Scheduling Multiple Agents in a Persistent Monitoring Task Using Reachability Analysis

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***Abstract*—We consider the problem of controlling the dynamic state of each of a finite collection of targets distributed in physical space using a much smaller collection of mobile agents. Each agent can attend to no more than one target at a given time, thus agents must move between targets to control the collective state, implying that the states of each of the individual targets are only controlled intermittently. We assume that the state dynamics of each of the targets are given by a linear, time-invariant, controllable system and develop conditions on the visiting schedules of the agents to ensure that the property of controllability is maintained in the face of the intermittent control. We then introduce constraints on the magnitude of the control input and a bounded disturbance into the target dynamics and develop a method to evaluate system performance under this scenario. Finally, we use this method to determine how the amount of time the agents spend at a given target before switching to the next in its sequence influences the control of the states of the entire collection of targets.**

***Index Terms*—multi-agent systems, networked control systems, controllability**

I. Introduction

Monitoring a finite number of targets distributed throughout a finite domain using a collection of cooperating mobile agents is a problem that maps to many interesting domains. The paradigm can be applied to various modern applications at length scales ranging from the macroscopic, such as in smart cities or surveillance settings, where a team must monitor specific locations in a large region for changes, intrusions, or other dynamic events [1]–[3], down to the microscopic, as in single particle tracking in molecular biology where the goal is to track multiple individual biological macromolecules using a small number of active sensors to understand their dynamics and their interactions [4], [5].

The class of problems we consider falls under the heading of persistent monitoring. Specifically, we assume there is a collection of mobile agents moving between fixed targets. Each of the targets has its own dynamic state that evolves over time and that can be influenced by an agent when it is visiting the target. While clearly related to both persistent surveillance and coverage, those problems require monitoring the entire domain rather than a finite set of targets in that domain. Under the persistent monitoring setting, there are two levels of design. The first is the design of the agents' schedule, that is the plan by which they sequence the targets they visit and determine how long to spend at each target. The second is the design of the controller that an agent will use at a particular target to steer the target's state as desired.

There is a significant body of work in the literature on the scheduling problem, following (at least) two approaches. Under the first approach, the targets are viewed as discrete tasks that are assigned to agents according to a designed sequence. The problem is often formulated using finite automata to describe target dynamics [2], [6]–[8] with the geometry of the target locations described by a graph where each vertex is a target (or task) and each edge carries a weight which is a function of the distance between the connected targets [9]. Solutions can be found by translating the framework into multiple integer programming problems [10]. This approach to the scheduling problem for the agents is strongly related to traveling salesman or vehicle routing problems [11]. With a schedule in place, the problem is reduced to optimizing a given cost function of the targets states as a function of the amount of time to spend at each target (and the controller to apply). Under some simplifying, but practically limiting, assumptions, this dwell time at each target can even be found analytically [12].

Under the second approach to the scheduling problem, the target space is described as a continuous domain rather than abstracted to a graph. One class of schemes uses parameterized curves to define the agent trajectories, transforming the problem into one of minimizing a given cost function of the target states over the trajectory parameters. Event-driven methods such as Infinitesimal Perturbation Analysis (IPA) can be applied to yield gradient-based approaches which, while they may get trapped in local optima, are scalable in terms of the number of targets and agents [10], [13]–[16].

Most existing methods under either the discrete or continuous formulation assume very simple dynamics for the targets' states. For example, in [10], [12], the state of each target is described using a single scalar variable whose evolution is described by a simple linear increase (in the absence of an agent) or decrease (when being attended by at least one agent). More generally, however, the dynamics of the targets may be described by a multi-variable state and the effect of an agent may not be to generate a simple decrease in the target state. In this work, the (multi-dimensional) state of each target evolves according to a linear, time-invariant (LTI) system. In this setting, the design of both the sequence schedule for the

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agents and the controller to be used when visiting a given target become interesting and coupled problems.

From the point of view of the targets, the mobile nature of agents leads to intermittent control. When no agent is at the target, the controller is assumed to be disconnected such that the input remains zero; when an agent arrives, the controller is connected, giving the agent the ability to affect the target dynamics. This intermittent connection can be modeled as the control of a large number of plants (the targets) using a small number of sensors and actuators over a limited bandwidth network (the agents). Formulating the model in this way, namely as a networked control problem, implies that the design of the visiting schedule of the agents to the target can affect the controllability (or reachability) of the combined control system as well as influence the design of any specific controller.

The question of reachability for LTI systems is well studied in the literature. A variety of tests have been developed, such as the Hautus test [17] and the Kalman rank condition, that use the controllability Gramian to provide a simple binary result as to whether the system is controllable or not. Such tests have also been developed for more complex systems, such as those with switched modes or more general hybrid models [18]–[20]. This basic question has also been studied under the networked control setting where the controller must be switched between systems [21]. These techniques, combined with the idea of *lifting*, were used in [22] to provide conditions on systems controlled over a network that ensure the reachability of the entire system is maintained even in the presence of periodic disconnections to the controller [23], [24]. Similar ideas were also developed in [25] where the connection to the controller was random rather than periodic.

Most of the work in this area, including our own previous efforts, assumed there were no constraints on the control inputs. In practice, of course, systems always face limitations on the magnitude of their control inputs as well as the presence of disturbances to the dynamics. Including disturbances places the problem into the category of robust control and previous work has focused on computing an *invariant set* for the control system by computing the *Minkowski sum* of both the input and disturbance signals [26], [27]. These results assume a continuous connection between the controller and the plant. In our problem, while the disturbance signal drives the target dynamics at every time step, independent of the presence or absence of any agent, control is available only intermittently due to the presence of one or more agents at any given target. The question of *degree of controllability*, that is, of how much control authority is needed, can be traced back at least to the 1970s where [28] proposed a measure based on the condition number of the controllability Gramian. The Hautus test was used in [29] to develop a measure based on the angles between the eigenvectors of the state and input matrices. In [30] and similar works [31], [32], the degree of controllability was measured based on a *recovery region* defined as the collection of states that can be brought back to the origin in a given finite time under limited control authority and methods to estimate both upper and lower bounds on the size of this region were developed. This was further extended in [33], [34] and later in [35] when disturbance rejection was introduced into the story.

In this paper, we build upon the results of our earlier work in [23], [24] to address the issue of reachability of a linear dynamic system i) under a periodically-connected control, ii) with constraints on the magnitude of the control input, and iii) in the presence of a disturbance. After formulating the problem in Sec. II, we focus in Sec. III on the unconstrained case and establish conditions on the periodic schedules that cause a loss of reachability for a given system and show how to alter such a schedule to regain reachability. In Sec. IV we bring back the control constraints and adopt a notion of the recovery region introduced in [30], modified for the discrete-time setting. Using the recovery region (as opposed to other approaches such as the Minkowski sum) allows us to directly compare the ability of the system to recover to the origin to the power of the disturbance to push the system away. By taking the ratio of the size of the recovery region to the size of the "escape" region , we establish a measure of the degree of reachability. Finally, we use this to determine the optimal (in terms of this degree of reachability) number of connected periods for a periodic sequence.

The main contributions of this work lie in the development of both qualitative and quantitative analyses of the timing of the schedule of an agent or agents in terms of their ability to control the state of a target or of multiple targets. This allows us to evaluate possible visiting schedules, eliminating those that do not maintain the property of reachability for the control of the state of the targets and ranking others based on their effectiveness in the presence of both disturbance and control constraints. This will be illustrated through an example in Sec. V. It is important to note that our results are independent of any particular cost function used to select an optimal schedule. Rather, they establish a feasible set (based on maintaining reachability) and at least a partial ranking based on their effectiveness.

II. Problem Formulation

Consider a group of $\mathcal{N}$ targets located in physical (2-D or 3-D) space, each with a state evolving in $\mathbb{R}^n$ according to

$$x(k+1) = Ax(k) + Bu(k) + d(k), \tag{1}$$

where the input $u(k) \in \mathbb{R}^m$ and disturbance $d(k) \in \mathbb{R}^n$ satisfy

$$u(k) \in \mathfrak{U},\ \mathfrak{U} = \left\{ u \in \mathbb{R}^m \middle| |u_i| \leq 1, i = 1, \dots, m \right\},$$
$$\|d(k)\|_2 \leq \delta,$$

with $\|\cdot\|_2$ indicating the standard Euclidean norm in $\mathbb{R}^n$ and δ a positive constant. We assume that A is invertible and that the pair $[A, B]$ is reachable. Note that n and m refer to the state and control dimensions, respectively, and are unrelated to the number of agents or targets. As described below, control is applied to the target by an agent. The need for the invertibility of A is related to the intermittent nature of the control arising from the limited duration of a visit by an agent and is elaborated on briefly later in this section. Note that the targets may each have different system matrices and bounds. In the sequel, we develop our analysis from the point of view of one of the $\mathcal{N}$ targets and thus omit any target index on the dynamics in (1) to avoid cluttering the notation.

There are $\mathcal{M}$ homogeneous agents that move in the physical space to visit the targets. When visiting a target, an agent dwells for some time to apply control to the system carried by this target before departing. We assume that the agents move in such a way that each target sees a periodic sequence of visits of period p. Inside this period, the target sees multiple interleaved "disconnected" and "connected" stages defined by whether an agent is present and applying control or not. The i^{th} disconnected and connected stages within one period have a duration of r_i and q_i time steps, respectively. We define $\bar{r} = (r_1, r_2 \ldots)$, $\bar{q} = (q_1, q_2, \ldots)$, and refer to this visiting sequence as a $(p, \bar{q}, \bar{r})$ policy. Since this policy affects the system's communication with its controller, we also refer to it as a 'communication policy' in the rest of this paper. We clearly have that the duration of the period is the sum of all the disconnected and connected steps $p = \sum_i r_i + \sum_i q_i$. For concreteness and without loss of generality, we assume that the period starts with a disconnected stage. Note that our definition of a $(p, \bar{q}, \bar{r})$ policy is analogous to a communication sequence in [21], although our definition allows for extended times in the sequence where control is completely disconnected. The need for invertibility of A follows from this view of a $(p, \bar{q}, \bar{r})$ policy as it is shown in [21] that lack of this property can lead to loss of reachability.

