Representations of the Necklace Braid Group: Topological and Combinatorial Approaches

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Abstract: The necklace braid group $\mathcal{NB}_n$ is the motion group of the $n + 1$ component necklace link $\mathcal{L}_n$ in Euclidean $\mathbb{R}^3$. Here $\mathcal{L}_n$ consists of n pairwise unlinked Euclidean circles each linked to an auxiliary circle. Partially motivated by physical considerations, we study representations of the necklace braid group $\mathcal{NB}_n$, especially those obtained as extensions of representations of the braid group $\mathcal{B}_n$ and the loop braid group $\mathcal{LB}_n$. We show that any irreducible $\mathcal{B}_n$ representation extends to $\mathcal{NB}_n$ in a standard way. We also find some non-standard extensions of several well-known $\mathcal{B}_n$-representations such as the Burau and LKB representations. Moreover, we prove that any local representation of $\mathcal{B}_n$ (i.e., coming from a braided vector space) can be extended to $\mathcal{NB}_n$, in contrast to the situation with $\mathcal{LB}_n$. We also discuss some directions for future study from categorical and physical perspectives.

1. Introduction

Topology, like many fields of mathematics, owes some of its early development to questions arising in physics. A classic example of this is the development of knot theory: Lord Kelvin and Tait, inspired by experiments of Helmholtz, theorized that atoms were knotted tubes of æther, distinguished by their knot type [43]. This theory was quickly dismissed, but Tait's tabulation of knot projections with few crossings is arguably the dawn of modern knot theory.

   Non-abelian statistics of anyons in two spatial dimensions has attracted considerable attention largely due to topological quantum computation [37,49]. Exchanging non-abelian anyons induces unitary representations of the braid group $\mathcal{B}_n$, which can yield
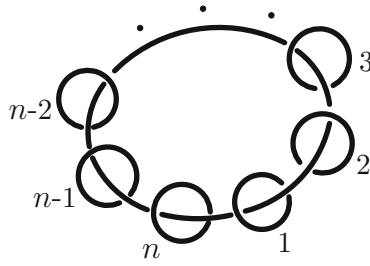
Fig. 1. The necklace $\mathcal{L}_n$ as seen from a generic observation point

braiding-only universal quantum computation models. Mathematically, this is a rich theory because the braid group acts faithfully on the fundamental group of the punctured plane. The well-studied framework of (2+1)-TQFTs can be used to systematically study these representations and their vast generalization to mapping class groups of punctured surfaces of any genus.

Naturally we would like to extend these ideas to 3-dimensional topological materials. Unfortunately $(3 + 1)$-TQFTs are not as well studied so we cannot obtain as explicit descriptions as in the 2-dimensional case. Instead, here we will study the relevant *motion groups* [24] and their representations from a more elementary algebraic perspective.

An extension of non-abelian statistics of point-like excitations to three spatial dimensions is not possible due to the spin-statistics theorem in its exchange statistics formulation: exchanging the positions of two indistinguishable particles changes their state vector by at most a sign. Mathematically, the motion group of n identical points in $\mathbb{R}^3$ is the symmetric group $\mathfrak{S}_n$ which leads to the possibility of parastatistics. Notice also that the fundamental group of $\mathbb{R}^3$ with n points deleted is trivial, so that motions of points cannot be detected in this way—one must consider the framing of paths to explain the spin-statistics theorem topologically. But in any case points in 3-dimensions are much less interesting than in 2-dimensions.

On the other hand, *loop* or closed string excitations occur naturally in condensed matter physics and string theory. The mathematical manifestation of this idea was considered in [9,28] with a study of local and low dimension representations of the loop braid group $\mathcal{LB}_n$: the group of motions of n oriented circles in $\mathbb{R}^3$. In this article we consider the related group $\mathcal{NB}_n$ of motions (up to isotopy) of a necklace $\mathcal{L}_n$: n unlinked oriented circles that are linked to another auxiliary oriented circle, see Figure 1. One compelling reason to undertake this study is that such a configuration may be more feasible physically than the free loop picture. Indeed, a number of proposals in this direction have appeared recently, see [12,25,31,47].

The motion group $\mathcal{NB}_n$, the *necklace braid group*, is described in [7], where it is identified with the fundamental group of the configuration space of $\mathcal{L}_n$. In $\mathcal{L}_n$ we fix a circle labelled 1, and order the n circles $1, \ldots, n$ in a counterclockwise fashion; and orient the auxiliary circle in the same counterclockwise way. The group $\mathcal{NB}_n$ includes elements $\sigma_1, \ldots, \sigma_n, \tau$ where σ_i is the motion, up to homotopy, of passing the ith circle through the $i + 1$st, while τ corresponds to shifting each circle one position in the counterclockwise direction. We use the function convention when composing elements of the motion group: fg means apply g then f. In fact $\mathcal{NB}_n$ is generated by these elements.

Although the realisation of $\mathcal{NB}_n$ as a motion group is fundamental to its utility in physical modelling, from a representation theory perspective, manipulating $\mathcal{NB}_n$ at the 'geometric topological' level of its definition as a motion group is relatively hard. Fortunately [7, Theorem 2.3] gives a presentation by abstract generators and relations that facilitates such manipulations. Next we discuss this 'combinatorial' realisation.

Theorem 1.1 ([7]). *We have a presentation of a group isomorphic to $\mathcal{NB}_n$ by abstract generators $\sigma_1, \ldots, \sigma_n, \tau$ satisfying:*

(B1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
(B2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \neq 1 \pmod{n}$,
(N1) $\tau \sigma_i \tau^{-1} = \sigma_{i+1}$ for $1 \leq i \leq n$
(N2) $\tau^{2n} = 1$

Here indices are taken modulo n, with $\sigma_{n+1} := \sigma_1$ and $\sigma_0 := \sigma_n$. $\quad\square$

Observe from the presentation in Theorem 1.1 that the subgroup generated by the σ_i for $1 \leq i \leq n - 1$ is a quotient of Artin's braid group $\mathcal{B}_n$. It is not hard to verify that (N1) and (N2) do not induce further relations among $\sigma_1, \ldots, \sigma_{n-1}$, so that, in fact, we have $\mathcal{B}_n < \mathcal{NB}_n$.

It also follows from the presentation that $\mathcal{NB}_n$ contains a normal subgroup isomorphic to the *affine braid group* (of type A) on n strands: $\mathcal{B}\tilde{A}_n \cong \langle \sigma_1, \ldots, \sigma_n \rangle \lhd \mathcal{NB}_n$. Notice that $[\mathcal{NB}_n : \mathcal{B}\tilde{A}_n] = 2n$ and $\mathcal{NB}_n = \mathcal{B}\tilde{A}_n \rtimes \langle \tau \rangle$. In particular, every element of $\mathcal{NB}_n$ may be written as $\tau^k \beta$ with $\beta \in \mathcal{B}\tilde{A}_n$.

The *annular* or *circular braid group* $\mathcal{CB}_n$ (the fundamental group of the configuration space of n points in an annulus) may be presented by generators as for $\mathcal{NB}_n$ in Theorem 1.1 but omitting relation (N2) (see e.g. [7]). Thus $\mathcal{NB}_n$ is isomorphic to a quotient of $\mathcal{CB}_n$.

$$\zeta : \mathcal{CB}_n \to \mathcal{NB}_n \tag{1.1}$$

Some of the relations (B1–N2) for $\mathcal{NB}_n$ are redundant: the following reduces the number of defining relations from $\frac{1}{2}n(n + 1) + 1$ to $2n - 1$.

Lemma 1.2. *The relations (N2), (N1), (B1) for $i = 1$ (i.e., $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$), and (B2) for $i = 1$ and $3 \leq j \leq n - 1$ (i.e., $\sigma_1 \sigma_j = \sigma_j \sigma_1$ for $3 \leq j \leq n - 1$), imply all relations of Theorem 1.1.*

Proof. Assuming (N1) gives us $\tau^{i-1} \sigma_1 \tau^{-i+1} = \sigma_i$ for all i where indices are taken modulo n. Thus $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$ implies that for any i:

$$\begin{aligned}
\sigma_i \sigma_{i+1} \sigma_i &= (\tau^{i-1} \sigma_1 \tau^{-i+1})(\tau^i \sigma_1 \tau^{-i})(\tau^{i-1} \sigma_1 \tau^{-i+1}) \\
&= \tau^{i-1} \sigma_1 \tau^1 \sigma_1 \tau^{-1} \sigma_1 \tau^{-i+1} \\
&= \tau^{i-1} \sigma_1 \sigma_2 \sigma_1 \tau^{-i+1} \\
&= \tau^{i-1} \sigma_2 \sigma_1 \sigma_2 \tau^{-i+1} \\
&= \tau^i \sigma_1 \tau^{-1} \sigma_1 \tau^1 \sigma_1 \tau^{-i} \\
&= (\tau^i \sigma_1 \tau^{-i})(\tau^{i-1} \sigma_1 \tau^{-i+1})(\tau^i \sigma_1 \tau^{-i}) \\
&= \sigma_{i+1} \sigma_i \sigma_{i+1}.
\end{aligned}$$

Next we verify (B2) assuming σ_1 commutes with σ_k with $3 \leq k \leq n - 1$. We may assume $n \geq j > i > 1$ and $|j - i| \not\equiv 1 \pmod{n}$.

$$\sigma_i \sigma_j = \tau^{i-1} \sigma_1 \tau^{-i+1} \tau^{i-1} \sigma_{j-i+1} \tau^{-i+1}$$

$$= \tau^{i-1} \sigma_1 \sigma_{j-i+1} \tau^{-i+1}$$
$$= \tau^{i-1} \sigma_{j-i+1} \sigma_1 \tau^{-i+1}$$
$$= \tau^{i-1} \sigma_{j-i+1} \tau^{-i+1} \tau^{i-1} \sigma_1 \tau^{-i+1} = \sigma_j \sigma_i.$$

□

Another easy observation is that $\sigma_i \mapsto (i\ i+1)$ (modulo n, so that $\sigma_n \mapsto (n\ 1)$) and $\tau \to (1\ 2\ \cdots\ n)$ gives a surjection $\varphi : \mathcal{NB}_n \to S_n$. The kernel of φ is the normal subgroup of motions that carry each circle back to their original positions: the *pure necklace braids*.

The motion group of n unlinked oriented circles in $\mathbb{R}^3$, i.e., the Loop braid group $\mathcal{LB}_n$ studied in [2] does not contain the necklace braid group, but it does contain a quotient of $\mathcal{NB}_n$ by a central subgroup of order 2 as we will see. This relationship is explored in Sect. 2.

1.1. Overview of paper. In Sect. 2 we lay down some basic facts about the relationships between $\mathcal{NB}_n$ and various other topologically constructed groups. In particular we develop a number of ways to construct representations of $\mathcal{NB}_n$ both by extending from $\mathcal{B}_n$ representations or by factoring through representations of the loop braid group. These ideas are applied in Sect. 3 to many well-known examples of $\mathcal{B}_n$ representations, while in Sect. 4 we focus on extending *local* $\mathcal{B}_n$ representations. It turns out to be much easier to construct local representations of $\mathcal{NB}_n$ than $\mathcal{LB}_n$. In Sect. 5 we address the 'brute' algebraic analysis of low-dimensional representations (in the spirit of [9,21,29]). In Sect. 6.1 we return closer to physical considerations. We discuss the construction of representations from braided fusion categories; and in Sect. 6.2 we explore more direct physical manifestations of $\mathcal{NB}_n$ using 'categorified' quantum spin chains.

