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Harnack inequalities in infinite dimensions

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Abstract

We consider the Harnack inequality for harmonic functions with respect to three types of infinite-dimensional operators. For the infinite-dimensional Laplacian, we show no Harnack inequality is possible. We also show that the Harnack inequality fails for a large class of Ornstein–Uhlenbeck processes, although functions that are harmonic with respect to these processes do satisfy an *a priori* modulus of continuity. Many of these processes also have a coupling property. The third type of operator considered is the infinite-dimensional analog of operators in Hörmander's form. In this case a Harnack inequality does hold.

Keywords: Harnack inequality; Abstract Wiener space; Ornstein–Uhlenbeck operator; Coupling; Infinite-dimensional processes

1. Introduction

The Harnack inequality is an important tool in analysis, partial differential equations, and probability theory. For over half a century there has been intense interest in extending the Harnack inequality to more general operators than the Laplacian, with seminal papers by Moser [24] and Krylov and Safonov [21]. See [20] for a survey of some recent work.

It is a natural question to ask whether the Harnack inequality holds for infinite-dimensional operators. If L is an infinite-dimensional operator and h is a function that is non-negative and

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harmonic in a ball with respect to the operator L and B_2 is a ball with the same center as B_1 but of smaller radius, does there exist a constant c depending on B_1 and B_2 but not on h such that

$$h(x) \leqslant ch(y)$$

for all $x, y \in B_2$?

When one considers the infinite-dimensional Laplacian, or alternatively the infinitesimal generator of infinite-dimensional Brownian motion, there is first the question of what one means by a ball. In this case there are two different norms present, one for a Banach space and one for a Hilbert space. We show that no matter what combination of definitions for B_1 and B_2 that are used, no Harnack inequality is possible. Our technique is to use estimates for Green functions for finite-dimensional Brownian motions and then to go from there to the infinite-dimensional Brownian motion.

For more on the potential theory of infinite-dimensional Brownian motion we refer to the classic work of L. Gross [19], as well as to [10,11,15,22,25,26]. V. Goodman [16,17] has several interesting papers on harmonic functions for the infinite-dimensional Laplacian.

We next turn to the infinite-dimensional Ornstein–Uhlenbeck process and its infinitesimal generator. See [13,22,28] for the construction and properties of these processes. In this case, the question of the definitions of B_1 and B_2 is not an issue.

We show that again, no Harnack inequality is possible. We again use estimates for the Green functions of finite-dimensional approximations, but unlike in the Brownian motion case, here the estimates are quite delicate.

We also establish two positive results for a large class of infinite-dimensional Ornstein–Uhlenbeck processes. First we show that functions that are harmonic in a ball are continuous and satisfy an *a priori* modulus of continuity.

Secondly, it is commonly thought that there is a close connection between coupling and the Harnack inequality. See [4] for an example where this connection is explicit. By coupling, we mean that given $B_2 \subset B_1$ with the same center but different radii and $x, y \in B_2$, it is possible to construct two Ornstein–Uhlenbeck processes X and Y started at x, y, respectively (by no means independent), such that the two processes meet (or couple) before either process exits B_1. Even though the Harnack inequality does not hold, we show that for a large class of Ornstein–Uhlenbeck processes it is possible to establish a coupling result.

Finally we turn to the infinite-dimensional analog of operators in Hörmander's form. These are operators of the form

$$Lf(x) = \sum_{j=1}^{n} \nabla_{A_j}^2 f(x),$$

where ∇_{A_j} is a smooth vector field. For these operators we are able to establish a Harnack inequality. To define a ball in this context we use a distance intimately tied to the vector fields $A_1, \dots, A_n$. In addition, we connect this distance to another distance introduced in [9] for Dirichlet forms, and later used in connection with parabolic Harnack inequalities in different settings in [27].

Our technique to prove the Harnack inequality for these operators in Hörmander's form is to employ methods developed by Bakry, Émery, and Ledoux. For general reviews on their approach with applications to functional inequalities see [1,23]. We prove a curvature-dimension inequality, derive a Li–Yau estimate from that, and then prove a parabolic Harnack inequality, from which the usual Harnack inequality follows. For this approach on Riemannian manifolds with Ricci curvature bounded below we refer to [3].

We are not the first to investigate Harnack inequalities for infinite-dimensional operators. In addition to the papers [9] and [8] mentioned above, they have been investigated by Bendikov and Saloff-Coste [7], who studied the related potential theory as well. Their context is quite different from ours, however, as they consider infinite-dimensional spaces which are close to finite-dimensional spaces, such as infinite products of tori. This allows them to modify some of the techniques used for finite-dimensional spaces.

We mention three open problems that we think are of interest:

1. Our positive result is for operators that are the infinite-dimensional analog of Hörmander's form, but we only have a finite number of vector fields. The corresponding processes need not live in any finite-dimensional Euclidean space, but one would still like to allow the possibility of there being infinitely many vector fields.

2. Are there any infinite-dimensional processes of the form Laplacian plus drift for which a Harnack inequality holds?

3. Restricting attention to the infinite-dimensional Ornstein–Uhlenbeck process, can one define B_1 and B_2 in terms of some alternate definition of distance such that the Harnack inequality holds?

The outline of our paper is straightforward. Section 2 considers infinite-dimensional Brownian motion, Section 3 contains our results on infinite Ornstein–Uhlenbeck processes, while our Harnack inequality for operators of Hörmander form appears in Section 4.

We use the letter c with or without subscripts for finite positive constants whose exact value is unimportant and which may change from place to place.

2. Brownian motion

We first prove a proposition that contains the key idea. Let $B^{(n)}(x,r)=\{y\in\mathbb{R}^n\colon |x-y|<r\}$, where $|x-y|=(\sum_{i-1}^n |x_i-y_i|^2)^{1/2}$.

Proposition 2.1. *Let $K>0$. For all n sufficiently large, there exists a function h_n which is non-negative and harmonic on its domain $B^{(n)}(0,1)$ and points $x_n, z_n\in B^{(n)}(0,1/2)$ such that*

$$\frac{h_n(z_n)}{h_n(x_n)}\geqslant K.$$

Proof. Let $G_n(x,y)=|x-y|^{2-n}$, a constant multiple of the Newtonian potential density on $\mathbb{R}^n$. Let $e_1=(1,0,\dots,0)$. If we set $h_n(x)=G_n(x,e_1)$, then it is well known that h_n is harmonic in $\mathbb{R}^n\setminus\{0\}$.

Let $x_n=0$ and $z_n=\frac{1}{4}e_1$. Both are in $B^{(n)}(0,1/2)$ and

$$\frac{h_n(z_n)}{h_n(x_n)}=\frac{(3/4)^{2-n}}{1^{2-n}}\geqslant K$$

if n is sufficiently large. □

Next we embed the above finite-dimensional example into the framework of infinite-dimensional Brownian motion.

Let (W, H, μ) be an abstract Wiener space, where W is a separable Banach space, H is a Hilbert space, and μ is a Gaussian measure. For background about abstract Wiener spaces, see [10] or [22]. We use $\|\cdot\|_H$ and $\|\cdot\|_W$ for the norms on H and W, respectively. We denote the inner product on H by $\langle \cdot, \cdot \rangle_H$.

The classical example of an abstract Wiener space has W equal to the continuous functions on $[0, 1]$ that are 0 at 0 and has H equal to the functions in W that are absolutely continuous and whose derivatives are square integrable. Another example that perhaps better illustrates what follows is to let H be the set of sequences $(x_1, x_2, \ldots)$ such that $\sum_i x_i^2 < \infty$ and let W be the set of sequences such that $\sum_i \lambda_i^2 x_i^2 < \infty$, where $\{\lambda_i\}$ is a fixed sequence with $\sum_i \lambda_i^2 < \infty$.

Let H_* be the set of $h \in H$ such that $\langle \cdot, h \rangle_H \in H^*$ extends to a continuous linear functional on W. Here H^* is the dual space of H, and is, of course, isomorphic to H. (We will continue to denote the continuous extension of $\langle \cdot, h \rangle_H$ to W by $\langle \cdot, h \rangle_H$.)

Next suppose that $P : H \to H$ is a finite rank orthogonal projection such that $PH \subset H_*$. Let $\{e_j\}_{j=1}^n$ be an orthonormal basis for PH and $\ell_j = \langle \cdot, e_j \rangle_H \in W^*$. Then we may extend P to a unique continuous operator from $W \to H$ (still denoted by P) by letting

$$Pw := \sum_{j=1}^{n} \langle w, e_j \rangle_H e_j = \sum_{j=1}^{n} \ell_j(w) e_j \quad \text{for all } w \in W. \tag{2.1}$$

For more details on these projections see [14].

Let $\mathrm{Proj}(W)$ denote the collection of finite rank projections on W such that $PW \subset H_*$ and $P|_H : H \to H$ is an orthogonal projection, i.e. P has the form given in (2.1). As usual a function $f : W \to \mathbb{R}$ is a (smooth) cylinder function if it may be written as $f = F \circ P$ for some $P \in \mathrm{Proj}(W)$ and some (smooth) function $F : \mathbb{R}^n \to \mathbb{R}$, where n is the rank of P. For example, let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis of H such that $e_n \in H_*$, and H_n be the span of $\{e_1, \ldots, e_n\}$ identified with $\mathbb{R}^n$. For each n, define $P_n \in \mathrm{Proj}(W)$ by

$$P_n : W \to H_n \subset H_* \subset H$$

as in (2.1).

For $t \geqslant 0$ let μ_t be the rescaled measure $\mu_t(A) := \mu_t(A/\sqrt{t})$ with $\mu_0 = \delta_0$. Then as was first noted by Gross in [19, p. 135] there exists a stochastic process B_t, $t \geqslant 0$, with values in W which is continuous a.s. in t with respect to the norm topology on W, has independent increments, and for $s < t$ has $B_t - B_s$ distributed as μ_{t-s}, with $B_0 = 0$ a.s. B_t is called standard Brownian motion on (W, μ).

Let $\mathcal{B}(W)$ be the Borel σ-algebra on W. If we set $\mu_t(x, A) := \mu_t(x - A)$, for $A \in \mathcal{B}(W)$, then it is well known that $\{\mu_t\}$ forms a family of Markov transition kernels, and we may thus view $(B_t, \mathbb{P}^x)$ as a strong Markov process with state space W, where $\mathbb{P}^x$ is the law of $x + B$. We do not need this fact in what follows, but want to point out that $B_n(t) := P_n B(t) \in P_n H \subset H \subset W$ give a natural approximation to $B(t)$ as is pointed out in [14, Proposition 4.6].

We denote the open ball in W of radius r centered at $x \in W$ by $B(x, r)$ and its boundary by $S_r(x)$. The first exit time of B_t from $B(0, r)$ will be denoted by τ_r. By [19, Remark 3.3] the exit time τ_r is finite a.s.

A set E is open in the fine topology if for each $x \in E$ there exists a Borel set $E_x \subset E$ such that $\mathbb{P}(\sigma_{E_x} > 0) = 1$, where σ_{E_x} is the first exit from E_x.