We consider two problems under this general scenario. In each case we begin with a simple $(p, \bar{q}, \bar{r})$ policy with a single disconnected stage of length r followed by a single connected stage of length q before extending results to more general policies with multiple disconnected and connected stages.

In the first problem, addressed in Sec. III, we explore under what conditions the reachability of the target is maintained under a $(p, \bar{q}, \bar{r})$ policy. To do so, we temporarily relax the bound on the control input and initially assume no disturbance. With these results in hand, we will then reintroduce the disturbance. Because the disturbance $d(k)$ in (1) is *a priori* unknown, reachability under the fully connected scenario (where there is always an agent at the target) implies only the ability to bring the state to within a ball of radius δ of the desired target state. Under a $(p, \bar{q}, \bar{r})$ policy, there are portions of the period where the system is uncontrolled and we explore the effect of this setting on the size of the final ball around the desired target state, using the notion of *lifting* [22] to address the periodic nature of the $(p, \bar{q}, \bar{r})$ policy. Lifting recasts the time-varying system (caused by the intermittent nature of control) to a time-invariant one where each time step describes the evolution of the original one over an entire period.

In the second problem, developed in Sec. IV, we reintroduce the bounds on the control signal and focus on bringing the target state back to the origin. To ensure control is needed, we assume A is unstable. Because the system is unstable, both the drift term $A(k)x(k)$ and the disturbance $d(k)$ tend to drive the system away from the origin. Over time and in the absence of control, the possible location of the system's state x lies within an expanding domain. The control can work to counteract this expansion to hold the system near the origin. In our setting, however, its effectiveness is limited by two factors: the bounds on the control signal and the $(p, \bar{q}, \bar{r})$ policy which forces the control actions to be applied for only a portion of the period p. This combination leads to a limited domain that can be brought back to the origin. Reachability is now defined as the expanding domain of possible state locations that can be recovered to the origin. Using lifting once again, we show that it is possible to design controls for a reachable system that keep it near the origin and explore the effect of the control bounds and the $(p, \bar{q}, \bar{r})$ policy on the size of region the system can be stabilized to.

III. Reachability analysis under periodic communication policy

We now address the first problem described in Sec. II. Throughout this section we relax the constraint on the control, allowing $u(k) \in \mathbb{R}^m$. We begin by ignoring the disturbance, focusing on the impact of the intermittent control. After establishing when the target system retains the property of reachability (and how it can be regained when it is lost), we extend the results to include the disturbance.

While the model for the target dynamics in (1) is linear and time-invariant (LTI), the introduction of intermittent but periodically-applied control through the $(p, \bar{q}, \bar{r})$ policy leads to a linear, time-varying system. Because the policy is periodic, we apply the idea of "lifting" [22] to transform the system back into an LTI one that incorporates the $(p, \bar{q}, \bar{r})$ policy. Define the new state $\hat{x}(k) = x(kp)$. The lifted version dynamics of the system (1) (ignoring the disturbance) are then

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}\hat{u}(k), \tag{2}$$

where

$$\begin{aligned} \hat{u}(k) &= \begin{bmatrix} u(kp) & u(kp+1) & \ldots & u(kp+q-1) \end{bmatrix}^T, \\ \hat{A} &= A^p, \\ \hat{B} &= \big[A^{r_1} \quad \cdots \quad A^{r_1+q_1-1} \quad A^{r_1+q_1+r_2} \quad \cdots \\ &\quad A^{r_1+q_1+r_2+q_2-1} \quad A^{r_1+q_1+r_2+q_2+r_3} \quad \cdots \big] B \end{aligned} \tag{3}$$

Note that k refers to a single step in the lifted system and to the number of periods that have elapsed in the original system.

The reachability matrix of (2) over k periods is

$$\mathfrak{R}_{kp} = \begin{bmatrix} \hat{B} & A^p\hat{B} & \cdots & A^{(k-1)p}\hat{B} \end{bmatrix}. \tag{4}$$

If there exist n linearly independent columns in this matrix, then the system is reachable within k periods. Recall that the original system (1) is reachable and let l denote the number of steps for the reachability matrix of (1) to achieve full rank. It is clear, then, that (2) is directly reachable with any $k \geq 1$, as long as $q \geq l$.

For $1 \leq q < l$, notice that (2) is LTI and consequently its reachability can be determined using the Hautus test [17] which states that an LTI MIMO system (A, B) is reachable if and only if

$$\text{rank}[\lambda I - A \| B] = n, \, \forall \, \lambda \in \mathbb{C},$$

where A is the state matrix, B is the input matrix, and for any two matrices P, Q with the same number of rows, $[P \parallel Q]$ is the concatenated matrix.

Theorem 3.1: Consider (1) with invertible $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, (A, B) a reachable pair, and without disturbance. Suppose the controller and plant are connected under a $(p, \bar{q}, \bar{r})$ policy with $q = \sum_i q_i$. Assume $q \geq 1$, $p \geq l$. Then the system will preserve its reachability if and only if $\forall \lambda \in \mathbb{C}$

$$\text{rank}\left[\lambda I - A^p \| \hat{B}\right] = n.$$

Proof: The result follows from the Hautus test. ■

With the general result in Thm. 3.1, our next goal is to establish specific conditions under which single input and multiple input systems lose reachability. In both cases we first limit ourselves to a policy consisting of only one connected period and one disconnected period such that $\bar{q} = q$ and $\bar{r} = r$. To distinguish it from the more general case, we denote this as a (p, q, r) policy. We then extend the results derived under a (p, q, r) policy to the general $(p, \bar{q}, \bar{r})$ setting.

A. Periodic communication on a single input plant

In [23] we considered a single input system under a $(p, 1, r)$ policy, where the system is connected for exactly one step and disconnected for r steps in a period of p steps. We showed that reachability is lost if and only if there are repeated eigenvalues in A^{r+1}. We now seek a similar result for a policy with arbitrary q. With a (p, q, r) policy allowing only one connected period within p steps, we write the dynamics of (2) as

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}_1\hat{u}(k), \quad (5)$$

where $\hat{B}_1 = [A^{p-1}B \quad A^{p-2}B \quad \cdots \quad A^r B]$.

Let $T \in \mathbb{C}^{n\times n}$ be an invertible matrix such that $J_A = TAT^{-1}$ is the Jordan form of A. The same operator T will also transform $\hat{A}$ to its Jordan form $J_{\hat{A}} = T\hat{A}T^{-1}$, since $\hat{A} = A^p$. Performing this similarity transform on the original system (1) and on the lifted system (5) yields

$$x_J(k+1) = J_A x_J(k) + TBu(k), \quad (6)$$
$$\tilde{x}(k+1) = J_{\hat{A}}\tilde{x}(k) + \tilde{B}\hat{u}(k) \quad (7)$$

where $\tilde{B} = T\hat{B}_1$. Clearly, by similarity, checking the reachability of (1) is equivalent to checking the reachability of (6), and of (5) the same as (7). Necessary and sufficient conditions for the loss of reachability of a single input plant connected under a (p, q, r) policy are given as follows:

Lemma 3.1: Consider (1) with $A \in \mathbb{R}^{n\times n}$, A invertible, $B \in \mathbb{R}^{n\times 1}$, (A, B) a reachable pair, and without disturbance. Suppose the controller and plant are connected under a (p, q, r) policy with $q \geq 1$, $p \geq l$. Then the system will lose reachability if and only if there exists at least one eigenvalue of A^p with a geometric multiplicity strictly greater than q.

Proof: Since the original system (1) is reachable in l steps, it is clear that (2) is reachable for $q \geq l$ and we need therefore consider only $q < l$.

Let π denote the number of Jordan blocks in J_A. Each block J_i, $i \in \{1, \ldots, \pi\}$, is associated with an eigenvalue λ_i. Since A is invertible, the λ_i are all non-zero. Further, the input matrix of (6) can be written as

$$TB = \begin{bmatrix}\beta_1 & \cdots & \beta_\pi\end{bmatrix}^T,$$

where each sub-vector β_i carries the same number of entries as the dimension of J_i. The last entry in every sub-vector is denoted as $\underline{\beta_i}$.

Since (A, B) is a reachable pair, the pair (J_A, TB) is also reachable. Then, by the Hautus test, there are n columns of

$$[\lambda I - J_A \| TB]$$

that are linearly independent from each other for all $\lambda \in \mathbb{C}$. Combined with the structure of the Jordan form, this implies that the eigenvalues associated to each of the Jordan blocks, $\{\lambda_1, ..., \lambda_\pi\}$, must be a set of distinct (non-repeating) elements. Furthermore, to ensure the Hautus text matrix above does not drop rank when $\lambda \in \{\lambda_1, ..., \lambda_\pi\}$, every $\underline{\beta_i}$, $i \in \{1, \ldots, \pi\}$, must be non-zero.

Now we analyze the Jordan form of the lifted system (7). The eigenvalues associated with the π Jordan blocks in $J_{\hat{A}}$ are $\{\lambda_1^p, ..., \lambda_\pi^p\}$. Similarly, the input matrix of (7) under the Jordan transformation can be written as

$$\tilde{B} = \begin{bmatrix}\tilde{\beta}_1 & \cdots & \tilde{\beta}_\pi\end{bmatrix}^T.$$

Since we have

$$\tilde{B} = T\hat{B}_1 = \begin{bmatrix}TA^{p-1}B & TA^{p-2}B & \cdots & TA^{p-q}B\end{bmatrix},$$

each sub-matrix $\tilde{\beta}_i$ can be written as

$$\tilde{\beta}_i = \begin{bmatrix}J_i^{p-1}\beta_i & J_i^{p-2}\beta_i & \cdots & J_i^{p-q}\beta_i\end{bmatrix}.$$

We denote the last row of $\tilde{\beta}_i$ as $\underline{\tilde{\beta}_i}$. By the structure of the Jordan blocks, we have

$$\underline{\tilde{\beta}_i} = \left[\lambda_i^{p-1}\underline{\beta_i} \;\; \lambda_i^{p-2}\underline{\beta_i} \;\; \cdots \;\; \lambda_i^{p-q}\underline{\beta_i}\right].$$

With this setup in place, we can show the necessity of the condition in the lemma for losing reachability. Let λ_i^p be any eigenvalue of A^p and suppose λ_i^p has a geometric multiplicity of $k \leq q$. Then, there exists (a unique) set of indices $i_1, ..., i_k \in [1, ..., \pi]$ satisfying $\lambda_{i_1}^p = ... = \lambda_{i_k}^p$. Since every $\underline{\beta_i}$ is non-zero and since the eigenvalues λ_i are non-zero and non-repeating, the vectors $\underline{\tilde{\beta}_{i_1}}, ..., \underline{\tilde{\beta}_{i_k}}$ are all nonzero and are linearly independent. Then, according to Thm. 6.8 in [36], $(J_{\hat{A}}, \tilde{B})$ is a controllable pair. By similarity, the lifted system (5) is also reachable and thus, by Thm. 3.1 the original system (1) under the (p, q, r) policy is as well.