2. $\mathcal{NB}_n$ Representations from $\mathcal{B}_n$ and $\mathcal{LB}_n$ Representations

The necklace braid group is closely related to both the ordinary braid group $\mathcal{B}_n$ and the loop braid group $\mathcal{LB}_n$, and both provide a rich source of representations. In this section we explore these relationships.

2.1. Relationship with $\mathcal{B}_n$. As noted, the braid group $\mathcal{B}_n$ on n strands is isomorphic to the subgroup of $\mathcal{NB}_n$ generated by $\sigma_1, \ldots, \sigma_{n-1}$. Notice that the absence of σ_n and τ obviates the consideration of indices modulo n in relations (B1) and (B2). In particular, any representation of $\mathcal{NB}_n$ restricts to a representation of $\mathcal{B}_n$.

In $\mathcal{B}_n$ define the *single twist*

$$\gamma = \sigma_1 \cdots \sigma_{n-1}. \tag{2.1}$$

The center of $\mathcal{B}_n$ is generated by the the full twist of the n strands: $\gamma^n = (\sigma_1 \cdots \sigma_{n-1})^n$.

$\mathcal{B}_n$ can be generated by γ and σ_1, as can be seen from the useful:

$$\gamma^k \sigma_1 \gamma^{-k} = \sigma_{k+1}, \quad 1 \le k \le n-2. \tag{2.2}$$

Defining

$$\sigma_n' := \gamma^{n-1} \sigma_1 \gamma^{1-n} = \gamma^{-1} \sigma_1 \gamma = \gamma \sigma_{n-1} \gamma^{-1} \tag{2.3}$$

we find that $\sigma_1, \ldots, \sigma_{n-1}, \sigma_n'$ satisfy the (modulo n) relations (B1) and (B2) above. Setting $\tau = \gamma$ we also verify (N1), but not (N2) in general. However, for certain representations we can take advantage of this close relationship to produce representations of $\mathcal{NB}_n$:

Lemma 2.1. *Let $\rho : \mathcal{B}_n \to GL(V)$ be any indecomposable finite-dimensional representation of $\mathcal{B}_n$ such that $\rho(\gamma^{2n}) = c_\rho \mathrm{Id}_V$ for some scalar c_ρ (for example any irreducible ρ). Then ρ extends to an indecomposable representation of $\mathcal{NB}_n$ by $\rho(\sigma_n) = \rho(\gamma \sigma_{n-1} \gamma^{-1})$ and*

$$\rho(\tau) = (c_\rho)^{-1/2n} \rho(\gamma).$$

Proof. Relations (B1) and (B2) are immediate, and relations (N1) follow from the centrality of γ^n. For (N2):

$$\rho(\tau^{2n}) = \left((c_\rho)^{-1/2n} \rho(\gamma) \right)^{2n} = \frac{1}{c_\rho} \rho(\gamma^{2n}) = Id_V.$$

In fact, since the braid relations are homogeneous we can rescale the $\rho(\sigma_i)$s by $\kappa \neq 0$ to obtain a new representation $\rho'(\sigma_i) = \kappa \rho(\sigma_i)$ of $\mathcal{B}_n$. Setting $\rho'(\tau) = \rho'(\gamma)$ the rescaling will not affect (B1), (B2), (N1), but can be chosen to cancel the scalar in $\rho(\gamma^{2n})$, and then we may define $\rho(\sigma_n)$ as above.

Now let ρ be any finite-dimensional representation of $\mathcal{B}_n$. It is a direct sum of indecomposable representations, ρ_i say, and there is a projection to each of these in $\mathrm{End}_{\mathcal{B}_n}(V)$. The sum of the projections each individually rescaled according to Lemma 2.1 gives a $\rho(\tau)$ and hence a representation of $\mathcal{NB}_n$. In particular:

Theorem 2.2. *Let $\rho : \mathcal{B}_n \to GL(V)$ be any completely reducible complex representation of $\mathcal{B}_n$ (for example any unitary or irreducible representation). Then there exists a $D \in \mathrm{End}_{\mathcal{B}_n}(V)$ such that defining*

$$\rho(\tau) = D\rho(\gamma), \quad \rho(\sigma_n) = \rho(\gamma)\rho(\sigma_{n-1})\rho(\gamma^{-1})$$

is a representation of $\mathcal{NB}_n$.

Proof. By complete reducibility, for some set of irreducibles $\{W_i\}_i$ we have $V \cong \bigoplus_i W_i$. Since γ^{2n} is central in $\mathcal{B}_n$, we have $\rho \mod_{W_i}(\gamma^{2n}) = c_i \mathrm{Id}_{W_i}$ for some $c_i \in \mathbb{C}$. Define $D = \bigoplus_i (c_i)^{\frac{-1}{2n}} \mathrm{Id}_{W_i} \in \mathrm{End}_{\mathcal{B}_n}(V)$ so that $(D\rho(\gamma))^{2n} = \mathrm{Id}_V$. Since D commutes with the operators $\rho(\sigma_i)$ for $1 \leq i \leq n-1$, it also commutes with $\rho(\gamma)$. Hence defining $\rho(\tau) = D\rho(\gamma)$ and $\rho(\sigma_n) = \rho(\gamma^{-1}\sigma_1\gamma)$ we see by the above discussion that all $\mathcal{NB}_n$ relations are satisfied and hence these assignments extend ρ to $\mathcal{NB}_n$. $\quad\square$

This approach bears some similarity with the notion of a standard extension found in [9], so we adopt this nomenclature and refer to any representation ρ of $\mathcal{NB}_n$ with $\rho(\tau) = A\rho(\gamma)$ and $[A, \rho(\sigma_i)] = 1$ for all i, a **standard extension**. Since $A = \rho(\tau\gamma^{-1})$, this operator is already in the image of ρ, so that $\rho(\mathcal{NB}_n)$ is generated by $\rho(\mathcal{B}_n)$ and A. In particular, $\rho(\mathcal{NB}_n)$ is a central product of the cyclic group generated by $\rho(\mathcal{B}_n)$ and the central operator A. From a topological perspective standard extensions do not fully exploit the $(3 + 1)$-dimensional nature of $\mathcal{NB}_n$, rather the interesting information they carry is already present in the braid group $\mathcal{B}_n$, which is related to $(2 + 1)$-dimensional topology.

Remark 2.3. By Schur's lemma a standard extension of an irreducible $\mathcal{B}_n$ representation has the form $\rho(\tau) = \lambda\rho(\gamma)$, where λ is a scalar. On the other hand, the standard extensions fit into a more general construction, observing that a representation φ of $\mathcal{B}_n$ lifts trivially to $\mathcal{CB}_n$ by letting $\tau \in \mathcal{CB}_n$ act by $\varphi(\gamma)$. Indeed, if $G = F \ltimes \langle y \rangle$ is a semi-direct product in which y acts by conjugation and y^n is central for some n then any representation of G for which the image of y^n is semisimple (diagonalizable) factors over the quotient $G/\langle y^n \rangle$. It is conceivable that more interesting representations of $\mathcal{NB}_n$ can be obtained from $\mathcal{CB}_n$ in this way, cf. Sect. 6.2.

Remark 2.4. Failure of complete reducibility does not preclude the existence of standard extensions: the representation of $\mathcal{B}_n$ defined by $\rho(\sigma_i) = J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ for all i is not completely reducible, yet $\rho(\gamma) = J^{n-1}$ commutes with $\rho(\sigma_i)$ so that $\rho(\tau) = Id = (J^{1-n})\rho(\gamma)$ is a standard extension.

Of course our existence Theorem does not tell us how to *construct* standard extensions. In this setting there is a computational distinction to be made between individual values of n (see Sect. 5), and a construction that starts with a braid group representation for all n and determines a D for each n. We will touch on the latter problem in Sect. 6.2, where we use integrable spin-chain methods.

Modular categories are a rich source of representations of $\mathcal{B}_n$, where motions of points in a disk lead to $\mathcal{B}_n$ representations on the Hilbert spaces obtained from the corresponding $(2 + 1)$TQFT. Categorical constructions of $(3 + 1)$TQFTs suggest that there should also be a way to obtain representations of $\mathcal{NB}_n$ and other motion groups of 1-dimensional submanifolds of 3-manifolds, by acting on appropriate morphism spaces.

2.2. Relationship between $\mathcal{NB}_n$ and $\mathcal{LB}_n$.

The motion class σ_i in $\mathcal{NB}_n$ can be implemented if the auxiliary circle is absent as can the motion τ. This suggests a relationship between $\mathcal{NB}_n$ and the loop braid group $\mathcal{LB}_n$ associated with the motions of an array of n *unlinked* loops in $\mathbb{R}^3$ studied in [2].

It will be useful to consider for a moment a groupoid Γ where $\mathcal{NB}_n$ and $\mathcal{LB}_n$ both belong (cf. e.g. [13, Ch.2]). A *link* is an embedding of some number of copies of the circle S^1 in $\mathbb{R}^3$ (hence a certain 1d submanifold of $\mathbb{R}^3$). For example consider the link $\mathcal{L}_n$ in Fig.1. Another example is the n-component unlink. Let a, b be two such embeddings. A motion of a link (in the motion set $\hom(a, b)$) is a smooth variation of one such embedding a into another b over an interval of time—a 2d submanifold of $\mathbb{R}^3 \times [0, 1] \subset \mathbb{R}^4$ 'starting' at $a \subset \mathbb{R}^3 \times \{0\}$ and 'ending' at $b \subset \mathbb{R}^3 \times \{1\}$. We may combine compatible motions in the obvious way [13, Ch.2 §1].

Two motions γ, γ' from a to b are *equivalent* if there is a continuous family [13, Ch.2 §2] in $\hom(a, b)$ starting in γ and ending in γ'. Under this equivalence the classes of $\hom(a, a)$ (denoted $\Gamma_a = \hom_\sim(a, a)$) form a group. The classes of all motions form a groupoid Γ with object set the set of links. Note that

Lemma 2.5. *Two groups Γ_a, Γ_b are isomorp hic if the object links are the same topological link.*

The loop braid group $\mathcal{LB}_n$ is the motion group $\hom_\sim(a, a)$ for any a that is topologically the n-unlink. We may visualize $\mathcal{LB}_n$ as follows: arrange the n loops as circles in the $xy-$plane along the x-axis and label them $1, \ldots, n$:

$$a \quad = \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc$$

Remark 2.6. Consider for comparison the n-loop arrangement obtained as follows. Starting from a single circle, add further circles that are rotations of it about an axis in the same plane but exterior to the circle:

$$b \quad = \quad$$

(2.4)

This is then precisely as in $\mathcal{L}_n$, but with the linking loop omitted. It will be clear, cf. e.g. [13], that this leads to a group Γ_b isomorphic to Γ_a.

