*. So suppose L is algebraic but not necessarily finite over L' . Fix a basis $\{\alpha_1, \dots, \alpha_n\}$ for L over L^p , and consider the intermediate field

$$L' \hookrightarrow L'(\alpha_1, \dots, \alpha_n) \hookrightarrow L.$$

Since each α_i is algebraic over L' , it follows that $\tilde{L} := L'(\alpha_1, \dots, \alpha_n)$ is finite over L' , so again $[\tilde{L} : \tilde{L}^p] = [L' : L'^p]$ by Proposition 4.6.1. Now observe that $\tilde{L}^p \subset L^p$, and so the L^p -linearly independent set $\{\alpha_1, \dots, \alpha_n\}$ is also linearly independent over \tilde{L}^p . This means that $[L : L^p] \leq [\tilde{L} : \tilde{L}^p]$ and hence $[L : L^p] \leq [L' : L'^p]$. This proves Lemma 5.8.

Finally, Proposition 5.6 is proved by applying Lemma 5.8 to the inclusion

$$L' = k(x_1, \dots, x_t) \hookrightarrow L = \kappa(v).$$

[Note that $\kappa(v)$ is F-finite, because $[\kappa(v) : (\kappa(v))^p] = f(v/v^p) \leq [K : K^p]$ from the general inequality (2.2.0.1).] So we get

$$f(v/v^p) = [\kappa(v) : (\kappa(v))^p] \leq [k(x_1, \dots, x_t) : (k(x_1, \dots, x_t))^p] = p^t[k : k^p].$$

□

Proof of Theorem 5.1. It only remains to prove the equivalence of (i) and (ii). First assume v is Abhyankar. Then its value group Γ_v is finitely generated and its residue field $\kappa(v)$ is finitely generated over k . According to Proposition 5.4, we have $e(v/v^p) = p^s$, where s is the rational rank of v . According to Proposition 5.6, we have $f(v/v^p) = p^t[k : k^p]$, where t is the transcendence degree of v . By definition of Abhyankar, $s + t = n$, where n is the transcendence degree of K/k . But then

$$e(v/v^p)f(v/v^p) = (p^s)(p^t)[k : k^p] = p^n[k : k^p] = [K : K^p].$$

By Theorem 4.3.1, we can conclude that v is F-finite.

Conversely, we want to prove that a valuation v with F-finite valuation ring R_v is Abhyankar. Let s denote the rational rank and t denote the transcendence degree of v . From Theorem 4.3.1, the F-finiteness of v gives

$$e(v/v^p)f(v/v^p) = [K : K^p] = p^n[k : k^p].$$

Using the bounds $p^s \geq e(v/v^p)$ and $p^t[k : k^p] \geq f(v/v^p)$ provided by Propositions 5.4 and 5.6, respectively, we substitute to get

$$(p^s)(p^t[k : k^p]) \geq e(v/v^p)f(v/v^p) = p^n[k : k^p].$$

It follows that $s + t \geq n$. Then $s + t = n$ by (2.3.1.1), and v is Abhyankar. □

6. F-REGULARITY

An important class of F-pure rings are the strongly F-regular rings. Originally, strongly F-regular rings were defined only in the Noetherian F-finite case. By definition, a Noetherian F-finite reduced ring R of prime characteristic p is strongly F-regular if for every non-zero-divisor c , there exists e such that the map

$$R \rightarrow F_*^e R \quad \text{sending } 1 \mapsto c$$

splits in the category of R -modules [21]. In this section, we show that by replacing the word "splits" with the words "is pure" in the above definition, we obtain a well-behaved notion of F-regularity in a broader setting. Hochster and Huneke themselves suggested, but never pursued, this possibility in [23, Remark 5.3].

Strong F-regularity first arose as a technical tool in the theory of *tight closure*: Hochster and Huneke made use of it in their deep proof of the existence of test elements [23]. Indeed, the original motivation for (and the name of) strong F-regularity was born of a desire to better understand **weak F-regularity**, the property of a Noetherian ring that all ideals are tightly closed. In many contexts, strong and weak F-regularity are known to be equivalent (see e.g. [30] for the graded case, [21] for the Gorenstein case) but it is becoming clear that at least for many applications, strong F-regularity is the more useful and flexible notion. Applications beyond tight closure include commutative algebra more generally [2, 5, 39, 36, 44], algebraic geometry [15, 18, 34, 42, 41], representation theory [8, 33, 35, 43] and combinatorics [4].

6.1. Basic Properties of F-pure regularity. We propose the following definition, intended to be a generalization of strong F-regularity to arbitrary commutative rings of characteristic p , not necessarily F-finite or Noetherian.

Definition 6.1.1. Let c be an element in a ring R of prime characteristic p . Then R is said to be **F-pure along c** if there exists $e > 0$ such that the R -linear map

$$\lambda_c^e : R \rightarrow F_*^e R \quad \text{sending } 1 \mapsto c$$

is a pure map of R -modules. We say R is **F-pure regular** if it is F-pure along every non-zero-divisor.

A ring R is F-pure if and only if it is F-pure along the element 1. Thus F-pure regularity is a substantial strengthening of F-purity, requiring F-purity along *all* non-zero-divisors instead of just along the unit.

Remark 6.1.2. (i) If R is Noetherian and F-finite, then the map $\lambda_c^e : R \rightarrow F_*^e R$ is pure if and only if it splits (by Lemma 2.5.1). So F-pure regularity for a Noetherian F-finite ring is the same as strong F-regularity.

(ii) If c is a zerodivisor, then the map λ_c^e is never injective for any $e \geq 1$. In particular, a ring is never F-pure along a zerodivisor.

(iii) The terminology "F-pure along c " is chosen to honor Ramanathan's closely related notion of "Frobenius splitting along a divisor" [35]. See [41].

The following proposition gathers up some basic properties of F-pure regularity for arbitrary commutative rings.