To establish the sufficiency of this lemma, we proceed by contradiction. Suppose, then, that there is at least one eigenvalue of A^p with a geometric multiplicity strictly greater than q; denote this entry as λ_{GM}^p. The matrix $[\lambda I - J_{\hat{A}} \| \tilde{B}]$ contains $n+q$ columns. If $\lambda = \lambda_{GM}^p$ then at least $q+1$ columns of this matrix will turn to zero and the rank of $[\lambda I - J_{\hat{A}} \| \tilde{B}]$ must be less than n. By the Hautus test, the system (7) is not reachable. By similarity, the system (2) is also not reachable and thus, by Thm. 3.1 the original system (1) under the (p, q, r) policy is not reachable as well. ■

Now consider the case of a $(p, \bar{q}, \bar{r})$ policy. If the system (1) has a single input, the $(p, \bar{q}, \bar{r})$ policy provides q columns to $\hat{B}$ in one period p. However, the q steps are not sequential (the blocks being separated by r_i disconnected steps), and therefore, while the q_i columns in the corresponding block of

$\hat{B}$ are linearly independent of each other, there is no guarantee that the columns in one block are linearly independent of those in another. As a result, a general $(p, \bar{q}, \bar{r})$ policy provides $q^* \leq q$ linearly independent columns in $\hat{B}$. With this we can extend the results based on repeated eigenvalues as follows.

Theorem 3.2: Consider system (1) with invertible $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, (A, B) a reachable pair, and without disturbance. Suppose the controller and plant are connected under a given $(p, \bar{q}, \bar{r})$ policy. Let q^* denote the number of independent columns in $\hat{B}$ for this policy. Then the system will lose reachability if and only if there exists at least one eigenvalue of A^p with a geometric multiplicity strictly greater than q^*.

Proof: By definition, $\hat{B}$ provides q^* independent columns. The proof then follows analogously to Lemma 3.1 with q^* playing the role of q. ∎

Since q^* must be determined from the reachability matrix while q is simply defined by the policy, it is easier to consider q directly. The fact that $q^* \leq q$ leads to the following corollaries.

Corollary 3.2.1: Consider (1) with $A \in \mathbb{R}^{n \times n}$, A invertible, $B \in \mathbb{R}^{n \times 1}$, (A, B) a reachable pair, and without disturbance. Suppose the controller and plant are connected under a $(p, \bar{q}, \bar{r})$ policy with $q \geq 1$, $p \geq l$. Then the system will lose reachability if A^p has at least one eigenvalue with a geometric multiplicity strictly greater than q.

Corollary 3.2.2: Consider (1) with $A \in \mathbb{R}^{n \times n}$, A invertible, $B \in \mathbb{R}^{n \times 1}$, (A, B) a reachable pair, and without disturbance. Suppose the controller and plant are connected under a $(p, \bar{q}, \bar{r})$ policy with $q \geq 1$, $p \geq l$. Then the system will preserve its reachability if A^p has no eigenvalue with a geometric multiplicity strictly greater than $\max(q_1, q_2, ...)$.

B. Periodic communication on a multiple input plant

In this section we extend the results of Sec.III-A to systems with multiple inputs connected under a general $(p, \bar{q}, \bar{r})$ policy. Lemma 3.1 established a condition for SISO systems under a (p, q, r) policy to maintain reachability that was based only on the multiplicity of the eigenvalues in A^p. Unlike Thm. 3.1, this result relied strongly on the SISO setting to establish that the columns of $\hat{B}_1$ were linearly independent of those of $\lambda I - \hat{A}$. In the multiple input setting, B has multiple columns. For any given column b_i, $i = 1, \ldots, m$, it is still true that if b_i is linearly independent of the columns of $\lambda I - A$ for all $\lambda \in \mathbb{C}$, then it will be linearly independent of the columns in $\lambda I - \hat{A}$ for all $\lambda \in \mathbb{C}$ as well. From this we can conclude that there is at least one column in each block of $[A^{p-1}B \| A^{p-2}B \| ... \| A^r B]$ that is linearly independent of the columns in $\lambda I - \hat{A}$.

However, the columns in each of the blocks of $\left[A^{p-1}B \| A^{p-2}B \| \cdots \| A^r B\right]$ are not necessarily linearly independent of one another. In general, the span of those columns may have a non-empty intersection with the span of the columns of $\lambda I - \hat{A}$. The degeneracy prevents us from finding a simple necessary and sufficient test such as was found for SISO systems. We instead pursue a sufficient condition only.

Theorem 3.3: Consider (1) with invertible $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, (A, B) a reachable pair, and without disturbance. Suppose the controller and plant are connected under a $(p, \bar{q}, \bar{r})$ policy with $q \geq 1$, $p \geq l$. Let $V_A = \{v_1, ..., v_n\}$ be the collection of eigenvalues of A (the state matrix from the original system) and let $V_{A^p} = \{v_1^p, ..., v_n^p\}$ be the collection of eigenvalues of $\hat{A} = A^p$. If

$$v_i^p \neq v_j^p \, \forall \, i, j \text{ such that } v_i \neq v_j,$$

then the system connected under a $(p, \bar{q}, \bar{r})$ policy will be reachable for any values of q_i satisfying $\sum_i q_i = q$.

Proof: Recall that the original system (1) is $l-$step reachable. That is, the reachability matrix of (1) achieves full rank in l steps. As a result, (2) is reachable for $q \geq l$.

Now consider $1 \leq q < l$. We first assume that there is only one connected step in the period p, independent of the actual values of $\bar{q}$, and show that the lemma holds with this *extra constraint*. The addition of additional connected steps may lead to reachability being achieved in fewer steps but will not alter the fact that reachability is preserved. We thus assume we have a $(p, q = 1, r = p - 1)$ policy.

The system can be lifted into an LTI version of the form in (2). As the reachability of the lifted system is independent of where we define the beginning of a period, we are free to consider any equivalent lifted system where the period begins at a different point. For convenience, then, in this proof we set the beginning of the period to coincide with the connected step. The lifted LTI system then has $\hat{A} = A^p$ and $\hat{B} = B$.

We again consider the Jordan form J_A computed through

$$J_A = TAT^{-1}$$

with $T \in \mathbb{C}^{n \times n}$. As before, similarity of the systems implies that establishing reachability can be done using the Jordan form. Recall that the Hautus test says that the system under a $(p, 1, r)$ policy will be reachable if and only if

$$\text{rank} \left[\lambda I - J_{\hat{A}} \| TB\right] = n, \ \forall \, \lambda \in \mathbb{C}.$$

Since the original system is, by assumption, reachable, we have

$$\text{rank} \left[\lambda I - J_A \| TB\right] = n$$

for all $\lambda \in \mathbb{C}$ and in particular for $\lambda \in V_A$. Consider now the matrix

$$\left[\lambda^p I - J_{\hat{A}} \| TB\right].$$

Since $J_{\hat{A}}$ is invertible, this matrix is guaranteed to have rank n except possibly when $\lambda^p \in V_{A^p}$. Set $\lambda = v_i$ Then the j^{th} column of $\lambda I - J_A$ and of $\lambda^p I - J_{\hat{A}}$ become

$$a_j = (0, ..., 0, (v_i - v_j), 0, ..., 0)^T,$$
$$a_j^p = \left(0, ..., 0, (v_i^p - v_j^p), 0, ..., 0\right)^T.$$

By the assumption in Thm. 3.3, the column a_j^p is all zero only if the column a_j is. Further, we can write

$$(v_i^p - v_j^p) = c(v_i - v_j),$$

where c is a constant defined by a bivariate polynomial in v_i and v_j. Since v_i and v_j are fixed, this implies that each column of $\lambda_i^p I - J_{\hat{A}}$ is a scalar multiple of the corresponding column of $\lambda_i I - J_A$. Thus the matrices $[\lambda I - J_A \| TB]$ and

$\left[\lambda^p I - J_{\hat{A}} \| TB\right]$ have exactly the same linearly independent columns. Since the original system was reachable, this yields

$$\text{rank}\left[\lambda I - J_{\hat{A}} \| TB\right] = n \tag{8}$$

for all $\lambda \in \mathbb{C}$.

■

The above results allow one to test whether a particular $(p, \bar{q}, \bar{r})$ policy will lead to loss of reachability. One way to interpret this result is to think of taking powers of A as a mapping operation that transforms the directions in the reachability matrix available through the system matrix. If this transformation does not lead to any additional degeneracy, as captured by the introduction of new repeated eigenvalues, then reachability will be preserved. If reachability is preserved, finding a controller for the system to achieve stability or other desired objectives can be achieved using standard algorithms (see, e.g. [21], [23]). We now briefly discuss how to regain reachability when it is lost. We focus on the general multiple input setting and a basic (p, q, r) policy; as before extension to the more general case is straightforward.

C. Regaining Reachability

When a given (p, q, r) policy fails to preserve a system's reachability, there are two options for regaining it. The first is to modify the system matrix to ensure the conditions of Thm. 3.3. In practice, of course, it can be difficult or impossible to change the system dynamics. The second option is to adjust the (p, q, r) policy. While the system may enforce a lower bound on the delay, it is reasonable to expect that it is possible to add delay. Perhaps surprisingly, this can lead to regaining reachability.