Keeping with arrangement a above, let s_i denote the interchange of loops i and $i+1$ via representative motions like this:
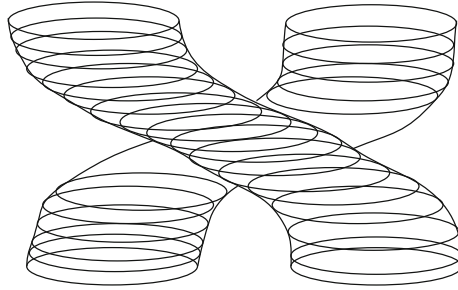
(NB. this is an overlaid 'movie' view; each still is a picture of two loops in a 3d space; time goes vertically). Denote by g_i the "leapfrog" motion of passing the ith loop under and through the $i+1$st loop followed by sliding the $i+1$st loop into the position previously occupied by the ith loop. The following projection is helpful to visualise this operation which is happening in 3 spacial and 1 (vertical) time dimension, (note that the self-intersections are just shadows of the projection):
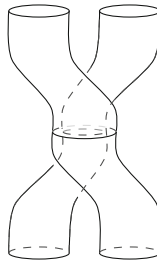
A presentation of $\mathcal{L}\mathcal{B}_n$ by abstract generators g_i, s_i (using the same symbols, to indicate how the homomorphism works) is (see e.g. [2]):

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \qquad g_i g_j = g_j g_i \quad |i-j| \neq 1 \tag{2.5}$$

$$s_i^2 = 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad s_i s_j = s_j s_i \quad |i-j| \neq 1 \tag{2.6}$$

$$s_i s_{i+1} g_i = g_{i+1} s_i s_{i+1}, \qquad g_i g_{i+1} s_i = s_{i+1} g_i g_{i+1}, \qquad g_i s_j = s_j g_i \text{ for } |i-j| > 1. \tag{2.7}$$

2.3. A group homomorphism. Note that Lemma 2.5 does not give an isomorphism between $\mathcal{LB}_n$ and $\mathcal{NB}_n$, and indeed they are not isomorphic. One way to see this is by taking their quotients by their commutator subgroups, i.e their abelianizations: $\mathcal{NB}_n^{ab} \cong \mathbb{Z} \times \mathbb{Z}_{2n}$ whereas $\mathcal{LB}_n^{ab} \cong \mathbb{Z} \times \mathbb{Z}_2$ for $n > 1$. Note that in the degenerate case $n = 1$ the group $\mathcal{NB}_1 \cong \mathbb{Z}_2$ and $\mathcal{LB}_1 = 1$. However our remark on the relationship between these constructions does lead to a beautiful homomorphism.

The precise relationship is the following:

Lemma 2.7. *There is a group homomorphism* $\zeta : \mathcal{NB}_n \to \mathcal{LB}_n$ *given by*

$$\tau \mapsto p := s_1 s_2 \cdots s_{n-1};$$

$\sigma_i \mapsto g_i$ *for* $i < n$*; and* $\sigma_n \mapsto p g_{n-1} p^{-1}$.

Proof. It is enough to check that the relations given in Lemma 1.2 are satisfied. Notice (B1) (for $\zeta(\sigma_1)$ and $\zeta(\sigma_2)$) as well as (B2) (for $\zeta(\sigma_1)$ and $\zeta(\sigma_j)$ with $2 < j < n$) are verified by (2.5) and $\zeta(\tau)^{2n} = 1$ follows from the symmetric group relations (2.6). We use (2.7) to see that

$$pg_i = s_1 \cdots (s_i s_{i+1} g_i) s_{i+2} \cdots s_{n-1} = s_1 \cdots s_{i-1} (g_{i+1} s_i s_{i+1}) \cdots s_{n-1} = g_{i+1} p$$

proving (N1) for $i \le n-2$. For $i = n-1$ (N1) is true by definition: $\zeta(\sigma_n) = p g_{n-1} p^{-1} = \zeta(\tau \sigma_{n-1} \tau^{-1})$. Moreover, $g_1 = p^{-n+1} g_n p^{n-1} = p g_n p^{-1}$ (since $p^n = 1$) which proves (N1) for $i = n$, completing the verification. $\square$

Theorem 2.8. *The kernel of* ζ *is the order* 2 *central subgroup* $\langle \tau^n \rangle$.

Proof. This is a direct consequence of [7, Lemma 3.1(2)] after observing that ζ is induced by the map from the necklace $\mathcal{L}_n$ to n free loops that forgets the auxiliary linking circle. $\square$

Note that for any representation $\rho : \mathcal{LB}_n \to GL(V)$ then $\rho \circ \zeta : \mathcal{NB}_n \to GL(V)$ is a representation of $\mathcal{NB}_n$.

Let us briefly review $\mathcal{LB}_n$ and see what this gets us in terms of representations. Present knowledge of $\mathcal{LB}_n$ representation theory is limited, but not zero (see e.g. [28] for a review). (NB the motion s_i is not possible in $\mathcal{NB}_n$).

Firstly there is a realisation of $\mathcal{LB}_n$ as the group of conjugating automorphisms of the the free group $F_n = \langle x_1, x_2, ..., x_n \rangle$ generated by braid and permutation automorphisms [20]:

$$g_i : \begin{cases} x_i & \mapsto x_{i+1} \\ x_{i+1} & \mapsto x_{i+1}^{-1} x_i x_{i+1} \\ x_j & \mapsto x_j \ \text{ otherwise} \end{cases} \qquad s_i : \begin{cases} x_i & \mapsto x_{i+1} \\ x_{i+1} & \mapsto x_i \\ x_j & \mapsto x_j \ \text{ otherwise} \end{cases}. \qquad (2.8)$$

While this is a faithful action for $\mathcal{LB}_n$, the induced action of $\mathcal{NB}_n$ is of course not faithful.

Local representations of $\mathcal{LB}_n$ are constructed in [28] which give rise to $\mathcal{NB}_n$ representations. However, there are many local representations of $\mathcal{NB}_n$ that do not extend to $\mathcal{LB}_n$ (cf. Sect. 4).

3. Extensions of Familiar $\mathcal{B}_n$ Representations

Having laid out the general theory in Sect. 2.1, in this section we provide concrete examples of $\mathcal{NB}_n$ representations obtained by extending some well-known representations of $\mathcal{B}_n$.

3.1. Extensions of standard representations. Recall (see e.g. [44]) that a *standard representation* of $\mathcal{B}_n$ is (β, V) such that

$$\beta(\sigma_i) = I_{i-1} \oplus \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}.$$

where I_k is the k dimensional identity, and $z \in \mathbb{C}\backslash\{0, 1\}$. Notice that the image of a standard representation is a group of monomial matrices, and is therefore virtually abelian. First we will state a theorem that deals with $n \geq 3$, and then we will state results for $n = 2$. We caution the reader that the established nomenclature is unfortunate as we must now discuss standard extensions of standard representations of $\mathcal{B}_n$.

Proposition 3.1. *For $n \geq 3$, any standard extension of the standard representation from $\mathcal{B}_n$ to $\mathbb{N}\mathcal{B}_n$ is of the form $\rho(\tau) = \lambda\beta(\gamma)$, where $\lambda \in \mathbb{C}$ such that $\lambda^{2n} = z^{-2(n-1)}$.*

Proof. From the fact that the standard representation is irreducible for $n \geq 3$ [44, Lemmas 5.3 and 5.4], we have that $\rho(\tau) = \lambda\beta(\gamma)$ for $\lambda \in \mathbb{C}\backslash\{0\}$. The fact that $\rho(\tau)^{2n} = \lambda^{2n}\beta(\gamma)^{2n} = I_n$, and $\beta(\gamma)^{2n} = z^{2(n-1)}I_n$ gives us that $\lambda^{2n} = z^{-2(n-1)}$. $\square$

Now considering $n = 2$, we want $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $AZ = ZA$ (where $\beta(\sigma_1) = Z = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$), $(AZ)^4 = I_2$, and $(AZ)^2 Z(AZ)^{-2} = Z$. The last equation comes from the fact that we want $\tau\sigma_2\tau^{-1} = \sigma_1$, and we are defining the image of σ_2 to be the image of $\tau\sigma_1\tau^{-1}$. From $AZ = ZA$ we get that $a = d$ and $b = zc$. Using this along with the other two equations, we get the following possibilities for $\rho(\tau) = AZ$:

$$\left\{ \pm I_2, \pm \begin{pmatrix} 1 & 0 \\ 0 & i(z)^{-1} \end{pmatrix}, \xi_4 \begin{pmatrix} 0 & \sqrt{z} \\ (\sqrt{z})^{-1} & 0 \end{pmatrix}, \pm\frac{1}{2} \begin{pmatrix} 1 \pm i & (1 \mp i)\sqrt{z} \\ (1 \mp i)\sqrt{z}^{-1} & 1 \pm i \end{pmatrix} \right\},$$

where ξ_4 is a choice of 4th root of unity.

Next let us consider non-standard extensions of the standard representation. This means we want a representation (ϕ, V), and $T \in \text{End}(V)$ such that $\phi(\tau) = T$ and $T \neq \lambda\beta(\gamma)$ (where λ is as defined in proposition 3.1). We know that we need $T^{2n} = I_V$, $T\beta(\sigma_i) = \beta(\sigma_{i+1})T$ for all $i = 1, \ldots, n-2$, and that $T^2\beta(\sigma_{n-1}) = \beta(\sigma_1)T^2$. Let $T = (t_{i,j})_{i,j=1}^n$. The later two relations give us that $t_{i,j} = 0$ if $j \not\equiv i - 1 \mod n$ and $t_{i,i-1} = t_{2,1}$ for all $i = 2, \ldots, n-1$. Hence T has the following block form: $\begin{pmatrix} 0 & a \\ tI_{n-1} & 0 \end{pmatrix}$. Then $T^{2n} = I_V$ gives us that $a = t^{-(n-1)}$, and therefore $T = \begin{pmatrix} 0 & t^{-n+1} \\ tI_{n-1} & 0 \end{pmatrix}$. If $t^{2n} = z^{-2(n-1)}$, we would have a standard extension. Hence, $t^{2n} \neq z^{-2(n-1)}$ would give us a non-standard extension. This gives us the following:

Theorem 3.2. *For $n \geq 3$, a representation, ϕ, of $\mathbb{N}\mathcal{B}_n$ is an extension of the standard representation, β, of $\mathcal{B}_n$ if $\phi(\sigma_i) = \beta(\sigma_i)$ for $i = 1, \ldots, n-1$, $\phi(\tau) = \begin{pmatrix} 0 & t^{-n+1} \\ tI_{n-1} & 0 \end{pmatrix}$*

(for $t \neq 0$), and $\phi(\sigma_n) = \phi(\tau\sigma_{n-1}\tau^{-1})$.
It should be noted that if $t^{2n} \neq z^{-2(n-1)}$, then the representation ϕ is not a standard extension of β, i.e., the image of τ is not a rescaling of that of the single twist γ.

