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Using order to reason about negative numbers: the case of Violet

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Abstract This case study illustrates how a 2nd-grade child, Violet, used an ordinal view of number to reason about positive and negative integers and arithmetic involving integers. Violet's ordinal view of number facilitated her ability to reason about and correctly solve some integer-related problems and constrained her solutions to others. We demonstrate how Violet's thinking evolved over time while she extended the properties of whole numbers and addition and subtraction to the integers. Using this case study as a basis, we propose a series of developmental milestones that build toward one's understanding of integers and integer arithmetic in an order-based way. We believe that understanding Violet's order-based reasoning can help us listen to other children.

Keywords Integers · Children's thinking · Negative numbers · Case study · Developmental milestones · Ordinal numbers

Which problem, $5+\square=3$ or $\square+5=3$, is more difficult for a typical elementary-grades child prior to school-based instruction on integers? Perhaps we should pose another question first: Are young children even able to solve problems like this prior to instruction? In our research, we have found children as young as 6 years old capable of engaging with and solving problems like those stated above. We have also found identifying the relative difficulty of integer-arithmetic tasks to be more complicated than considering problem features such as location of the unknown, number choice, or the presence of action (though these features do

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affect children's strategies). For example, in the domain of whole numbers, start-unknown problems such as □+3=5 are generally more difficult for children than change-unknown problems such as 3+□=5 because they are harder to directly model (Carpenter, Fennema, Franke, Levi & Empson, 1999). But in the domain of integers, whether a problem is more difficult than another or even solvable for a child seems to be influenced heavily by his or her underlying conception(s) of number (as ordinal, cardinal, or formal) and his or her ability to extend that conception from whole numbers to integers (Bishop et al., 2013). In this paper we consider one such view of number in detail—an order-based understanding of number—in investigating the integer understandings of one second-grade student, Violet. In the following section we describe the theoretical perspectives guiding our work, provide a brief rationale for the importance of understanding children's intuitive understandings of number, particularly negative numbers,[1] and identify different understandings of number that children typically develop and encounter in their early childhood experiences, both within and outside of school.

1 Conceptual framework

1.1 Theoretical perspective

We ground our work in constructivist notions of thinking and learning while acknowledging the critical role that semiotic systems and tools play in shaping the understandings that learners construct. Our stance is that children's mathematical thinking differs from a teacher's, a mathematician's, or even an average adult's mathematical thinking. We build on the work of other scholars whose approach to studying teaching and learning is based on the constructivist principles that children have existing knowledge and experiences they bring with them into the classroom, upon which they continue to build (e.g., see Carpenter et al., 1999; Fuson, 1992; Fuson, Smith, & Lo Cicero, 1997; Steffe, 1994, 2002; Steffe & Olive, 2010). "Learning is influenced by the knowledge that children start with; … educators and educational researchers need to understand the nature of the knowledge that children possess prior to instruction and how that knowledge influences what they learn." (Carpenter & Peterson, 1988, p. 83). Our integers research grows out of the goal of helping students to learn mathematics with understanding and the belief that students have intuitive ideas and previous knowledge they build on while they learn. But the understandings that learners develop are mediated by tools and sign systems. Thus, we also draw on the Vygotskian idea of *mediated activity* (1978; see also Wertsch, 1991); our stance is that children's mathematical thinking is mediated by their interactions with the tools at their disposal and features of tasks with which they engage. We are concerned here with the ways that different tools, including the number line, interact with tasks to support children in making sense of integers.

1.2 Why integers?

Compared to mathematical topics such as rate and proportionality or place value, integers, in general, and children's informal and intuitive thinking about integers, in particular, have been

[1] We use the terms *negative number* and *negative integer* as synonyms in this paper because the child in this study consistently referred to *integers* less than zero as negative numbers. She did not have occasion to reason about noninteger values less than zero. We acknowledge that, mathematically speaking, the set of negative integers is a subset of negative numbers, but in this paper we do not discuss nonwhole-rational numbers or irrational numbers that are less than zero.