Let f be a locally bounded, Borel measurable, finely continuous, real-valued function f whose domain is an open set in W. Then f is harmonic if

$$f(x) = \int_{S_r(0)} f(x+y)\pi_r(dy) \tag{2.2}$$

for any r such that the closure of $B(x, r)$ is contained in the domain of f, where

$$\pi_r(dy) = \mathbb{P}^0(B_{\tau_r} \in dy).$$

Let f be a real-valued function on W. We can consider $F(h) = f(x+h)$ as a function on H. If F has the Fréchet derivative at 0, we say that f is H-differentiable. Similarly we can define the second H-derivative D^2, and finally

$$\Delta f(x) := \operatorname{tr} D^2 f(x)$$

whenever $D^2 f(x)$ exists and of trace class.

The following properties can be found in [16, Theorems 1, 2, 3].

Theorem 2.2. *Let (W, H, μ) be an abstract Wiener space.*

(1) *A harmonic function on W is infinitely H-differentiable. The second derivative of a harmonic function at each point of its domain is a Hilbert–Schmidt operator.*
(2) *If a harmonic function on W satisfies a uniform Lipschitz condition in a neighborhood of a point x, then the Laplacian of u exists at x and $(\Delta u)(x) = 0$.*

Remark 2.3. So far the theory of harmonic functions in infinite dimensions may not seem that different from the finite-dimensional case. There are, however, striking differences. For example, Goodman [16, Proposition 4] shows there exists a harmonic function that is not continuous with respect to the topology of W. In view of the previous theorem, however, it is smooth with respect to the topology of H.

Let (W, H, μ) be an abstract Wiener space. Denote by $G_n(x, z)$ the function on $\mathbb{R}^n \times \mathbb{R}^n$ defined by $G_n(x, z) = |x - z|^{2-n}$. Consider $P_n \in \operatorname{Proj}(W)$ as defined by (2.1), and define the cylinder function $g_n(w) := G_n(P_n w, P_n z)$ for any $w \in W$ and $z = e_1$.

Proposition 2.4. *The function g_n is harmonic on W away from the set $\{w \in W\colon\ P_n w = e_1\} = \{w \in W\colon\ e_1(w) = 1\}$.*

Proof. We need to check that g_n is locally bounded, Borel measurable, finely continuous, and (2.2) holds with f replaced by g_n for all $r > 0$ whenever the closure of $B_r(x)$ is contained in the domain of g_n. One can show that g_n is locally bounded, Borel measurable, and finely continuous similarly to [16, p. 455].

Now we check the last part. Suppose $x \notin \{w \in W\colon\ P_n w = e_1\}$,

$$\int_{S_r(0)} g_n(x+y)\pi_r(dy) = \int_{S_r(0)} G_n \circ P_n(x+y)\pi_r(dy)$$
$$= \mathbb{E}^x\big(G_n \circ P_n(B_{\tau_r})\big)$$
$$= \mathbb{E}^x\big(G_n \circ P_n(P_n B_{\tau_r})\big).$$

Note that $P_n B_t$ is a martingale, and τ_r is a stopping time, and we would like to use the optional stopping time theorem. We need to point out here that $e_1 \in H_* \subset H$ and therefore $P_n e_1 = e_1$. So if we choose $r < 1/2\|e_1\|_{W^*}$, then $e_1 \notin P_n B(0,r)$. Indeed, if there is a $w \in B(0,r)$ such that $P_n w = e_1$, then $e_1(w) = \langle w, e_1\rangle = 1$. But

$$\big|e_1(w)\big| \leqslant \|e_1\|_{W^*}\|w\|_W < r\|e_1\|_{W^*} < \frac{1}{2}$$

which is a contradiction. Thus G_n is harmonic in $P_n B(0,r) \subseteq P_n H \cong \mathbb{R}^n$ and therefore

$$\int_{S_r(0)} g_n(x+y)\pi_r(dy) = G_n(P_n x) = g_n(x). \qquad \square$$

Our main theorem of this section is now simple.

Theorem 2.5. *For each n there exist functions g_n that are non-negative and harmonic in the ball of radius 1 about 0 with respect to the norm of W and points x, z in the ball of radius $1/2$ about 0 with respect to the norm of H such that*

$$\frac{g_n(z)}{g_n(x)} \to \infty$$

as $n \to \infty$. In particular, the Harnack inequality fails.

Proof. We let g_n be as above and $x = 0$ and $z = \frac{1}{4}e_1$ for all n. Our result follows by combining Propositions 2.1 and 2.4. $\square$

3. Ornstein–Uhlenbeck process

Let H be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and corresponding norm $|\cdot|$. Define

$$\|f\|_0 := \sup_{x\in H}\big|f(x)\big|.$$

Recall (see [13]) that for an arbitrary positive trace class operator Q on H and $a \in H$ there exists a unique measure $N_{a,Q}$ on $\mathcal{B}(H)$ such that

$$\int_H e^{i\langle h,x\rangle} N_{a,Q}(dx) = e^{i\langle a,h\rangle - \frac{1}{2}\langle Qh,h\rangle}, \quad h \in H.$$

We call such $N_{a,Q}(dx)$ a Gaussian measure with mean a and covariance Q. It is easy to check that

$$\int_H x N_{a,Q}(dx) = a,$$

$$\int_H |x-a|^2 N_{a,Q}(dx) = \operatorname{Tr} Q,$$

$$\int_H \langle x-a, y\rangle\langle x-a, z\rangle N_{a,Q}(dx) = \langle Qy, z\rangle, \quad \text{and}$$

$$\frac{dN_{b,Q}}{dN_{a,Q}}(dy) = e^{-\frac{1}{2}|Q^{-1/2}(a-b)|^2 + \langle Q^{-1/2}(y-a), Q^{-1/2}(b-a)\rangle}.$$

We consider the Ornstein–Uhlenbeck process in a separable Hilbert space H. The process in question is a solution to the stochastic differential equation

$$dZ_t = -AZ_t\,dt + Q^{1/2}\,dW_t, \quad Z_0 = x, \tag{3.1}$$

where A is the generator of a strongly continuous semigroup e^{-At} on H, W is a cylindrical Wiener process on H, and $Q : H \to H$ is a positive bounded operator. The solution to (3.1) is given by

$$Z_t^x = e^{-At}x + \int_0^t e^{-A(t-s)} Q^{1/2}\,dW_s.$$

The corresponding transition probability is defined as usual by $(P_t f)(x) = \mathbb{E} f(Z_t^x)$, $f \in \mathcal{B}_b(H)$, where $\mathcal{B}_b(H)$ are the bounded Borel measurable functions on H. It is known that the law of Z_t is a Gaussian measure centered at $e^{-At}x$ with covariance

$$Q_t = \int_0^t e^{-A(t-s)} Q e^{-A^*(t-s)}\,ds,$$

which we called $N_{e^{-tA}x, Q_t}(dy)$. Note that for the corresponding parabolic equation in H to be well-posed we need a basic assumption on Q_t to be non-negative and trace-class for all $t > 0$ [13, p. 99].

We assume the controllability condition

$$e^{-At}(H) \subset Q_t^{1/2}(H) \quad \text{for all } t > 0 \tag{3.2}$$

holds. As is described in [13, p. 104], under the condition (3.2) the stochastic differential equation in question has a classical solution. We define

$$\Lambda_t := Q_t^{-1/2} e^{-tA}, \quad t > 0, \tag{3.3}$$

where $Q_t^{-1/2}$ is the pseudo-inverse of $Q_t^{1/2}$. By the closed graph theorem we see that Λ_t is a bounded operator in H for all $t > 0$.

Suppose $Q = I$, the identity operator, and A is a self-adjoint invertible operator on H, then

$$Q_t = \int_0^t e^{-sA} e^{-sA^*} x \, ds = \int_0^t e^{-2sA} \, ds = \frac{1}{2} A^{-1}\bigl(I - e^{-2tA}\bigr), \quad t \geqslant 0.$$

If in addition we assume that A^{-1} is trace-class, then there is an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of H and the corresponding eigenvalues a_n such that

$$Ae_n = a_n e_n, \quad a_n > 0, \ a_n \uparrow \infty, \qquad \sum_{n=1}^{\infty} a_n^{-1} < \infty.$$

Then Q_t is diagonal in the orthonormal basis $\{e_n\}_{n=1}^{\infty}$:

$$Q_t e_n = \frac{t(e^{2ta_n} - 1)}{2ta_n e^{2ta_n}} e_n.$$

Then Q_t is trace class with

$$\operatorname{Tr} Q_t = \sum_{n=1}^{\infty} \frac{t(e^{2ta_n} - 1)}{2ta_n e^{2ta_n}} \leqslant \sum_{n=1}^{\infty} \frac{1}{2a_n} = \frac{\operatorname{Tr} A^{-1}}{2} < \infty.$$

Now we see that

$$\Lambda_t e_n = \frac{\sqrt{2ta_n}}{t^{1/2}\sqrt{e^{2ta_n} - 1}} e_n,$$

and so $|\Lambda_t x| \leqslant |x|/\sqrt{t}$. This proves the following proposition.

Proposition 3.1. *Assume that $Q = I$ and A^{-1} is trace-class. Then the operator Q_t is a trace-class operator on H and $\|\Lambda_t\| \leqslant 1/\sqrt{t}$.*

Using the properties of Gaussian measures, we see that the Ornstein–Uhlenbeck semigroup can be described by the following Mehler formula

$$(P_t f)(x) = \int_H f\bigl(z + e^{-tA}x\bigr) N_{0,Q_t}(dz). \tag{3.4}$$

3.1. Modulus of continuity for harmonic functions

We establish an *a priori* modulus of continuity for harmonic functions.

Lemma 3.2. *Suppose* (3.2) *is satisfied. If* f *is a bounded Borel measurable function on* H *and* $t>0$*, there exists a constant* $c(t)$ *not depending on* f *such that*

$$\left|P_t f(x) - P_t f(y)\right| \leqslant c\|f\|_0 |x-y|, \quad x, y \in H. \tag{3.5}$$

Moreover, for any $u \in H$*,*

$$D_u P_t f(x) \leqslant \left(P_t f^2(x)\right)^{1/2} \|\Lambda_t u\|^2.$$

Proof. Consider $N_{0,Q_t}(dz)$, a centered Gaussian measure with covariance Q_t. By the Cameron–Martin theorem the transition probability

$$P_t^x(dz) = N_{e^{-tA}x, Q_t}(dz)$$

has a density with respect to $N_{0,Q_t}(dz)$ given by

$$J_t(x,z) := \frac{N_{e^{-tA}x, Q_t}(dz)}{N_{0,Q_t}(dz)} = \exp\left(\left\langle \Lambda_t x, Q_t^{-1/2} z\right\rangle - \frac{1}{2}|\Lambda_t x|^2\right). \tag{3.6}$$

Thus

$$(P_t f)(x) = \int_H J_t(x,z) f(z) N_{0,Q_t}(dz). \tag{3.7}$$

Now we can use (3.7) to estimate the derivative $D_u P_t f$ for any $u \in H$, by

$$\begin{aligned}
D_u P_t f(x) &= \int_H \left\langle \Lambda_t u, Q_t^{-1/2}\left(z - e^{-At}x\right)\right\rangle f(z) J_t(x,z) N_{0,Q_t}(dz) \\
&= \int_H \left\langle \Lambda_t u, Q_t^{-1/2} z\right\rangle f\left(z + e^{-At}x\right) N_{0,Q_t}(dz) \\
&\leqslant \left(P_t f^2(x)\right)^{1/2} \left(\int_H \left|\left\langle \Lambda_t u, Q_t^{-1/2} z\right\rangle\right|^2 N_{0,Q_t}(dz)\right)^{1/2} \\
&= \left(P_t f^2(x)\right)^{1/2} \|\Lambda_t u\|^2.
\end{aligned}$$

Note that Λ_t is bounded, therefore for bounded measurable functions f we see that $P_t f$ is uniformly Lipschitz, and therefore strong Feller. □

Assumption 3.3. We now suppose $Q = I$ and that A is diagonal in an orthonormal basis $\{e_n\}_{n=1}^\infty$ of H with eigenvalues a_n being a sequence of positive numbers. Moreover, we assume that $a_n/n^p \to \infty$ for some $p > 3$.

