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Local cohomology and base change[☆]

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ABSTRACT

Let $X \xrightarrow{f} S$ be a morphism of Noetherian schemes, with S reduced. For any closed subscheme Z of X finite over S , let j denote the open immersion $X \setminus Z \hookrightarrow X$. Then for any coherent sheaf \mathcal{F} on $X \setminus Z$ and any index $r \geq 1$, the sheaf $f_*(R^r j_* \mathcal{F})$ is generically free on S and commutes with base change. We prove this by proving a related statement about local cohomology: Let R be Noetherian algebra over a Noetherian domain A , and let $I \subset R$ be an ideal such that R/I is finitely generated as an A -module. Let M be a finitely generated R -module. Then there exists a non-zero $g \in A$ such that the local cohomology modules $H_I^r(M) \otimes_A A_g$ are free over A_g and for any ring map $A \rightarrow L$ factoring through A_g , we have $H_I^r(M) \otimes_A L \cong H_{I \otimes_A L}^r(M \otimes_A L)$ for all r .

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1. Introduction

In his work on maps between local Picard groups, Kollár was led to investigate the behavior of certain cohomological functors under base change [12]. The following theorem directly answers a question he had posed:

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Theorem 1.1. *Let $X \xrightarrow{f} S$ be a morphism of Noetherian schemes, with S reduced. Suppose that $Z \subset X$ is closed subscheme finite over S , and let j denote the open embedding of its complement U . Then for any coherent sheaf \mathcal{F} on U , the sheaves $f_*(R^r j_* \mathcal{F})$ are generically free and commute with base change for all $r \geq 1$.*

Our purpose in this note is to prove this general statement. Kollár himself had proved a special case of this result in a more restricted setting [12, Thm. 78].

We pause to say precisely what is meant by *generically free and commutes with base change*. Suppose \mathcal{H} is a functor which, for every morphism of schemes $X \rightarrow S$ and every quasi-coherent sheaf \mathcal{F} on X , produces a quasi-coherent sheaf $\mathcal{H}(\mathcal{F})$ on S . We say $\mathcal{H}(\mathcal{F})$ is *generically free* if there exists a dense open set S^0 of S over which the \mathcal{O}_S -module $\mathcal{H}(\mathcal{F})$ is free. If in addition, for every change of base $T \xrightarrow{p} S^0$, the natural map

$$p^* \mathcal{H}(\mathcal{F}) \rightarrow \mathcal{H}(p_X^* \mathcal{F})$$

of quasi-coherent sheaves on T is an isomorphism (where p_X is the induced morphism $X \times_S T \rightarrow X$), then we say that $\mathcal{H}(\mathcal{F})$ is *generically free and commutes with base change*. See [12, §72].

Remark 1.2. We do not claim the $r = 0$ case of Theorem 1.1; in fact, it is false. For a counterexample, consider the ring homomorphism splitting $\mathbb{Z} \hookrightarrow \mathbb{Z} \times \mathbb{Q} \twoheadrightarrow \mathbb{Z}$. The corresponding morphism of Noetherian schemes

$$Z = \operatorname{Spec}(\mathbb{Z}) \hookrightarrow X = \operatorname{Spec}(\mathbb{Z} \times \mathbb{Q}) \rightarrow S = \operatorname{Spec} \mathbb{Z}$$

satisfies the hypothesis of Theorem 1.1. The open set $U = X \setminus Z$ is the component $\operatorname{Spec} \mathbb{Q}$ of X . The coherent sheaf determined by the module \mathbb{Q} on U is not generically free over \mathbb{Z} , since there is no open affine subset $\operatorname{Spec} \mathbb{Z}[\frac{1}{n}]$ over which \mathbb{Q} is a free module. [In this case, the map j is affine, so the higher direct image sheaves $R^p j_* \mathcal{F}$ all vanish for $p > 0$.]

On the other hand, if f is a map of finite type, then the $r = 0$ case of Theorem 1.1 can be deduced from Grothendieck's Lemma on Generic freeness; see Lemma 4.1.