Proposition 6.1.3. *Let R be a commutative ring of characteristic p , not necessarily Noetherian or F -finite.*

- (a) *If R is F -pure along some element, then R is F -pure. More generally, if R is F -pure along a product cd , then R is F -pure along the factors c and d .*
- (b) *If R is F -pure along some element, then R is reduced.*
- (c) *If R is an F -pure regular ring with finitely many minimal primes, and $S \subset R$ is a multiplicative set, then $S^{-1}R$ is F -pure regular. In particular, F -pure regularity is preserved under localization in Noetherian rings, as well as in domains.*
- (d) *Let $\varphi : R \rightarrow T$ be a pure ring map which maps non-zerodivisors of R to non-zerodivisors of T . If T is F -pure regular, then R is F -pure regular. In particular, if $\varphi : R \rightarrow T$ is faithfully flat and T is F -pure regular, then R is F -pure regular.*
- (e) *Let R_1, \dots, R_n be rings of characteristic p . If $R_1 \times \dots \times R_n$ is F -pure regular, then each R_i is F -pure regular.*

The proof of Proposition 6.1.3 consists mostly of applying general facts about purity to the special case of the maps λ_c^e . For the convenience of the reader, we gather these basic facts together in one lemma:

Lemma 6.1.4. *Let A be an arbitrary commutative ring A , not necessarily Noetherian nor of characteristic p .*

- (a) *If $M \rightarrow N$ and $N \rightarrow Q$ are pure maps of A -modules, then the composition $M \rightarrow N \rightarrow Q$ is also pure.*
- (b) *If a composition $M \rightarrow N \rightarrow Q$ of A -modules is pure, then $M \rightarrow N$ is pure.*
- (c) *If B is an A -algebra and $M \rightarrow N$ is pure map of A -modules, then $B \otimes_A M \rightarrow B \otimes_A N$ is a pure map of B -modules.*
- (d) *Let B be an A -algebra. If $M \rightarrow N$ is a pure map of B -modules, then it is also pure as a map of A -modules.*
- (e) *An A -module map $M \rightarrow N$ is pure if and only if for all prime ideals $\mathcal{P} \subset A$, $M_{\mathcal{P}} \rightarrow N_{\mathcal{P}}$ is pure.*
- (f) *A faithfully flat map of rings is pure.*
- (g) *If (Λ, \leq) is a directed set with a least element λ_0 , and $\{N_\lambda\}_{\lambda \in \Lambda}$ is a direct limit system of A -modules indexed by Λ and $M \rightarrow N_{\lambda_0}$ is an A -linear map, then $M \rightarrow \varinjlim_\lambda N_\lambda$ is pure if and only if $M \rightarrow N_\lambda$ is pure for all λ .*
- (h) *A map of modules $A \rightarrow N$ over a Noetherian local ring (A, m) is pure if and only if $E \otimes_A A \rightarrow E \otimes_A N$ is injective where E is the injective hull of the residue field of R .*

Proof of Lemma 6.1.4. Properties (a)-(d) follow easily from the definition of purity and elementary properties of tensor product. As an example, let us prove (d). If P is an A -module, we want to show that $P \otimes_A M \rightarrow P \otimes_A N$ is injective. The map of B -modules

$$(P \otimes_A B) \otimes_B M \rightarrow (P \otimes_A B) \otimes_B N$$

is injective by purity of $M \rightarrow N$ as a map of B -modules. Using the natural A -module isomorphisms $(P \otimes_A B) \otimes_B M \cong P \otimes_A M$ and $(P \otimes_A B) \otimes_B N \cong P \otimes_A N$, we conclude that $P \otimes_A M \rightarrow P \otimes_A N$ is injective in the category of A -modules.

Property (e) follows from (c) by tensoring with A_p and the fact that injectivity of a map of modules is a local property. Property (f) follows from [7, I.3.5, Proposition 9(c)]. Properties (g) and (h) are proved in [24, Lemma 2.1]. \square

Proof of Proposition 6.1.3. (a) Multiplication by d is an R -linear map, so by restriction of scalars also

$$F_*^e R \xrightarrow{\times d} F_*^e R$$

is R -linear. Precomposing with λ_c^e we have

$$R \xrightarrow{\lambda_c^e} F_*^e R \xrightarrow{\times d} F_*^e R \quad \text{sending } 1 \mapsto cd,$$

which is λ_{cd}^e . Our hypothesis that R is F-pure along cd means that there is some e for which this composition is pure. So by Lemma 6.1.4(b), it follows also that λ_c^e is pure. That is, R is F-pure along c (and since R is commutative, along d). The second statement follows since F-purity along the product $c \times 1$ implies R is F-pure along 1. So some iterate of Frobenius is a pure map, and so F-purity follows from Lemma 6.1.4(b).

(b) By (a) we see that R is F-pure. In particular, the Frobenius map is pure and hence injective, so R is reduced.

(c) Note R is reduced by (b). Let $\alpha \in S^{-1}R$ be a non-zerodivisor. Because R has finitely many minimal primes, a standard prime avoidance argument shows that there exists a non-zerodivisor $c \in R$ and $s \in S$ such that $\alpha = c/s$ (a minor modification of [20, Proposition on Pg 57]). By hypothesis, R is F-pure along c . Hence there exists $e > 0$ such that the map $\lambda_c^e : R \rightarrow F_*^e R$ is pure. Then the map

$$\lambda_{c/1}^e : S^{-1}R \longrightarrow F_*^e(S^{-1}R) \quad \text{sending } 1 \mapsto c/1$$

is pure by 6.1.4(e) and the fact that $S^{-1}(F_*^e R) \cong F_*^e(S^{-1}R)$ as $S^{-1}R$ -modules (the isomorphism $S^{-1}(F_*^e R) \cong F_*^e(S^{-1}R)$ is given by $r/s \mapsto r/s^{p^e}$). Now the $S^{-1}R$ -linear map

$$\ell_{1/s} : S^{-1}R \rightarrow S^{-1}R \quad \text{sending } 1 \mapsto 1/s$$

is an isomorphism. Applying F_*^e , we see that

$$F_*^e(\ell_{1/s}) : F_*^e(S^{-1}R) \rightarrow F_*^e(S^{-1}R) \quad \text{sending } 1 \mapsto 1/s$$

is also an isomorphism of $S^{-1}R$ -modules. In particular, $F_*^e(\ell_{1/s})$ is a pure map of $S^{-1}R$ -modules. So purity of

$$F_*^e(\ell_{1/s}) \circ \lambda_{c/1}^e$$

follows by 6.1.4(a). But $F_*^e(\ell_{1/s}) \circ \lambda_{c/1}^e$ is precisely the map

$$\lambda_{c/s}^e : S^{-1}R \rightarrow F_*^e(S^{-1}R) \quad \text{sending } 1 \mapsto c/s.$$