Note that under this assumption A^{-1} is trace-class for $p > 3$, and therefore by Proposition 3.1 the operator Q_t is trace-class as well. We need the following lemma.

Lemma 3.4. *Suppose X_t is an Ornstein–Uhlenbeck process with Q and A satisfying Assumption 3.3. Let $r > q > 0$ and $\varepsilon > 0$. Then there exists t_0 such that*

$$\mathbb{P}^x\Big(\sup_{s\leqslant t_0}|X_s| > r\Big) \leqslant \varepsilon, \quad x \in B(0,q).$$

Proof. We first consider the nth component of X_s. Taking the stopping time τ identically equal to t_0, the main theorem of [18, Theorem 2.5] tells us that

$$\mathbb{E}\sup_{s\leqslant t_0}\big|X_s^n\big| \leqslant \frac{c\sqrt{\log(1+a_n t_0)}}{\sqrt{a_n}}.$$

Then by Chebyshev's inequality,

$$\mathbb{P}\Big(\sup_{s\leqslant t_0}\big|X_s^n\big| \geqslant d_n\Big) \leqslant \frac{c\sqrt{\log(1+a_n t_0)}}{d_n\sqrt{a_n}} \tag{3.8}$$

for any positive real number d_n.

Choose $\delta > 0$ small so that $(p-1)/2 > 1+\delta$. Take $d_n = C(r-q)n^{-1/2-\delta}$, where C is chosen so that $C^2\sum_{n=1}^{\infty} n^{-1-2\delta} = 1$. Then $\mathbb{P}(\sup_{s\leqslant t_0}|X_s^n| \geqslant d_n)$ is summable in n, and if we choose n_0 large enough,

$$\sum_{n=n_0}^{\infty}\mathbb{P}\Big(\sup_{s\leqslant t_0}\big|X_s^n\big| \geqslant d_n\Big) < \varepsilon/2.$$

By taking t_0 smaller if necessary, we then have

$$\sum_{n=1}^{\infty}\mathbb{P}\Big(\sup_{s\leqslant t_0}\big|X_s^n\big| \geqslant d_n\Big) < \varepsilon.$$

Suppose $|x| \leqslant q$ and we start the process at x. By symmetry, we may assume each coordinate of x is non-negative. Since

$$|X_s| \leqslant |X_s - x| + |x|,$$

we observe that in order for the process to exit the ball $B(0,r)$ before time t_0, for some coordinate n we must have $|X_s^n|$ increasing by at least d_n. The probability of this happening is largest when $x_n = 0$. But the probability that for some n we have $|X_s^n|$ increasing by at least d_n in time t_0 is bounded by ε. $\square$

Theorem 3.5. *Suppose X_t is an Ornstein–Uhlenbeck process with Q and A satisfying Assumption 3.3. If h is a bounded harmonic function in the ball $B(0,1)$, there is a constant c such that*

$$\left|h(x)-h(y)\right| \leqslant c\|h\|_0|x-y|, \quad x,y \in B(0,1/2). \tag{3.9}$$

Proof. Let $\varepsilon>0$ and let τ be the exit time from $B(0,1)$. By Lemma 3.4 we can choose t_0 such that

$$\mathbb{P}^x(\tau<t_0)<\varepsilon, \quad x\in B(0,1/2).$$

If h is harmonic in $B(0,1)$ and $x,y\in B(0,1/2)$,

$$h(x)=\mathbb{E}^x h(X_\tau)=\mathbb{E}^x\left[h(X_\tau);\tau<t_0\right]+\mathbb{E}^x\left[h(X_\tau);\tau\geqslant t_0\right].$$

The first term is bounded by $\|h\|_0\varepsilon$. By the Markov property the second term is equal to

$$\mathbb{E}^x\left[\mathbb{E}^{X_{t_0}}h(X_\tau);\tau\geqslant t_0\right]=\mathbb{E}^x\left[h(X_{t_0});\tau\geqslant t_0\right],$$

which differs from $P_{t_0}h(x)$ by at most $\|h\|_0\varepsilon$. We have a similar estimate for $h(y)$. Therefore by Lemma 3.2,

$$\left|h(x)-h(y)\right|\leqslant\left|P_{t_0}h(x)-P_{t_0}h(y)\right|+4\|h\|_0\varepsilon\leqslant c(t_0)|x-y|\|h\|_0+4\|h\|_0\varepsilon.$$

This proves the uniform modulus of continuity. □

Remark 3.6. We remark that the constant c in the statement of Theorem 3.5 depends on r. Moreover, there does not exist a constant c independent of z_0 such that (3.9) holds for $x,y\in B(z_0,r/2)$ when h is harmonic in $B(z_0,r)$. It is not hard to see that this is the case even for the two-dimensional Ornstein–Uhlenbeck process.

3.2. *Counterexample to the Harnack inequality*

As we have seen, the transition probabilities for the Ornstein–Uhlenbeck process Z_t are

$$P_t^x(dz):=N_{e^{-tA}x,Q_t}(dz).$$

Suppose now that $Q=I$ and A satisfy Assumption 3.3 with $p=1$, but also that a_n is an increasing sequence with A^{-1} being a trace-class operator on H. As examples of such a_n, we can take $a_n=n^p$ for $p>1$.

Denote by P_n the orthogonal projection on $H_n:=\operatorname{Span}\{e_1,\dots,e_n\}$. Then

$$P_t^{P_nx}(dP_nz):=p_n(t,P_nx,P_nz)\,dz,$$

where

$$p_n(t,P_nx,P_nz)=\prod_{j=1}^n\left(\frac{1}{2\pi}\frac{2a_j}{1-e^{-2a_jt}}\right)^{1/2}\exp\left(-\frac{2a_j(z_j-e^{-a_jt}x_j)^2}{2(1-e^{-2a_jt})}\right).$$

We would like to consider the Green function h_n with pole at $z_n = 4e_n$ for Z_t killed when Z_t^1 exceeds 6 in absolute value. We use a killed process to insure transience. We will show that

$$\frac{h_n(x_n)}{h_n(0)} \to \infty$$

as $n \to \infty$, where $x_n = e_n$. The key is to estimate the Green function

$$h_n(x,z) := \int_0^\infty \tilde{p}_n(t, P_n x, P_n z)\,dt,$$

where $\tilde{p}_n$ is the density for the killed process. We will prove an upper estimate on $h_n(0, z_n)$ and a lower estimate on $h_n(x_n, z_n)$.

First we need the following lemma.

Lemma 3.7. *Let $a > 0$ and let Y_t be a one-dimensional Ornstein–Uhlenbeck process that solves the stochastic differential equation*

$$dY_t = dB_t - aY_t\,dt,$$

where B_t is a one-dimensional Brownian motion and $a > 0$. Let $\widetilde{Y}$ be Y killed on first exiting $[-6, 6]$, let $q(t,x,y)$ be the transition densities for Y, and let $\tilde{q}(t,x,y)$ be the transition densities for $\widetilde{Y}$.

(1) *There exist constants c and β such that*

$$\tilde{q}(t,0,0) \leqslant ce^{-\beta t}, \quad t \geqslant 1.$$

(2) *We have*

$$\frac{\tilde{q}(t,0,0)}{q(t,0,0)} \to 1$$

as $t \to 0$.

Proof. The transition densities of $\widetilde{Y}$ with respect to the measure $e^{-x^2/2}\,dx$ are symmetric and by Mercer's theorem can be written in the form

$$\sum_{i=1}^\infty e^{-\beta_i t}\varphi_i(x)\varphi_i(y)$$

with $0 < \beta_1 \leqslant \beta_2 \leqslant \beta_3 \leqslant \cdots$. Here the β_i are the eigenvalues and the φ_i are the corresponding eigenfunctions for the Sturm–Liouville problem

$$\begin{cases} Lf(x) = \frac{1}{2}f''(x) - af'(x) = -\beta f(x), \\ f(-6) = f(6) = 0. \end{cases}$$

See [6, Chapter IV, Section 5] for details. (1) is now immediate.

Let U be the first exit of Y from $[-6, 6]$. Using the strong Markov property at U, we have the well-known formula

$$q(t,0,0) = \tilde{q}(t,0,0) + \int_0^t \mathbb{E}^0\big[q(t-s, Y_s, 0); U \in ds\big].$$

Using symmetry, this leads to

$$q(t,0,0) = \tilde{q}(t,0,0) + \int_0^t q(t-s,6,0)\mathbb{P}^0(U \in ds). \tag{3.10}$$

Now by the explicit formula for $q(r,x,y)$, we see that $q(t-s,6,0)$ is bounded in s and t and so the second term on the right-hand side of (3.10) is bounded by a constant times $\mathbb{P}^0(U \leqslant t)$, which tends to 0 as $t \to 0$. On the other hand, $q(t,0,0) \sim (2\pi t)^{-1/2} \to \infty$ as $t \to 0$. (2) now follows by dividing both sides of (3.10) by $q(t,0,0)$. □

We now proceed to an upper estimate for the Green function.