For the commutative algebraists, we record the following version of the main result, which is essentially just the statement in the affine case:

Corollary 1.3. *Let A be a reduced Noetherian ring. Let R be a Noetherian A -algebra with ideal $I \subset R$ such that the induced map $A \rightarrow R/I$ is finite. Then for any Noetherian R module M , the local cohomology modules $H_I^i(M)$ are generically free and commute with base change over A for all $i \geq 0$. Explicitly, this means that there exists an element g not in any minimal prime of A such that the modules $H_I^i(M) \otimes_A A_g$ are free over A_g , and that for any algebra L over A_g , the natural map*

$$H_I^i(M) \otimes_A L \rightarrow H_I^i(M \otimes_A L)$$

is an isomorphism.

Some version of [Theorem 1.1](#) may follow from known statements in the literature, but looking through works of Grothendieck ([8], [7], [6]) and [3], I am not able to find it; nor presumably could Kollár. After this paper was written, I did find a related statement due to Hochster and Roberts [11, Thm. 3.4] in a special case, not quite strong enough to directly answer Kollár’s original question; furthermore, my proof is different and possibly of independent interest. In any case, there may be value in the self-contained proof here, which uses a relative form of Matlis duality proved here using only basic results about local cohomology well-known to most commutative algebraists.

2. Restatement of the problem

In this section, we show that [Theorem 1.1](#) reduces to the following special statement:

Theorem 2.1. *Let A be a Noetherian domain. Let $R = A[[x_1, \dots, x_n]]$ be a power series ring over A , and M a finitely generated R -module. Denote by I the ideal $(x_1, \dots, x_n) \subset R$. Then the local cohomology modules*

$$H_I^i(M)$$

are generically free over A and commute with base change for all i .

For basic definitions and properties of local cohomology modules, we refer to [7].

For the remainder of this section, we show how [Theorem 1.1](#) and [Corollary 1.3](#) follow from [Theorem 2.1](#).

First, [Theorem 1.1](#) is local on the base. Because the scheme S is reduced, it is the finite union of its irreducible components, each of which is reduced and irreducible, so it suffices to prove the result on each of them. Thus we can immediately reduce to the case where $S = \operatorname{Spec} A$, for some Noetherian domain A . Because Z is finite over S , we have that Z is affine too.

We now reduce to the case where X is affine as well. There is no loss of generality in assuming Z is a proper closed subset of X . The coherent sheaf \mathcal{F} on the non-empty open set U extends to a coherent sheaf on X , which we also denote by \mathcal{F} . To simplify notation, let us denote the sheaf $R^r j_* \mathcal{F}$ by \mathcal{H} , which we observe vanishes outside the closed set Z . Thus each section is annihilated by a power of the ideal \mathcal{I}_Z of Z , which means \mathcal{H} has the structure of a module over the sheaf of rings $\widehat{\mathcal{O}}_{X,Z} = \varprojlim_t \frac{\mathcal{O}_X}{\mathcal{I}_Z^t}$; put differently, the quasi-coherent \mathcal{O}_X -module \mathcal{H} can be viewed as a sheaf on the formal scheme \widehat{X}^Z (whose underlying topological space is the affine closed set Z , finite over S). The sheaf of abelian groups $\mathcal{H} = R^r j_* \mathcal{F}$ (for $r \geq 1$) is supported on and defined locally on Z , which means that to check whether \mathcal{H} is generically free and commutes with base

change over S , we can restrict to any open set X_0 containing all the generic points of the components of Z . In particular, we can choose such X_0 to be affine, thus reducing the proof of [Theorem 1.1](#) to the case where both X and S are affine.

We can now assume that $X \rightarrow S$ is the affine map of affine schemes corresponding to a ring homomorphism $A \rightarrow T$. In this case the closed set Z is defined by some ideal I of T . Because Z is finite over $S = \operatorname{Spec} A$, the composition $A \rightarrow T \rightarrow T/I$ is a finite integral extension of A . The coherent sheaf \mathcal{F} on U extends to a coherent sheaf \mathcal{F} on X , which corresponds to a finitely generated T -module M . Since $X = \operatorname{Spec} T$ is affine, we have natural identifications for $r \geq 1$

$$R^r j_* \mathcal{F} = H^r(X \setminus Z, \mathcal{F}) = H_I^{r+1}(M)$$

of modules over T [[7, Cor. 1.9](#)]. Thus we have reduced [Theorem 1.1](#) to showing that if T is a Noetherian algebra over a Noetherian domain A and I is any ideal such that T/I is finitely generated as an A -module, then for any finitely generated T -module M , the modules $H_I^{r+1}(M)$ are generically free and commute with base change over A for $r \geq 1$. In fact, we will be able to show this for all indices $r \geq -1$.