Theorem 3.4: Consider (1) with invertible $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, (A, B) a reachable pair, and without disturbance. Suppose the controller and plant are connected under a (p, q, r) policy with $q \geq 1$, $p \geq l$ and suppose further that, under this policy the system loses its reachability. Then there exists a new policy with the same q but with extended delay $\hat{r} > r$ such that under this policy the system's reachability is preserved.

Proof: The conditions for sufficiency of losing reachability discussed in Thm. 3.3 are based on the repeated eigenvalues in the system matrix A^p. Let the integer periods leading to a loss of reachability be called 'critical periods'. Since there are a finite number of eigenvalues to test, there are a finite number of minimal critical periods, such that all the other critical periods are natural multiple of these integers. Let $\hat{p}$ be the least common multiple of the currently known minimal critical periods. Then $A^{\hat{p}+1}$ does not satisfy any of the conditions in Thm. 3.3 and a $(\hat{p}+1, q, \hat{r})$ policy with $\hat{r} = \hat{p} + 1 - q$ will preserve the properties for the original system. ■

We note that this is a sufficient condition; there may be an $\hat{r}$ shorter than the one based on the least common multiple of $\hat{p}$ such that the system regains reachability. While it is possible that the longer r will lead to a control direction directly aligned with how the system needs to be moved, in general the effect of the unstable system matrix, as well as any disturbance, will dominate and thus the increase in the period is almost certain to lead to an increase in the magnitude of control needed.

D. Impact of the disturbance

Given that the system (2) is reachable for a given $(p, \bar{q}, \bar{r})$ policy, the presence of the unknown but bounded disturbance simply implies that the system can be driven only to a bounded domain containing the desired state. After one period p counted from the time step kp, the size of this domain is determined by

$$\hat{D}(kp) = \sum_{i=1}^{p} A^{p-i} d(kp + i - 1). \tag{9}$$

A worse-case upper bound can be established by assuming the disturbance takes its maximum magnitude δ at each step. Let s_A^* denote the maximum singular value of A. We note that $s_A^* > |\lambda^*| > 1$ where λ^* is the spectral radius of A. Then, the state after p steps is guaranteed to lie inside a ball of radius

$$\sum_{i=1}^{p} (s_A^*)^{i-1} \delta \tag{10}$$

centered on the desired state. Eqn. (10) demonstrates (the perhaps obvious) fact that longer periods amplify the effect of the disturbance. Thus, while it may be necessary to extend the period in order to regain reachability of the system under a given $(p, \bar{q}, \bar{r})$ policy, this comes at the cost of poorer performance in terms of the size of the disturbance ball.

IV. Reachability analysis with constrained control input and with disturbance rejection

In Sec. III, we ignored the constraint on the control input. We now bring back that limitation and investigate its impact on an appropriate notion of reachability. We view the dynamic system shown in (1) as a competition between two effects: the accumulated effect of the input on each step to bring the system states back to the origin (referred to as its *recovery power*), and the accumulated effect of the disturbance (referred to as its *escaping power*). In Sec. IV-A we first ignore the disturbance and focus on the recovery power, measuring it using an approach introduced in [30] based on the concept of a *recovery region*, a subset of the state space such that every element in the subset can be brought back to the origin through the accumulated input over a fixed, finite number of time steps.We develop a method to calculate a lower bound on the size of this set and then, in Sec. IV-B, re-introduce the disturbance to analyze the competing effects of control and disturbance to determine whether a given system will remain reachable.

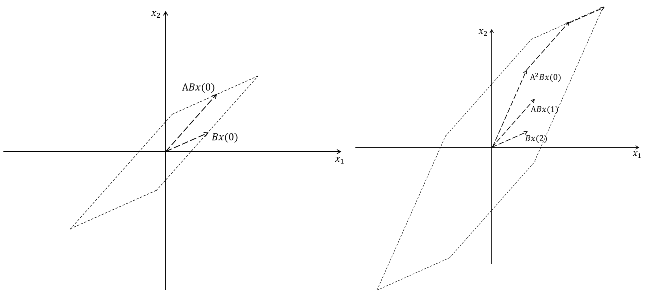
As before, the system dynamics are described by (1) under a given $(p, \bar{q}, \bar{r})$ policy. We also follow the same sequence as before, beginning with a simple (p, q, r) policy (that is, one with a single connected and single disconnected stage) before extending to the more general $(p, \bar{q}, \bar{r})$ setting.

A. Measurement of the recovery region

We will start with the definition of recovery region for a system with bounded input.

Definition 4.1: Consider a system of the form (1) under a given (p, q, r) policy. The *recovery region* is defined as

$$\begin{aligned}\mathfrak{S}(q) = \{&x \in \mathbb{R}^n \mid \exists\, u(k), k = 0, \ldots, p-1,\ p > q > 0, \\ &\text{with } u(k) = 0, k = 0, \ldots, p-q-1 \text{ and } u(k) \in \mathfrak{U}, \\ &k = p-q, \ldots, p-1,\ \text{steering the state to } x(p) = 0\}.\end{aligned}$$



(a) After two steps. (b) After three steps.

Fig. 1: Recovery region of a 2-D system with bounded input.

This recovery region is an expanding convex polytope, illustrated in Fig. 1 for a two-dimensional system with single input after two and three steps. We use the radius of the largest inscribed ball of this polytope as a measure of this system's reachability and refer to this as the *recovery distance*, $\rho_r^*(q)$. It is given by

$$\rho_r^*(q) = \max_{x \in \mathfrak{S}(q)} \|x\| \tag{11}$$

$$\text{such that } \forall y \in \mathbb{R}^n,\ \text{ if } \|y\| \leq \rho_r^*(q),\ \text{ then } y \in \mathfrak{S}(q).$$

While (11) makes clear the notion of the recovery distance, it is not particularly useful for calculating its value. In the sequel, we develop a method for calculating $\rho_r^*(q)$ for discrete-time systems.

Given a finite $\rho_r^*(q)$, there is at least one initial condition with norm larger than this recovery distance from which the system cannot be brought back to the origin. This implies that there is (at least) one direction such that any initial condition in this direction with norm greater than $\rho_r^*(q)$ cannot be brought to the origin in q steps. As illustrated in Fig. 2, we denote this *critical direction* with the vector v_c (choosing one arbitrarily if there are multiple possibilities) and set $\|v_c\| = \rho_r^*(q)$. Our calculation of $\rho_r^*(q)$ proceeds by considering the effect of the inputs over all q steps on this direction.

Geometrically, v_c points to where the inscribed ball is tangent to a facet of the convex polytope. This facet is a co-dimension one linear subspace and as such is spanned by $n-1$ linearly independent directions. By construction of the recovery region, these directions must be from those generated by the system over the q steps. Collecting these directions as columns of a single $n \times mq$ matrix, we define

$$\mathfrak{D}(q) = \{\ B \quad AB \quad \cdots \quad A^{q-1}B\ \}. \tag{12}$$

Recalling that the input is bounded with $u_{(k)} \in \mathfrak{U}$, we have

$$\rho_r^*(q) = \sum_{j=1}^{q} \sum_{l=1}^{m} \|P_{v_c}(A^{j-1}b_l)\|,$$

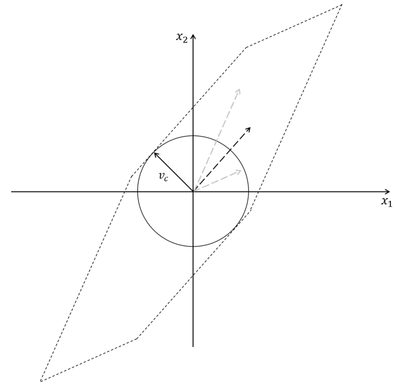


Fig. 2: Inscribed ball of the recovery region and critical direction v_c.

where P_{v_c} is the projection operator onto the direction of v_c.

Clearly, if there are no more than $n-1$ linearly independent directions in $\mathfrak{D}(q)$, then the system is not reachable over q steps and $\rho_r^*(q)$ equals zero. If, on the other hand, $\mathfrak{D}(q)$ is full rank, then the facet that is orthogonal to the critical direction v_c is spanned by some choice of $n-1$ columns of the matrix.

Let $\mathfrak{I}(q)$ denote the collection of all choices of $n-1$ columns of $\mathfrak{D}_q$. Since $\mathfrak{D}_q$ has mq columns, there are $\binom{mq}{n-1}$ different subsets in $\mathfrak{I}(q)$. Let $\mathfrak{I}(q)_j$ denote the j^{th} entry of the collection and let $\hat{v}_k$ denote the unit vector that is orthogonal to all entries of the collection. Then the recovery distance can be found by

$$\rho_r^*(q) = \min_{\mathfrak{I}_k(q)} \sum_{j=1}^{q} \sum_{l=1}^{m} \|P_{v_k}(A^{j-1}b_l)\|. \tag{13}$$

With this notion of reachability, we turn our attention back to system (1) under a given (p, q, r) communication policy. Using the lifting method described in Sec. III yields the system

$$\bar{x}(k+1) = A^p \bar{x}(k) + \bar{B}\bar{u}(k)$$

where $\bar{B} = [B, AB, ..., A^{q-1}B]$. Since we are interested in the situation where control is needed to steer the system to the origin, we assume that A (and thus A^p) is unstable. Let us focus on one step of the lifted system (that is, on one period p) and consider how the recovery distance changes as a function of the number of connected steps q.

For the sake of simplicity, we focus on state matrices that are diagonalizable over the field of reals. In addition, the mathematical arguments are simplest in the two-dimensional single input setting and thus we present them in that context, extending to the more general case in Appendix. A.