Proposition 3.8. *There are constants $K > 0$ and $c > 0$ such that*

$$h_n(0,z) \leqslant K c^n a_n^{n/2} e^{-16a_n}.$$

Proof. First for $x = 0$ and $z = 4e_n$ we have

$$p_n(t, P_n 0\,, P_n z) = \prod_{j=1}^n \left(\frac{1}{2\pi}\frac{2a_j}{1-e^{-2a_j t}}\right)^{1/2} \exp\left(-\frac{16a_n}{1-e^{-2a_n t}}\right).$$

Step 1. Let t be in the interval $0 < t \leqslant \frac{1}{2a_n} < 1$. Then

$$\prod_{j=1}^n \left(\frac{1}{2\pi}\frac{2a_j}{1-e^{-2a_j t}}\right)^{1/2} \leqslant \left(\frac{1}{t\pi}\right)^{n/2},$$

where we used the fact that a_n is an increasing sequence. For any t we have

$$\frac{16a_n}{1-e^{-2a_n t}} \geqslant \frac{8}{t};$$

therefore for $0 < t < \frac{1}{2a_n}$,

$$p_n(t,0,4e_n) \leqslant e^{-8/t}\left(\frac{1}{t\pi}\right)^{n/2}.$$

The right-hand side has its maximum at $\frac{16}{n}$ which is larger than $\frac{1}{2a_n}$ for all large enough n by our assumptions on Q and A. Thus we can estimate the right-hand side by its value at the endpoint $\frac{1}{2a_n}$:

$$p_n(t,0,4e_n) \leqslant e^{-16a_n}\left(\frac{2a_n}{\pi}\right)^{n/2}, \quad 0<t\leqslant \frac{1}{2a_n}.$$

Step 2. Let t be in the interval $\frac{1}{2a_n}<t\leqslant 1$. Denote by n_0 the index for which $\frac{1}{2a_{n_0+1}}<t\leqslant \frac{1}{2a_{n_0}}$. As before

$$\begin{aligned}
&\prod_{j=1}^{n}\left(\frac{1}{2\pi}\frac{2a_j}{1-e^{-2a_jt}}\right)^{1/2}\exp\left(-\frac{16a_n}{1-e^{-2a_nt}}\right)\\
&\quad\leqslant \left(\frac{1}{t\pi}\right)^{n_0/2}\prod_{j=n_0+1}^{n}\left(\frac{1}{2\pi}\frac{2a_j}{1-e^{-2a_jt}}\right)^{1/2}\exp\left(-\frac{16a_n}{1-e^{-2a_nt}}\right)\\
&\quad\leqslant e^{-16a_n}\left(\frac{1}{t\pi}\right)^{n_0/2}\prod_{j=n_0+1}^{n}\left(\frac{1}{2\pi}\frac{2a_j}{1-e^{-2a_jt}}\right)^{1/2}.
\end{aligned}$$

There is constant c independent of n such that

$$\frac{1}{2\pi}\frac{2a_j}{1-e^{-2a_jt}}\leqslant ca_j\leqslant ca_n, \quad j=n_0+1,\dots,n.$$

Since $1/t<2a_n$, there is a constant c such that

$$\prod_{j=1}^{n}\left(\frac{1}{2\pi}\frac{2a_j}{1-e^{-2a_jt}}\right)^{1/2}\exp\left(-\frac{16a_n}{1-e^{-2a_nt}}\right)\leqslant c^n a_n^{n/2}e^{-16a_n}.$$

Step 3. For $t>1$ the transition density of the killed process can be estimated by

$$\prod_{j=2}^{n}\left(\frac{1}{2\pi}\frac{2a_j}{1-e^{-2a_jt}}\right)^{1/2}\exp\left(-\frac{16a_n}{1-e^{-2a_nt}}\right)e^{-\beta t}$$

for some $\beta>0$, using Lemma 3.7(1). Similarly to Step 2,

$$\tilde{p}(t,0,4e_n)\leqslant c_1^{n-1}a_n^{(n-1)/2}e^{-16a_n}e^{-\beta t}$$

for some constant c_1. Thus we have that there is a constant $c>0$ such that

$$\tilde{p}(t,0,4e_n)\leqslant \begin{cases} c^n a_n^{n/2}e^{-16a_n}, & 0<t<1,\\ c^n a_n^{n/2}e^{-16a_n}e^{-\beta t}, & 1<t.\end{cases}$$

Integrating over t from 0 to ∞ yields the result. □

We now obtain the lower bound for the Green function.

Proposition 3.9. *Let $x = e_n$. There are constants $M > 0$, $c > 0$ and $\varepsilon > 0$ such that*

$$h_n(x, z) \geqslant M c^n e^{-16a_n} a_n^{n/2} \frac{e^{\varepsilon a_n}}{a_n}.$$

Proof. For $x = e_n$ and $z = 4e_n$ we have

$$p_n(t, P_n x, P_n z) = \prod_{j=1}^{n} \left(\frac{1}{2\pi} \frac{2a_j}{1 - e^{-2a_j t}} \right)^{1/2} \exp\left(-\frac{a_n(4 - e^{-a_n t})^2}{(1 - e^{-2a_n t})} \right).$$

Observe that

$$\prod_{j=1}^{n} \left(\frac{1}{2\pi} \frac{2a_j}{1 - e^{-2a_j t}} \right)^{1/2} \geqslant \left(\frac{1}{2\pi t} \right)^{n/2}.$$

Consider t in the interval $[1/a_n, 2/a_n]$. When n is large, $2/a_n \leqslant 1$. Set $v = e^{-a_n t}$, so that $v \in [1/e^2, 1/e]$ when $t \in [1/a_n, 2/a_n]$. Note that

$$16 - \frac{(4 - v)^2}{1 - v^2} > 0$$

for $v \in [0, 8/17] \supset [1/e^2, 1/e]$, so there is a constant $\varepsilon > 0$ such that

$$16 - \frac{(4 - v)^2}{1 - v^2} > \varepsilon, \quad v \in \left[1/e^2, 1/e\right].$$

Thus

$$\exp\left(-\frac{a_n(4 - e^{-a_n t})^2}{(1 - e^{-2a_n t})} \right) \geqslant e^{-16a_n + \varepsilon a_n}.$$

We now apply Lemma 3.7(2) and obtain

$$\begin{aligned} h_n(x, z) &\geqslant \int_{1/a_n}^{2/a_n} \tilde{p}_n(t, P_n x, P_n z)\, dt \\ &\geqslant e^{-16a_n + \varepsilon a_n} c_2^n \int_{1/a_n}^{2/a_n} t^{-n/2}\, dt \\ &= e^{-16a_n + \varepsilon a_n} c_3^n a_n^{n/2 - 1} \left(\frac{1 - 2^{-\frac{n}{2}+1}}{\frac{n}{2} - 1} \right). \end{aligned}$$

Thus we have

$$h_n(x,z) \geqslant Mc^n e^{-16a_n} a_n^{n/2} \frac{e^{\varepsilon a_n}}{a_n}. \qquad \square$$

Theorem 3.10. *Let $K > 0$. There exist functions h_n harmonic and non-negative on $B(0,4)$ and points x_n in $B(0,2)$ such that*

$$\frac{h_n(x_n)}{h_n(0)} \geqslant K$$

for all n sufficiently large. Thus the Harnack inequality does not hold for the Ornstein–Uhlenbeck process.

Proof. The embedding of the finite-dimensional functions h_n into the Hilbert space framework is done similarly to the proof of Theorem 2.5, but is simpler here as there is no Banach space W to worry about. We leave the details to the reader. The theorem then follows by combining Propositions 3.8 and 3.9. $\square$

3.3. Coupling

It is commonly thought that coupling and the Harnack inequality have close connections. Therefore it is interesting that there are infinite-dimensional Ornstein–Uhlenbeck processes that couple even though they do not satisfy a Harnack inequality.

We now consider the infinite-dimensional Ornstein–Uhlenbeck defined as in the previous subsection, but with $a_n = n^p$ and $p = 6$. We have the following theorem. Given a process X, let $\tau_X(r) = \inf\{t\colon |X_t| \geqslant r\}$.

Theorem 3.11. *Let $x_0, y_0 \in B(0,1)$. We can construct two infinite-dimensional Ornstein–Uhlenbeck processes X_t and Y_t such that $X_0 = x_0$ a.s., $Y_0 = y_0$ a.s., and if $\mathbb{P}^{x_0,y_0}$ is the joint law of the pair (X,Y), then*

$$\mathbb{P}^{x_0,y_0}\big(T_C < \tau_X(2) \wedge \tau_Y(2)\big) > 0,$$

where $T_C = \inf\{t\colon X_t = Y_t\}$.

Proof. Let $W_j^X(t)$, $W_j^Y(t)$, $j = 1,2,\dots,$ all be independent one-dimensional Brownian motions. Let

$$dX_t^j = dW_j^X(t) - a_j X_t^j\,dt, \quad X_0^j = x_0^j,$$

and the same for Y_t^j, where we replace dW_j^X by dW_j^Y and x_0 by y_0. Let $T_C^j = \inf\{t\colon X^j(t) = Y^j(t)\}$. We define

$$\overline{Y}^j(t) = \begin{cases} Y^j(t), & t < T_C^j; \\ X^j(t), & t \geqslant T_C^j. \end{cases}$$

Let $\mathbb{P}^x$ be the law of X when starting at x and similarly for $\mathbb{P}^y$. Define $\mathbb{P}^{x^j}$ to be the law of $X^j(t)$ started at x^j and so on. Use Lemma 3.4 to choose t_0 small such that

$$\sup_{x,y\in B(0,1)} \mathbb{P}^{x,y}\big(\tau_X(5/4)\wedge\tau_Y(5/4)\leqslant t_0\big)\leqslant 1/4.$$

Our first step is to show

$$\sum_{j=1}^{\infty}\mathbb{P}^{x^j,y^j}\big(T_C^j>t_0\big)<\infty. \tag{3.11}$$

The law of $X^j_{t_0/2}$ under $\mathbb{P}^{x^j}$ is that of a normal random variable with mean $e^{-a_j t_0/2}x^j$ and variance $(1-e^{-a_j t_0/2})/2a_j$. If A_j^X is the event where $X^j(t_0/2)$ is not in $[-a_j^{-1/4}, a_j^{-1/4}]$, then standard estimates using the Gaussian density show that $\sum_j \mathbb{P}^{x_j}(A_j^X)$ is summable. The same holds if we replace X by Y.

Suppose $|x_j'|, |y_j'|\leqslant a_j^{-1/4}$. Let

$$Z^j(t)=\big(x_j'-y_j'\big)+\big(W_j^X(t)-W_j^Y(t)\big)-a_j\int_0^t Z_j(s)\,ds. \tag{3.12}$$

Now Z^j is again a one-dimensional Ornstein–Uhlenbeck process, but with the Brownian motion replaced by $\sqrt{2}$ times a Brownian motion. Using (3.12) the probability that Z_t does not hit 0 before time $t_0/2$ is less than or equal to the probability that $\sqrt{2}$ times a Brownian motion does not hit 0 before time $t_0/2$. This latter probability is less than or equal to

$$c\big|x_j'-y_j'\big|/\sqrt{t_0/2}\leqslant 2ca_j^{-1/4}/\sqrt{t_0/2},$$

which is summable in j.

Let B_j be the event $(T_C^j>t_0/2)$. We can therefore conclude that if $|x_j'|, |y_j'|\leqslant a_j^{-1/4}$, then $\mathbb{P}^{x_j',y_j'}(B_j)$ is summable in j.