(d) Let $c \in R$ be a non-zerodivisor. Then $\varphi(c)$ is a non-zero divisor in T by hypothesis. Pick $e > 0$ such that the map $\lambda_{\varphi(c)}^e : T \rightarrow F_*^e T$ is a pure map of T -modules. By 6.1.4(f) and 6.1.4(a),

$$R \xrightarrow{\varphi} T \xrightarrow{\lambda_{\varphi(c)}^e} F_*^e T$$

is a pure map of R -modules. We have commutative diagram of R -linear maps

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & T \\ \downarrow \lambda_c^e & & \downarrow \lambda_{\varphi(c)}^e \\ F_*^e R & \xrightarrow{F_*^e(\varphi)} & F_*^e T \end{array}$$

The purity of λ_c^e follows by 6.1.4(b). Note that if φ is faithfully flat, then it is pure by 6.1.4(f) and maps non-zero-divisors to non-zero-divisors.

(e) Let $R := R_1 \times \cdots \times R_n$. Consider the multiplicative set

$$S := R_1 \times \cdots \times R_{i-1} \times \{1\} \times R_{i+1} \times \cdots \times R_n.$$

Since $S^{-1}R \cong R_i$, it suffices to show that $S^{-1}R$ is F-pure regular. So let $\alpha \in S^{-1}R$ be a non-zero-divisor. Note that we can select $u \in R$ and $s \in S$ such that u is a non-zero-divisor and $\alpha = u/s$. So we can now repeat the proof of (c) verbatim to see that $S^{-1}R$ must be pure along α . \square

Remark 6.1.5. It is worth observing in Definition 6.1.1, that if the map λ_c^e is a pure map, then λ_c^f is also a pure map for all $f \geq e$. Indeed, to see this note that it suffices to show that λ_c^{e+1} is pure. We know R is F-pure by 6.1.3(a). So Frobenius

$$F : R \rightarrow F_* R$$

is a pure map of R -modules. By hypothesis,

$$\lambda_c^e : R \rightarrow F_*^e R$$

is pure. Hence 6.1.4(d) tell us that

$$F_*(\lambda_c^e) : F_* R \rightarrow F_*(F_*^e R)$$

is a pure map of R -modules. Hence the composition

$$R \xrightarrow{F} F_* R \xrightarrow{F_*(\lambda_c^e)} F_*(F_*^e R) \quad \text{sending } 1 \mapsto c$$

is a pure map of R -modules by 6.1.4(a). But $F_*(F_*^e R)$ as an R -module is precisely $F_*^{e+1} R$. So

$$\lambda_c^{e+1} : R \rightarrow F_*^{e+1} R.$$

is pure.

Example 6.1.6. The polynomial ring over \mathbb{F}_p in infinitely many variables (localized at the obvious maximal ideal) is an example of a F-pure regular ring which is not Noetherian.

6.2. Relationship of F-pure regularity to other singularities. We show that our generalization of strong F-regularity continues to enjoy many important properties of the more restricted version.

Theorem 6.2.1. (C.f. [21, Theorem 3.1(c)]) *A regular local ring, not necessarily F-finite, is F-pure regular.*

Proof. Let (R, m) be a regular local ring. By Krull's intersection theorem we know that

$$\bigcap_{e>0} m^{[p^e]} = 0.$$

Since R is a domain, the non-zero divisors are precisely the non-zero elements of R . So let $c \in R$ be a non-zero element. Choose e such that $c \notin m^{[p^e]}$. We show that the map

$$\lambda_c^e : R \rightarrow F_*^e R; \quad 1 \mapsto c$$

is pure.

By Lemma 6.1.4, it suffices to check that for the injective hull E of the residue field of R , the induced map

$$\lambda_c^e \otimes id_E : R \otimes_R E \rightarrow F_*^e R \otimes_R E$$

is injective, and for this, in turn, we need only check that the socle generator is not in the kernel.

Recall that E is the direct limit of the injective maps

$$R/(x_1, \dots, x_n) \xrightarrow{x} R/(x_1^2, \dots, x_n^2) \xrightarrow{x} R/(x_1^3, \dots, x_n^3) \xrightarrow{x} R/(x_1^4, \dots, x_n^4) \longrightarrow \dots$$

where x_1, \dots, x_n is a minimal set of generators for m , and the maps are given by multiplication by $x = \prod_{i=1}^d x_i$. So the module $F_*^e R \otimes_R E$ is the direct limit of the maps

$$R/(x_1^{p^e}, \dots, x_n^{p^e}) \xrightarrow{x^{p^e}} R/(x_1^{2p^e}, \dots, x_n^{2p^e}) \xrightarrow{x^{p^e}} R/(x_1^{3p^e}, \dots, x_n^{3p^e}) \xrightarrow{x^{p^e}} R/(x_1^{4p^e}, \dots, x_n^{4p^e}) \longrightarrow \dots$$

which remains injective by the faithful flatness of $F_*^e R$. The induced map $\lambda_c^e \otimes id_E : E \rightarrow F_*^e R \otimes_R E$ sends the socle (namely the image of 1 in R/m) to the class of c in $R/m^{[p^e]}$, so it is non-zero provided $c \notin m^{[p^e]}$. Thus for every non-zero c in a regular local (Noetherian) ring, we have found an e , such that the map λ_c^e is pure. So regular local rings are F-pure regular. \square

Proposition 6.2.2. *An F-pure regular ring is normal, that is, it is integrally closed in its total quotient ring.*

Proof. Take a fraction r/s in the total quotient ring integral over R . Then clearing denominators in an equation of integral dependence, we have $r \in \overline{(s)}$, the integral closure of the ideal (s) . This implies that there exists an h such that $(r, s)^{n+h} = (s)^n (r, s)^h$ for all n [32, p64]. Setting $c = s^h$, this implies $cr^n \in (s)^n$ for all large n . In particular, taking $n = p^e$, we see that class of r modulo (s) is in the kernel of the map induced by tensoring the map

$$(6.2.2.1) \quad R \rightarrow F_*^e R \quad \text{sending } 1 \mapsto c$$

with the quotient module $R/(s)$. By purity of the map (6.2.2.1), it follows that $r \in (s)$. We conclude that r/s is in R and that R is normal. \square

6.3. Connections with Tight Closure. In his lecture notes on tight closure [20], Hochster suggests another way to generalize strong F-regularity to non-F-finite (but Noetherian) rings using tight closure. We show here that his generalized strong F-regularity is the same as F-pure regularity for local Noetherian rings.