To get to the power series case, we first observe that for all i , every element of $H_I^i(M)$ is annihilated by some power of I . This means that $H_I^i(M)$ has the structure of a module over the I -adic completion \hat{T}^I . There is no loss of generality in replacing T and M by their I -adic completions \hat{T}^I and \hat{M}^I —the module $H_I^i(M)$ is canonically identified with $H_{I\hat{T}^I}^i(\hat{M}^I)$. So without loss of generality, we assume that T is I -adically complete.

Now, [Lemma 2.2](#) below guarantees that T is a module-finite algebra over a power series ring $A[[x_1, \dots, x_n]]$. So the finitely generated T -module M is also a finitely generated module over $A[[x_1, \dots, x_n]]$, and the computation of local cohomology is the same viewed over either ring. This means that to prove [Theorem 1.1](#), it would suffice to prove [Theorem 2.1](#). It only remains to prove [Lemma 2.2](#).

Lemma 2.2. *Let A be a Noetherian ring. Let T be a Noetherian A -algebra containing an ideal I such that the composition of natural maps $A \rightarrow T \twoheadrightarrow T/I$ is finite. Then there is a natural ring homomorphism from a power series ring*

$$A[[x_1, \dots, x_n]] \rightarrow \hat{T}^I := \varprojlim_t T/I^t$$

which is also finite.

Proof of Lemma. Fix generators y_1, \dots, y_n for the ideal I of T . Consider the A -algebra homomorphism

$$A[x_1, \dots, x_n] \rightarrow T \quad x_i \mapsto y_i.$$

We will show that this map induces a ring homomorphism

$$A[[x_1, \dots, x_n]] \rightarrow \hat{T}^I$$

which is finite. First note that for each natural number t , there is a naturally induced ring homomorphism

$$\frac{A[x_1, \dots, x_n]}{(x_1, \dots, x_n)^t} \rightarrow \frac{T}{I^t} \quad (1)$$

sending the class $\overline{x_i}$ to the class $\overline{y_i}$.

Claim. For every t , the map (1) is finite. Indeed, if t_1, \dots, t_d are elements of T whose classes modulo I are A -module generators for T/I , then the classes of t_1, \dots, t_d modulo I^t are generators for T/I^t as a module over $A[x_1, \dots, x_n]/(x_1, \dots, x_n)^t$.

The claim is straightforward to prove using induction on t and the exact sequence

$$0 \rightarrow I^t/I^{t+1} \rightarrow T/I^{t+1} \rightarrow T/I^t \rightarrow 0.$$

We leave these details to the reader.

Now to prove the lemma, we take the direct limit of the maps (1). Since at every stage, the target is generated over the source by the classes of t_1, \dots, t_d , also in the limit, \hat{T}^I will be generated over $A[[x_1, \dots, x_n]]$ by the images of t_1, \dots, t_d . So the induced ring homomorphism $A[[x_1, \dots, x_n]] \rightarrow T$ is finite. \square

Having reduced the proof of the main results discussed in the introduction to [Theorem 2.1](#), the rest of the paper focuses on the local cohomology statement in the special case. Our proof of [Theorem 2.1](#) uses an A -relative version of Matlis duality to convert the problem to an analogous one for finitely generated modules over a power series ring, where it will follow from the theorem on generic freeness. This relative version of Matlis duality might be of interest to commutative algebraists in other contexts, and holds in greater generality than what we develop here. To keep the paper as straightforward and readable as possible, we have chosen to present it only in the case we need to prove the main result. Some related duality is worked out in [\[4\]](#).

3. A relative Matlis dual functor

3.1. Matlis duality

We first recall the classical Matlis duality in the complete local (Noetherian) case.

Let (R, m) be a complete local ring, and let E be an injective hull of its residue field R/m . The Matlis dual functor $\text{Hom}_R(-, E)$ is an exact contravariant functor on the category of R -modules. It takes each Noetherian R -module (i.e., one satisfying the ascending chain condition) to an Artinian R -module (i.e., one satisfying the descending

chain condition) and vice-versa. Moreover, for any Artinian or Noetherian R -module \mathcal{H} , we have a natural isomorphism $\mathcal{H} \rightarrow \text{Hom}_R(\text{Hom}_R(\mathcal{H}, E), E)$. That is, the Matlis dual functor defines an equivalence of categories between the category of Noetherian and the category of Artinian R -modules. See [7], [2, Thm. 3.2.13] or [10] for more on Matlis duality.