3.2. Extensions of the reduced Burau representation.

As a reminder, the reduced Burau representation ρ of $\mathcal{B}_n$ is a $n-1$ dimensional representation defined as follows:

$$\rho(\sigma_1) = \begin{pmatrix} -t & 0 & \\ -1 & 1 & \\ & & I_{n-3} \end{pmatrix}, \rho(\sigma_i) = \begin{pmatrix} I_{i-2} & & & \\ & 1 & -t & 0 & \\ & 0 & -t & 0 & \\ & 0 & -1 & 1 & \\ & & & & I_{n-i-2} \end{pmatrix}, \rho(\sigma_{n-1}) = \begin{pmatrix} I_{n-3} & & \\ & 1 & -t \\ & 0 & -t \end{pmatrix}$$

where t is a nonzero complex number. The reduced Burau representation is irreducible if $1 + t + t^2 + \cdots + t^{n-1} \neq 0$. This means (as stated in remark 2.3) that any standard extension has $\rho(\tau) = \lambda \rho(\gamma)$.

Proposition 3.3. *Any standard extension ρ of $\mathcal{NB}_n$ of the reduced Burau representation (with $1 + t + \cdots + t^{n-1} \neq 0$) has the form $\rho(\tau) = \lambda \rho(\gamma)$ where $\lambda^{2n} = t^{-2n}$.*

Proof. From the above, we have that $\rho(\tau) = \lambda \rho(\gamma)$ for some scalar λ. From the fact that $\rho(\tau)^{2n} = \lambda^{2n} \rho(\gamma)^{2n} = I_V$, and $\rho(\gamma)^{2n} = t^{2n} I_V$. We are given that $\lambda^{2n} = t^{-2n}$. □

3.3. Extensions of the Lawrence–Krammer–Bigelow representation.

Let V be a $\binom{n}{2}$ dimensional vector space with basis $v_{i,j}$ ($1 \leq i, j \leq n$). Assuming that the order of the indices do not matter, and that t, q are nonzero complex numbers, the Lawrence-Krammer-Bigelow (LKB) representation is defined as:

$$\sigma_i v_{i,i+1} = tq^2 v_{i,i+1}$$
$$\sigma_i v_{j,k} = v_{j,k} \qquad \text{for } \{i, i+1\} \cap \{j, k\} = \emptyset$$
$$\sigma_i v_{i+1,j} = v_{i,j} \qquad \text{for } j \neq i, i+1$$
$$\sigma_i v_{i,j} = tq(q-1)v_{i,i+1} + (1-q)v_{i,j} + q v_{i+1,j} \qquad \text{if } i+1 < j$$
$$\sigma_i v_{j,i} = (1-q)v_{j,i} + q v_{j,i+1} + q(q-1)v_{i,i+1} \qquad \text{if } j < i.$$

It can be computed that $\gamma v_{i,j} = \begin{cases} tq^2 v_{i,i+1} & \text{if } j = n \\ q^2 v_{i+1,j+1} & \text{if } j < n \end{cases}$. Repeating this gives that $\gamma^n v_{i,j} = tq^{2n} v_{i,j}$. Therefore, a standard extension to $\mathcal{NB}_n$ given by $\tau \mapsto \kappa \gamma$ where κ is a scalar, we find $\kappa = \omega_{2n}(t^{-1/n}q^{-2})$, with ω_{2n} a $2n$-th root of unity. For $n = 3$ and $n = 4$, these are the only standard extensions. Notice that if ω_{2n} is an nth root of unity, τ^n would be in the kernel. Thus the standard extension would not be faithful.

3.3.1. Nonstandard extensions.

For $n = 2$, since any LKB representation is 1 dimensional, any extension is standard.

For $n = 3$, with the additional assumption that $q \neq 1$ and α a choice of cube root of $\pm t^{-1}$, the following give a nonstandard extension of the LKB representation:

$$\tau \mapsto \alpha \begin{pmatrix} 0 & (q^2 - q + 1)q^{-2} & -(q-1)q^{-2} \\ 0 & -(q-1)q^{-1} & q^{-1} \\ tq^2 & (q-1)(tq^2 - q + 1)q^{-1} & (q-1)q^{-1} \end{pmatrix}.$$

If we choose α to be the cube root of t^{-1}, then the extension is not faithful (as τ^3 would be in the kernel).

For $n = 4$, again, let $q \neq 1$ and $\beta \in \{\pm\sqrt{\pm t}, (-t^2)^{\frac{1}{4}}\}$. Then we obtain a nonstandard extension of the LKB representation by having the image of τ be

$$\beta \begin{pmatrix} 0 & 0 & (q^4t)^{-1}(q^3 - q + 1) & 0 & -(q^4t)^{-1}p & -(q^4t)^{-1}p \\ 0 & 0 & -(q^3t)^{-1}p & 0 & (q^3t)^{-1}(q^2 - q + 1) & -(q^3t)^{-1}p \\ 0 & 0 & -(q^2t)^{-1}p & 0 & -(q^2t)^{-1}p & (q^2t)^{-1} \\ q^2 & 0 & (q^3t)^{-1}p(q^3t - q + 1) & 0 & (q^3t)^{-1}(q^3 - 2q^2 + 2q - 1) & -(q^3t)^{-1}p^2 \\ 0 & q^2 & (q^2t)^{-1}p(q^3t - q + 1) & 0 & -(q^2t)^{-1}p^2 & (q^2t)^{-1}p \\ 0 & 0 & -(qt)^{-1}p^2 & q^2 & (qt)^{-1}(q^3t - q^2(t + 1) + 2q - 1) & (qt)^{-1}p \end{pmatrix},$$

where $p = q - 1$. Similar to the case with $n = 3$, if we choose $\beta = \pm\sqrt{\pm t}$, then τ^4 is in the kernel of the extension. Hence it would not be faithful.

4. Local Representations

One source of matrix representations of $\mathcal{B}_n$ is through braided vector spaces (BVS): these are pairs (R, V) where V is a vector spaces and $R \in \mathrm{Aut}(V^{\otimes 2})$ satisfies the Yang-Baxter equation (on $V^{\otimes 3}$)

$$(R \otimes I_V)(I_V \otimes R)(R \otimes I_V) = (I_V \otimes R)(R \otimes I_V)(I_V \otimes R).$$

The assignment $\rho^R(\sigma_i) = I_V^{\otimes(i-1)} \otimes R \otimes I_V^{\otimes(n-i-1)}$ then gives a representation of $\mathcal{B}_n$ on $V^{\otimes n}$. This is an example of a *local* representation: each generator has non-trivial action only on two (adjacent) copies of V.

Remark 4.1. Note that in general ρ^R may not lift to a representation of $\mathcal{LB}_n$: in [28, Proposition 3.3] BVSs of *group-type* are shown to lift to a loop braided vector space but the general case is open.

By Theorem 2.2, ρ^R has a standard extension (as long as it is completely reducible). As above, the image of the standard extension of ρ^R to $\mathcal{NB}_n$ does not carry much more information than ρ^R itself. However, there is another extension of ρ^R, using the BVS obtained from the symmetric group. Namely, define the flip operator $P(x \otimes y) = y \otimes x$ on $V \otimes V$. Then we have:

Theorem 4.2. *Suppose (R, V) is a BVS and ρ^R the corresponding $\mathcal{B}_n$ representation. For $n \geq 3$, setting $\rho^R(\tau) = (P \otimes I_V^{\otimes n-2}) \cdots (I_V^{\otimes n-2} \otimes P)$ defines an extension of ρ^R to $\mathcal{NB}_n$.*

Proof. Again using Lemma 1.2 it is enough to check (N1) and (N2): the fact that (R, V) is a BVS gives (B1) for $i = 1$ and (B2) for $i = 1$ and $3 \leq j \leq n - 1$ immediately.

From our definition of $\rho^R(\tau)$, we have that $(N2)$ is satisfied (in fact, $\rho^R(\tau)$ has order n).

The key computation is to show that (N1) holds. For this it is sufficient to show that

$$(P \otimes I)(I \otimes P)(R \otimes I)(P \otimes I)(I \otimes P) = (I \otimes R).$$

Comparing the two operators on a pure tensor of basis elements $v_1 \otimes v_2 \otimes v_3$ we obtain

$$(P \otimes I)(I \otimes P)[R(v_2 \otimes v_3) \otimes v_1] \quad \text{and} \quad v_1 \otimes R(v_2 \otimes v_3)$$

for the left- and right-hand sides respectively, which are clearly equal. This completes the proof. $\square$

Remarks 4.3.   • Obviously the operator $\rho^R(\tau)$ is not local in the strict sense: it acts non-trivially on all tensor factors. However, its action does not mix vectors within the tensor factors, it only permutes them globally.

   • We think the following is an interesting question: given R, how much bigger is the image $\rho^R(\mathcal{N}\mathcal{B}_n)$ than $\rho^R(\mathcal{B}_n)$? Note that while the subgroup generated by $\sigma_1, \ldots, \sigma_n$ has index $2n$ in $\mathcal{N}\mathcal{B}_n$ [7] $\mathcal{B}_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ has infinite index. If $|\rho^R(\mathcal{B}_n)| < \infty$ is $|\rho^R(\mathcal{N}\mathcal{B}_n)| < \infty$?

   • In the degenerate $n = 2$ case we have $\mathcal{B}_2 \cong \mathbb{Z}$, so that *any* $R \in \mathrm{Aut}(V^{\otimes 2})$ gives a representation of $\mathcal{B}_2$. Setting $\rho^R(\tau) = P$ defines an extension to $\mathcal{N}\mathcal{B}_2$ if R is symmetric in the standard product basis of $V \otimes V$: we have that $\rho^R(\sigma_1) = R$, $\rho^R(\tau) = P$, and $\rho^R(\sigma_2) = PRP$. The only relation that needs checking is $\rho^R(\sigma_1\sigma_2\sigma_1) = \rho^R(\sigma_2\sigma_1\sigma_2)$ i.e., $RPRPR = PRPRPRP$ which is satisfied if R is symmetric i.e., $PRP = R$. Note that $(B1)$ does not hold for every R satisfying the Yang-Baxter equation, as the following example illustrates.

Example 4.4. Consider $\dim V = 2$ and $R = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$. This gives us that

$$\rho^R(\sigma_1\sigma_2\sigma_1)(e_2 \otimes e_1) \neq \rho^R(\sigma_2\sigma_1\sigma_2)(e_2 \otimes e_1).$$

Thus mapping $\rho^R(\tau) = P$ does not define an extension of ρ^R from $\mathcal{B}_2$ to $\mathcal{N}\mathcal{B}_2$.

For the matrix R of this example we have verified the following conjecture for $n \leq 5$, showing that the image of $\mathcal{N}\mathcal{B}_n$ can be significantly larger than that of $\mathcal{B}_n$.

Conjecture 4.5. Given the above R and $n \geq 3$, $|\rho^R(\mathcal{N}\mathcal{B}_n)| = n|\rho^R(B\tilde{A}_n)| = n2^n|\rho^R(\mathcal{B}_n)|$.

4.1. Gaussian braided vector spaces.

From the above discussion, any unitary BVS provides a local representation of $\mathcal{B}_n$, which can be extended to a representation of $\mathcal{N}\mathcal{B}_n$ in two ways: (1) the standard extension (Lemma 2.1) and (2) the n-cycle extension of Theorem 4.2. In this subsection we consider extensions of the Gaussian representations first studied by Jones [27] and analyzed in [22]. As the matrix representations are somewhat unwieldy, we take a more algebraic approach.