Although Hochster and Huneke introduced tight closure only in Noetherian rings, we can make the same definition in general for an arbitrary ring of prime characteristic p . Let $N \hookrightarrow M$ be R -modules. The **tight closure** of N in M is an R -module N_M^* containing N . By definition, an element $x \in M$ is in N_M^* if there exists $c \in R$, not in any minimal prime, such that for all sufficiently large e , the element $c \otimes x \in F_*^e R \otimes_R M$ belongs to the image of the module $F_*^e R \otimes_R N$ under the natural map $F_*^e R \otimes_R N \rightarrow F_*^e R \otimes_R M$ induced

by tensoring the inclusion $N \hookrightarrow M$ with the R -module F_R^e . We say that N is tightly closed in M if $N_M^* = N$.

Definition 6.3.1. Let R be a Noetherian ring of prime characteristic p . We say that R is *strongly F-regular* in the sense of Hochster if, for any pair of R modules $N \hookrightarrow M$, $N_M^* = N$.

The next result compares F-pure regularity with strongly F-regularity in the sense of Hochster:

Proposition 6.3.2. *Let R be an arbitrary commutative ring of prime characteristic. If R is F-pure regular, then N is tightly closed in M for any pair of R modules $N \subset M$. The converse also holds if R is Noetherian and local.*

Proof. (i) Suppose $x \in N_M^*$. Equivalently the class \bar{x} of x in M/N is in $0_{M/N}^*$. So there exists c not in any minimal prime such that $c \otimes \bar{x} = 0$ in $F_*^e R \otimes_R M/N$ for all large e . But this means that the map

$$R \rightarrow F_*^e R \text{ sending } 1 \mapsto c$$

is not pure for any e , since the naturally induced map

$$R \otimes M/N \rightarrow F_*^e R \otimes M/N$$

has $1 \otimes \bar{x}$ in its kernel.

For the converse, let $c \in R$ be not in any minimal prime. We need to show that there exists some e such that the map $R \rightarrow F_*^e R$ sending 1 to c is pure. Let E be the injective hull of the residue field of R . According to Lemma 6.1.4(i), it suffices to show that there exists an e such that after tensoring E , the induced map

$$R \otimes E \rightarrow F_*^e R \otimes E$$

is injective. But if not, then a generator η for the socle of E is in the kernel for every e , that is, for all e , $c \otimes \eta = 0$ in $F_*^e R \otimes E$. In this case, $\eta \in 0_E^*$, contrary to our hypothesis that all modules are tightly closed. \square

Remark 6.3.3. We do not know whether Proposition 6.3.2 holds in the non-local case. Indeed, we do not know if F-pure regularity is a local property: if R_m is F-pure regular for all maximal ideals m of R , does it follow that R is F-pure regular? If this were the case, then our argument above extends to arbitrary Noetherian rings.

Remark 6.3.4. A Noetherian ring of characteristic p is **weakly F-regular** if N is tightly closed in M for any pair of Noetherian R modules $N \subset M$. Clearly F-pure regular implies weakly F-regular. The converse is a long standing open question in the F-finite Noetherian case. For valuation rings, however, our arguments show that weak and pure F-regularity are equivalent (and both are equivalent to the valuation ring being Noetherian); See Corollary 6.5.4.

6.4. Elements along which F-purity fails. We now observe an analog of the splitting prime of Aberbach and Enescu [1]; See also [47, 4.7].

Proposition 6.4.1. *Let R be a ring of characteristic p , and consider the set*

$$\mathcal{I} := \{c \in R : R \text{ is not F-pure along } c\}.$$

Then \mathcal{I} is closed under multiplication by R , and $R - \mathcal{I}$ is multiplicatively closed. In particular, if \mathcal{I} is closed under addition, then \mathcal{I} is a prime ideal (or the whole ring R).

Proof of Proposition 6.4.1. We first note that \mathcal{I} is closed under multiplication by elements of R . Indeed, suppose that $c \in \mathcal{I}$ and $r \in R$. Then if $rc \notin \mathcal{I}$, we have that R is F-pure along rc , but this implies R is F-pure along c by Proposition 6.1.3(a), contrary to $c \in \mathcal{I}$.

We next show that the complement $R \setminus \mathcal{I}$ is a multiplicatively closed set (if non-empty). To wit, take $c, d \notin \mathcal{I}$. Because R is F-pure along both c and d , we have that there exist e and f such such the maps

$$R \xrightarrow{\lambda_c^e} F_*^e R \text{ sending } 1 \mapsto c, \quad \text{and} \quad R \xrightarrow{\lambda_d^f} F_*^f R \text{ sending } 1 \mapsto d$$

are both pure. Since purity is preserved by restriction of scalars (Lemma 6.1.4(d)), we also have that

$$F_*^e R \xrightarrow{F_*^e(\lambda_d^f)} F_*^e F_*^f R = F_*^{e+f} R$$

is pure. Hence the composition

$$R \xrightarrow{\lambda_c^e} F_*^e R \xrightarrow{\lambda_d^f} F_*^e F_*^f R \text{ sending } 1 \mapsto c^{p^e} d$$

is pure as well (Lemma 6.1.4(a)). This means that $c^{p^e} d$ is not in \mathcal{I} , and since \mathcal{I} is closed under multiplication, neither is cd . Note also that if $R \setminus \mathcal{I}$ is non-empty, then $1 \in R \setminus \mathcal{I}$ by Proposition 6.1.3(a). Thus $R \setminus \mathcal{I}$ is a multiplicative set.

