

A RATIONAL MAP WITH INFINITELY MANY POINTS OF DISTINCT ARITHMETIC DEGREES

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ABSTRACT. Let $f: X \dashrightarrow X$ be a dominant rational self-map of a smooth projective variety defined over $\overline{\mathbb{Q}}$. For each point $P \in X(\overline{\mathbb{Q}})$ whose forward f -orbit is well-defined, Silverman introduced the arithmetic degree $\alpha_f(P)$, which measures the growth rate of the heights of the points $f^n(P)$. Kawaguchi and Silverman conjectured that $\alpha_f(P)$ is well-defined and that, as P varies, the set of values obtained by $\alpha_f(P)$ is finite. Based on constructions of Bedford–Kim and McMullen, we give a counterexample to this conjecture when $X = \mathbb{P}^4$.

1. INTRODUCTION

Let $f: X \dashrightarrow X$ be a dominant rational map of a smooth projective variety defined over $\overline{\mathbb{Q}}$. We let I_f denote the indeterminacy locus of f , and $X_f(\overline{\mathbb{Q}})$ denote the set of $\overline{\mathbb{Q}}$ -points of X whose forward f -orbit is well-defined, i.e. those $P \in X(\overline{\mathbb{Q}})$ such that $f^n(P) \notin I_f$ for all $n \geq 0$. To each point $P \in X_f(\overline{\mathbb{Q}})$, Silverman [Sil14] introduced the following quantity which measures the arithmetic growth rate of $f^n(P)$. Fix an ample divisor H on X and a logarithmic Weil height function $h_H: X(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}$ for H . Letting $h_H^+(P) = \max(h_H(P), 1)$, consider the quantities

$$\underline{\alpha}_f(P) = \liminf_{n \rightarrow \infty} h_H^+(f^n(P))^{1/n}, \quad \overline{\alpha}_f(P) = \limsup_{n \rightarrow \infty} h_H^+(f^n(P))^{1/n}.$$

Kawaguchi and Silverman proved in [KS16b, Proposition 12] that these quantities are independent of the choice of ample divisor H . When $\underline{\alpha}_f(P) = \overline{\alpha}_f(P)$, the *arithmetic degree* $\alpha_f(P)$ is defined to be the common limit. Kawaguchi and Silverman made the following conjecture and proved it in the case when f is a morphism [KS16a, Theorem 3].

Conjecture 1 ([KS16b, Conjecture 6abc]). *If $P \in X_f(\overline{\mathbb{Q}})$, then the limit $\alpha_f(P)$ exists. Moreover,*

$$\{\alpha_f(Q) \mid Q \in X_f(\overline{\mathbb{Q}})\}$$

is a finite set of algebraic integers.

We prove the following result which gives a counterexample to Conjecture 1.

Theorem 2. *Let $f: \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ be the birational map defined by*

$$[X; Y; Z; A; B] \mapsto [XY + AX; YZ + BX; XZ; AX; BX].$$

Then there exists a sequence of points $P_n \in X_f(\overline{\mathbb{Q}})$ for which $\alpha_f(P_n)$ exists, and $\{\alpha_f(P_n)\}_n$ is an infinite set.

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The strategy we use to prove Theorem 2 is actually inspired by another conjecture of Kawaguchi and Silverman [KS16b, Conjecture 6d], namely that if $P \in X_f(\overline{\mathbb{Q}})$ and P has Zariski dense orbit under f , then $\alpha_f(P)$ is equal to the first dynamical degree $\lambda_1(f)$. Consider a family $\pi: X \rightarrow T$ and a dominant rational map $f: X \dashrightarrow X$ which preserves fibers and induces a dominant rational map $f_t: X_t \dashrightarrow X_t$ on every fiber. For generic values of t , the first dynamical degrees $\lambda_1(f)$ and $\lambda_1(f_t)$ agree, but it is possible to have a countable union of subvarieties $\mathcal{T} \subset T$ such that $\lambda_1(f_t) < \lambda_1(f)$ for all $t \in \mathcal{T}$, and for which infinitely many distinct values arise as $\lambda_1(f_t)$. Suppose that for all $t \in \mathcal{T}$ we can find $P_t \in X_t(\overline{\mathbb{Q}})$ whose forward orbit under f_t is well-defined and Zariski dense in X_t . Then we would expect that $\alpha_f(P_t) = \alpha_{f_t}(P_t) = \lambda_1(f_t)$. Since the set of $\lambda_1(f_t)$ is infinite, this would achieve infinitely many different values for α_f .