Consider therefore a system with

$$A = \begin{bmatrix} (1+\alpha)\lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad B = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix},$$

with $|\lambda| \geq 1$, $\alpha > 0$ and where β_1 and β_2 are arbitrary non-zero real numbers. (Note that zero is not allowed because (A, B) is a controllable pair.) The lifted system with a (p, q, r) policy ($q > 2$), yields

$$\mathfrak{D} = \left\{ \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \begin{bmatrix} (1+\alpha)\lambda\beta_1 \\ \lambda\beta_2 \end{bmatrix}, ..., \begin{bmatrix} (1+\alpha)^{q-1}\lambda^{q-1}\beta_1 \\ \lambda^{q-1}\beta_2 \end{bmatrix} \right\}.$$

The critical direction v_c is orthogonal to one of the vectors in $\mathfrak{D}$. A vector orthogonal to the k^{th} vector in $\mathfrak{D}$ is

$$v_k = \begin{bmatrix} \beta_2 \\ -(1+\alpha)^{k-1}\beta_1 \end{bmatrix}.$$

The magnitude of the projection of any vector in $\mathfrak{D}$ on the direction of v_k can be calculated to be

$$\begin{aligned}\|P_{v_k}(\mathfrak{D}_j)\| &= \left|\frac{\mathfrak{D}_j \cdot v_k}{\|v_k\|}\right| \\ &= \frac{|\left((1+\alpha)^{j-1} - (1+\alpha)^{k-1}\right)\lambda^{j-1}\beta_1\beta_2|}{\sqrt{(1+\alpha)^{2(k-1)}\beta_1^2 + \beta_2^2}},\end{aligned}$$

and the sum of projections of the inputs over the q steps is

$$\rho_r^k(q) = \sum_{j=1}^{q} \|P_{v_k}(\mathfrak{D}_j)\|. \tag{14}$$

A different selection of k yields different v_k and alters the sum value. Using these results in (13) and denoting the index that minimizes the sum of these projections as k_q, we get

$$\rho_r^*(q) = \sum_{j=1}^{q} \frac{|\left((1+\alpha)^{j-1} - (1+\alpha)^{k_q-1}\right)\lambda^{j-1}\beta_1\beta_2|}{\sqrt{(1+\alpha)^{2(k_q-1)}\beta_1^2 + \beta_2^2}}. \tag{15}$$

Equation (15) describes a monotone increasing sequence indexed by q. Unfortunately, because there is no analytical formula for k_q, we cannot determine an analytical transition rule from $\rho_r^*(q)$ to $\rho_r^*(q+1)$. Note that (14) actually defines a collection of sequences in q with each sequence distinguished by choice of k (which may itself be a function of q). Below we consider a specific choice, namely the one given by setting $k = q$ to yield $\rho_r^q(q)$. We show that this sequence dominates the one we are actually interested in, namely the recovery distance, $\rho_r^*(q)$ where, as illustrated in Fig.3, a dominating sequence is one that is larger at every index. The figure also shows that at small q, the behavior of $\rho_r^*(q)$ is non-trivial while that of $\rho_r^q(q)$ is much simpler, yet at large q both approach a simple exponential. Analysis of the simpler, dominating sequence will provide insight into the rate of growth of the recovery distance.

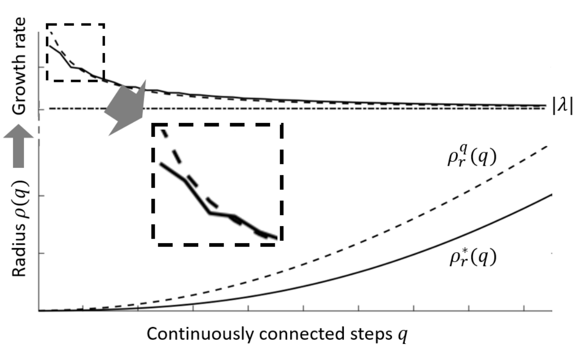


Fig. 3: (top) Growth rates of the sequences $\rho_r^q(q)$ (dashed) and $\rho_r^*(q)$ (solid). (bottom) the sequences themselves. At small q, the recover distance $\rho_r^*(q)$ grows in a complicated manner, converging at large q to simple exponential growth with the same rate as $\rho_r^q(q)$.

We begin by showing the optimizing k cannot be unity.

Lemma 4.1: For any q, $\rho_r^1(q) > \rho_r^*(q)$ always holds.

Proof: We simply compare $\rho_r^1(q)$ and $\rho_r^2(q)$

$$\rho_r^1(q) = \sum_{j=1}^{q} \frac{|\left((1+\alpha)^{j-1} - 1\right)\lambda^{j-1}\beta_1\beta_2|}{\sqrt{\beta_1^2 + \beta_2^2}},$$

$$\rho_r^2(q) = \sum_{j=1}^{q} \frac{|\left((1+\alpha)^{j-1} - (1+\alpha)\right)\lambda^{j-1}\beta_1\beta_2|}{\sqrt{(1+\alpha)^2\beta_1^2 + \beta_2^2}}.$$

It is clear from inspection that $\rho_r^2 < \rho_r^1$. ∎

Lemma 4.1 implies that $k_q - 1 > 0$. We will take advantage of this in later proofs. We now establish a lower bound on the relative size of two sequential elements in one of these sequences.

Lemma 4.2: For any k, we have

$$\frac{\rho_r^{k+1}(q+1)}{\rho_r^k(q)} > |\lambda|.$$

Proof: Writing out the q^{th} term in the sequence ρ_r^k yields

$$\rho_r^k(q) = \sum_{j=1}^{q} \frac{|\left((1+\alpha)^{j-1} - (1+\alpha)^{k-1}\right)\lambda^{j-1}\beta_1\beta_2|}{\sqrt{(1+\alpha)^{2(k-1)}\beta_1^2 + \beta_2^2}}, \tag{16}$$

while the $(q+1)^{st}$ term in ρ_r^{k+1} is

$$\begin{aligned}\rho_r^{k+1}(q+1) &= \sum_{j=1}^{q+1} \frac{|\left((1+\alpha)^{j-1} - (1+\alpha)^{k}\right)\lambda^{j-1}\beta_1\beta_2|}{\sqrt{(1+\alpha)^{2k}\beta_1^2 + \beta_2^2}} \\ &= \sum_{j=2}^{q+1} \frac{|\left((1+\alpha)^{j-2} - (1+\alpha)^{k-1}\right)\lambda^{j-1}\beta_1\beta_2|}{\sqrt{(1+\alpha)^{2(k-1)}\beta_1^2 + \left(\frac{\beta_2}{1+\alpha}\right)^2}} \qquad \text{(17a)} \\ &\quad + \frac{|\left((1+\alpha)^{k} - 1\right)\lambda^{j-1}\beta_1\beta_2|}{\sqrt{(1+\alpha)^{2k}\beta_1^2 + \beta_2^2}}, \qquad \text{(17b)}\end{aligned}$$

Note that the portion (17b) is positive and that (17a) can be rewritten as

$$\sum_{j=1}^{q} \frac{|\left((1+\alpha)^{j-1} - (1+\alpha)^{k-1}\right)\lambda^{j}\beta_1\beta_2|}{\sqrt{(1+\alpha)^{2(k-1)}\beta_1^2 + \left(\frac{\beta_2}{1+\alpha}\right)^2}} > |\lambda| \cdot \rho_r^k(q) \tag{18}$$

Thus since $|\lambda| \geq 1$, $\rho_r^{k+1}(q+1) > |\lambda|\rho_r^k(q)$ holds. ∎

Next we formally state the dominating sequence before considering some of its properties.

Lemma 4.3: The sequence ρ_r^q dominates the sequence ρ_r^*. That is

$$\rho_r^*(q) \leq \rho_r^q(q), \quad \forall\, q.$$

Proof: This is immediate from the definition of $\rho_r^*(q)$ in (13) as the smallest distance at every q. ∎

Lemma 4.4: The rate of increase of the sequence $\rho_r^q(q)$ is always greater than $|\lambda|$ and converges to $|\lambda|$ as $q \to \infty$.

Proof: The rate of increase is given by

$$\frac{\rho_r^{q+1}(q+1)}{\rho_r^q(q)}.$$

By Lemma 4.2 this is strictly greater than $|\lambda|$. To calculate the limit as $q \to \infty$ we first write

$$\rho_r^q(q) = \sum_{j=1}^{q} \frac{|\left((1+\alpha)^{q-1} - (1+\alpha)^{j-1}\right)\lambda^{j-1}\beta_1\beta_2|}{\sqrt{(1+\alpha)^{2q-2}\beta_1^2+\beta_2^2}}$$

$$= \frac{\left|\left((1+\alpha)^{q-1}\sum_{j=1}^{q}|\lambda|^{j-1} - \sum_{j=1}^{q}(1+\alpha)^{j-1}|\lambda|^{j-1}\right)\beta_1\beta_2\right|}{\sqrt{(1+\alpha)^{2q-2}\beta_1^2+\beta_2^2}}$$

$$= \left[\frac{(1+\alpha)^{q-1}\frac{|\lambda|^q-1}{|\lambda|-1} - \frac{(1+\alpha)^q|\lambda|^q-1}{(1+\alpha)|\lambda|-1}}{\sqrt{(1+\alpha)^{2q-2}\beta_1^2+\beta_2^2}}\right]|\beta_1\beta_2|.$$

Similarly,

$$\rho_r^{q+1}(q+1) = \left[\frac{(1+\alpha)^q\frac{|\lambda|^{q+1}-1}{|\lambda|-1} - \frac{(1+\alpha)^{q+1}|\lambda|^{q+1}-1}{(1+\alpha)|\lambda|-1}}{\sqrt{(1+\alpha)^{2q}\beta_1^2+\beta_2^2}}\right]|\beta_1\beta_2|.$$

Taking limit of the ratio of the numerators, we get

$$\lim_{q\to\infty}\frac{(1+\alpha)^q\frac{|\lambda|^{q+1}-1}{|\lambda|-1} - \frac{(1+\alpha)^{q+1}|\lambda|^{q+1}-1}{(1+\alpha)|\lambda|-1}}{(1+\alpha)^{q-1}\frac{|\lambda|^q-1}{|\lambda|-1} - \frac{(1+\alpha)^q|\lambda|^q-1}{(1+\alpha)|\lambda|-1}} = (1+\alpha)|\lambda|.$$

Looking now at the denominators, we get

$$\lim_{q\to\infty}\frac{\sqrt{(1+\alpha)^{2q}\beta_1^2+\beta_2^2}}{\sqrt{(1+\alpha)^{2q-2}\beta_1^2+\beta_2^2}} = 1+\alpha$$

Combining these we get that the limit of the ratios converges to $|\lambda|$ and the lemma is established. ■

Alongside these properties of the sequence $\rho_r^q(q)$, we also establish the following property of the sequence $\rho_r^*(q)$.