3.2. A relative version of Matlis duality

Let A be a domain. Let R be an A -algebra, with ideal $I \subset R$ such that R/I is finitely generated as an A -module. Define the *relative Matlis dual functor* to be the functor

$$\{R\text{-modules}\} \rightarrow \{R\text{-modules}\}$$

$$M \mapsto M^{\vee_A} := \varinjlim_t \text{Hom}_A(M/I^t M, A).$$

We also denote M^{\vee_A} by $\text{Hom}_A^{cts}(M, A)$, since it is the submodule of $\text{Hom}_A(M, A)$ consisting of maps continuous in the I -adic topology. That is, $\text{Hom}_A^{cts}(M, A)$ is the R -submodule of $\text{Hom}_A(M, A)$ consisting of maps $\phi : M \rightarrow A$ satisfying $\phi(I^t M) = 0$ for some t .

Proposition 3.1. *Let R be a Noetherian algebra over a Noetherian domain A , with ideal $I \subset R$ such that R/I is finitely generated as an A -module.*

(1) *The functor $\text{Hom}_A^{cts}(-, A)$ is naturally equivalent to the functor*

$$M \mapsto \text{Hom}_R(M, \text{Hom}_A^{cts}(R, A)).$$

(2) *The functor preserves exactness of sequences*

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

of finitely generated R -modules, provided that the modules $M_3/I^n M_3$ are (locally) free A -modules for all $n \gg 0$.

Remark 3.2. If $A = R/I$ is a field, then the relative Matlis dual specializes to the usual Matlis dual functor $\text{Hom}_R(-, E)$, where E is the injective hull of the residue field of R at the maximal ideal I (denoted here now m). Indeed, one easily checks that $\text{Hom}_A^{cts}(R, A)$ is an injective hull of R/m . To wit, the R -module homomorphism

$$R/m \rightarrow \text{Hom}_A^{cts}(R, A) \quad \text{sending} \quad r \bmod m \mapsto \begin{cases} R \rightarrow A \\ s \mapsto rs \bmod m \end{cases}$$

is a maximal essential extension of R/m .

Proof of Proposition. Statement (1) follows from the adjointness of tensor and Hom, which is easily observed to restrict to the corresponding statement for modules of continuous maps.

For (2), we need to show the sequence remains exact after applying the relative Matlis dual functor. The functor $\text{Hom}_A(-, A)$ preserves left exactness: that is,

$$0 \rightarrow \text{Hom}_A(M_3, A) \rightarrow \text{Hom}_A(M_2, A) \rightarrow \text{Hom}_A(M_1, A) \quad (2)$$

is exact. We want to show that, restricting to the submodules of *continuous maps*, we also have exactness at the right. That is, we need the exactness of

$$0 \rightarrow \text{Hom}_A^{cts}(M_3, A) \rightarrow \text{Hom}_A^{cts}(M_2, A) \rightarrow \text{Hom}_A^{cts}(M_1, A) \rightarrow 0. \quad (3)$$

The exactness of (3) at all spots except the right is easy to verify using the description of a continuous map as one annihilated by a power of I .

To check exactness of (3) at the right, we use the Artin Rees Lemma [1, 10.10]. Take $\phi \in \text{Hom}_A^{cts}(M_1, A)$. By definition of continuous, we know ϕ is annihilated by I^n for sufficiently large n . By the Artin–Rees Lemma, there exists t such that for all $n \geq t$, we have $I^{n+t}M_2 \cap M_1 \subset I^n M_1$. This means we have a surjection

$$M_1/(I^{n+t}M_2 \cap M_1) \twoheadrightarrow M_1/I^n M_1.$$

Therefore the composition

$$M_1/I^{n+t}M_2 \cap M_1 \twoheadrightarrow M_1/I^n M_1 \rightarrow A$$

gives a lifting of ϕ to an element ϕ' in $\text{Hom}_A(M_1/I^{n+t}M_2 \cap M_1, A)$.