As in [22], we define $ES(m, n-1)$ as the algebra generated by $u_1, \ldots, u_{n-1}$ with the relations $u_i^m = 1$, $u_i u_{i+1} = q^2 u_{i+1} u_i$, and $u_i u_j = u_j u_i$ if $|i - j| > 1$ where $q = \begin{cases} e^{2\pi i/m}, & \text{if } m \text{ odd} \\ e^{\pi i/m}, & \text{if } m \text{ even} \end{cases}$. Setting $\varphi_n(\sigma_i) = \sum_{j=0}^{m-1} q^{j^2} u_i^j$ defines a homomorphism $\varphi_n : \mathcal{B}_n \to ES(m, n-1)$. To get a braided vector space from $ES(m, n-1)$ it is enough to find a vector space V and $U \in \mathrm{Aut}(V^{\otimes 2})$ such that the map $u_i \mapsto I_V^{\otimes i-1} \otimes U \otimes I_V^{\otimes n-i-1}$ defines a representation of $ES(m, n-1)$ on $V^{\otimes n}$. Let $V \cong \mathbb{C}^m$, with standard basis $\{e_i | 0 \leq i \leq m-1\}$. Define $e_{i+m} = e_i$, and $U \in \mathrm{End}(V^{\otimes 2})$ by $U(e_i \otimes e_j) = q^{j-i} e_{i+1} \otimes e_{j+1}$. In [22] it was shown that $u_i \mapsto U_i := I^{\otimes i-1} \otimes U \otimes I^{\otimes n-i-1}$ gives a $*$-algebra homomorphism (where $u_i^* = u_i^{-1}$) from $ES(m, n-1)$ to $\mathrm{End}(V^{\otimes n})$. It was also shown that $R := \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} q^{j^2} U^j$ is a unitary operator. Composing with φ_n we obtain a unitary representation $\Phi_n : \mathcal{B}_n \to \mathrm{Aut}(V^{\otimes n})$, where $\Phi_n(\sigma_i) = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} q^{j^2} U_i^j$.

To extend this idea to $\mathcal{NB}_n$ first we extend $ES(m, n - 1)$ to another algebra, $NES(m, n)$. We define $NES(m, n)$ to be the algebra generated by $u_1, \ldots, u_{n-1}, t$ subject to the following relations:

(1) $u_i^m = 1 = t^n$
(2) $u_i u_{i+1} = q^2 u_{i+1} u_i$ for all $1 \leq i \leq n - 2$
(3) $u_i u_j = u_j u_i$ if $|i - j| \neq 1$,
(4) $t u_i t^{-1} = u_{i+1}$ for all $1 \leq i \leq n - 2$

where q is either an mth or $2m$ root of unity as above.

Notice that $NES(m, n)$ is *nearly* a semidirect product of $ES(m, n-1)$ with $\mathbb{Z}_n$, except that $t u_{n-1} t^{-1}$ is not in $NES(m, n - 1)$. To make the connection to $\mathcal{NB}_n$ clearer, and to remedy this defect we introduce a useful auxiliary generator to obtain a presentation with more familiar modulo n relations:

Lemma 4.6. *If we define $u_n := t u_{n-1} t^{-1}$ then u_n satisfies (1) above, relations (2) and (4) above hold modulo n and the condition in (3) can be replaced with $|i - j| \not\equiv 1$ mod n.*

Proof. Since $t^{n-2} u_1 t^{-n+2} = u_{n-1}$ and $t^n = 1$ we have $t^{n-1} u_1 t^{1-n} = u_n$ proving that (4) holds modulo n. Next we see that

$$u_{n-1} u_n = t u_{n-2} u_{n-1} t^{-1} = q^2 t u_{n-1} t^{-1} t u_{n-2} t^{-1} = q^2 u_{n-1} u_n$$

as desired, with $u_n u_1 = q^2 u_1 u_n$ verified similarly. For (3) it is enough to check that u_n commutes with u_{n-2} (with $n \geq 4$). This is also straightforward:

$$u_n u_{n-2} = t u_{n-1} t^{-1} t u_{n-3} t^{-1} = t u_{n-3} u_{n-1} t^{-1} = u_{n-2} u_n.$$

$\square$

Observe that the algebra $NES(m, n)$ is a finite dimensional semisimple algebra over $\mathbb{Q}(q)$. Indeed, $NES(m, n)$ is essentially a group algebra. Next we show that $\mathcal{NB}_n$ admits a representation in $NES(m, n)$.

Theorem 4.7. *The map $\hat{\varphi}_n : \mathcal{NB}_n \to NES(m, n)^*$ by $\sigma_i \mapsto R_i(m) = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} q^{j^2} u_i^j$ and $\tau \mapsto t$ is a group homomorphism.*

Proof. As shown in [22, Proposition 3.1], the relation $\hat{\varphi}_n(\sigma_1 \sigma_2 \sigma_1) = \hat{\varphi}_n(\sigma_2 \sigma_1 \sigma_2)$, and $\hat{\varphi}_n(\sigma_1 \sigma_j) = \hat{\varphi}_n(\sigma_j \sigma_1)$ for $2 < j < n$ are true. From the definition of t, $\hat{\varphi}_n(\tau \sigma_i \tau^{-1}) = \hat{\varphi}_n(\sigma_{i+1})$ and $\hat{\varphi}_n(\tau)^{2n} = 1$. Thus we have $\hat{\varphi}_n$ is a representation of $\mathcal{NB}_n$ into $NES(m, n)$. $\square$

To obtain a representation of $\mathcal{NB}_n$ again let $V \cong \mathbb{C}^m$, with standard basis $\{e_0, \ldots, e_{m-1}\}$ with $e_{i+m} = e_i$. Now define $U, T \in \text{End}(V^{\otimes 2})$ by $U(e_i \otimes e_j) = q^{j-i} e_{i+1} \otimes e_{j+1}$ and $T(e_i \otimes e_j) = e_j \otimes e_i$. Further define, for any $n \geq 2$, elements of $\text{Aut}(V^{\otimes n})$:

$$X := (T \otimes I^{\otimes n-2})(I \otimes T \otimes I^{n-3}) \cdots (I^{\otimes n-2} \otimes T)$$

and $U_i := I^{\otimes i-1} \otimes U \otimes I^{\otimes n-i-1}$ for each $1 \leq i \leq n - 1$. Notice that

$$X(e_{i_1} \otimes \cdots \otimes e_{i_n}) = e_{i_n} \otimes e_{i_1} \otimes \cdots \otimes e_{i_{n-1}}.$$

Proposition 4.8. *The map Ψ given on generators by $u_i \mapsto U_i$ and $t \mapsto X$ defines a representation of $NES(m, n)$ on $V^{\otimes n}$.*

Observe that we only need to verify relations (1)-(4) above.

Proof. It is clear that U_i commutes with U_j if $|i - j| \neq 1$ proving (3). Also,

$$U^m(e_i \otimes e_j) = \begin{cases} q^{(m-j)(j-i)}q^{(j-i-m)(j-i)}q^{(j-i)i}e_i \otimes e_j = e_i \otimes e_j & i \leq j \\ q^{(m-i)(j-i)}q^{(i-j-m)(j-i)}q^{(j-i)j}e_i \otimes e_j = e_i \otimes e_j & i > j. \end{cases}$$

Notice that X has order n, hence (1) is satisfied. A straightforward calculation shows that $XU_iX^{-1} = U_{i+1}$, i.e., (4). From this we see that for (2) it is sufficient to check the last relation on $NES(m, n)$ for U_1, U_2 with $n = 3$, which has been verified in [42]: $U_1U_2(e_i \otimes e_j \otimes e_k) = q^2 U_2U_1(e_i \otimes e_j \otimes e_k)$. □

Observe that $\Psi \circ \hat{\varphi} : \mathcal{NB}_n \rightarrow \mathrm{Aut}(V^{\otimes n})$ gives a local representation of $\mathcal{NB}_n$. The case $m = 2$ has the realisation given in Example 4.4, and conjecturally has finite image. More generally, in [22] it is shown that the restricted image $\hat{\varphi}(\mathcal{B}_n)$ in $ES(m, n - 1)$ is finite. We wish to follow a similar approach to show that the image of $\hat{\varphi}_n(\mathcal{NB}_n)$ is also finite. Notice that the monomials in $NES(m, n)$ have the following normal form: $t^\alpha u_1^{\alpha_1} \cdots u_n^{\alpha_n}$ where $0 \leq \alpha < n$ and $0 \leq \alpha_i < m$. In fact, we see that these $n(m)^n$ monomials form a basis for $NES(m, n)$ over $\mathbb{Q}(q)$. The structure of $NES(m, n)$ is more complicated than $ES(m, n - 1)$, which is actually simple for n odd and has exactly m simple components for n even [42]. We let $\hat{\varphi}_n(\mathcal{NB}_n) \subset NES(m, n)$ act on the span of $\hat{U} = \{u_1^{\alpha_1} \cdots u_n^{\alpha_n}\}$ by conjugation. Since conjugation by t obviously permutes this spanning set,

We first show that the conjugation action of $R_i(m)$ also permutes this set. The same approach as in [22] works here: (note we may omit the scalar $\frac{1}{\sqrt{m}}$ in $R_i(m)$ in these calculations):

$$qu_i^{-1}u_{i+1}R_i(m) = qu_i^{-1}u_{i+1}\sum_{j=0}^{m-1} q^{j^2}u_i^j = q^{-1}u_{i+1}u_i^{-1}\sum_{j=0}^{m-1} q^{j^2}u_i^j$$

$$= q^{-1}\sum_{j=0}^{m-1} q^{j^2}u_{i+1}u_i^{j-1} = q^{-1}\sum_{j=0}^{m-1} q^{j^2}(q^{-2(j-1)})u_i^{j-1}u_{i+1}$$

$$= \sum_{j=0}^{m-1} q^{-1}q^{j^2}q^{-2j+2}u_i^{j-1}u_{i+1}$$

$$= \left(\sum_{j=0}^{m-1} q^{(j-1)^2}u_i^{j-1}\right)u_{i+1} = R_i(m)u_{i+1}$$

and

$$q^{-1}u_{i-1}u_i R_i(m) = qu_iu_{i-1}\sum_{j=0}^{m-1} q^{j^2}u_i^j$$

$$= qu_i\sum_{j=0}^{m-1} q^{j^2}u_{i-1}u_i^j = qu_i\sum_{j=0}^{m-1} q^{j^2}q^{2j}u_i^ju_{i-1}$$

$$= \sum_{j=0}^{m-1} q q^{j^2} q^{2j} u_i^{j+1} u_{i-1}$$

$$= \left(\sum_{j=0}^{m-1} q^{(j+1)^2} u_i^{j+1} \right) u_{i-1} = R_i(m) u_{i-1}.$$

This shows that $R_i(m) u_{i+1} R_i(m)^{-1} = q u_i^{-1} u_{i+1}$ and $R_i(m) u_{i-1} R_i(m)^{-1} = q^{-1} u_{i-1} u_i$.
Thus conjugation by $R_i(m)$ permutes the spanning set $\hat{U}$ up to scalars that are roots of
unity (i.e., powers of q). Thus $\hat{\varphi}_n(\mathcal{NB}_n)$ is finite modulo the center. The subalgebra of
$NES(m,n)$ generated by $\hat{\varphi}_n(\mathcal{NB}_n)$ is semisimple, so that the faithful representation of
$\hat{\varphi}_n(\mathcal{NB}_n)$ on $NES(m,n)$ decomposes into full matrix algebras. Thus any element x of
the center of $\hat{\varphi}_n(\mathcal{NB}_n)$ acts via a scalar matrix on each irreducible subrepresentation.
But since the generators of $\hat{\varphi}_n(\mathcal{NB}_n)$ have determinant a root of unity (of degree m or
n), the scalar x is also a root of unity of degree only depending on m and n (indeed
the degree of each irreducible representation depends only on m, n). Thus the center of
$\hat{\varphi}_n(\mathcal{NB}_n)$ is a finite group and has finite index, so $\hat{\varphi}_n(\mathcal{NB}_n)$ is a finite group.