Now use the Markov property at time $t_0/2$ on the event $(A_X^j)^c\cap(A_Y^j)^c$ to obtain

$$\begin{aligned}
&\mathbb{P}^{x_j,y_j}\big(T_C^j>t_0, \big(A_X^j\big)^c\cap\big(A_Y^j\big)^c\big)\\
&\quad=\mathbb{E}^{x_j,y_j}\big[\mathbb{P}^{X_j(t_0/2),Y_j(t_0/2)}\big(T_C^j>t_0/2\big); \big(A_X^j\big)^c\cap\big(A_Y^j\big)^c\big]\\
&\quad\leqslant\Big(\sup_{|x_j'|,|y_j'|\leqslant a_j^{-1/4}}\mathbb{P}^{x_j',y_j'}\big(T_C^j>t_0/2\big)\Big)\mathbb{P}^{x_j,y_j}\big(\big(A_X^j\big)^c\cap\big(A_Y^j\big)^c\big).
\end{aligned}$$

Therefore

$$\mathbb{P}^{x_j,y_j}\big(T_C^j>t_0, \big(A_X^j\big)^c\cap\big(A_Y^j\big)^c\big)$$

is summable in j. Since we already know that $\mathbb{P}^{x_j,y_j}(A^j_X)$ and $\mathbb{P}^{x_j,y_j}(A^j_Y)$ are summable in j, we conclude that (3.11) holds.

Now choose j_0 such that

$$\sum_{j=j_0+1}^{\infty} \mathbb{P}^{x^j,y^j}\big(T^j_C \geqslant t_0\big) < 1/4.$$

Choose ε such that $(1+\varepsilon)^{j_0} \leqslant 5/4$. We will show that there exists a constant c_1 such that for each $j \leqslant j_0$ we have

$$\mathbb{P}^{x^j,y^j}\big(T^j_C < \tau_X(1+\varepsilon) \wedge \tau_Y(1+\varepsilon)\big) \geqslant c_1. \tag{3.13}$$

We know that with probability at least $1/2$, for each $j > j_0$ each pair $(X^j(t), \overline{Y}^j(t))$ couples before (X, Y) exits $B(0, 5/4)$. Once we have (3.13), we know that with probability at least c_1, the pair $(X^j(t), \overline{Y}^j(t))$ couples before exiting $[-1-\varepsilon, 1+\varepsilon]$ for $j \leqslant j_0$. Hence, using independence, with probability at least $c_1^{j_0}$ we have that for all $j \leqslant j_0$, each pair $(X^j(t), \overline{Y}^j(t))$ couples before either $X^j(t)$ or $Y^j(t)$ exits the interval $[-1-\varepsilon, 1+\varepsilon]$. Using the independence again, we have coupling with probability at least $c_1^{j_0}/2$ of X and Y before either exits the ball of radius $\sqrt{2}(5/4) < 2$.

To show (3.13), on the interval $[-1-\varepsilon, 1+\varepsilon]$, the drift term of the Ornstein–Uhlenbeck process is bounded, so by using the Girsanov theorem, it suffices to show with positive probability W^X_j hits W^Y_j before either exits $[-1-\varepsilon, 1+\varepsilon]$. The pair $(W^X_j(t), W^Y_j(t))$ is a two-dimensional Brownian motion started inside the square $[-1, 1]^2$ and we want to show that it hits the diagonal $\{y = x\}$ before exiting the square $[-1-\varepsilon, 1+\varepsilon]^2$ with positive probability. This follows from the support theorem for Brownian motion. See, e.g., [5, Theorem I.6.6]. □

4. Operators in Hörmander form

We let $C_b(H)$ denote the set of bounded continuous functions on H with the supremum norm and $C^n_b(H)$ the space of n times continuously Fréchet differentiable functions with all derivatives up to order n being bounded. $C^{0,1}_b(H)$ will be the space of all Lipschitz continuous functions with

$$\|f\|_{0,1} := \sup_x \big|f(x)\big| + \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|}.$$

Finally, $C^{1,1}_b(H)$ will be the space of Fréchet differentiable functions f with continuous and bounded derivatives such that Df is Lipschitz continuous; we use the norm

$$\|f\|_{1,1} = \|f\|_{0,1} + \sup_{x\neq y} \frac{|Df(x)-Df(y)|_{H^*}}{|x-y|}.$$

Suppose H is a separable Hilbert space, and $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis in H. We set

$$(\partial_j f)(x) := (D_{e_j} f)(x).$$

4.1. Stochastic differential equation

Let $m \geqslant 1$ and suppose $A^1, \dots, A^m$ are bounded maps from H to H. Let $A := (A^1, \dots A^m)$. We assume that

$$a_i^k(x) := \langle A^k(x), e_i \rangle > 0 \quad \text{for any } x \in H, \tag{4.1}$$

and that we have $a_i \in C_b^{1,1}(H)$ with

$$\|A^k\|_{1,1}^2 := \sum_{i=1}^{\infty} \|a_i^k\|_{1,1}^2 < \infty. \tag{4.2}$$

For any $f \in C_b^1(H)$ we define

$$(\nabla_{A^k} f)(x) := \sum_{i=1}^{\infty} a_i^k(x)(\partial_i f)(x),$$
$$(\nabla_A f)(x) := \big((\nabla_{A^1} f)(x), \dots, (\nabla_{A^m} f)(x)\big).$$

Note that

$$\big|(\nabla_{A^k} f)(x)\big|^2 \leqslant \left(\sum_{i=1}^{\infty} \big|a_i^k(x)\big|^2\right)\left(\sum_{i=1}^{\infty} \big|(\partial_i f)(x)\big|^2\right)$$
$$\leqslant \|A^k\|_{1,1}^2 \big|(Df)(x)\big|^2,$$

so $\nabla_{A^k} f$ and $\nabla_A f$ are well defined for $f \in C_b^1(H)$.

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F}_t$, $t \geqslant 0$, satisfying the usual conditions, that is, $\mathcal{F}_0$ contains all null sets in $\mathcal{F}$, and $\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$ for all $t \in [0, T]$. Suppose $W_t = (W_t^1, \dots, W_t^m)$ is a Wiener process on H^m with covariance operator $Q = (Q^1, \dots, Q^m)$. We assume that each Q^k, $k = 1, \dots, m$ is a non-negative trace-class operator on H such that

$$Q^k e_i = \lambda_i^k e_i, \quad \text{with } \lambda_i^k > 0 \quad \text{and} \quad \sum_{i=1}^{\infty} \lambda_i^k = 2, \quad k = 1, \dots, m.$$

We consider a stochastic differential equation such that the infinitesimal generator of the solution is $L = \sum_{k=1}^m (\nabla_{A^k})^2$.

Define $B(x) := (B^1(x), \dots, B^m(x))$, $x \in H$ as a linear operator from H to H^m by

$$\langle B^k(x)h, e_i \rangle := a_i^k(x), \quad \text{for any } h \in H,\ k = 1, \dots, m,$$

and $F : H \to H^m$ by

$$\langle F^k(x), e_i \rangle := \sum_{j=1}^{\infty} a_j^k(x)\partial_j a_i^k(x), \quad k = 1, \dots, m.$$

We can also re-write B and F as

$$B(x)(h_1,\dots,h_m) = A(x), \quad \text{for any } (h_1,\dots,h_m) \in H^m,$$
$$F(x) = \left(\sum_{i=1}^{\infty} \nabla_{A^1} a_i^1(x) e_i, \dots, \sum_{i=1}^{\infty} \nabla_{A^m} a_i^m(x) e_i \right).$$

Theorem 4.1.

(1) *Suppose X_0 is an H^m-valued random variable. Then the stochastic differential equation*

$$X_t = X_0 + \int_0^t B(X_s)\, dW_s^T + \int_0^t F(X_s)\, ds,$$

has a unique solution (up to a.s. equivalence) among the processes satisfying

$$\mathbb{P}\left(\int_0^T |X_t|_{H^m}^2\, dt < \infty \right) = 1.$$

(2) *If in addition $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$, then there is a constant $C_T > 0$ such that*

$$\mathbb{E}|X_t|^2 \leqslant C_T \mathbb{E}|X_0|^2.$$

(3) *Suppose $f \in C_b^2(H)$. Then $v(t,x) := \mathbb{E}(f(X_t^x)) = P_t f(x)$ is in $C_b^{1,2}(H)$ and is the unique solution to the following parabolic equation*

$$\partial_t v(t,x) = Lv, \quad t > 0,\ x \in H^m,$$
$$v(0,x) = f(x),$$

where L is the operator

$$\begin{aligned}
(Lf)(x) &:= \sum_{k=1}^{m} (\nabla_{A^k} \nabla_{A^k} f)(x) \\
&= \sum_{k=1}^{m} \sum_{j=1}^{\infty} a_j^k(x) \partial_j \left(\sum_{i=1}^{\infty} a_i^k(x) \partial_i f(x) \right) \\
&= \sum_{k=1}^{m} \sum_{i,j=1}^{\infty} a_i^k a_j^k \partial_{ij}^2 f(x) + \sum_{k=1}^{m} \sum_{i,j=1}^{\infty} a_j^k(x) \partial_j a_i^k(x) \partial_i f(x), \quad x \in H.
\end{aligned}$$

Proof. For simplicity of notation we take $m = 1$, and write A^1 for A with corresponding functions a_j. The proof for the general case is very similar.

In this case $B(x)$, $x \in H$, is a linear operator on H defined by

$$\langle B(x)h, e_i\rangle := a_i(x), \quad \text{for any } h \in H,$$

and $F : H \to H$ by

$$\langle F(x), e_i\rangle := \sum_{j=1}^{\infty} a_j(x)\partial_j a_i(x),$$

or equivalently $B(x)e_j = A(x)$, $F(x) = \sum_{i,j}^{\infty} a_j(x)\partial_j a_i(x)e_i$.

According to [12, Theorem 7.4], for this stochastic differential equation to have a unique mild solution it is enough to check that

(a) $B(x)(\cdot)$ is a measurable map from H to the space L_2^0 of Hilbert–Schmidt operators from $Q^{1/2}H$ to H;
(b) $\|B(x) - B(y)\|_{L_2^0} \leqslant C|x - y|$, $x, y \in H$;
(c) $\|B(x)\|_{L_2^0}^2 \leqslant K(1 + |x|^2)$, $x \in H$;
(d) F is Lipschitz continuous on H and $|F(x)| \leqslant L(1 + |x|^2)$, $x \in H$.

Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis of H. Then $\{\lambda_j^{1/2}e_j\}_{j=1}^{\infty}$ is an orthonormal basis of $Q^{1/2}H$. First observe that since A is bounded we have

$$\|B(x)\|_{L_2^0}^2 = \sum_{i,j=1}^{\infty} \left|\langle B(x)\lambda_j^{1/2}e_j, e_i\rangle\right|^2,$$

$$\left|A(x)\right|^2 \sum_{j=1}^{\infty} \lambda_j = 2\left|A(x)\right|^2 \leqslant C,$$

and similarly

$$\left\|B(x) - B(y)\right\|_{L_2^0} \leqslant \|A\|_{1,1}|x - y|.$$

The last estimate implies

$$\left\|B(x)\right\|_{L_2^0} \leqslant \max\{C, \left|B(0)\right|\}\left(1 + |x|\right)$$

which proves (a) and (c). We also have

$$\begin{aligned}\left|F(x) - F(y)\right|^2 &= \sum_{i=1}^{\infty}\langle F(x) - F(y), e_i\rangle^2 \\ &= \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_j(x)\partial_j a_i(x) - a_j(y)\partial_j a_i(y)\right)^2\end{aligned}$$

$$\leqslant 2\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\big(a_j(x)-a_j(y)\big)\partial_j a_i(x)\right)^2$$

$$+2\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}a_j(y)\big(\partial_j a_i(x)-\partial_j a_i(y)\big)\right)^2$$

$$\leqslant 2\left(\sum_{j=1}^{\infty}\big(a_j(x)-a_j(y)\big)^2\right)\left(\sum_{i,j=1}^{\infty}\big(\partial_j a_i(x)\big)^2\right)$$

$$+2\left(\sum_{j=1}^{\infty}\big(a_j(y)\big)^2\right)\left(\sum_{i,j=1}^{\infty}\big(\partial_j a_i(x)-\partial_j a_i(y)\big)^2\right).$$

Now we can use our assumptions on A to see that

$$\sum_{j=1}^{\infty}\big(a_j(x)-a_j(y)\big)^2 \leqslant \sum_{j=1}^{\infty}\|a_i\|_{1,1}^2|x-y|^2=\|A\|_{1,1}^2|x-y|^2,$$

$$\sum_{j=1}^{\infty}\big|a_j(y)\big|^2\leqslant\|A\|_{1,1}^2,$$

$$\sum_{i,j=1}^{\infty}\big|\partial_j a_i(x)\big|^2=\sum_{i=1}^{\infty}\big|Da_i(x)\big|^2\leqslant\|A\|_{1,1}^2,\quad\text{and}$$

$$\sum_{i,j=1}^{\infty}\big(\partial_j a_i(x)-\partial_j a_i(y)\big)^2=\sum_{i=1}^{\infty}\big|Da_i(x)-Da_i(y)\big|^2$$

$$\leqslant\sum_{i=1}^{\infty}\|a_i\|_{1,1}^2|x-y|^2\leqslant\|A\|_{1,1}^2|x-y|^2,$$

which gives Lipschitz continuity for F. Finally the estimate for $|F(x)|$ follows from the Lipschitz continuity of F together with boundedness of A in a similar fashion to what we did for B.

Assertion (2) follows directly from [12, Theorem 9.1]. Assertion (3) follows from [12, Theorem 9.16] which says that $P_t f$ is the solution to the parabolic type equation with operator

$$Lv=\frac{1}{2}\operatorname{tr}v_{xx}\big(B(x)Q^{1/2},B(x)Q^{1/2}\big)+\langle v_x,F(x)\rangle$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}v_{xx}\big(B(x)Q^{1/2}e_n,B(x)Q^{1/2}e_n\big)+\left\langle v_x,\sum_{i,j}^{\infty}a_j(x)\partial_j a_i(x)e_i\right\rangle$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\lambda_n v_{xx}\left(\sum_{i=1}^{\infty}a_i(x)e_i,\sum_{j=1}^{\infty}a_j(x)e_j\right)+\sum_{i,j}^{\infty}a_j(x)\partial_j a_i(x)\langle v_x,e_i\rangle$$

$$= \sum_{i,j=1}^{\infty} a_i(x)a_j(x)v_{xx}(e_i,e_j) + \sum_{i,j}^{\infty} a_j(x)\partial_j a_i(x)\langle v_x, e_i\rangle$$

$$= \sum_{i,j=1}^{\infty} a_i(x)a_j(x)\partial_{ij}^2 v + \sum_{i,j}^{\infty} a_j(x)\partial_j a_i(x)\partial_i v. \qquad \square$$

Remark 4.2. Denote

$$L^k f := \nabla_{A^k}^2 f = \sum_{i,j=1}^{\infty} a_i^k(x)a_j^k(x)\partial_{ij}^2 f + \sum_{i,j}^{\infty} a_j^k(x)\partial_j a_i^k(x)\partial_i f,$$

where $k = 1, \dots, m$. Suppose $f \in C_b^2(H)$. Then

$$\begin{aligned}
\left|\left(L^k f\right)(x)\right|^2 &\leqslant \sum_{i,j=1}^{\infty} \left|a_i^k a_j^k(x)\right|^2 \sum_{i,j=1}^{\infty} \left|\partial_{ij}^2 f(x)\right|^2 \\
&\quad + \sum_{j=1}^{\infty} \left|a_j^k(x)\right|^2 \sum_{j=1}^{\infty} \left|\sum_{i=1}^{\infty} \partial_j a_i^k(x)\partial_i f(x)\right|^2 \\
&\leqslant \left\|A^k\right\|_{1,1}^4 \|f\|_2^2 + \left\|A^k\right\|_{1,1}^2 \sum_{i,j=1}^{\infty} \left|\partial_j a_i^k(x)\right|^2 \sum_{i=1}^{\infty} \left|\partial_i f(x)\right|^2 \\
&\leqslant 2\left\|A^k\right\|_{1,1}^4 \|f\|_2^2,
\end{aligned}$$

and therefore L^k is well defined on $C_b^2(H)$, and so is $L = \sum_{k=1}^m L_k$.

4.2. *Curvature-dimension inequality*

We can write

$$L = \sum_{k=1}^{m} L_k = \sum_{k=1}^{m} \nabla_{A^k}^2.$$

For any $f, g \in C_b^2(H)$ we define

$$\Gamma(f,g) := \frac{1}{2}\left(L(fg) - fL(g) - gL(f)\right), \tag{4.3}$$

$$\Gamma_2(f) := \frac{1}{2}L\left(\Gamma(f,f)\right) - \Gamma(f, Lf). \tag{4.4}$$

Theorem 4.3. *For any $f, g \in C_b^2(H)$,*

$$\Gamma(f,g) = \sum_{k=1}^{m} (\nabla_{A^k} f)(\nabla_{A^k} g), \tag{4.5}$$

$$\Gamma_2(f) = \sum_{k,l=1}^{m} \Gamma^{(k)}(\nabla_{A^l} f), \tag{4.6}$$

where

$$\Gamma^{(k)}(f) := (\nabla_{A^k} f)^2.$$

Proof. Note that for functions $f, g \in C_b^2(H)$,

$$\begin{aligned} L_k(fg) &= f L_k(g) + g L_k(f) + 2\left(\sum_i a_i^k \partial_i f\right)\left(\sum_j a_j^k \partial_j g\right) \\ &= f L_k(g) + g L_k(f) + 2(\nabla_{A^k} f)(\nabla_{A^k} g), \end{aligned} \tag{4.7}$$

and therefore

$$L(fg) = f L(g) + g L(f) + 2\sum_{k=1}^{m} (\nabla_{A^k} f)(\nabla_{A^k} g). \tag{4.8}$$

Hence

$$\Gamma(f, g) = \frac{1}{2}\bigl(L(fg) - f L(g) - g L(f)\bigr) = \sum_{k=1}^{m} (\nabla_{A^k} f)(\nabla_{A^k} g),$$

and in particular $\Gamma(f) := \Gamma(f, f) = \sum_{k=1}^{m} (\nabla_{A^k} f)^2$. Before we find $\Gamma_2(f)$ we need the following calculation:

$$\begin{aligned} [L_k, \partial_i] &:= (L_k \partial_i - \partial_i L_k) f \\ &= \sum_{jm} \bigl(a_j^k \partial_j a_m\bigr) \partial_{im}^2 f + \sum_{jm} a_j^k a_m^k \partial_{ijm}^3 f \\ &\quad - \partial_i \left(\sum_{jm} a_j^k \partial_j a_m^k \partial_m f + \sum_{jm} a_j^k a_m^k \partial_{jm}^2 f \right) \\ &= -\sum_{jm} \bigl(\partial_i a_j^k \partial_j a_m^k + a_j^k \partial_{ij}^2 a_m^k\bigr) \partial_m f - 2 \sum_{jm} \bigl(a_m^k \partial_i a_j^k\bigr) \partial_{jm}^2 f. \end{aligned} \tag{4.9}$$

Use (4.9) to see that

$$\begin{aligned} \sum_i a_i^l \bigl([L_k, \partial_i] f\bigr) &= -\sum_m \left(\sum_{ij} \bigl(a_i^l \partial_i a_j^l \partial_j a_m^l + a_i^l a_j^l \partial_{ij}^2 a_m^l\bigr) \right) \partial_m f \\ &\quad - 2 \sum_{ijm} \bigl(a_i^l a_m^l \partial_i a_j^l\bigr) \partial_{jm}^2 f \\ &= -\sum_m \bigl(L_k a_m^l\bigr) \partial_m f - 2 \sum_{ijm} \bigl(a_i^l a_m^l \partial_i a_j^l\bigr) \partial_{jm}^2 f \end{aligned}$$

$$= -\sum_m \bigl(L_k a^l_m\bigr)\partial_m f - 2\sum_j \Bigl(\sum_i a^l_i \partial_i a^l_j\Bigr)\Bigl(\sum_m a^l_m \partial^2_{mj} f\Bigr)$$
$$= -\sum_m \bigl(L_k a^l_m\bigr)\partial_m f - 2\sum_j \bigl(\nabla_{A^l} a^l_j\bigr)(\nabla_{A^l}\partial_j f). \tag{4.10}$$

Now we can deal with $\Gamma_2(f)$. We use (4.8) in the first line.

$$\frac{1}{2}L\bigl(\Gamma(f)\bigr) = \frac{1}{2}\sum_{k=1}^m L_k\bigl(\Gamma(f)\bigr) = \frac{1}{2}\sum_{k=1}^m L_k\Biggl(\sum_{l=1}^m (\nabla_{A^l} f)^2\Biggr)$$
$$= \sum_{k,l=1}^m \bigl((\nabla_{A^l} f)(L_k \nabla_{A^l} f) + \Gamma^{(k)}(\nabla_{A^l} f)\bigr).$$

The second term in $\Gamma_2(f)$ is

$$\Gamma(f, Lf) = \sum_{l=1}^m (\nabla_{A^l} f)(\nabla_{A^l} Lf) = \sum_{k,l=1}^m (\nabla_{A^l} f)(\nabla_{A^l} L_k f).$$

Thus

$$\Gamma_2(f) = \sum_{k,l=1}^m (\nabla_{A^l} f)\bigl([L_k, \nabla_{A^l}]f\bigr) + \sum_{k,l=1}^m \Gamma_k(\nabla_{A^l} f).$$

By (4.7) we have

$$[L_k, \nabla_{A^l}]f = L_k\Biggl(\sum_{j=1}^\infty a^l_j \partial_j f\Biggr) - \sum_{j=1}^\infty a^l_j \partial_j L_k f$$
$$= \sum_{j=1}^\infty L_k\bigl(a^l_j\bigr)\partial_j f + \sum_{j=1}^\infty a^l_j L_k \partial_j f + 2\sum_{j=1}^\infty \bigl(\nabla_{A^k} a^l_j\bigr)(\nabla_{A^k}\partial_j f)$$
$$- \sum_{j=1}^\infty a^l_j \partial_j L_k f$$
$$= \sum_{j=1}^\infty L_k\bigl(a^l_j\bigr)\partial_j f + \sum_{j=1}^\infty a^l_j [L_k, \partial_j] f + 2\sum_{j=1}^\infty \bigl(\nabla_{A^k} a^l_j\bigr)(\nabla_{A^k}\partial_j f).$$

We can use (4.10) to see that $[L_k, \nabla_{A^l}]f = 0$ for $k, l = 1, \ldots, m$. Thus (4.6) holds. □

Corollary 4.4. *L satisfies the curvature-dimension inequality* CD$(0, m)$

$$\Gamma_2(f) \geqslant \frac{1}{m}(Lf)^2. \tag{4.11}$$

Moreover, for $m = 1$ *we have* $\Gamma_2(f) = (Lf)^2$.