Finally, if \mathcal{I} is closed under addition (and $\mathcal{I} \neq R$), we conclude that \mathcal{I} is a prime ideal since it is an ideal whose complement is a multiplicative set. \square

Remark 6.4.2. If R is a Noetherian local domain, then the set \mathcal{I} of Proposition 6.4.1 can be checked to be closed under addition (see, for example, [47, 4.7] for the F-finite case). Likewise, for valuation rings, the set \mathcal{I} is also an ideal: we construct it explicitly in the next section. However, for an arbitrary ring, \mathcal{I} can fail to be an ideal. For example, under suitable hypothesis, the set \mathcal{I} is also the union of the centers of F-purity in the sense of Schwede, hence in this case, \mathcal{I} is a finite union of ideals but not necessarily an ideal in the non-local case; see [38].

6.5. F-pure regularity and Valuation Rings. In this subsection we characterize valuation rings that are F-pure regular. The main result is:

Theorem 6.5.1. *A valuation ring is F-pure regular if and only if it is Noetherian. Equivalently, a valuation ring is F-pure regular if and only if it is a field or a DVR.*

A key ingredient in the proof is the following theorem about the set of elements along which V fails to be F-pure (C.f. Definition 6.1.1):

Theorem 6.5.2. *The set of elements c along which a valuation ring (V, m) fails to be F-pure is the prime ideal*

$$\mathcal{Q} := \bigcap_{e>0} m^{[p^e]}.$$

Proof. First, take any $c \in \mathcal{Q}$. We need to show that V is not F-pure along c , that is, that the map

$$\lambda_c^e : V \rightarrow F_*^e V \text{ sending } 1 \mapsto c$$

is not pure for any e . Because $c \in m^{[p^e]}$, we see that tensoring with $\kappa := V/m$ produces the zero map. So λ_c^e is not pure for any e , which means V is not F-pure along c .

For the other inclusion, let $c \notin m^{[p^e]}$ for some $e > 0$. We claim that $\lambda_c^e : V \rightarrow F_*^e V$ is pure. Apply Lemma 6.1.4(g) to the set Σ of finitely generated submodules of $F_*^e V$ which contain c . Note Σ is a directed set under inclusion with a least element, namely the V -submodule of $F_*^e V$ generated by c , and $F_*^e V$ is the direct limit of the elements of Σ . It suffices to show that if $T \in \Sigma$, then

$$\lambda_T : V \rightarrow T \text{ sending } 1 \mapsto c$$

is pure. But T is free since it is a finitely generated, torsion-free module over a valuation ring (Lemma 3.2). Since $c \notin m^{[p^e]}$, by the V module structure on T , we get $c \notin mT$. By Nakayama's Lemma, we know c is part of a free basis for T . So λ_T splits, and is pure in particular.

Now that we know that the set of elements along which R is not F-pure is an ideal, it follows that it is a prime ideal from Proposition 6.4.1. \square

Corollary 6.5.3. *For a valuation ring (V, m) of characteristic p , define $\mathcal{Q} := \bigcap_{e>0} m^{[p^e]}$. Then the quotient V/\mathcal{Q} is a F-pure regular valuation ring. Furthermore, V is F-pure regular if and only if \mathcal{Q} is zero.*

Proof. The second statement follows immediately from Theorem 6.5.2. For the first, observe that V/\mathcal{Q} is a domain since \mathcal{Q} is prime. So ideals of V/\mathcal{Q} inherit the total ordering under inclusion from V , and V/\mathcal{Q} is a valuation ring whose maximal \overline{m} ideal satisfies $\bigcap_{e>0} \overline{m}^{[p^e]} = 0$. So V/\mathcal{Q} is F-pure regular. \square

Corollary 6.5.4. *For a valuation ring, F-pure regularity is equivalent to all ideals (equivalently, the maximal ideal) being tightly closed.*

Proof. Proposition 6.3.2 ensures that F-pure regularity implies all ideals are tightly closed. For the converse, note that if there is some non-zero c in $\bigcap_{e>0} m^{[p^e]}$, then $1 \in m^*$. So for any proper ideal m , the condition that $m^* = m$ implies that $\bigcap_{e>0} m^{[p^e]} = 0$. In particular, if the maximal ideal of a valuation ring V is tightly closed, then Corollary 6.5.3 implies that V is F-pure regular. \square

Proof of Theorem 6.5.1. First observe that if V is a field or DVR, then it is F-pure regular. Indeed, every map of modules over a field is pure (since all vector space maps split). And a DVR is a one dimensional regular local ring, so it is F-pure regular by Theorem 6.2.1.

Conversely, we show that if (V, m) is F-pure regular, its dimension is at most one. Suppose (V, m) admits a non-zero prime ideal $P \neq m$. Choose $x \in m \setminus P$, and a non-zero element $c \in P$. The element c cannot divide x^n in V , since in that case we would have $x^n \subset (c) \subset P$, but P is a prime ideal not containing x . It follows from the definition of a valuation ring, then, that x^n divides c for all n . This means in particular that $c \in (x)^{[p^e]} \subset m^{[p^e]}$ for all e . So $c \in \mathcal{Q}$. According to Theorem 6.5.2, R is not F-pure regular.

It remains to show that an F-pure regular valuation ring V of dimension one is discrete. Recall that the value group Γ of V is (order isomorphic to) an additive subgroup of \mathbb{R} [32, Theorem 10.7].

We claim that Γ has a least positive element. To see this, let η be the greatest lower bound of all positive elements in Γ . First observe that η is strictly positive. Indeed, for fixed

$c \in m$, the sequence $\frac{v(c)}{p^e}$ consists of positive real numbers approaching zero as e gets large. If Γ contains elements of arbitrarily small positive values, then we could find $x \in V$ such that

$$0 < v(x) < \frac{v(c)}{p^e}.$$

But then $0 < v(x^{p^e}) < v(c)$, which says that $c \in (x)^{[p^e]} \subset m^{[p^e]}$ for all e . This contradicts our assumption that V is F-pure along c (again, using Theorem 6.5.2).