4.2. Quaternionic representation. Similar to the Gaussian Braided Vector Space, the
idea is to take a finite group (in this case Q_8) and consider the group algebra with
n copies of the group, where the generators will interact with 'close' neighbours in a
specific way, but commute with 'far' neighbours. Let $q = e^{2i\pi/6}$ and $[\,,\,]$ denote the
group commutator. Similar to the Gaussian case, we take the algebra Q_n defined in [41],
and define $\mathfrak{Q}_n$ to be (almost) a semi-direct product of Q_n with $\mathbb{Z}_n$ the algebra generated
by $t, u_1, \ldots, u_{n-1}, v_1, \ldots, v_{n-1}$ with the following relations:

(1) $u_i^2 = v_i^2 = -1$ for all i,
(2) $[u_i, v_j] = -1$ if $|i - j| < 2$,
(3) $[u_i, v_j] = 1$ if $|i - j| \geq 2$,
(4) $[u_i, u_j] = [v_i, v_j] = 1 = t^n$,
(5) $t u_i t^{-1} = u_{i+1}$, and $t v_i t^{-1} = v_{i+1}$ for all i.

As in the Gaussian case, we have the following lemma.

Lemma 4.9. *Defining, in $\mathfrak{Q}_n$, $u_n := t u_{n-1} t^{-1}$ and $v_n := t v_{n-1} t^{-1}$, v_n, u_n satisfy relations* (1) − (5) *with indices* $\mod n$ *(defining $v_0 = v_n$ and $v_{n+1} = v_1$).*

Proof. We must check that u_n and v_n also satisfy the relations (1)–(5). For (1), note that
$u_n^2 = (t u_{n-1} t^{-1})^2 = t u_{n-1}^2 t^{-1} = -1 = t v_{n-1}^2 t^{-1} = (t v_{n-1} t^{-1})^2 = v_n^2$. The relation
(5) follows from the definition of u_n, v_n and $t^n = 1$; $t u_n t^{-1} = t(t u_{n-1} t^{-1}) t^{-1} =$
$t(t^{n-1} u_1 t^{-n+1}) t^{-1} = t^n u_1 t^{-n} = u_1$ and similarly $t v_n t^{-1} = v_1$. For (2) and (3), we
will first consider $[u_n, v_j]$. Doing so gives the following:

$$[u_n, v_j] = [t u_{n-1} t^{-1}, v_j] = t u_{n-1} t^{-1} v_j t u_{n-1}^{-1} t^{-1} v_j^{-1}$$

$$= t u_{n-1} v_{j+1} u_{n-1}^{-1} v_{j+1}^{-1} t^{-1}$$

$$= t[u_{n-1}, v_{j+1}] t^{-1}.$$

Similarly $[u_j, v_n] = t[u_{j+1}, v_{n-1}] t^{-1}$. Thus the relation (3) holds for $|i - j| \mod n \geq$
2. For (2) we need to check the above equation, $j = 1, n - 1, n$.

$$[u_n, v_1] = [t^{-1} u_1 t, v_1] = t^{-1} [u_1, v_2] t = -1$$

$$[u_n, v_{n-1}] = [tu_{n-1}t^{-1}, tv_{n-2}t^{-1}] = t[u_{n-1}, v_{n-2}]t^{-1} = -1$$
$$[u_n, v_n] = [tu_{n-1}t^{-1}, tv_{n-1}t^{-1}] = t[u_{n-1}, v_{n-1}]t^{-1} = -1.$$

Hence (2) holds for $|i - j| \mod n < 2$. Lastly (4) holds from $[u_n, u_j] = t[u_{n-1}, u_j]t^{-1}$ and $[v_n, v_j] = t[v_{n-1}, v_j]t^{-1}$. $\square$

Theorem 4.10. *The map $\xi_n : \mathcal{NB}_n \to \mathcal{Q}_n^\times$ given by $\xi_n(\sigma_i) = \frac{-1}{2q}(1 + u_i + v_i + u_i v_i)$ and $\xi_n(\tau) = t$ defines a group homomorphism.*

Proof. Similarly as in the Gaussian case above, we use Lemma 1.2 and [41] to reduce to checking $\xi_n(\tau \sigma_i \tau^{-1}) = \xi_n(\sigma_{i+1})$ and $[\xi_n(\tau)]^{2n} = 1$. These are both immediate from the (last two) relations in $\mathcal{Q}_n$. $\square$

Interestingly, $\mathcal{Q}_n$ does not have an obvious local representation. Instead, we obtain a 3-local representation, (see [23, Theorem 5.28]). On the other hand, we can easily show that the image $\xi_n(\mathcal{NB}_n)$ is a finite group. First we show that,the conjugation action on the subalgebra $\hat{\mathcal{Q}}_n$ generated by $u_1, \ldots, u_n, v_1, \ldots, v_n$ is finite as follows. Observe that $\hat{\mathcal{Q}}_n$ is spanned by monomials of the form

$$u_1^{\epsilon_1} \cdots u_n^{\epsilon_n} v_1^{\nu_1} \cdots v_n^{\nu_n}$$

where nonzero $\epsilon_i, \nu_i \in \{0, \pm 1\}$. The action of $\xi_n(\tau) = t$ obviously permutes this generating set. We can now compute, with $k = i \pm 1$:

$$
\begin{aligned}
u_i \xi_n(\sigma_i) &= \frac{-1}{2q} u_i(1 + u_i + v_i + u_i v_i) = \frac{-1}{2q}(u_i - 1 + u_i v_i - v_i) \\
&= \frac{-1}{2q}(v_i u_i v_i + u_i v_i u_i v_i + u_i v_i + u_i u_i v_i) = \frac{-1}{2q}(v_i + u_i v_i + 1 + u_i)u_i v_i) \\
&= \xi_n(\sigma_i)u_i v_i
\end{aligned}
$$

and

$$
\begin{aligned}
v_i \xi_n(\sigma_i) &= \frac{-1}{2q}v_i(1 + u_i + v_i + u_i v_i) = \xi_n(\sigma_i)u_i \\
u_k \xi_n(\sigma_i) &= \frac{-1}{2q}u_k(1 + u_i + v_i + u_i v_i) = \xi_n(\sigma_i)u_k v_i \\
v_k \xi_n(\sigma_i) &= \frac{-1}{2q}v_k(1 + u_i + v_i + u_i v_i) = \xi_n(\sigma_i)(-u_i v_i v_k).
\end{aligned}
$$

Thus the conjugation action of $\xi_n(\mathcal{NB}_n)$ permutes a spanning set up to roots of unity so that $\xi_n(\mathcal{NB}_n)$ is finite modulo its center.

Now again, as in the Gaussian case we can see that the $\mathcal{Q}_n$ is a finite dimensional semisimple algebra and the restriction to the center of $\xi_n(\mathcal{NB}_n)$ on any irreducible subrepresentation of the faithful regular representation gives a scalar of finite order, hence $\xi_n(\mathcal{NB}_n)$ has finite center and is thus a finite group.

5. Extending Low-Dimensional $\mathcal{B}_n$ Representations

There are gaps in the irreducible representation degrees of $\mathcal{B}_n$ (see [29] and references therein): for example $\mathcal{B}_n$ has no irreducible representations of dimension $2 \leq d \leq n-3$, for $n \geq 5$.

Lemma 5.1. *Let (ρ, V) to be an irreducible $\mathcal{NB}_n$ representation, $w \in V$, and $0 < \alpha$ be minimal such that $\tau^{-\alpha} w \in span\{\tau^{-\gamma} w | 0 \leq \gamma < \alpha\}$. Then $\tau^{-(\alpha+1)} w \in span\{\tau^{-\gamma} w | 0 \leq \gamma < \alpha\}$.*

Proof. We have that $\tau^{-\alpha} w = \sum\limits_{i=0}^{-\alpha+1} a_i \tau^{-i} w$ for a_i scalars. Then

$$\tau^{-(\alpha+1)} w = \tau^{-1}(\tau^{-\alpha} w) = \tau^{-1}\left(\sum_{i=0}^{\alpha-1} a_i \tau^{-i} w\right)$$

$$= \sum_{i=0}^{\alpha-1} a_i \tau^{-i-1} w$$

$$= \left(\sum_{i=1}^{\alpha-1} a_{i-1} \tau^{-i} w\right) + \tau^{-\alpha} w$$

which is in the span of $\{\tau^{-\gamma} w | 0 \leq \gamma < \alpha\}$. $\square$

Theorem 5.2. *Let $n \geq 5$ and (ρ, V) be an irreducible $\mathcal{NB}_n$ representation. If $\dim V = n - 2$, then $\phi = \rho|_{\mathcal{B}_n}$ is also irreducible.*

Proof. Assume that (V, ρ) is an irreducible $\mathcal{NB}_n$ representation and to the contrary that $\rho|_{\mathcal{B}_n}$ is not irreducible. So we have that there exists a proper nonempty subspace W (of minimal dimension, $1 \leq q < n - 2$) of V such that $(\phi|_W, W)$ is a $\mathcal{B}_n$ representation. Note that W being minimal dimension guarantees that $(\phi|_W, W)$ is irreducible. Since $n \geq 5$, we have that the only irreducible representations of $\mathcal{B}_n$ of dimension strictly less than $n - 2$ are 1-dimensional. Let W be spanned by the vector $w \in V$. Since W is a $\mathcal{B}_n$ invariant space, we have that $\sigma_i w = \lambda_i w$ for all $1 \leq i \leq n - 1$ (i.e., w is an eigenvector for all $\rho(\sigma_i)$). Since the σ_i are conjugate to each other, we have that w is an eigenvector of the same eigenvalue. Since $\tau^{2n} = 1$ and V is irreducible, we also have that $\rho(\tau)^{\pm n} = \pm 1$. Note that $M = \{w, \tau^{-1} w, \cdots, \tau^{-n} w\}$ is linearly dependent and that the span of M is τ invariant. We may extract a basis $\beta = \{\tau^{-\alpha_1} w, \tau^{-\alpha_2} w, \ldots, \tau^{-\alpha_k} w | 0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq n\}$ for $Q = span(M)$. Since Q is τ invariant, we may instead use the basis $\beta' = \tau^{\alpha_1} \beta$. This gives us that $w \in \beta'$. From the above lemma, we obtain $\beta' = \{w, \tau^{-1} w, \ldots, \tau^{-\alpha} w\}$ (where $\alpha < n - 2$) is a basis for Q. The restriction that $\alpha < n - 2$ is from the fact that Q is a subspace of V, and therefore $\dim Q \leq n - 2$.