There are a few issues one must handle in order to turn the above strategy into a counterexample to Conjecture 1. First, we must produce a suitable map f , and ensure that there are points P_t with dense orbit under f_t and whose orbits avoid the indeterminacy of f . Second, one would expect that $\alpha_{f_t}(P_t) = \lambda_1(f_t)$ but this requires a proof. The easiest way to show this is to work in a case where [KS16b, Conjecture 6d] is already known to hold. For this reason, we consider a family of surface maps with f birational and where f_t extends to an automorphism of a birational model of X_t , so that we can appeal to [Kaw08, KS14], which proves that $\alpha_f(P) = \lambda_1(f)$ in this case. We implement this strategy based on constructions of Bedford–Kim [BK06] and McMullen [McM07].

2. PROOF OF THEOREM 2

We begin by taking the strategy described in the introduction and codifying it as the following result.

Proposition 3. *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$, and $\pi: X \rightarrow T$ be a projective morphism of $\overline{\mathbb{Q}}$ -varieties with two-dimensional fibers. Let $f: X \dashrightarrow X$ be a birational map defined over T and suppose there is an infinite sequence of parameters $t_n \in T(\overline{\mathbb{Q}})$ satisfying the following:*

- (1) *for each n , there exists a birational model $\pi_{t_n}: \tilde{X}_{t_n} \rightarrow X_{t_n}$ so that f_{t_n} extends to an automorphism $\tilde{f}_{t_n}: \tilde{X}_{t_n} \rightarrow \tilde{X}_{t_n}$;*
- (2) *for each n , there exists a $\overline{\mathbb{Q}}$ -point P_n of X_{t_n} , contained in the open set where π_{t_n} is an isomorphism, and with well-defined f -orbit that is Zariski dense in X_{t_n} ;*
- (3) *the set of values $\lambda_1(f_{t_n})$ is infinite.*

Then the set of values of $\alpha_f(P_n)$ is infinite.

Proof. Fix an ample divisor H on X . Since H restricts to an ample on X_t , we see $\alpha_f(P) = \alpha_{f_t}(P)$ for all $P \in X_t(\overline{\mathbb{Q}})$ such that the arithmetic degree is well-defined. So to complete the proof, it is enough to show $\alpha_{f_{t_n}}(P_n) = \lambda_1(f_{t_n})$.

Let $\tilde{P}_n$ be the unique point of $\tilde{X}_{t_n}$ with $\pi_{t_n}(\tilde{P}_n) = P_n$. We have $\alpha_{f_{t_n}}(P_n) = \alpha_{\tilde{f}_{t_n}}(\tilde{P}_n)$ by [MSS17, Theorem 3.4], and $\lambda_1(f_{t_n}) = \lambda_1(\tilde{f}_{t_n})$ by [Dan17, Theorem 1.(2)] and the discussion that follows it. Since $\tilde{f}_{t_n}$ is a surface automorphism and $\tilde{P}_n$ has dense orbit, [KS14, Theorem 2c] tells us that $\alpha_{\tilde{f}_{t_n}}(\tilde{P}_n) = \lambda_1(\tilde{f}_{t_n})$, completing the proof. $\square$

We next use a construction due in various guises to Bedford–Kim [BK06] and McMullen [McM07]. The relation between these two constructions is explained in the introduction of

[BK09] as well as their remark on page 578. We collect the relevant facts from these papers in the following proposition.