Now that we know the greatest lower bound η of Γ is positive, it remains to show that $\eta \in \Gamma$. Choose ϵ such that $0 < \epsilon < \eta$. If $\eta \notin \Gamma$, we know $\eta < v(y)$ for all $y \in m$. Since η is the greatest lower bound, we can find y such that

$$\eta < v(y) < \eta + \epsilon,$$

as well as x such that

$$\eta < v(x) < v(y) < \eta + \epsilon.$$

Then

$$0 < v(y/x) < \epsilon < \eta,$$

contradicting the fact that η is a lower bound for Γ . We conclude that $\eta \in \Gamma$, and that Γ has a least positive element.

It is now easy to see using the Archimedean axiom for real numbers that the ordered subgroup Γ of \mathbb{R} is generated by its least positive element η . In particular, Γ is order isomorphic to \mathbb{Z} . We conclude that V is a DVR. \square

Remark 6.5.5. For a valuation ring (V, m) of dimension $n \geq 1$, our results show that in general

$$\mathcal{Q} = \bigcap_{e \in \mathbb{N}} m^{[p^e]}$$

is a prime ideal of height at least $n - 1$. It is easy to see that the situation where V/\mathcal{Q} is a DVR arises if and only if m is principal, which in turn is equivalent to the value group Γ having a least positive element. For example, this is the case for the lex valuation in Example 4.4.2. It is not hard to check that \mathcal{Q} is uniformly F-compatible ideal in the sense of Schwede [38] (see also [44, 3.1] for further discussion of uniformly F-compatible ideals), generalizing of course to the non-Noetherian and non-F-finite setting. A general investigation of uniformly F-compatible ideals appears to be fruitful, and is being undertaken by the first author.

6.6. Split F-regularity. Of course, there is another obvious way³ to adapt Hochster and Huneke's definition of strongly F-regular to arbitrary rings of prime characteristic p :

Definition 6.6.1. A ring R is **split F-regular** if for all non-zero divisors c , there exists e such that the map $R \rightarrow F_*^e R$ sending 1 to c splits as a map of R -modules.

Since split maps are pure, a split F-regular ring is F-pure regular. Split F-regular rings are also clearly Frobenius split. On the other hand, Example 4.5.1 shows that a discrete valuation ring need not be Frobenius split, so split F-regularity is strictly stronger than F-pure regularity. In particular, not every regular local ring is split F-regular, so split F-regularity should not really be considered a class of "singularities" even for Noetherian rings.

³This generalization is used for cluster algebras in [4] for example.

Remark 6.6.2. In Noetherian rings, split F-regularity is very close to F-pure regularity. For example, if R is an F-pure regular Noetherian domain whose fraction field is F-finite, then the only obstruction to split F-regularity is the splitting of Frobenius. This is a consequence of Theorem 4.2.4, which tells us R is F-finite if it is Frobenius split, and Lemma 2.5.1, which tells us F-split and F-pure are the same in F-finite Noetherian rings.

Corollary 6.6.3. *For a discrete valuation ring V whose fraction field is F-finite, the following are equivalent:*

- (i) V is split F-regular;
- (ii) V is Frobenius split;
- (iii) V is F-finite;
- (iv) V is free over V^p ;
- (v) V is excellent.

Moreover, if K is a function field over an F-finite ground field k , and V is a valuation of K/k , then (i)-(v) are equivalent to V being a divisorial valuation ring.

Proof. All this has been proved already. Recall that a DVR is a regular local ring, so it is always F-pure regular and hence split F-regular if it is F-finite. Also, the final statement follows from Theorem 5.1 because an Abyhankar valuation of rational rank one is necessarily divisorial, and a divisorial valuation of a functional field over an F-finite field is necessarily F-finite. \square

To summarize: A valuation ring is F-pure regular if and only if it is Noetherian, and split F-regular (under the additional assumption that its fraction field is F-finite) if and only if it is excellent.

7. CONCLUDING REMARKS

We have argued that for valuation rings, F-purity and F-pure regularity (a version of strong F-regularity defined using pure maps instead of split maps) are natural and robust properties. We have also seen that the conditions of Frobenius splitting and split F-regularity are more subtle, and that even regular rings can fail to satisfy these.

For Noetherian valuation rings in F-finite fields, we have seen that the Frobenius splitting property is equivalent to F-finiteness and also to excellence, but we do not know what happens in the non-Noetherian case: does there exist an example of a (necessarily non-Noetherian) Frobenius split valuation ring of an F-finite field that is *not* F-finite? By Corollary 5.2, a possible strategy could be to construct a Frobenius split valuation ring in a function field whose value group is infinitely generated. For example, can one construct an F-split valuation in $\mathbb{F}_p(x, y)$ with value group \mathbb{Q} ? On the other hand, perhaps Frobenius splitting is equivalent to F-finiteness (just as in the Noetherian case). One might then ask whether a generalized version of Theorem 4.1.1 holds for arbitrary fields: is a valuation ring Frobenius split if and only if Frobenius is free?

We propose that F-pure regularity is a more natural generalization of strong F-regularity to the non-F-finite case than a suggested generalization of strong F-regularity using tight closure due to Hochster. We have seen that F-pure regularity implies Hochster's notion, and

that they are equivalent for local Noetherian rings. However, we do not know whether F-pure regularity is a local notion: if R_m is F-pure regular for all maximal ideals, does it follow that R is F-pure regular? We expect this to be true but the standard arguments are insufficient to prove it. (In the Noetherian F-finite case, this is well known; C.f. [23, Theorem 5.5(a)]. Furthermore, the answer is affirmative for excellent rings with F-finite total quotient rings by Proposition 2.6.1.) If true, then F-pure regularity would be equivalent to all modules being tightly closed in the Noetherian case. More generally, might F-pure regularity be equivalent to the property that all modules are tightly closed even in the non-Noetherian case? Or even that all ideals are tightly closed? An affirmative answer to this last question would imply that strong and weak F-regularity are equivalent.