Lemma 4.5: The recovery distance $\rho_r^*(q)$ is monotonically increasing in q with

$$\frac{\rho_r^*(q+1)}{\rho_r^*(q)} > |\lambda|.$$

Proof: Let the indices yielding the recovery distance after q and $q+1$ steps be denoted by k_q and k_{q+1} respectively. By Lemma 4.1, we have $k_q > 1$ and $k_{q+1} > 1$. Thus $\rho_r^{k_{q+1}-1}(q)$ exists. Now, by Lemma 4.2 we have

$$\rho_r^{k_{q+1}}(q+1) > |\lambda| \cdot \rho_r^{k_{q+1}-1}(q). \quad (19)$$

Since $\rho_r^{k_{q+1}}(q+1) = \rho_r^*(q+1)$ and $\rho_r^{k_{q+1}-1}(q) \geq \rho_r^*(q)$ (by the definition of $\rho_r^*(q)$), we have

$$\rho_r^*(q+1) > |\lambda| \cdot \rho_r^*(q) \quad (20)$$

and the Lemma is established. ■

With Lemmas 4.3-4.5 in hand, we are able to show that the growth of ρ^* is bounded by a particular function of q.

Theorem 4.1: For the sequence $\rho_r^*(q)$ and for arbitrarily small $\epsilon > 0$, we can always find two indexes $q_1(\epsilon), q_2(\epsilon) \in \mathbb{N}$ with $q_1(\epsilon) > q_2(\epsilon)$, such that

$$|\lambda|^{q-q_2(\epsilon)} < \frac{\rho_r^*(q)}{\rho_r^*(q_2(\epsilon))} < (|\lambda|+\epsilon)^{q-q_2(\epsilon)}, \forall q > q_1(\epsilon). \quad (21)$$

Proof: The left side of (21) follows directly from Lemma 4.5. We now establish the right side. From Lemma 4.4 we have that, for any ϵ, an index $q_3(\epsilon)$ can be found such that

$$\frac{\rho_r^{q+1}(q+1)}{\rho_r^q(q)} < |\lambda| + \frac{\epsilon}{2}, \forall q > q_3(\epsilon). \quad (22)$$

Also, according to Lemma 4.5 we have $\lim_{q\to\infty}\rho_r^*(q) = \infty$. Thus, there exists an index $q_2(\epsilon)$ such that

$$\rho_r^*(q_2(\epsilon)) > \rho_r^{q_3(\epsilon)}(q_3(\epsilon)) \quad (23)$$

According to Lemma 4.3, we know that $q_2(\epsilon) > q_3(\epsilon)$. Therefore,

$$\frac{\rho_r^*(q)}{\rho_r^*(q_2(\epsilon))} < \frac{\rho_r^{q+1}(q)}{\rho_r^{q_3(\epsilon)}(q_3(\epsilon))} < (|\lambda|+\frac{\epsilon}{2})^{q-q_3(\epsilon)}, \forall q > q_2(\epsilon). \quad (24)$$

Since $|\lambda| > 1$ and $\epsilon > 0$, there always exists $q_1(\epsilon) > q_2(\epsilon)$, such that

$$(|\lambda| + \frac{\epsilon}{2})^{q-q_3(\epsilon)} < (|\lambda|+\epsilon)^{q-q_2(\epsilon)}, \forall q > q_1(\epsilon), \quad (25)$$

establishing the right side of (21) and thus the Theorem. ■

Theorem 4.1 shows that the exponential growth rate of the sequence $\rho_r^*(q)$ can be upper bounded by any rate arbitrarily close to but larger than $|\lambda|$. This will be useful in the next section where we compare the size of the recovery region with the size of the accumulated disturbance.

The extension to the general multiple-input setting is straightforward but messy and is relegated to the Appendix. The main concept established here holds, namely that the growth rate becomes aribtrarily close to the smallest magnitude eigenvalue of the system matrix. For simplicity, we will continue to refer to this rate simply as λ. We now reintroduce the disturbance.

B. Disturbance rejection

In Sec. IV-A we showed that the radius of the recovery region grows with time and the growth rate gets arbitrarily close to $|\lambda|$. Now we are interested in the cumulative effect of the disturbance. From the system dynamics under the effect of the disturbance, each direction grows exponentially, giving rise to an ellipsoid. We bound this ellipse by the largest radius, denoted as ρ_e, and refer to the ball as the *escaping ball*. Since the disturbance is independent of the presence of control, we consider the effect after an entire period p rather than just after the connected portion of q steps. Since we are focused on the disturbance we assume the system begins from the origin. Then, from the system dynamics, the growth of ρ_e is defined as in (10)

$$\rho_e(p) = \sum_{i=1}^{p}\delta(s_A^*)^{i-1} = \delta\frac{(s_A^*)^p - 1}{s_A^* - 1}. \quad (26)$$

We remark upon the effect of non-zero initial conditions later in Remark 1.

With these two results, we can compare the recovery region with the escaping ball and make a conservative estimate of the system's ability of disturbance rejection.

Theorem 4.2: The growth rate of the sequence describing the escaping ball, $\rho_e(p), p = 1, 2, ...,$ lies within$[s_A^*, s_A^* + 1]$. Furthermore, as $p \to \infty$, the sequence converges to an exponentially growing sequence $c(s_A^*)^p$, for some constant c that depends on the system parameters.

Proof: We have

$$\frac{\rho_e(p+1)}{\rho_e(p)} = \frac{(s_A^*)^{p+1} - 1}{(s_A^*)^p - 1}.$$

The growth rate is monotonically decreasing since $s_A^* > 1$, and therefore

$$\frac{\left((s_A^*)^{p+2} - 1\right)\left((s_A^*)^p - 1\right)}{\left((s_A^*)^{p+1} - 1\right)^2} < 1.$$

Hence the growth rate of $\rho_e(p)$ is upper bounded by its value at $p = 1$. Thus,

$$\frac{\rho_e(p+1)}{\rho_e(p)} \leq \frac{(s_A^*)^2 - 1}{s_A^* - 1} = s_A^* + 1.$$

Taking the limit yields

$$\lim_{p\to\infty} \frac{(s_A^*)^{p+1} - 1}{(s_A^*)^p - 1} = s_A^*, \text{ for } s_A^* > 1.$$

Since the sequence growth rate converges to s_A^*, the sequence converges to some exponentially growing sequence $c(s_A^*)^p$, and the lemma is established. ∎

It was shown in the previous section (and the Appendix) that the size of the recovery ball grows with q at a growth rate asymptotically close to $|\lambda|$. We refer to the radius of this ball as the "recovery radius". We then define the ratio of the recovery radius over the radius of the escaping ball as the "recovery-escape ratio",

$$\mathfrak{r}_{(p,q,r)} = \frac{\rho_r^*(q)}{\rho_e(p)}.$$

Theorem 4.3: A system under a given (p, q, r) policy and in the face of both limited control authority and disturbance is reachable if

$$\mathfrak{r}_{(p,q,r)} \geq 1.$$

Proof: The proof follows immediately from the above discussion. ∎

As a function of q for a fixed number of disconnected steps r, we find

$$\lim_{q\to\infty} \mathfrak{r}_{(p,q,r)} = \lim_{q\to\infty} \frac{\kappa|\lambda|^q}{c(s_A^*)^p} = \frac{\kappa}{c(s_A^*)^r} \lim_{q\to\infty} \left(\frac{|\lambda|}{s_A^*}\right)^q.$$

It is obvious that, because $|\lambda| < |\lambda^*| \leq s_A^*$, this sequence $\mathfrak{r}_{(p,q,r)}$ approaches an exponentially decreasing sequence when q is large enough. However, for small q, this ratio may increase since the growth rate of $\rho_e(p)$ is upper bounded and the growth rate of $\rho_r^*(q)$ may initially be large enough to dominate $\rho_e(p)$. From this we conclude that there exists a finite $\bar{q}$, such that for $q > \bar{q}$, the recovery-escape ratio is decreasing (see Fig. 5) and that there is an optimal number of connected steps $q^* \leq \bar{q}$ that maximizes $\mathfrak{r}_{(p,q,r)}$.

We note, however, that with a fixed number of disconnected steps, the choice of q directly defines the total period p. As discussed previously, the presence of the disturbance implies the system can only be guaranteed to be stabilized to a ball around the origin with the size of the ball increasing with the period using in the lifting. Therefore, choosing the number of connected steps is a trade off between maximizing the recovery-escape ratio and minimizing the size of that ball.

We note also that the above analysis holds for a system that is always connected since this can be represented using a $(q, q, 0)$ policy. This in turn implies that there is an optimal lifting period that maximizes the recovery-escape ratio. Passing to a lifted system implies a periodic measure of the system's states rather than a measurement at each time step. The *optimal* lifting period here maximizes the recovery-escape ratio. However, longer lifting periods may increase the size of the ball to which the system can be guaranteed to be brought back to (while still maximizing the ratio). Thus, in practice, there is likely a trade-off between these two measures of performance.

Remark 1: The above derivation assume a zero initial condition on the system. A non-zero initial state simply increases the growth of the system (since A is assumed to be unstable). To make a conservative estimate of the impact of this state, we consider the worst case scenario and redefine the escape radius as $(s_A^*)^p|x(0)| + \sum_{i=1}^{p} \delta(s_A^*)^{i-1}$. As before, the system is reachable if the corresponding recovery-escape ratio is greater than one.

Remark 2: The only difference for a system under a general $(p, \bar{q}, \bar{r})$ policy is that the recovery-escape ratio must be calculated by considering all possible combinations of the control directions spanned during the connected steps.

V. EXAMPLES

In this section we provide two examples to illustrate our major results from Sec. III and IV.

We consider a collection of targets arrayed in $\mathbb{R}^2$ as shown in Fig. 4, each being periodically visited by some agents according to pre-selected visiting sequences. We focus our analysis on the target highlighted in gray in the figure. The edge weights indicate the time steps an agent needs to travel between the targets. For simplicity, we take a simple (p, q, r) communication policy on this target, that is, we assume that in one period there is a single visit to this target. We then consider two different policies. These could be defined, for example, by a single agent moving through the targets along two different paths, as shown in Fig. 4, or more generally by multiple agents whose individual paths collectively lead to a given policy. The number on each target specifies the time steps an agent spends with the target while visiting. The first sequence has a total period of $p = 30$ steps while the second has a period of $p = 34$ steps.