Now note that for $n \gg 0$, we have exact sequences

$$0 \rightarrow M_1/M_1 \cap I^{n+t}M_2 \rightarrow M_2/I^{n+t}M_2 \rightarrow M_3/I^{n+t}M_3 \rightarrow 0,$$

which are split over A by our assumption that $M_3/I^{n+t}M_3$ is projective. Thus

$$\begin{aligned} 0 \rightarrow \text{Hom}_A(M_3/I^{n+t}M_3, A) &\rightarrow \text{Hom}_A(M_2/I^{n+t}M_2, A) \\ &\rightarrow \text{Hom}_A(M_1/M_1 \cap I^{n+t}M_2, A) \rightarrow 0 \end{aligned} \quad (4)$$

is also split exact. This means we can pull $\phi' \in \text{Hom}_A(M_1/I^{n+t}M_2 \cap M_1, A)$ back to some element $\tilde{\phi}$ in $\text{Hom}_A(M_2/I^{n+t}M_2, A)$. So our original continuous map $M_1 \xrightarrow{\phi} A$ is the restriction of some map $M_2 \xrightarrow{\tilde{\phi}} A$ which satisfies $\tilde{\phi}(I^{n+t}M_2) = 0$. This exactly says the continuous map ϕ on M_1 extends to a continuous map $\tilde{\phi}$ of M_2 . That is, the sequence (3) is exact. \square

Remark 3.3. If M_3 is a Noetherian module over a Noetherian algebra R over the Noetherian domain A , then the assumption that $M_3/I^n M_3$ is locally free for all n holds generically on A —that is, after inverting a single element of A . See [Lemma 4.2](#).

4. Generic freeness

We briefly review Grothendieck’s idea of generic freeness, and use it to prove that the relative Matlis dual of a Noetherian R -module is generically free over A (under suitable Noetherian hypothesis on R and A).

Let M be a module over a commutative domain A . We say that M is *generically free* over A if there exists a non-zero $g \in A$, such that $M \otimes_A A_g$ is a free A_g -module, where A_g denotes the localization of A at the element g . Likewise, a collection \mathcal{M} of A -modules is *simultaneously generically free* over A if there exists a non-zero $g \in A$, such that $M \otimes_A A_g$ is a free for all modules $M \in \mathcal{M}$. Note that any finite collection of generically free modules is always simultaneously generically free, since we can take g to be the product the g_i that work for each of the M_i .

Of course, finitely generated A -modules are always generically free. Grothendieck’s famous **Lemma on Generic Freeness** ensures that many other modules are as well:

Lemma 4.1 ([\[5, 6.9.2\]](#)). *Let A be a Noetherian domain. Let M be any finitely generated module over a finitely generated A -algebra T . Then M is generically free over A .*

We need a version of Generic Freeness for certain infinite families of modules over more general A -algebras:

Lemma 4.2. *Let A be any domain. Let T be any Noetherian A -algebra, and $I \subset T$ any ideal such that T/I is finite over A . Then for any Noetherian T -module M , the family of modules*

$$\{M/I^n M \mid n \geq 1\}$$

is simultaneously generically free over A . That is, after inverting a single element of A , the modules $M/I^n M$ for all $n \geq 1$ become free over A .

Remark 4.3. We will make use of [Lemma 4.2](#) in the case where $T = A[[x_1, \dots, x_n]]$.

Proof. If M is finitely generated over T , then the associated graded module

$$gr_I M = M/IM \oplus IM/I^2 M \oplus I^2 M/I^3 M \oplus \dots$$

is finitely generated over the associated graded ring $gr_I T = T/I \oplus I/I^2 \oplus I^2/I^3 \oplus \dots$, which is a homomorphic image of a polynomial ring over T/I . Hence $gr_I T$ is a finitely generated A -algebra. Applying the Lemma of generic freeness to the $gr_I T$ -module $gr_I M$,

we see that after inverting a single non-zero element of g of A , the module $gr_I M$ becomes A -free. Since $gr_I M$ is graded over $gr_I T$ and A acts in degree zero, clearly its graded pieces are also free after tensoring with A_g . We can thus replace A by A_g for suitable g , and assume that the $I^n M/I^{n+1} M$ are A_g -free for all $n \geq 0$.