Let $0 \leq \gamma \leq \alpha < n - 2$. This means that $\sigma_{\gamma+1} \in \mathcal{B}_n$. From the relation $\tau^\gamma \sigma_1 \tau^{-\gamma} = \sigma_{\gamma+1}$, we get that $\sigma_1 \tau^{-\gamma} w = \tau^{-\gamma} \sigma_{\gamma+1} w = \lambda(\tau^\gamma w)$. This gives us that Q is also σ_1 invariant. Hence Q is both τ and σ_1 invariant, and therefore $\mathcal{NB}_n$ invariant. Since V was irreducible, we have that $Q = V$. This means that β' is also a basis for V, and in this basis, $\rho(\sigma_1) = \lambda \cdot id_V$. Hence

$$\rho(\sigma_2) = \rho(\tau)\rho(\sigma_1)\rho(\tau^{-1}) = \lambda\rho(\tau)\rho(\tau^{-1}) = \lambda \cdot id_V = \rho(\sigma_1).$$

This gives us that $\rho(\tau)\rho(\sigma_1) = \rho(\sigma_1)\rho(\tau)$. Which would give us a contradiction that (ρ, V) is not an irreducible $\mathcal{NB}_n$ representation. $\square$

We can now show that $\mathcal{NB}_n$ also has gaps in its irreducible representation degrees:

Corollary 5.3. *For $n \geq 5$, the only irreducible representations of $\mathcal{NB}_n$ of dimension at most $n - 3$ are 1-dimensional.*

Proof. Assume to the contrary that (ρ, V) is an irreducible representation of $\mathcal{NB}_n$ with $2 \leq \dim V < n - 2$. Note that $\rho|_{\mathcal{B}_n}$ can not be irreducible, since $\mathcal{B}_n$ has no irreducible representations of dimension between 2 and $n - 3$. Hence there exists a 1 dimensional subrepresentaion of $\mathcal{B}_n$. Following the proof of Theorem 5.2, we would get that $\rho(\mathcal{NB}_n)$ is abelian, and therefore not irreducible. $\square$

The following theorem is similar to the one above it, but it has the additional assumption that $\rho|_{\mathcal{B}_n}$ is completely reducible.

Theorem 5.4. *Let $n \geq 5$ and (ρ, V) be an irreducible $\mathcal{NB}_n$ representation. If $\dim V = n - 1$ and $\rho|_{\mathcal{B}_n}$ is completely reducible, then $\rho|_{\mathcal{B}_n}$ is also irreducible.*

Proof. Assume to the contrary that $\rho|_{\mathcal{B}_n}$ is completely reducible and not irreducible. Then we have two possibilities, $V = \bigoplus_{i=1}^{n-1} W_i$ or $V = W \oplus U$ where W, W_i are all 1-dimensional subrepresentations, and U is an n-2 irreducible subrepresentation of V for $\mathcal{B}_n$. In either case, we have the existence of a 1-dimensional subrepresentation. From here, we follow the proof of Theorem 5.2, and note that the inequality that $\alpha < n - 2$ becomes $\alpha \leq n - 2$. However, this still ensures that Q is σ_1 invariant, because again, $\sigma_{n-1} = \tau^{n-2}\sigma_1\tau^{-n+2}$. So again, we would get the contradiction that (ρ, V) is not an irreducible representation of $\mathcal{NB}_n$. $\square$

Remark 5.5. Consider the unreduced Burau representation, which is defined as follows:

$$\sigma_i \mapsto \begin{pmatrix} I_{i-1} & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & & I_{n-i-1} \end{pmatrix}.$$ It can easily be verified that the $(n \times 1)$ vector of all 1's

is fixed by all of the σ_i's. Consider the mapping $\tau \to \begin{pmatrix} 0 & a^{n-1} \\ \frac{1}{a} \cdot I_{n-1} & 0 \end{pmatrix}$ where I_{n-1} is the

$n - 1 \times n - 1$ identity. It can be checked that $\rho(\tau^{-1}\sigma_i\tau) = \rho(\sigma_{i+1})$ and $\rho(\tau)^{2n} = I_n$. Thus it gives us a representation of $\mathcal{NB}_n$. Note that any invariant subspace of $\rho(\mathcal{NB}_n)$ will also be invariant under $\rho(\mathcal{B}_n)$. It is known that the Burau representation is reducible with invariant subspaces of dimension 1 and $n - 1$. The 1 dimensional subrepresentation is spanned by the vector of 1's. The other is the subspace of $\mathbb{C}^n$ of all vectors whose entries add up to 0. If $a \neq 1$, then we get that the vector of 1's is not fixed by τ. If $a^n \neq 1$, then we get that τ does not fix the $n - 1$-dimensional subspace. This means that if $\mathcal{NB}_n$ has no invariant subspaces. Therefore if $a^n \neq 1$, then the extension of the Burau representation described above is an irreducible representation of $\mathcal{NB}_n$ whose restriction to $\mathcal{B}_n$ is reducible. As this shows, there exist irreducible representations of $\mathcal{NB}_n$ of dimension n whose restriction to $\mathcal{B}_n$ is no longer irreducible.

5.1. Irreducible representations of dimension 2. From the fact that τ has order $2n$, we may assume that we have chosen a basis for V such that $\rho(\tau) = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$ where t_1, t_2 are $2n^{th}$ roots of unity. As stated before, for all $n \geq 5$, there are no irreducible 2 dimensional representations. This means we need only consider $n = 2, 3,$ and 4. Since

Table 1. Dimension 2 representations of $\mathcal{NB}_n$, $2 \leq n \leq 4$. Here ω is a primitive 3rd root of unity

$\rho(\sigma_1)$	$\rho(\tau)$	Restrictions
$\begin{pmatrix} a & 1 \\ a^2 - ad + d^2 & d \end{pmatrix}$	$\begin{pmatrix} -t_2 & 0 \\ 0 & t_2 \end{pmatrix}$	$a \neq d$
$\begin{pmatrix} a & 1 \\ -a^2 + ad - d^2 & d \end{pmatrix}$	$\pm\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \pm\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$	$a \neq d, n = 2$
$\begin{pmatrix} a & 1 \\ \frac{-1}{2}(a^2 - ad + d^2) & d \end{pmatrix}$	$\pm\begin{pmatrix} e^{\pm i\frac{\pi}{3}} & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 0 \\ 0 & e^{\pm i\frac{\pi}{3}} \end{pmatrix},$ $\pm\begin{pmatrix} e^{i\frac{2\pi}{3}} & 0 \\ 0 & e^{i\frac{\pi}{3}} \end{pmatrix}, \pm\begin{pmatrix} e^{i\frac{\pi}{3}} & 0 \\ 0 & e^{i\frac{2\pi}{3}} \end{pmatrix},$	$a \neq d, n = 3$
$\begin{pmatrix} \omega d & 1 \\ c & d \end{pmatrix}$	$\pm\begin{pmatrix} 1 & 0 \\ 0 & e^{\pm i\frac{2\pi}{3}} \end{pmatrix}, \pm\begin{pmatrix} e^{\pm i\frac{2\pi}{3}} & 0 \\ 0 & e^{\mp i\frac{2\pi}{3}} \end{pmatrix}$	$d \neq 0, c \neq \omega d^2, n = 3$

we are wanting irreducible reps, we have that $t_1 \neq t_2$. Similarly we have that $\rho(\sigma_1)$ is not upper or lower triangular, as otherwise $(1, 0)$ or $(0, 1)$ would generate an invariant subspace. Due to rescaling, we may assume that $\rho(\sigma_1) = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$. Since we do not want diagonal or triangular matrices, this means that $c \neq 0$. Wanting our representations to be irreducible, we also have at least one of a or d are nonzero.

Proposition 5.6. *Any irreducible dimension 2 representation of $\mathcal{NB}_2$, $\mathcal{NB}_3$ or $\mathcal{NB}_4$ is isomorphic to one of those forms in Table 1.*

6. Representations from Topological Physics

The category-theoretic approach to TQFTs as well as its statistical-mechanical predecessor suggest that similar techniques should yield physically relevant necklace braid group representations. Here we discuss some related ideas from braided fusion categories and spin chain models that to illustrate this direction.

6.1. Representations from braided fusion categories. From any (unitary) braided fusion category $\mathcal{C}$ Walker and Wang [52] construct a $(3+1)$-TQFT. In the extreme case of modular $\mathcal{C}$ their construction is degenerate. In the other extreme case of a symmetric braided fusion category of the form $\text{Rep}(G)$ for a finite group G one recovers the Dijkgraaf-Witten theory [16]. The most interesting case is neither symmetric nor modular. As we expect a $(3+1)$TQFT to provide representations of motion groups of links in $\mathbb{R}^3$ or S^3, we briefly explore this possibility directly from the categorical perspective.

Given a braided fusion category $\mathcal{C}$ we obtain from any object X a representation of $\mathcal{B}_n$ on $\text{End}(X^{\otimes n})$, via the braiding $c_{X,X}$ (we use the notation of [42], and assume $\mathcal{C}$ is strictly associative for notational convenience). More specifically, the map $\sigma_i \mapsto \text{Id}_X^{\otimes i-1} \otimes c_{X,X} \otimes \text{Id}^{\otimes n-i-1}$ defines a group homomorphism $\mathcal{B}_n \to \text{Aut}(X^{\otimes n})$, which we may regard as a *categorical* representation of $\mathcal{B}_n$. If $\{X_i\}$ is a complete set of representatives of the isomorphism classes then $\text{Aut}(X^{\otimes n})$ acts faithfully on $\mathcal{H}_{X,n} := \bigoplus_i \text{Hom}(X_i, X^{\otimes n})$ by composition, which then gives us an honest linear representation $(\rho_X, \mathcal{H}_{X,n})$ of $\mathcal{B}_n$. Are there categorical representations of $\mathcal{NB}_n$? That is, group homomorphisms $\mathcal{NB}_n \to \text{Aut}(Y(n))$ for some objects $Y(n)$ depending on n? As the representations $(\rho_X, \mathcal{H}_{X,n})$ are completely reducible we may extend them to $\mathcal{NB}_n$ in the standard way as in Theorem 2.2

by rescaling the image of γ. It would be interesting to find categorical non-standard extensions.

One cannot expect categorical $\mathcal{B}_n$ representations to admit extensions to the loop braid group $\mathcal{LB}_n$. Of course if the braiding $c_{X,X}$ is symmetric: $c_{X,X}^2 = \mathrm{Id}_{X \otimes X}$ then we can extend $(\rho_X, \mathcal{H}_{X,n})$ to $\mathcal{LB}_n$ by $s_i \to \sigma_i$. However, the map $\sigma_i \mapsto s_i$ savagely reduces $\mathcal{LB}_n$ to the symmetric group S_n, so this is not an interesting choice.