We first consider a dynamic system with unlimited input, and then one with input constraints and disturbance is analyzed. We show that in both cases the reachability analysis developed in this paper can be used to evaluate the two possible sequences.

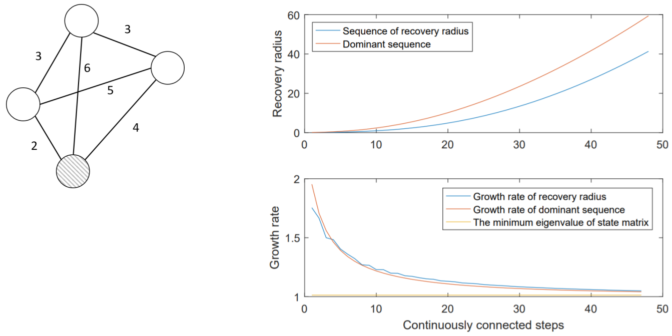


Fig. 4: (left) Layout of targets. The shaded one is the target under consideration. (middle and right) Two possible agent sequences, giving rise to two different communication policies.

A. Example with unlimited input

Consider the single-input system with dynamics given by

$$x(k+1) = Ax(k) + bu(k) \quad x \in \mathbb{R}^3, \quad u(k) \in \mathbb{R}$$

where the state matrix A has eigenvalues $[0.5+0.866i, 0.5-0.866i, 1.2269]$. For the purposes of analyzing the communication sequences, the particular choice of b is unimportant other than assuming the pair (A, b) is reachable.

For any period that is a multiple of 3, the eigenvalues of A^p are $[\pm 1, \pm 1, 1.2269^p]$. The multiplicity of the first eigenvalue has increased from one to two and thus such periodicity will cause a loss of reachability. Since the first communication sequence has $p = 30$, application of Thm. 3.1 implies the resulting lifted system will lose reachability. The second sequence, however, does not lead to repeated eigenvalues in A^p and thus preserves reachability of the system.

Furthermore, according to Theorem 3.4, Sequence I can regain its reachability by simply adding one step into the period. This can be achieved either by increasing the number of connected steps or by delaying the agent by one step on any one of the transitions.

B. Example with limited input and disturbance

Consider the single-input system

$$\begin{gathered} x(k+1) = Ax(k) + Bu(k) + d(k), \quad x \in \mathbb{R}^3 \\ u(k) \in \mathfrak{U}, \mathfrak{U} = \{u \in \mathbb{R} \| |u_1| \leq 1\}, \\ \|d(k)\|_2 \leq 0.0001, \quad k = 1, 2, \cdots, \end{gathered}$$

where the state matrix A is a normal matrix with eigenvalues $[1.0136; 1.0580; 1.2269]$, and the input matrix is $B = [1.8186; 1.8987; 1.1723]^T$. The operator norm of A is therefore equal to the spectral radius 1.2269.

Consider the first communication sequence. The time the agents spend while not visiting the target is fixed at $r = 23$ (the number of disconnected steps). Fig. 5 shows both the recovery and escape radii growing with visiting steps q on this target. The plot starts at $q = 3$ since this system requires at least three steps to establish its reachability, and the recovery radii at $q = 1$ and $q = 2$ are both zero. Notice that the recovery ratio is initially larger than the escape ratio but, for large enough q, the disturbance dominates. As expected, the ratio drops with large q. The ratio plot clearly indicates that, for this example, the system is guaranteed to be able to overcome the effect of the disturbance if the number of connected steps in the period is in the range $q \in (7, 22)$ and thus sequence I is a feasible choice.

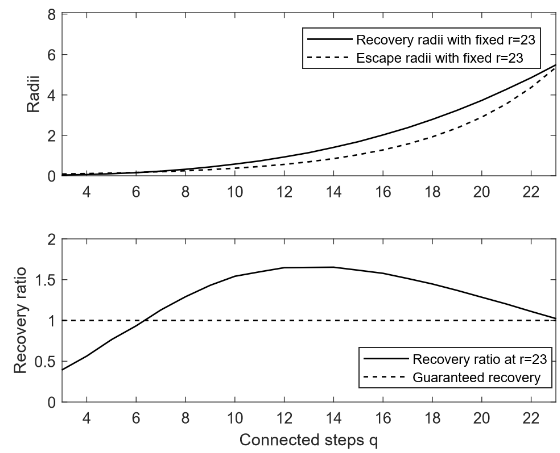


Fig. 5: Comparing the escape and recovery radii growing with visiting steps for sequence I. The ratio τ exceeds 1 at $q = 7$ and reaches maximum at $q = 14$.

For Sequence II, the time steps that the agent is absent from the target is 27. The comparison of both radii and the recovery ratio are shown in Fig. 6. Under this sequence, the recovery ratio never grows to 1, and a full recovery is not guaranteed.

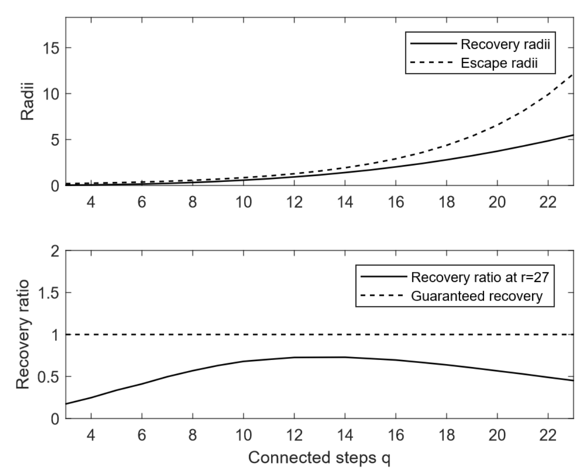


Fig. 6: Comparing the escape and recovery radii growing with visiting steps for sequence II. The ratio never exceeds one .

It is obvious that longer non-connected time results in a smaller recovery ratio. Fig. 7 shows both the recovery ratios at $r = 25$ and $r = 26$. We can tell from this figure that $r = 25$ is the critical number, such that a target being not visited for a longer time than 25 steps, the recovery is not guaranteed.

VI. Conclusion

This paper considers the scenario of a collection of agents moving among a (larger) collection of targets. Each target has

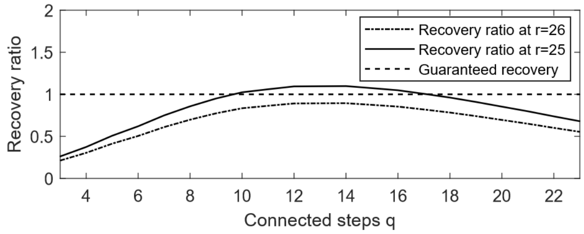


Fig. 7: Comparing recovery ratio at $r = 25$ and $r = 26$. A full recovery cannot be guaranteed for any sequence that requires this target to be not attended for more than 25 steps.

a state evolving according to a discrete time linear system that could be controlled only when being attended by an agent. Under the assumption that a target's dynamics are reachable when being attended to continuously, we establish conditions such that the intermittent nature of the control, due to the need of the agents to move among the targets, does not cause a loss of reachability of the system. We then consider the effect of a finite disturbance and a constrained input in the face of this intermittent control, showing that there is a range of periodicity on the visits to a target that guarantee the state of the system can be driven to a finite ball at the end of each period.

A possible extension to this work is to use the results in a persistent monitoring system with mobile targets. Given a pre-selected sequence of agents visiting mobile targets in the target space, the periodicity will change with the evolving geometry. So long as the target motion is slow enough with respect to the periodicity of the visiting sequence, our results can be applied on a per-cycle basis.

Appendix

A. Calculating $\rho_r^*(q)$ in the general MIMO case

In the main text, we developed the calculation of $\rho_r^*(q)$ and the analysis of its growth rate in the two dimensional, single input setting. In this appendix, we show how to extend those results to the general n-dimensional multiple input setting. We begin with systems whose state matrix is diagonalizable over the reals and then move to systems that need a Jordan form. We first consider the calculation of the recovery radius and then use that to generalize Lemmas 4.1 through 4.4.

Consider, then, an n-dimensional system with matrices

$$A = \operatorname{diag}\left(\lambda_1, \lambda_2, \dots, \lambda_n\right),$$
$$B = \begin{bmatrix} \beta_{1,1}, & \beta_{2,1}, & \cdots & \beta_{m,1} \\ \beta_{1,2}, & \beta_{2,2}, & \cdots & \beta_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{1,n}, & \beta_{2,n}, & \cdots & \beta_{m,n} \end{bmatrix}.$$

As before, we are only interested in the unstable portion of the system and thus we assume $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| > 1$ where, without loss of generality, we have ordered the eigenvalues. Over q continuously connecting steps (with $q > n$), the matrix $\mathfrak{D}(q)$ has qm columns, given by

$$\mathfrak{D}(q) = \left\{ \begin{bmatrix} \beta_{1,1} \\ \beta_{1,2} \\ \vdots \\ \beta_{1,n} \end{bmatrix}, \cdots, \begin{bmatrix} \beta_{m,1} \\ \beta_{m,2} \\ \vdots \\ \beta_{m,n} \end{bmatrix}, \begin{bmatrix} \lambda_1\beta_{1,1} \\ \lambda_2\beta_{1,2} \\ \vdots \\ \lambda_n\beta_{1,n} \end{bmatrix}, \cdots, \right.$$
$$\left. \begin{bmatrix} \lambda_1\beta_{m,1} \\ \lambda_2\beta_{m,2} \\ \vdots \\ \lambda_n\beta_{m,n} \end{bmatrix}, \cdots, \begin{bmatrix} \lambda_1^{q-1}\beta_{1,1} \\ \lambda_2^{q-1}\beta_{1,2} \\ \vdots \\ \lambda_n^{q-1}\beta_{1,n} \end{bmatrix}, \cdots, \begin{bmatrix} \lambda_1^{q-1}\beta_{m,1} \\ \lambda_2^{q-1}\beta_{m,2} \\ \vdots \\ \lambda_n^{q-1}\beta_{m,n} \end{bmatrix} \right\}.$$