Now consider the short exact sequences

$$0 \rightarrow I^n M/I^{n+1} M \rightarrow M/I^{n+1} M \rightarrow M/I^n M \rightarrow 0, \quad (5)$$

for each $n \geq 1$. We already know that $M/I^1 M$ and $I^n M/I^{n+1} M$ for all $n \geq 1$ are free (over A_g), so by induction, we conclude that the sequences (5) are all split over A_g for every n . In particular, the modules $M/I^n M$ are also free over A_g for all $n \geq 1$. The lemma is proved. \square

Proposition 4.4. *Let A be a Noetherian domain. Let R be any Noetherian A -algebra with ideal $I \subset R$ such that R/I is a finitely generated A -module. Then for any Noetherian R -module M , the relative Matlis dual*

$$\mathrm{Hom}_A^{cts}(M, A)$$

is a generically free A -module. Furthermore, if $g \in A$ is a non-zero element such that $A_g \otimes_A \mathrm{Hom}_A^{cts}(M, A)$ is free over A_g , then for any base change $A \rightarrow L$ factoring through A_g , the natural map

$$\mathrm{Hom}_A^{cts}(M, A) \otimes_A L \rightarrow \mathrm{Hom}_L^{cts}(M \otimes_A L, L)$$

is an isomorphism of $R \otimes_A L$ -modules, functorial in M .

Proof. We can invert one element of A so that each $M/I^t M$ is free over A ; replace A by this localization. We now claim that the A -module

$$\mathrm{Hom}_A^{cts}(M, A) = \varinjlim_t \mathrm{Hom}_A \left(\frac{M}{I^t M}, A \right)$$

is free. Indeed, since each $M/I^t M$ is free and has finite rank over A , its A -dual $\mathrm{Hom}_A \left(\frac{M}{I^t M}, A \right)$ is also free of finite rank. The direct limit is also A -free because the maps in the limit system are all split over A and injective. Indeed, if some finite A -linear combination of $f_i \in \mathrm{Hom}_A^{cts}(M, A)$ is zero, then that same combination is zero considered as elements of the free-submodule $\mathrm{Hom}_A \left(\frac{M}{I^t M}, A \right)$ of homomorphisms in $\mathrm{Hom}_A^{cts}(M, A)$ killed by a large power of I . It follows that $\mathrm{Hom}_A^{cts}(M, A)$ is free over A , as desired.

The result on base change follows as well, since tensor commutes with direct limits and with dualizing a finitely generated free module. \square

Remark 4.5. We can interpret Proposition 4.4 as saying that *generically on A* , the relative Matlis dual functor (applied to Noetherian R -modules) is exact and commutes with base change.

5. Relative local duality and the proof of the main theorem

The proof [Theorem 2.1](#) and therefore of our main result answering Kollár’s question, follows from a relative version of Local Duality:

Theorem 5.1. *Let R be a power series ring $A[[x_1, \dots, x_n]]$ over a Noetherian domain A , and let M be a finitely generated R -module. Set $I \subset R$ to be the ideal generated by x_1, \dots, x_n . Then, after replacing A by its localization at one element, there is a functorial isomorphism for all i*

$$H_I^i(M) \cong [\mathrm{Ext}_R^{n-i}(M, R)]^{\vee_A}.$$

To prove [Theorem 5.1](#), we need the following lemma.

Lemma 5.2. *Let R be a power series ring $A[[x_1, \dots, x_n]]$ over a ring A . There is a natural R -module isomorphism $\mathrm{Hom}_A^{cts}(R, A) \cong H_I^n(R)$, where $I = (x_1, \dots, x_n)$. In particular, the relative Matlis dual functor can also be expressed*

$$M \mapsto \mathrm{Hom}_R(M, H_I^n(R)).$$

Proof. We recall that $H_I^i(R)$ is the i -th cohomology of the extended Čech complex K^\bullet on the elements x_1, \dots, x_n . This is the complex

$$0 \rightarrow R \xrightarrow{\delta_1} R_{x_1} \oplus R_{x_2} \cdots \oplus R_{x_n} \rightarrow \bigoplus_{i < j} R_{x_i x_j} \rightarrow \cdots \xrightarrow{\delta_n} R_{x_1 x_2 \cdots x_n} \rightarrow 0$$

where the maps are (sums of) suitably signed localization maps. In particular, $H_I^n(R)$ is the cokernel of δ_n , which can be checked to be the free A -module on (the classes of) the monomials $x_1^{a_1} \cdots x_n^{a_n}$ with all $a_i < 0$.¹