Proposition 4. *Let $X = \mathbb{P}^2 \times \mathbb{A}^2$ and consider the map $f: X \dashrightarrow X$ whose fiber over $(a, b) \in \mathbb{A}^2$ is given in affine coordinates by $f_{a,b}(x, y) = (y + a, \frac{y}{x} + b)$. There is a sequence $t_n = (a_n, b_n) \in \mathbb{A}^2(\overline{\mathbb{Q}})$ indexed by the integers $n \geq 10$ with the following properties:*

- (1) *the first dynamical degree $\lambda_1(f_{t_n})$ is given by the largest real root δ_n of the polynomial $x^{n-2}(x^3 - x - 1) + x^3 + x^2 - 1$;*
- (2) *the numbers δ_n increase monotonically in n to $\delta_* \approx 1.32472\dots$, the real root of $x^3 - x - 1$;*
- (3) *there is an f_{t_n} -invariant cuspidal cubic curve $C_{t_n} \subset \mathbb{P}^2$ with cusp q_{t_n} which is invariant under f_{t_n} ;*
- (4) *there is a birational model $\pi_{t_n}: \tilde{X}_{t_n} \rightarrow \mathbb{P}^2$ such that f_{t_n} extends to an automorphism of $\tilde{X}_{t_n}$; specifically, π_{t_n} is a blow-up at n points in the smooth locus of C_{t_n} ;*
- (5) *the point q_{t_n} is not contained in the indeterminacy locus of f ;*
- (6) *the derivative of f_{t_n} at q_{t_n} is given in suitable coordinates by $\begin{pmatrix} \delta_n^{-2} & 0 \\ 0 & \delta_n^{-3} \end{pmatrix}$.*

Proof. First note that the indeterminacy locus of f is $\{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\} \times \mathbb{A}^2$.

Let $t_n = (a_n, b_n)$ be as on page 39 of [McM07]. Let $p_1 = (0 : 0 : 1)$, $p_2 = (1 : 0 : 0)$, $p_3 = (0 : 1 : 0)$, and $p_{4+i} = f_{t_n}^i(a_n : b_n : 1)$ for $0 \leq i \leq n - 4$. By construction (see §7), the p_j lie in the smooth locus of a cuspidal cubic curve C_{t_n} , and letting $\pi_{t_n}: \tilde{X}_{t_n} \rightarrow \mathbb{P}^2$ be the blow-up at the p_j , the map f_{t_n} extends to an automorphism $\tilde{f}_{t_n}$ of $\tilde{X}_{t_n}$.¹ Moreover, $\tilde{f}_{t_n}$ preserves an irreducible curve $Y_n \subset \tilde{X}_{t_n}$ in the complete linear system of the anti-canonical bundle, and $C_{t_n} = \pi_{t_n}(Y_n)$. Since the cusp q_{t_n} of C_{t_n} is not a smooth point of the curve, q_{t_n} is necessarily distinct from the p_j . In particular, π_{t_n} is an isomorphism in a neighborhood of q_{t_n} . Since Y_n is preserved by $\tilde{f}_{t_n}$, we see q_{t_n} is fixed by $\tilde{f}_{t_n}$ and hence f_{t_n} . Finally, q_{t_n} is not in the indeterminacy locus of f as $q_{t_n} \notin \{p_1, p_2, p_3\}$. This handles statements (3)–(5).

By equation (9.1) of [McM07], the derivative of f_{t_n} at q_{t_n} has eigenvalues $\lambda_1(f_{t_n})^{-2}$ and $\lambda_1(f_{t_n})^{-3}$ so (6) will follow upon showing $\lambda_1(f_{t_n}) = \delta_n$ in (1).

Lastly, taking into account differences in notation explained in the remark on page 578 of [BK09], we see (a_n, b_n) belongs to the locus V_{n-3} as defined in their equation (0.2). Statements (1) and (2) then follow from [BK06, Theorem 2] by taking $\alpha = (a, 0, 1)$ and $\beta = (b, 1, 0)$. $\square$

We now prove the main result.