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ERRATUM TO FROBENIUS AND VALUATION RINGS

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ABSTRACT. Theorem 5.1 is corrected in the paper *Frobenius and valuations rings*, DOI: 10.2140/ant.2016.10.1057.

The characterization of F-finite valuation rings of function fields (Theorem 5.1) is incorrect. The corrected characterization is

Theorem 0.1. *Let K be a finitely generated extension of an F-finite ground field k . Let V be any non-trivial valuation ring of K/k . Then V is F-finite if and only if V is divisorial.*

A proof of this corrected theorem follows a complete accounting of affected statements in the paper. Notation, Theorem and page numbers are as in [2].

1. LIST OF AFFECTED STATEMENTS

The source of the error is Proposition 2.2.1 (pg. 1061), which is misquoted from the original source, [1]. Specifically, the last sentence in the statement of Proposition 2.2.1 should read: *Furthermore, equality holds if the integral closure of R_ν in L is a finitely generated R_ν -module, not if and only if.* The statement *is* correct, however, under the additional assumption that the valuation ring R_ν is *Noetherian* [1, VI, §8.5, Remark (1)].

This error has consequences in the following statements from the paper:

THEOREM 4.3.1 should read: *Let V be a valuation ring of an F-finite field K of prime characteristic p . If V is F-finite, then*

$$[\Gamma : p\Gamma][\kappa : \kappa^p] = [K : K^p],$$

where Γ is the value group and κ is the residue field of V .

The paper had incorrectly stated the converse, which *does* hold if the valuation ring is Noetherian [1, VI, §8.5, Remark (1)].

COROLLARY 4.3.2 should read: *With hypotheses as in Thm 4.3.1, if $[K : K^p] = [\kappa : \kappa^p]$, then V is F-finite.* See Section 2 for the proof. The original statement holds as stated if V is Noetherian.

EXAMPLES 4.4.1 AND 4.4.2 (pg. 1069): The computations here are correct but the conclusions are not. Rather, Theorem 0.1 ensures these valuation rings are **not** F-finite.

EXAMPLE 4.5.1 (pg. 1070) is correct as stated, as it follows from the correct implication of Theorem 4.3.1.

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PROPOSITION 4.6.1 (pg. 1071): This statement is correct. However, the proof of (iii) is not, because it cited the incorrect implication of Proposition 2.2.1. The corrected proof is in Section 3 below.

THEOREM 5.1 (pg. 1072): The equivalence of (ii) and (iii) is correct, and both imply (i). Under the additional assumption that the valuation is discrete, also (i) implies (ii) and (iii). The proof of Theorem 5.1 in the paper proves:

Theorem 1.1. *Let K be a finitely generated field extension of an F -finite ground field k of characteristic p . The following are equivalent for a valuation v on K/k :*

- (i) *The valuation v is Abhyankar.*
- (ii) $[\Gamma : p\Gamma][\kappa : \kappa^p] = [K : K^p]$

where Γ is the value group, and κ is the residue field of v .

COROLLARY 5.2 (pg. 1072) is correct as stated; it is a consequence of Theorem 0.1.

COROLLARY 6.6.3 (pg. 1086) is correct as stated; the proof should invoke Theorem 0.1 instead of Theorem 5.1.

The remaining statements in Section 5 (Remark 5.3, Proposition 5.4, Lemma 5.5, Proposition 5.6, Lemma 5.7, Lemma 5.8) and Section 6 are correct, because they do not rely on Lemma 2.2.1 or its consequences.

2. PROOF OF THEOREM 0.1

Lemma 2.1. *Let (V, \mathfrak{m}) be a valuation ring of prime characteristic p . If \mathfrak{m} is not principal, then*

$$\mathfrak{m} = \mathfrak{m}^{[p]}.$$

Proof. Take any $x \in \mathfrak{m}$. It suffices to show that there exists $y \in \mathfrak{m}$ such that $y^p | x$. Note that since \mathfrak{m} is not principal, the value group Γ of the corresponding valuation v does not have a smallest element > 0 . This means we can choose $\beta_0 \in \Gamma$ such that $0 < \beta_0 < v(x)$, and then also choose $\beta_1 \in \Gamma$ such that

$$0 < \beta_1 < \min\{\beta_0, v(x) - \beta_0\}.$$

Let $y_1 \in \mathfrak{m}$ such that $v(y_1) = \beta_1$. Then

$$v(y_1^2) = 2\beta_1 < \beta_0 + (v(x) - \beta_0) = v(x).$$

Thus, $y_1^2 | x$. Repeating this argument inductively, we can find $y_n \in \mathfrak{m}$ such that $y_n^2 | y_{n-1}$ for all $n \in \mathbb{N}$. For $n > \log_2 p$, we have $y_n^{2^n}$, and hence y_n^p , divides x . \square

Lemma 2.2. *Let $(V, \mathfrak{m}, \kappa)$ be valuation ring of characteristic p . Then the dimension of $V/\mathfrak{m}^{[p]}$ over κ^p is*

- (a) $[\kappa : \kappa^p]$ if \mathfrak{m} is not finitely generated.
- (b) $p[\kappa : \kappa^p]$ if \mathfrak{m} is finitely generated.

Proof. Consider the short exact sequence of κ^p -vector spaces

$$(1) \quad 0 \rightarrow \mathfrak{m}/\mathfrak{m}^{[p]} \rightarrow V/\mathfrak{m}^{[p]} \rightarrow \kappa \rightarrow 0.$$

If \mathfrak{m} is not finitely generated, then Lemma 2.1 implies that $\mathfrak{m}/\mathfrak{m}^{[p]} = 0$, and (a) follows. Otherwise, \mathfrak{m} is principal, so $\mathfrak{m}^{[p]} = \mathfrak{m}^p$ and we have a filtration

$$\mathfrak{m} \supsetneq \mathfrak{m}^2 \supsetneq \cdots \supsetneq \mathfrak{m}^{p-1} \supsetneq \mathfrak{m}^{[p]} = \mathfrak{m}^p.$$

Since $\mathfrak{m}^i/\mathfrak{m}^{i+1} \cong \kappa$, we see that

$$\dim_{\kappa^p}(\mathfrak{m}/\mathfrak{m}^{[p]}) = (p-1)[\kappa : \kappa^p].$$

From the short exact sequence (1), $\dim_{\kappa^p}(V/\mathfrak{m}^{[p]}) = p[\kappa : \kappa^p]$, proving (b). \square

Proof of Theorem 0.1. If V is divisorial, it is a localization of a finitely generated algebra over the F -finite ground field k , hence it is F -finite.