We index the columns of $\mathfrak{D}(q)$ with two indices, the first selecting the group of columns with the same power of λ and the second selecting a particular column in that group. For example,

$$\mathfrak{D}(q)(i,j) = \begin{bmatrix} \lambda_1^{i-1}\beta_{j,1} & \lambda_2^{i-1}\beta_{j,2} & \cdots & \lambda_n^{i-1}\beta_{j,n} \end{bmatrix}^T.$$

Notice that $\mathfrak{D}(q+1)$ is formed by appending additional columns to $\mathfrak{D}(q)$. Thus

$$\mathfrak{D}(q+1)(i,j) = \mathfrak{D}(q)(i,j), \quad i = 1, \dots, q,\ j = 1, \dots, m.$$

Similar to the two dimensional setting, the critical direction v_c should be orthogonal to a subset of $n-1$ of the vectors in $\mathfrak{D}(q)$. We again denote the collection of all such possible subsets as $\mathfrak{I}(q)$ and index each of the $\binom{mq}{n-1}$ subsets in this collection as $\mathfrak{I}(q)(\alpha)$. Generically, one of these subsets can be written as the matrix

$$\mathfrak{I}(q)(\alpha) = \begin{bmatrix} \lambda_1^{k_1-1}\beta_{j_1,1} & \dots & \lambda_1^{k_{n-1}-1}\beta_{j_{n-1},1} \\ \lambda_2^{k_1-1}\beta_{j_1,2} & \dots & \lambda_2^{k_{n-1}-1}\beta_{j_{n-1},2} \\ \vdots & \vdots & \vdots \\ \lambda_{n-1}^{k_1-1}\beta_{j_1,n-1} & \dots & \lambda_{n-1}^{k_{n-1}-1}\beta_{j_{n-1},n-1} \\ \lambda_n^{k_1-1}\beta_{j_1,n} & \dots & \lambda_n^{k_{n-1}-1}\beta_{j_{n-1},n} \end{bmatrix}$$

for some choice of $k_1, \dots, k_{n-1}$ and $j_1, \dots, j_{n-1}$. Thus, $\mathcal{I}(q)(\alpha)$ contains $n-1$ columns. Let

$$v_q(\alpha) = \begin{bmatrix} v_{q,1}(\alpha) & v_{q,2}(\alpha) & \cdots & v_{q,n}(\alpha) \end{bmatrix}^T$$

denote a vector that is orthogonal to the $n-1$ vectors in $\mathfrak{I}(q)(\alpha)$. As a specific choice, we set

$$v_{q,i}(\alpha) = \det \mathfrak{I}_i(q)(\alpha) \tag{27}$$

where $\mathfrak{I}_i(q)(\alpha)$ is constructed by eliminating the i^{th} row in $\mathfrak{I}(q)(\alpha)$.

As in the 2-D setting, we look at the sum of the projections of all of the input directions in $\mathfrak{D}(q)$ onto a given $v_q(\alpha)$,

$$\rho_r^\alpha(q) = \sum_{\forall\kappa,\forall j} \|P_{v_q(\alpha)}\left(\mathfrak{D}(q)(\kappa,j)\right)\|,$$

where $P_{v_q(\alpha)}$ is the corresponding projection operator onto $v_q(\alpha)$. The critical direction is the minimum among all these. That is,

$$\rho_r^*(q) = \min_{\forall\alpha} \sum_{\forall\kappa,\forall j} \|P_{v_q(\alpha)}\left(\mathfrak{D}(q)(\kappa,j)\right)\|$$

Now we establish how the conclusion of Theorem 4.1 (bounding the growth rate of the sequence) holds for the n

dimensional case. Following Lemma 4.5, it is easy to show that the growth rate of $\rho_r^*(q)$ is larger than $|\lambda_n|$ (the smallest magnitude eigenvalue). We now establish a dominating sequence $\tilde{\rho}(q)$ and show that the growth rate of the dominating sequence converges to $|\lambda_n|$ as $q \to \infty$.

For any given subset $\mathfrak{I}(q)(\alpha)$, expressed as a matrix as shown above, we define

$$\mathfrak{I}^+(q)(\alpha) = A\mathfrak{I}(q)(\alpha).$$

Note that because every element of $\mathfrak{I}^+(q)$ belongs to the collection $\mathfrak{D}(q+1)$, we have that

$$\mathfrak{I}^+(q)(\alpha) \in \mathfrak{I}(q+1).$$

Let $v_q^+(\alpha)$ denote a vector that is orthogonal to all the vectors in $\mathfrak{I}^+(q)$.

We start building this sequence by randomly choosing one direction form all possible $v_q(\alpha)$ and projecting the inputs of all m directions provided by the input matrix B and through all q steps onto this selected direction. The sum of all the projections is denoted as $\tilde{\rho}_r(q)$, and measures the accumulated effect of the inputs over q steps on this direction. The next entry in this sequence is the summed projections onto the direction of $v_q^+(\alpha)$ of the input directions available over $q+1$ steps, and so on. That is,

$$\tilde{\rho}_r(q) = \sum_{i=1,\dots,q,\forall j} \|P_{v_q(\alpha)}\left(\mathfrak{D}(q)(i,j)\right)\|,$$
$$\tilde{\rho}_r(q+1) = \sum_{i=1,\dots,q+1,\forall j} \|P_{v_q^+(\alpha)}\left(\mathfrak{D}(q+1)(i,j)\right)\|.$$

Obviously for any q we have

$$\rho_r^*(q) \leq \tilde{\rho}_r(q).$$

The growth rate of $\tilde{\rho}_r(q)$ can be calculated as

$$\frac{\tilde{\rho}_r(q+1)}{\tilde{\rho}_r(q)} = \frac{\sum_{i,j}\|P_{v_q^+(\alpha)}\left(\mathfrak{D}(q+1)(i,j)\right)\|}{\sum_{i,jj}\|P_{v_q(\alpha)}\left(\mathfrak{D}(q)(i,j)\right)\|}.$$

For any given i, j we have

$$\begin{aligned}
&\|P_{v_q^+(\alpha)}\left(\mathfrak{D}(q+1)(i+1,j)\right)\| \\
=&\frac{|\lambda_1|^\kappa \beta_{j,1} v_{q,1}^+ + \cdots + |\lambda_n|^\kappa \beta_{j,n} v_{q,n}^+}{\sqrt{\left(v_{q,1}^+\right)^2 + \left(v_{q,2}^+\right)^2 + \cdots + \left(v_{q,n}^+\right)^2}} \\
=&\prod_{i=1}^n |\lambda_i| \frac{|\lambda_1|^{\kappa-1}\beta_{j,1}v_{q,1} + \cdots + |\lambda_n|^{\kappa-1}\beta_{j,n}v_{q,n}}{\sqrt{\left(v_{q,1}^+\right)^2 + \left(v_{q,2}^+\right)^2 + \cdots \left(v_{q,n}^+\right)^2}}.
\end{aligned}$$

Note that to simplify the notation somewhat, we have omitted the dependence of v_q on the index α in the expressions. With this result we calculate

$$\begin{aligned}
&\frac{\|P_{v_q^+(\alpha)}\left(\mathfrak{D}(q+1)(i+1,j)\right)\|}{\|P_{v_q(\alpha)}\left(\mathfrak{D}(q)(i,j)\right)\|} \\
=&\prod_{i=1}^n |\lambda_i| \sqrt{\frac{\left(v_{q,1}\right)^2 + \left(v_{q,2}\right)^2 + \cdots + \left(v_{q,n}\right)^2}{\left(v_{q,1}^+\right)^2 + \left(v_{q,2}^+\right)^2 + \cdots + \left(v_{q,n}^+\right)^2}}
\end{aligned}$$

Using $v_{q,i}^+ = \det \mathfrak{I}_i^+(q)$ yields

$$v_{q,i}^+ = v_{q,i} \prod_{j=1\dots n, j\neq i} |\lambda_j|$$

Since all $v_{q,l}(\alpha) > 1$ and since $|\lambda_1| > |\lambda_2| ... > |\lambda_n|$, we have

$$\begin{aligned}
&\lim_{q\to\infty} \frac{\|P_{v_q^+(\alpha)}\left(\mathfrak{D}(q+1)(\kappa+1,j)\right)\|}{\|P_{v_q(\alpha)}\left(\mathfrak{D}(q)(\kappa,j)\right)\|} \\
=&\prod_{i=1}^n |\lambda_i| \cdot \frac{1}{|\lambda_1| \cdot |\lambda_2| \cdot \ldots \cdot |\lambda_{n-1}|} = |\lambda_n|
\end{aligned}$$

and

$$\frac{\|P_{v_q^+(\alpha)}\left(\mathfrak{D}(q+1)(\kappa+1,j)\right)\|}{\|P_{v_q(\alpha)}\left(\mathfrak{D}(q)(\kappa,j)\right)\|} > |\lambda_n|$$

Therefore we have

$$\lim_{q\to\infty} \frac{\tilde{\rho}_r(q+1)}{\tilde{\rho}_r(q)} = |\lambda_n|$$

and

$$\frac{\tilde{\rho}_r(q+1)}{\tilde{\rho}_r(q)} > |\lambda_n|.$$

The rest of the proof holds as in the two dimensional case.

The analysis above is established on the case of a diagonalizable A. Following a similar discussion using the Jordan form, one can show that

$$\frac{\|P_{v_q^+(\alpha)}\left(\mathfrak{D}(q+1)(\kappa+1,j)\right)\|}{\|P_{v_q(\alpha)}\left(\mathfrak{D}(q)(\kappa,j)\right)\|} > \frac{q}{q+1}|\lambda_n|$$

if A has a repeating eigenvalue. The rest of the proof holds as before since $\frac{q}{q+1}$ goes to 1 as $q \to \infty$.

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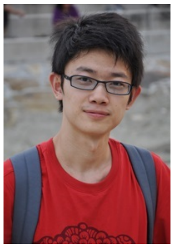

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