Proof of Theorem 2. We keep the notation of Proposition 4. By construction, f gives a rational self-map of $\mathbb{A}^4$ sending (x, y, a, b) to $(y + a, \frac{y}{x} + b, a, b)$. Taking projective coordinates $[X; Y; Z; A; B]$ on $\mathbb{P}^4$, our map extends to the birational map $f: \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ given by

$$[X; Y; Z; A; B] \mapsto [XY + AX; YZ + BX; XZ; AX; BX].$$

To prove the theorem, we apply Proposition 3. Condition (1) of the proposition is met by virtue of Proposition 4 (4), and condition (3) follows from Proposition 4 (1) and (2). So we need only find $P_n \in X_{t_n}(\overline{\mathbb{Q}})$ whose forward orbit under f is well-defined and Zariski dense in $X_{t_n} = \mathbb{P}^2$, and for which P_n lies in the locus where π_{t_n} is an isomorphism.

¹For reference, McMullen denotes C_{t_n} , $\tilde{X}_{t_n}$, and $\tilde{f}_{t_n}$ by X_n , S_n , and F_n , respectively.

Notice that by Proposition 4 (1) and (2), for each $n \geq 10$ we have $\lambda_1(f_{t_n}) = \delta_n \geq \delta_{10} > 1$. From [Dan17, Theorem 1.(2)], we see $\lambda_1(\tilde{f}_{t_n}) = \lambda_1(f_{t_n}) > 1$. Theorem 1.1 (1) and Lemma 2.4 (1) of [Zha10] then show there are only finitely many $\tilde{f}_{t_n}$ -periodic curves.

By Proposition 4 (3) and (6), q_{t_n} is an attracting fixed point of f_{t_n} . Fixing a metric d on $\mathbb{P}^2(\mathbb{C})$, we find that there exists an analytic open set $U_n \subset \mathbb{P}^2(\mathbb{C})$ containing q_{t_n} for which $f_{t_n}(U_n) \subseteq U_n$ and for which there exists a constant $C < 1$ so that for any u in U_n , we have $d(f_{t_n}(u), q_{t_n}) < C d(u, q_{t_n})$. In particular, the set U_n does not contain any f_{t_n} -periodic point other than q_{t_n} . By (4) and (5), we can choose U_n so that it avoids the indeterminacy locus of f and such that $\pi_{t_n} : \tilde{U}_n = \pi_{t_n}^{-1}(U_n) \rightarrow U_n$ is an isomorphism.

Let $\tilde{P}_n$ be any $\overline{\mathbb{Q}}$ -point of $\tilde{U}_n \setminus \bigcup_{C \text{ is } \tilde{f}_{t_n}\text{-periodic}} C$, and $P_n = \pi_{t_n}(\tilde{P}_n)$. Notice that the f -orbit of P_n is contained in U_n , so the orbit is well-defined and contained in the locus over which π_{t_n} is an isomorphism. By construction, $\tilde{P}_n$ is not contained in any $\tilde{f}_{t_n}$ -periodic curve. At last, since $\tilde{P}_n$ lies in $\tilde{U}_n$, it is not $\tilde{f}_{t_n}$ -periodic. Since $\tilde{P}_n$ is not periodic and does not lie on any $\tilde{f}_{t_n}$ -periodic curve, it must have Zariski dense orbit under $\tilde{f}_{t_n}$, so that P_n has dense orbit under f_{t_n} . $\square$

Remark 5. One can imagine various corrections to Conjecture 1 to circumvent the counterexample of Theorem 2. For example, one might ask that the map $f : X \dashrightarrow X$ does not preserve any fibration. This does not seem sufficient, however. Indeed, the map $g : \mathbb{P}^5 \dashrightarrow \mathbb{P}^5$ defined by

$$[X; Y; Z; A; B; C] \mapsto [XY + AX; YZ + BX; XZ; AX + CY; BX + CZ; C^2]$$

does not appear to preserve a fibration, but the hyperplane $C = 0$ is g -invariant, and the restriction of g to this hyperplane is the map $f : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ of Theorem 2. One might instead attempt to correct Conjecture 1 by requiring either:

- (1) There is no subvariety $Z \subset X$ such that $f|_Z$ preserves a fibration; or
- (2) The points P_n are of bounded degree over $\mathbb{Q}$.

We know of no counterexamples in these settings.

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