Now define an explicit R -module isomorphism Φ from $H_I^n(R)$ to $\mathrm{Hom}_A^{cts}(R, A)$ by sending the (class of the) monomial x^α to the map $\phi_\alpha \in \mathrm{Hom}_A^{cts}(R, A)$:

$$R \xrightarrow{\phi_\alpha} A$$

$$x_1^{b_1} \cdots x_n^{b_n} \mapsto \begin{cases} 1 & \text{if } \alpha_i + b_i = -1 \text{ for all } i \\ 0 & \text{otherwise} \end{cases}$$

Since $\{x^\beta \mid \beta \in \mathbb{N}^n\}$ is an A -basis for R , the map ϕ_α is a well-defined A -module map from R to A , and since it sends all but finitely many x^β to zero, ϕ_α is I -adically continuous. Thus the map $H_I^n(R) \xrightarrow{\Phi} \mathrm{Hom}_A^{cts}(R, A)$ is an A -module homomorphism; in fact, Φ is an

¹ See page 226 of [\[9\]](#), although I have included one extra map $\delta_1 : R \rightarrow \bigoplus R_{x_i}$ sending $f \mapsto (\frac{f}{x_1}, \dots, \frac{f}{x_n})$ in order to make the complex exact on the left, and my ring is a power series ring over A instead of a polynomial rings over a field. This is also discussed in [\[7\]](#) page 22.

A -module isomorphism, since it defines a bijection between the A -basis $\{x^\alpha \mid \alpha_i < 0\}$ for $H_I^n(R)$ and the dual basis for $\text{Hom}_A^{cts}(R, A)$ corresponding to the free basis of monomials for R over A . It is easy to check that Φ is also R -linear. Thus Φ identifies the R -modules $H_I^n(R)$ and $\text{Hom}_A^{cts}(R, A)$. \square

Proof of Theorem 5.1. We proceed by proving that both modules are generically isomorphic to a third, namely $\text{Tor}_{n-i}^R(M, H_I^n(R))$.

First, recall how to compute $H_I^i(M)$. Let K^\bullet be the extended Čech complex on the elements x_1, \dots, x_n

$$0 \rightarrow R \xrightarrow{\delta_1} R_{x_1} \oplus R_{x_2} \cdots \oplus R_{x_n} \rightarrow \bigoplus_{i < j} R_{x_i x_j} \rightarrow \cdots \xrightarrow{\delta_n} R_{x_1 x_2 \cdots x_n} \rightarrow 0.$$

This is a complex of flat R -modules, all free over A , exact at every spot except the right end. Thus it is a flat R -module resolution of the local cohomology module $H_I^n(R)$. The local cohomology module $H_I^i(M)$ is the cohomology of this complex after tensoring over R with M , that is

$$H_I^i(M) = \text{Tor}_{n-i}^R(M, H_I^n(R)).$$

On the other hand, let us compute the relative Matlis dual of $\text{Ext}_R^{n-i}(M, R)$. Let P_\bullet be a free resolution of M over R . The module $\text{Ext}_R^\bullet(M, R)$ is the cohomology of the complex $\text{Hom}_R(P_\bullet, R)$. We would like to say that the computation of the cohomology of this complex commutes with the relative Matlis dual functor, but the best we can say is that this is true *generically on A* . To see this, we will apply Lemma 4.2 to the following finite set of R -modules:

- For $i = 0, \dots, n$, the image D_i of the i -th map of the complex $\text{Hom}_R(P_\bullet, R)$;
- For $i = 0, \dots, n$, the cohomology $\text{Ext}_R^{n-i}(M, R)$ of the same complex.

Lemma 4.2 guarantees that the modules

$$D_i/I^t D_i \quad \text{and} \quad \text{Ext}_R^{n-i}(M, R)/I^t \text{Ext}_R^{n-i}(M, R)$$

are all simultaneously generically free over A for all $t \geq 1$. This allows us to break up the complex $A_g \otimes_A \text{Hom}_R(P_\bullet, R)$ into many short exact sequences, split over A_g , which satisfy the hypothesis of Proposition 3.1(2) (using A_g in place of A and $A_g \otimes_A R$ in place of R). It follows that the computation of cohomology of $\text{Hom}_R(P_\bullet, R)$ commutes with the relative Matlis dual functor (generically on A).