For the converse, let \mathfrak{m} be the maximal ideal and κ the residue field of V . Since V is F -finite, by (the corrected) Theorem 4.3.1

$$(2) \quad [\Gamma : p\Gamma][\kappa : \kappa^p] = [K : K^p],$$

where Γ is the value group of V . Thus, the valuation is Abhyankar by Theorem 1.1 above. To show it is divisorial, we need to show it is discrete.

Because the value group of an Abhyankar valuation is finitely generated, $\Gamma \cong \mathbb{Z}^{\oplus s}$, for some integer $s \geq 1$. Hence it suffices to show that $s = 1$.

Since V is F -finite, we know V is free over V^p of rank $[K : K^p]$. Tensoring with the residue field κ^p of V^p , we have that $V/\mathfrak{m}^{[p]}$ is also a free κ^p -module of rank $[K : K^p]$. Thus from (2), we have

$$(3) \quad \dim_{\kappa^p} V/\mathfrak{m}^{[p]} = [K : K^p] = [\Gamma : p\Gamma][\kappa : \kappa^p] = |\mathbb{Z}^{\oplus s}/p\mathbb{Z}^{\oplus s}|[\kappa : \kappa^p] = p^s[\kappa : \kappa^p].$$

But now since $p^s \neq 1$, Lemma 2.2 forces $s = 1$, and the proof is complete. \square

Remark 2.3. Theorem 0.1 does not hold without *some* assumption on the field K . For instance, if K is perfect, Frobenius will be an isomorphism (hence a finite map) for *any* valuation ring. The proof of Theorem 0.1 does show that an F -finite valuation ring with finitely generated value group must be Noetherian (hence discrete), without any restriction on its fraction field. Furthermore, the proof also shows that a valuation ring cannot be F -finite if its value group Γ satisfies $[\Gamma : p\Gamma] > p$.

Proof of REVISED COROLLARY 4.3.2: In general, $[\Gamma : p\Gamma][\kappa : \kappa^p] \leq [K : K^p]$ by [1, VI, §8.1, Lemma 2]. Hence our hypothesis forces $[\Gamma : p\Gamma] = 1$. Thus $\Gamma = p\Gamma$, so that Γ can not have a smallest positive element, which means that the maximal ideal \mathfrak{m} of V is not finitely generated. Lemma 2.2(a) now ensures $\dim_{\kappa^p}(V/\mathfrak{m}^{[p]}) = [\kappa : \kappa^p] = [K : K^p]$. Then V is F -finite by [1, VI, §8.5, Theorem 2(c)]. \square

3. PROOF OF **Proposition 4.6.1(iii)**:

We recall **Proposition 4.6.1(iii)**: Let $K \hookrightarrow L$ be a finite extension of F-finite fields of characteristic p . Let w be a valuation on L and v its restriction to K . Then the valuation ring of v is F-finite if and only if the valuation ring of w is F-finite.

Lemma 3.1. *With notation as in Proposition 4.6.1(iii), the maximal ideal of the valuation ring of v is finitely generated if and only if the maximal ideal of the valuation ring of w is finitely generated.*

Proof. For ideals in a valuation ring, finite generation is the same being principal. Principality of the maximal ideal is equivalent to the value group having a smallest element > 0 . Thus, it suffices to show that the value group Γ_v of v has this property if and only if Γ_w does.

Assume Γ_w has a smallest element $g > 0$. We claim that for each $t \in \mathbb{N}$, the only positive elements of Γ_w less than tg are $g, 2g, \dots, (t-1)g$. Indeed, suppose $0 < h < tg$. Since g is smallest, $g \leq h < tg$, whence $0 \leq h - g < (t-1)g$. So by induction, $g - h = ig$ for some $i \in \{0, 1, \dots, t-2\}$, and hence h is among $g, 2g, \dots, (t-1)g$.

Now, because $[\Gamma_w : \Gamma_v] \leq [L : K] < \infty$ by [1, VI, §8.1, Lemma 2], every element of Γ_w/Γ_v is torsion. Let n be the smallest positive integer such that $ng \in \Gamma_v$. We claim that ng is the smallest positive element of Γ_v . Indeed, the only positive elements smaller than ng in Γ_w are $g, 2g, \dots, (n-1)g$, and none of these are in Γ_v by our choice of n .

Conversely, if Γ_v has a smallest element $h > 0$, then the set

$$S := \{g \in \Gamma_w : 0 < g < h\}$$

is finite because for distinct g_1, g_2 in this set, their classes in Γ_w/Γ_v are also distinct, while Γ_w/Γ_v is a finite group. Then the smallest positive element of Γ_w is the smallest element of S , or h if S is empty. \square

Proof of Proposition 4.6.1(iii). A necessary and sufficient condition for the F-finiteness of a valuation ring $(V, \mathfrak{m}, \kappa)$ with F-finite fraction field K is, by [1, VI, §8.5, Theorem 2(c)], that

$$(4) \quad \dim_{\kappa^p}(V/\mathfrak{m}^{[p]}) = [K : K^p].$$

Lemma 2.2 gives a formula for $\dim_{\kappa^p}(V/\mathfrak{m}^{[p]})$ in terms of $[\kappa : \kappa^p]$ that depends on whether the maximal ideal is finitely generated, which is the same for v and w by Lemma 3.1. Proposition 4.6.1 (i) and (ii) tells us that both $[K : K^p] = [L : L^p]$ and $[\kappa_v : \kappa_v^p] = [\kappa_w : \kappa_w^p]$. Thus Lemma 2.2 and Equation 4 guarantee that the valuation ring of v is F-finite if and only if the valuation ring of w is F-finite. \square

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