The annular braid group $\mathcal{CB}_n$ does admit such representations as follows (cf. [38]). Let X, Y be objects in a braided fusion category, and set

$$R_i = I_Y \otimes I_X^{\otimes i-1} \otimes R_{X,X} \otimes I_X^{\otimes n-i-1} \in \mathrm{End}(Y \otimes X^{\otimes n})$$

for $1 \leq i \leq n-1$, $\alpha_{Y,X} = (R_{X,Y} R_{Y,X} \otimes I_X^{\otimes n-1})$ and $T_{Y,X} = \alpha_{Y,X} R_1 \cdots R_{n-1}$. Then $\sigma_i \to R_i$ and $\tau \to T_{Y,X}$ satisfy (B1), (B2) and (N1) after defining $R_n = T_{Y,X} R_{n-1} T_{Y,X}^{-1}$. Notice that (B1) and (B2) are immediate for small i but we must check that $R_{n-1} R_n R_{n-1} = R_n R_{n-1} R_n$ and $R_1 R_n R_1 = R_n R_1 R_n$ and the far-commutation for R_n with $R_2, \ldots, R_{n-2}$. Observing that $\alpha_{Y,X}$ commutes with R_j for $2 \leq j \leq n$, relation (N1) can be seen in the following way:

$$T_{Y,X} R_i$$
$$= \alpha_{Y,X} R_1 \cdots R_{i-1} R_i R_{i+1} R_i \cdots R_{n-1} = \alpha_{Y,X} R_1 \cdots R_{i-1} R_{i+1} R_i R_{i+1} \cdots R_{n-1}$$
$$= \alpha_{Y,X} R_{i+1} R_1 \cdots R_{i-1} R_i R_{i+1} \cdots R_{n-1} = R_{i+1} \alpha_{Y,X} R_1 \cdots R_{n-1}$$
$$= R_{i+1} T_{Y,X}.$$

For such a representation to factor over $\mathcal{NB}_n$ one needs $(T_{Y,X})^{2n} = \mathrm{Id}_{Y \otimes X^{\otimes n}}$. Of course if this is required for all n then $T_{Y,X}^2 = \mathrm{Id}_{Y,X}$–which is precisely the condition that X and Y centralise each other in the sense of Müger [36]. If Y is transparent [8] then this condition is satisfied for all X. Indeed, in [12] a 3 loop configuration with fermion interactions is considered, which would be a particular case of this set up.

Example 6.1. Consider the Ising braided fusion category with simple objects **1**, ψ and σ. Setting $Y = \psi$ (the Majorana fermion) and $X = \sigma$ (the Ising anyon) we find that $c_{X,Y} c_{Y,X} = -\mathrm{Id}_{Y \otimes X}$ where $Y \otimes X \cong \sigma$ so that $\mathrm{Id}_{Y \otimes X} \in \mathrm{End}(\sigma) \cong \mathbb{C}$. In particular $\alpha_{Y,X} = -\mathrm{Id}_{Y \otimes X^{\otimes n}}$, and we can easily adjust the scalar to ensure $(T_{Y,X})^{2n} = \mathrm{Id}$. Since $Y \otimes X^{\otimes n} \cong X^{\otimes n}$ this does not provide a very interesting example.

6.2. Necklaces and spin chains.

Beautiful classes of representations of $\mathcal{B}_n$ are constructed in statistical mechanics, both from open spin chains and from transfer matrices. Drawn suitably, a necklace has some similarity with a 'thickened' *periodic* spin chain.
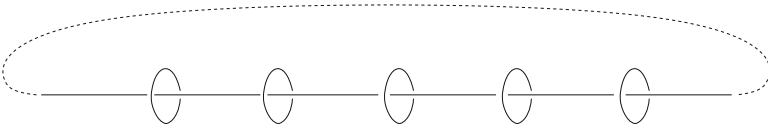
The main superficial difference with $\mathcal{B}_n$ is that there is no algebraic way to make this chain open. This in turn corresponds to the fact that, unlike the natural inclusion of $\mathcal{B}_{n-1}$ in $\mathcal{B}_n$, and similarly for $\mathcal{LB}_n$, $\mathcal{NB}_{n-1}$ does not include in an analogous way in $\mathcal{NB}_n$.

Consider for example the n-site XXZ spin chain [30]. Here we associate the leapfrog motion σ_i to the local spin-spin interaction and, formally, the motion τ to the periodic

symmetry of the chain. Naively this gives $\tau^n = 1$. But the physical system may be endowed with what is sometimes called a cohomology seam, generally localised in a boundary condition, (cf. e.g. [6,35]) so that the image of τ may generate groups of various orders.

Let us start by using this construction to discuss Theorem 2.2 a little more. We proceed as follows. Firstly, a neat way to construct the spin chain itself is from an XXZ TQFT (cf. e.g. [18,26,30,33,49]—the 'bordisms' here are plane-embedded 1-manifolds) with parameter $q = t^2$. This system is monoidally generated by

$$\mathsf{u} = \boxed{\smile} \mapsto (0, -t, t^{-1}, 0) \qquad \text{and} \qquad 1_1 = \boxed{| \,|} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Setting $U = \mathsf{u}^t \mathsf{u}$ (hereafter we will simply identify elements with their images in the category $Vect$ as above, so that u^t means transpose); and, for n given, setting U_i to be U localised in the i-th position in an n-fold tensor product in the usual way, we recall firstly that

$$H = \sum_{i=1}^{n-1} U_i$$

is the 'open' XXZ Hamiltonian, and also that

$$g_i = 1 - qU_i \tag{6.1}$$

obeys the braid relation. Here we write ρ for this representation (notationally suppressing the dependencies on q and n).

For $n = 2$ there is nothing to check here, and yet it is not quite trivial viewed in the context of Theorem 2.2. Factoring through (6.1) and ρ, the image of $\gamma^{n=2}$ is simply

$$\rho(g_1^2) = \begin{pmatrix} 1 & & \\ & 1 - q^2 + q^4 & -q(1 - q^2) \\ & -q(1 - q^2) & q^2 \\ & & & 1 \end{pmatrix}$$

with eigenvalues $1, q^4$. If $q^4 \neq 1$ or $q^2 = 1$ then the Theorem applies directly (and gives a very uninteresting representation). If $q = i$, however, the Theorem does not apply directly (all eigenvalues are 1 but the matrix is not the identity matrix, so the representation of $\mathcal{B}_n$ is not completely reducible—indeed it contains a representation isomorphic to that in Remark 2.4, for $n = 2$, as a direct summand). (The case of $q = i$ exhibits non-complete-reducibility for general even n. We start with $n = 2$ to postpone unhelpfully non-trivial algebra.) We note for future reference that the 'uninteresting' representation with $q^4 = -1$ (to which the Theorem formally does apply) obeys $\rho(\tau^{2n}) = g_1^4 = 1$ *without need of any renormalisation*. Let us call this eventuality ($D = 1$ in Theorem 2.2) a *flat* standard extension.

In general the problem of computing the spectrum of γ^n here is a kind of elementary integrable system (see e.g. [33]). By centrality it acts by a scalar on each indecomposable summand of tensor space. These summands are indexed by the integers l congruent to

n mod.2 (the XXZ charge or number of propagating lines as for example in [5]), so the complete spectrum is given by

$n \setminus l$	0	1	2	3	4	5	6
0	1						
1		1					
2	q^4		1				
3		q^6		1			
4	q^{12}		q^8		1		
5		q^{16}		q^{10}		1	
6	q^{24}		q^{20}		q^{12}		1
7		q^{30}		q^{24}		q^{14}	1

(the general pattern for larger n will be clear). A necessary condition for $\rho(\gamma^{2n}) = 1$ is, of course, that the only eigenvalue is 1. We see that for this to be true for all n (indeed for it to be true for $n = 5$) we require $q^4 = 1$. (And to be true for all even n we require $q^8 = 1$.) The case $q^2 = 1$ factors through the symmetric group, so the more interesting case is $q^2 = -1$.

Proposition 6.2. *Setting $q^2 = -1$ then (a) for all odd n, ρ gives a representation of NB_n via $\rho(\tau) = \rho(\gamma)$ (i.e. with trivial D). Furthermore $\rho(\tau^n) \neq 1$. (b) for even n we never get a representation this way, i.e. we do not get a flat standard extension.*

Proof. (a) For odd n these representations of B_n are completely reducible by [33, §7.3 Th.2], and every eigenvalue of $\rho(\gamma^{2n})$ takes the form q^{4m} for some m. Meanwhile the spectrum of $\rho(\gamma^n)$ always contains 1 and q^{2m} with m odd.
(b) For even n these representations are not completely reducible (they are faithful on a non-semisimple quotient algebra). It remains to show in particular that $\rho(\gamma^{2n})$ has a non-trivial Jordan form. To see this note that $j = U^{\otimes n/2}$ is both central and radical in the quotient algebra; that γ expressed in the bordism basis contains j with non-vanishing coefficient; and that this holds also for any power of γ.

The difference between odd and even cases is well-known in the XXZ setting, but intriguing here. It again suggests to use a cohomology seam in the manner of [6, 35].

Closed boundary conditions have been studied extensively in the XXZ and indeed the wider spin chain setting (see e.g. [19] for recent references). The seam approach extends the TQFT by an operator that acts only on the first position—represented in 'bordisms' by a blob [17, 35]. A simple example of this is

$$\mathsf{f} = \boxed{\bullet} \mapsto \frac{1}{a + a^{-1}} \begin{pmatrix} a & 1 \\ 1 & a^{-1} \end{pmatrix}$$

(always localised in the first position in the tensor product). Such an extension introduces a new 'boundary' parameter into the chain, normally given [35] by the value of the 'topological loop' scalar

$$y_{\mathsf{f}} = \mathsf{u}(\mathsf{f} \otimes 1_1)\mathsf{u}^t = \text{} = \frac{at^2 + (at^2)^{-1}}{(a + a^{-1})} \tag{6.2}$$

(N.B. bottom-to-top stacking convention). Such an extension then allows the construction of a 'seamed' braid translation operator

$$\beta = (1 + x\mathsf{f})\gamma \tag{6.3}$$

where $x = \frac{q - q^{-1}}{q^{-1} - y_{\mathsf{f}}}$. Just as with γ itself in (2.3) we define $g_0 = \beta g_{n-1}\beta^{-1}$ whereupon

$$\beta g_i \beta^{-1} = g_{i+1} \tag{6.4}$$

with indices understood periodically [35]. Indeed this holds not only in ρ but in the setting of the algebra $b_n(q, a)$ of abstract generators g_i and f. Thus we have the following.

Theorem 6.3. *Applying the D-matrix method of Theorem 2.2 to the generators g_i and the braid translator β we obtain a two-parameter representation of $\mathcal{NB}_n$ whenever the corresponding complete reducibility condition is satisfied. The reducibility condition is satisfied, for example, when the parameters q, a are indeterminate—i.e. on a Zariski open subset of parameter space.*

Proof. By (6.4) $\zeta \mapsto \beta$ gives a representation of $\mathcal{CB}_n$. Since ζ^n is central in $\mathcal{CB}_n$ the proof of Theorem 2.2 generalises. The Zariski open property follows from [35]. □

The system required to determine the eigenvalues of β^n, and hence determine the D matrix explicitly, is substantially more involved than the γ case above. We will discuss this elsewhere.

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