Proof. Note that by the Cauchy–Schwarz inequality

$$\sum_{k,l=1}^{m} \Gamma_k(\nabla_{A^l} f) = \sum_{k,l=1}^{m} (\nabla_{A^k}\nabla_{A^l} f)^2 \geqslant \frac{1}{m}\left(\sum_{k=1}^{m} \nabla^2_{A^k} f\right)^2 = \frac{1}{m}(Lf)^2.$$

Therefore

$$\Gamma_2(f) \geqslant \sum_{k,l=1}^{m} (\nabla_{A^l} f)\big([L_k, \nabla_{A^l}]f\big) + \frac{1}{m}(Lf)^2. \qquad \square$$

We need chain rules for the operators Γ and Γ_2.

Proposition 4.5. *Let Ψ be a C^∞ function on $\mathbb{R}$ and suppose f is in the domain of L. Then*

$$L\Psi(f) = \Psi'(f)Lf + \Psi''(f)\Gamma(f, f), \tag{4.12}$$

$$\Gamma\big(\Psi(f), g\big) = \Psi'(f)\Gamma(f, g), \tag{4.13}$$

$$\begin{aligned}\Gamma_2\big(\Psi(f)\big) &= \big(\Psi''(f)\big)^2\big(\Gamma(f)\big)^2 + \big(\Psi'(f)\big)^2\Gamma_2(f) \\ &\quad + \Psi'(f)\Psi''(f)\Gamma\big(f, \Gamma(f)\big).\end{aligned} \tag{4.14}$$

Proof. Suppose $\Psi \in C^\infty(\mathbb{R})$. Recall that we can write L as $Lf = \sum_{k=1}^m L_k = \sum_{k=1}^m \nabla^2_{A^k} f$, where $\nabla_{A^k} f := \sum_{i=1}^\infty a_i^k \partial_i f$. It is clear that

$$\nabla_{A^k}\big(\Psi(f)\big) = \Psi'(f)\nabla_{A^k} f. \tag{4.15}$$

Then

$$\begin{aligned}\nabla_{A^k}\nabla_{A^k}\big(\Psi(f)\big) &= \nabla_{A^k}\big(\Psi'(f)\big)\nabla_{A^k} f + \Psi'(f)\nabla_{A^k}(\nabla_{A^k} f) \\ &= \Psi''(f)(\nabla_{A^k} f)^2 + \Psi'(f)\nabla_{A^k}(\nabla_{A^k} f) \\ &= \Psi'(f)L_k f + \Psi''(f)\Gamma_k(f),\end{aligned}$$

which implies (4.12) by Theorem 4.3.

Now we can easily show (4.13). Indeed, using (4.15) we have

$$\begin{aligned}\Gamma_k\big(\Psi(f), g\big) &= \big(\nabla_{A^k}\Psi(f)\big)(\nabla_{A^k} g) \\ &= \Psi'(f)(\nabla_{A^k} f)(\nabla_{A^k} g) = \Psi'(f)\Gamma_k(f, g).\end{aligned}$$

In particular, (4.13) implies

$$\Gamma\big(\Psi(f)\big) = \big(\Psi'(f)\big)^2\Gamma(f).$$

Now we would like to prove (4.14). First, using (4.13) twice we see that

$$\Gamma\big(\Psi(f)\big) = \big(\Psi'(f)\big)^2 \Gamma(f). \tag{4.16}$$

By (4.8) and (4.12)

$$\begin{aligned}\frac{1}{2}L\Gamma\big(\Psi(f)\big) &= \frac{1}{2}\Gamma(f)L\big(\big(\Psi'(f)\big)^2\big) + \frac{1}{2}\big(\Psi'(f)\big)^2 L\Gamma(f) + \Gamma\big(\big(\Psi'(f)\big)^2, \Gamma(f)\big)\\ &= \Psi'(f)\Psi''(f)(Lf)\Gamma(f) + \big(\big(\Psi''(f)\big)^2 + \Psi'(f)\Psi'''(f)\big)\big(\Gamma(f)\big)^2\\ &\quad + \frac{1}{2}\big(\Psi'(f)\big)^2 L\Gamma(f) + 2\Psi'(f)\Psi''(f)\Gamma\big(f, \Gamma(f)\big).\end{aligned}$$

Now use (4.8) and (4.14) repeatedly to obtain

$$\begin{aligned}\Gamma\big(\Psi(f), L\Psi(f)\big) &= \Gamma\big(\Psi(f), \Psi'(f)Lf\big) + \Gamma\big(\Psi(f), \Psi''(f)\Gamma(f)\big)\\ &= \big(\Psi'(f)\big)^2 \Gamma(f, Lf) + \Psi'(f)\Psi''(f)(Lf)\Gamma(f)\\ &\quad + \Psi'(f)\Psi''(f)\Gamma\big(f, \Gamma(f)\big) + \Psi'(f)\Psi'''(f)\big(\Gamma(f)\big)^2.\end{aligned}$$

Note that we also used the fact that

$$\Gamma(f, gh) = g\Gamma(f, h) + h\Gamma(f, h).$$

Combining these two calculations gives (4.14). □

Corollary 4.6. *By (4.14) with $\Psi(x) = \log x$, $x > 0$, and $g > 0$ we see that*

$$\Gamma_2(\log g) = \frac{(\Gamma(g))^2}{g^4} - \frac{\Gamma(g, \Gamma(g))}{g^3} + \frac{\Gamma_2(g)}{g^2}. \tag{4.17}$$

4.3. Li–Yau estimate

The following is the Li–Yau estimate in our context. In this proof we follow an argument in [2], which they used to prove a finite-dimensional logarithmic Sobolev inequality for heat kernel measures.

Theorem 4.7.

$$L(\log P_t f) > -\frac{1}{2t}. \tag{4.18}$$

Proof. By (4.13) with $\Psi(x) = \log x$, $x > 0$, $f > 0$, and $0 \leqslant s \leqslant t$,

$$\Gamma(P_{t-s}f) := \Gamma(P_{t-s}f, P_{t-s}f) = (P_{t-s}f)^2 \Gamma(\log P_{t-s}f).$$

Define for $f > 0$,

$$\varphi(s) := P_s\big(P_{t-s} f\,\Gamma(\log P_{t-s} f)\big) = P_s\left(\frac{\Gamma(P_{t-s} f)}{P_{t-s} f}\right).$$

Then with $g := P_{t-s} f$ and $\partial_s g = -Lg$ we see that by (4.12) and (4.13),

$$\begin{aligned}
\varphi'(s) &= \partial_s\left(P_s\left(\frac{\Gamma(g)}{g}\right)\right)\\
&= P_s\left(L\left(\frac{\Gamma(g)}{g}\right) - \frac{2\Gamma(g, Lg)}{g} + \frac{\Gamma(g)Lg}{g^2}\right)\\
&= P_s\left(L\Gamma(g)g + \Gamma(g)L\left(\frac{1}{g}\right) + 2\Gamma\left(\Gamma(g), \frac{1}{g}\right) - \frac{2\Gamma(g, Lg)}{g} + \frac{\Gamma(g)Lg}{g^2}\right)\\
&= P_s\left(\Gamma(g)\left(\frac{2\Gamma(g)}{g^3} - \frac{Lg}{g^2}\right) - \frac{2\Gamma(\Gamma(g), g)}{g^2} + \frac{L\Gamma(g) - 2\Gamma(g, Lg)}{g} + \frac{\Gamma(g)Lg}{g^2}\right)\\
&= 2P_s\left(\frac{(\Gamma(g))^2}{g^3} - \frac{\Gamma(g, \Gamma(g))}{g^2} + \frac{\Gamma_2(g)}{g}\right) = 2P_s\big(g\Gamma_2(\log g)\big)
\end{aligned}$$

by (4.17). We use the curvature-dimension inequality (4.11) to obtain

$$\varphi'(s) \geqslant \frac{2}{m} P_s\big(g(L\log g)^2\big). \tag{4.19}$$

In particular, this means that φ is non-decreasing, and therefore

$$\varphi(0) = P_t f\,\Gamma(\log P_t f) \leqslant P_t\big(f\,\Gamma(\log f)\big) = \varphi(t).$$

Using the chain rule (4.13) we get

$$P_t f\,\Gamma(\log P_t f) = \frac{\Gamma(P_t f)}{P_t f} \leqslant P_t\left(\frac{\Gamma(f)}{f}\right) = P_t\big(f\,\Gamma(\log f)\big).$$

This inequality together with (4.12) gives

$$P_t f\,L(\log P_t f) = LP_t f - \frac{\Gamma(P_t f)}{P_t f} \geqslant LP_t f - P_t\left(\frac{\Gamma(f)}{f}\right) = P_t\big(f\,L(\log f)\big).$$

Thus

$$P_t f\,L(\log P_t f) \geqslant P_t\big(f\,L(\log f)\big). \tag{4.20}$$

We need more information about φ to complete the proof. Our expression for φ' can be rewritten using the chain rule (4.12) as

$$\varphi'(s) = P_s\big(g(L\log g)^2\big) = P_s\left(\frac{1}{g}\left(Lg - \frac{\Gamma(g)}{g}\right)^2\right).$$

Note that since $g > 0$ we have

$$P_s\left(Lg - \frac{\Gamma(g)}{g}\right) = P_s\left(\sqrt{g}\left(\frac{1}{\sqrt{g}}\left(Lg - \frac{\Gamma(g)}{g}\right)\right)\right)$$
$$\leqslant (P_s g)^{1/2}\left(P_s\left(\frac{1}{g}\left(Lg - \frac{\Gamma(g)}{g}\right)^2\right)\right)^{1/2},$$

so

$$P_s\left(\frac{1}{g}\left(Lg - \frac{\Gamma(g)}{g}\right)^2\right) \geqslant \frac{(P_s(Lg - \frac{\Gamma(g)}{g}))^2}{P_s g}.$$