Thus, after replacing A by a localization at one element, $\text{Ext}_R^{n-i}(M, R)]^{\vee_A}$ is the cohomology of the complex

$$\text{Hom}_R(\text{Hom}_R(P_\bullet, R), H_I^n(R)).$$

In general, for any finitely generated projective module P and any module H (over any Noetherian ring R), the natural map

$$P \otimes H \rightarrow \operatorname{Hom}(\operatorname{Hom}(P, R), H)$$

sending a simple tensor $x \otimes h$ to the map which sends $f \in \operatorname{Hom}(P, R)$ to $f(x)h$, is an isomorphism, functorially in P and H . Thus we have a natural isomorphism of complexes

$$P_{\bullet} \otimes H_I^n(R) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(P_{\bullet}, R), H_I^n(R)),$$

and so $[\operatorname{Ext}^{n-i}(M, R)]^{\vee A}$ is identified with $\operatorname{Tor}_{n-i}(M, H_I^n(R))$, which as we saw is identified with $H_I^i(M)$.

Since all identifications are functorial, we have proven the relative local duality $H_I^i(M) \cong [\operatorname{Ext}^{n-i}(M, R)]^{\vee A}$, generically on A . \square

We can finally finish the proof of [Theorem 1.1](#), and hence the main result:

Proof of Theorem 2.1. Let R be a power series ring over a Noetherian domain A , and let M be any Noetherian R -module. We need to show that the local cohomology modules $H_I^i(M)$ are generically free and commute with base change over A .

In light of [Proposition 4.4](#), we can accomplish this by showing that $H_I^i(M)$ is the relative Matlis dual of a Noetherian R -module, generically on A . But this is guaranteed by the relative local duality theorem [Theorem 5.1](#), which guarantees that

$$H_I^i(M) \cong \operatorname{Ext}_R^{n-i}(M, R)^{\vee A}$$

generically on A . \square

Remark 5.3. One could obviously develop the theory of relative Matlis duality, especially [Theorem 5.1](#), further; I wrote down only the simplest possible case and the simplest possible statements needed to answer Kollár’s question as directly as possible.

References

- [1] M.F. Atiyah, I.G. MacDonald, *Introduction to Commutative Algebra*, Addison Wesley Series in Mathematics, Addison Wesley, London, 1969.
- [2] Winfred Burns, Jürgen Herzog, *Cohen-Macaulay Rings*, Cambridge Series in Advanced Mathematics, vol. 39, Cambridge University Press, 1993.
- [3] Brian Conrad, *Grothendieck Duality and Base Change*, Lecture Notes in Mathematics, vol. 1750, Springer, 2001.
- [4] W.G. Dwyer, J.P.C. Greenlees, S. Iyengar, Duality in algebra and topology, *Adv. Math.* 200 (2) (1 March 2006) 357–402.
- [5] Alexander Grothendieck, Jean Dieudonné, *Éléments de Géométrie Algébrique*, Chapter IV, part 2, Inst. Hautes Études Sci. Publ. Math. 24 (1965).
- [6] Alexander Grothendieck, Michele Raynaud, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*, in: Yves Laszlo (Ed.), *Documents Mathématiques (Paris)*, vol. 4, Société Mathématique de France, Paris, ISBN 978-2-85629-169-6, 2005, arXiv:math/0511279, [1968], MR 2171939.

- [7] Robin Hartshorne, Local Cohomology. A Seminar Given by A. Grothendieck, Harvard University, Fall, 1961, Lecture Notes in Mathematics, vol. 41, Springer-Verlag, Berlin, New York, 1967, MR0224620.
- [8] Robin Hartshorne, Residues and Duality: Lecture Notes of a Seminar on the Work of A. Grothendieck, Given at Harvard 1963/64, Lecture Notes in Mathematics, vol. 20, Springer Verlag, 1966.
- [9] Robin Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, 2006.
- [10] Mel Hochster, Lecture notes on local cohomology, unpublished, from his University of Michigan website <http://www.math.lsa.umich.edu/~hochster/615W11/loc.pdf>.
- [11] Mel Hochster, Joel Roberts, The purity of Frobenius and local cohomology, *Adv. Math.* 21 (1976) 117–172.
- [12] János Kollár, Maps between local Picard groups, arXiv:1407.5108v2, 2014.