Since $\varphi(s) = P_s(\frac{\Gamma(g)}{g})$, the last estimate becomes

$$\varphi'(s) \geqslant 2\frac{(P_s Lg - \varphi(s))^2}{P_s g}.$$

Now use the definition of g and the fact that L and P_s commute to see that $P_s g = P_t f$, so we have that for $0 \leqslant s \leqslant t$,

$$\varphi'(s) \geqslant 2\frac{(LP_t f - \varphi(s))^2}{P_t f} = 2\frac{(\varphi(s) - LP_t f)^2}{P_t f}.$$

Thus for all s such that $\varphi'(s) > 0$ we have

$$-\partial_s\left(\frac{1}{\varphi(s) - LP_t f}\right) \geqslant \frac{2}{P_t f} > 0.$$

By (4.19) we know that $\varphi'(s) \geqslant 0$, and by integrating this estimate from 0 to t, we obtain

$$\frac{1}{\varphi(0) - LP_t f} - \frac{1}{\varphi(t) - LP_t f} \geqslant \frac{2t}{P_t f}.$$

That is,

$$\frac{\varphi(t) - \varphi(0)}{(\varphi(0) - LP_t f)(\varphi(t) - LP_t f)} \geqslant \frac{2t}{P_t f} > 0.$$

Since φ is non-decreasing, the numerator on the left is non-negative. Since the right-hand side of the estimate is positive, no matter what the sign of the denominator on the left, the following estimate holds:

$$\varphi(t) - \varphi(0) \geqslant \frac{2t}{P_t f}\big(\varphi(0) - LP_t f\big)\big(\varphi(t) - LP_t f\big).$$

Similarly to the proof of (4.20)

$$\varphi(0) - LP_t f = \frac{\Gamma(P_t f)}{P_t f} - LP_t f = -P_t f L(\log P_t f),$$
$$\varphi(t) - LP_t f = P_t\left(\frac{\Gamma(f)}{f}\right) - LP_t f = -P_t\big(f L(\log f)\big).$$

Finally we have

$$P_t f L(\log P_t f) \geqslant P_t\big(f L(\log f)\big)\big(1 + 2t L(\log P_t f)\big). \tag{4.21}$$

Now we are ready to prove (4.18). We only need to check (4.18) when $L(\log P_t f) < 0$. In this case, by (4.20)

$$P_t\big(f L(\log f)\big) < 0,$$

and therefore (4.21) implies

$$1 + 2t L(\log P_t f) > 0. \qquad \square$$

Corollary 4.8. *For $f > 0$,*

$$-\partial_t(\log P_t f) < \frac{1}{2t} - \Gamma(\log P_t f).$$

Proof. By (4.12) and (4.16),

$$\begin{aligned} L(\log P_t f) &= \frac{LP_t f}{P_t f} - \frac{\Gamma(P_t f)}{(P_t f)^2} \\ &= \frac{\partial_t P_t f}{P_t f} - \Gamma(\log P_t f) \\ &= \partial_t(\log P_t f) - \Gamma(\log P_t f) > -\frac{1}{2t}. \qquad \square \end{aligned}$$

4.4. Distances

For the purposes of the next subsection we need to introduce several distances related to the gradient ∇_A. A natural distance as described in [1] is:

$$d(x, y) := \sup_{\{f:\, \Gamma(f) \leqslant 1\}} \big(f(y) - f(x)\big), \quad x, y \in H.$$

We will need another distance which is better suited for the proof of the parabolic Harnack inequality, and it will turn out that this distance is equal to the one we have just defined. First we note that for any $x \in H$ there is a smooth path $\gamma_A : [0, \infty) \to H^m$ (possibly defined only on a finite subinterval $[0, T]$ of $\mathbb{R}_+$) such that

$$\dot{\gamma}_A(t) = A\big(\gamma_A(t)\big), \qquad \gamma_A(0) = x. \tag{4.22}$$

This is equivalent to solving a system of ordinary differential equations, which gives γ_A implicitly as the solution to

$$x_j + \int \frac{d\gamma_j}{a_j(\gamma)} = t.$$

Using the assumption that $a_j > 0$, we can determine γ_A as a function of t.

An admissible component of x is defined as

$$V_A(x) := \{\gamma_A(s), \text{ where } s \in [0, T], \ \dot{\gamma_A}(s) = A(\gamma_A(s)), \ \gamma_A(0) = x\}$$

as described by (4.22).

Example 4.9. Suppose $a_j(x) = c_j$. Then γ is a straight line, and so V_A is a straight line through x in the direction of $(c_1, c_2, \ldots)$. In particular, if $H = \mathbb{R}^2$, and $a_1(x) = 1$ and $a_2(x) = 0$, then V_A is a horizontal line through x.

Definition 4.10. Let $x \in H$, and define

$$d_{\text{arc}}(x, y) := \begin{cases} T_y, & y \in V_A(x); \\ +\infty, & y \notin V_A(x), \end{cases}$$

where the path γ_A is described by (4.22) with $\gamma_A(T_y) = y$.

Remark 4.11. Note that our assumptions on A are essential for the definition of the distance function d_{arc} as we use the ordinary differential equations (4.22) to find γ_A.

Theorem 4.12. *For any $x, y \in H$,*

$$d(x, y) = d_{\text{arc}}(x, y).$$

Proof. Fix $x \in H$. We will consider the case when $d_{\text{arc}}(x, y) = \infty$ or $d(x, y) = \infty$ later, so for now we assume that both distances are finite.

Let γ be any path connecting x and y with $\gamma(s) = y$. Note that since $d_{\text{arc}}(x, y) < \infty$, we have $y \in V_A(x)$. Then

$$d(x, y) = \sup_{\{f:\, \Gamma(f) \leqslant 1\}} \big(f(y) - f(x)\big) = \sup_{\{f:\, \Gamma(f) \leqslant 1\}} \int_0^s \langle \nabla f|_{\gamma(t)}, \dot{\gamma}(t)\rangle \, dt. \tag{4.23}$$

Choosing f_A such that $\nabla f_A = \frac{A}{|A|^2}$, then

$$\Gamma(f_A) = |\nabla_A f_A|^2 = \langle \nabla f_A, A\rangle^2 = 1,$$

and therefore for the function f_A,

$$d(x,y) \geqslant f_A(y) - f_A(x) = \int_0^{T_y} \langle \nabla f_A, \dot{\gamma_A}(t) \rangle \, dt = \int_0^{T_y} 1 \, dt = T_y = d_{\mathrm{arc}}(x,y).$$

Again, by (4.23),

$$\begin{aligned} d(x,y) &= \sup_{\{f:\, \Gamma(f) \leqslant 1\}} \int_0^{T_y} \langle \nabla f|_{\gamma_A(t)}, \dot{\gamma_A}(t) \rangle \, dt \\ &= \sup_{\{f:\, \Gamma(f) \leqslant 1\}} \int_0^{T_y} \langle \nabla f|_{\gamma_A(t)}, \gamma_A(\gamma(t)) \rangle \, dt \\ &= \sup_{\{f:\, \Gamma(f) \leqslant 1\}} \int_0^{T_y} \nabla_A f|_{\gamma_A(t)} \, dt \leqslant \int_0^{T_y} 1 \, dt = d_{\mathrm{arc}}(x,y). \end{aligned} \tag{4.24}$$

Finally we want to show that both distances are infinite for the same y. Define a function

$$f_N(z) := \begin{cases} 0, & z \in V_A(x); \\ N, & z \notin V_A(x) \end{cases}$$

for some N. Note that $\Gamma(f_N) = 0$. Suppose $d_{\mathrm{arc}}(x,y) = \infty$, so $f_N(y) = N$. Then

$$d(x,y) \geqslant f_N(y) - f_N(x) = N.$$

By taking $N \to \infty$ we see that $d(x,y) = +\infty$.

Next suppose that $d(x,y) = \infty$. Then there are functions f_N with $\Gamma(f_N) \leqslant 1$ such that $f_N(y) - f_N(x) \to +\infty$ as $N \to \infty$. Similarly to (4.24) (if we assume that $d_{\mathrm{arc}}(x,y) < \infty$ to find γ_A) we see that

$$+\infty = \lim_{N \to \infty} f_N(y) - f_N(x) \leqslant T_y = d_{\mathrm{arc}}(x,y),$$

and therefore $d_{\mathrm{arc}}(x,y) = +\infty$. □

4.5. The parabolic Harnack inequality

Theorem 4.13. *Suppose u is a positive solution to the heat equation*

$$\partial_t u = Lu, \qquad u(0,\cdot) = f.$$

Then for any $0 \leqslant t_1 < t_2 \leqslant 1$ and x, y in the same admissible component, say, $V_A(x)$, we have

$$\log u(t_1,x) - \log u(t_2,y) \leqslant \frac{T_x^2}{4(t_2 - t_1)} + \frac{1}{2} \log \frac{t_2}{t_1},$$

where T_x is defined in Definition 4.10.

Proof. The proof is standard. Let $u(t,x) := P_t f(x)$ for a positive function $f \in C_b^2(H)$. Then by Theorem 4.1, u is the solution to the heat equation

$$\partial_t g = Lg, \quad g(0,\cdot) = f.$$

Denote $g(t,x) := \log u(t,x)$. Let $t_2 > t_1 \geqslant 0$, $x, y \in H$. Since $y \in V_A(x)$, we can find a smooth path $\gamma_A : [0, T_y] \to H^m$ such that $\gamma(0) = y$, $\gamma(T_x) = x$, and $\dot{\gamma}(t) = A(\gamma(t))$. Define $\sigma : [0, T_x] \to [t_1, t_2] \times H^m$ by $\sigma(s) := (t_2 - \frac{t_2 - t_1}{T_x} s, \gamma(s))$. Note that $\sigma(0) = (t_2, y)$ and $\sigma(T_x) = (t_1, x)$. Then

$$\begin{aligned}
g(t_1,x) - g(t_2,y) &= g\big(\sigma(0)\big) - g\big(\sigma(T_x)\big) \\
&= \int_0^{T_x} \frac{d}{ds} g\big(\sigma(s)\big)\, ds \\
&= \int_0^{T_x} \left(\langle \nabla g, \dot{\gamma_A} \rangle - \left(\frac{t_2 - t_1}{T_x} \right) \partial_t g\big(\sigma(s)\big) \right) ds \\
&\leqslant \int_0^{T_x} \nabla_A f|_{\gamma_A(s)}\, ds - \int_0^{T_x} \frac{t_2 - t_1}{T_x} \Gamma(g) + \frac{1}{2} \int_0^{T_x} \frac{(t_2 - t_1)}{T_x t_2 - (t_2 - t_1)s}\, ds
\end{aligned}$$

by Corollary 4.8. Note that $\Gamma(g) = |\nabla_A g|^2$, so

$$\nabla_A f - \frac{t_2 - t_1}{T_x} \Gamma(g) \leqslant \frac{T_x}{4(t_2 - t_1)},$$

where we used the elementary estimate $ax - bx^2 \leqslant a^2/4b$ for $b > 0$ with $x = \nabla_A g$. Finally, we have

$$g(t_1,x) - g(t_2,y) \leqslant \frac{T_x^2}{4(t_2 - t_1)} + \frac{1}{2} \log \frac{t_2}{t_1}. \qquad \square$$

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