studied very little (Kieran, 2007; National Research Council, 2001). Our goal is to identify how young children reason about integer-related tasks and how their reasoning develops over time by building on the small but growing body of existing research on students' integer reasoning (e.g., Behrend & Mohs, 2006; Bishop, Lamb, Philipp, Schappelle, & Whitacre 2011; Peled, 1991; Peled, Mukhopadhyay, & Resnick, 1989; Whitacre et al., 2012; Wilcox, 2008). In doing so, we hope to generate new knowledge to inform the teaching and learning of mathematics in the domain of integers. Knowledge of children's integer reasoning will enable teachers, researchers, and parents to better assess children's current understandings in-the-moment to formulate appropriate responses and questions that build coherently on important ideas and help to mature student thinking. Without frameworks or models of children's reasoning in the content domain of integers, assessing, for example, the sophistication of a child's strategy or knowing how to support a learner is difficult. This research is designed to fill this void. Further, this knowledge will enable curriculum developers to design tasks and sequence instruction by not only considering the mathematical landscape but also building from children's intuitions.

1.3 Three understandings of number

Broadly speaking, children will encounter and develop three underlying views of number in their school experiences: an ordinal, a cardinal, and a formal understanding of number (Bishop et al., 2013; Lakoff & Núñez, 2000; see also Baroody & Wilkins, 1999; Clements & Sarama, 2007; Fuson, 1992 for discussions of cardinal and ordinal meanings of number). We believe that students need *each* of these understandings to reason robustly about integers and that, at times, proficient students may draw on more than one view of number to reason about a single integer problem. We describe each view in the paragraphs that follow.

The idea of order is a basic principle of our number system. Children's initial experiences with ordering occur when they learn to count and reason about *before* and *after* and *smaller* and *greater*. We define an *ordinal* (or positional) view of number[2] as related to the idea of ordering relations wherein learners impose some kind of ordering on **Z** (the set of all integers) and then use the ordinal, or positional, nature of numbers in their strategies. In this view, numbers are sequenced and ordered (e.g., −3 is the number greater than −4 and less than −2) but not necessarily related to a countable amount or quantity (see also Clements & Sarama, 2007).

A second understanding of number is as a countable quantity that we call a *cardinal* view of number. A number's cardinality is the number of objects it represents; as such, this way of understanding number is tied to numeration, counting, and the idea of magnitude. We recognize that within the realm of whole numbers a relationship exists between a counting act and the cardinality of a set of objects (e.g., "One, 2, 3, 4; I see four bears there."). By separating cardinal and ordinal meanings of number, we suggest **not** that these are independent meanings for children but that in extending children's mathematics to the entire set of integers, one must attend to new issues and questions that arise about cardinal and ordinal meanings of number.

In the third view of number, as a *formal* entity, students approach numbers in an algebraic way, generalizing from what they already know to be true about whole numbers and operations on them. In this view, numbers can be treated abstractly, as objects that obey rules and relationships. This formal approach to number echoes Kaput's (1998) description of algebraic reasoning; he identified five forms of algebraic reasoning, two of which are "the study of structures and systems abstracted from computations and relations," and "generalizing and

[2] We define an *ordinal view of number* more broadly than the more traditional definition of ordinal numbers as acquiring and using the words *first, second, third, etc.* to indicate position in a series.

formalizing patterns and constraints" (p. 26). When children extend their understanding of numbers to new domains (from whole number to integers or from whole numbers to rational numbers), they have opportunities to look for and make use of underlying structures. For example, a child might use a formal understanding of number to reason about the expression $-5--5$ by extending principles he or she has discovered about whole numbers (that any number subtracted from itself is zero) to negative numbers. This type of *generalization* is an example of a formal approach to number.

We do not claim that any one view of number is superior to the others. In fact, as mentioned earlier, we believe that students need *each* of these understandings to reason soundly about integers and that different views of number support students to reason productively about integers in different ways. For example, consider the problems $-7-\square=-5$ and $-9+5=\square$. The former could easily be solved using a cardinal view of number, whereas the latter could be solved using an ordinal view. We have seen children who view numbers as cardinal and as having magnitude think about the problem $-7-\square=-5$ as follows:

> Well this one I need little cubes. (Child counts out a stack of 5 cubes. Then child takes 2 more cubes.) … [It] would be like real numbers, but just add the minus sign … You just do 7 plus, well actually, 7 minus 2 equals 5. That's the answer for real numbers. So I just added a negative to all of them, and there is my answer.

Here we see a child treat negative numbers like "real" numbers to productively reason about and solve problems involving negative integers. In particular, he treated −7, −5, and −2 as having magnitude and representing a countable number of objects that could be represented with cubes. By treating −5 as a set of five countable objects similar to the way that 5 also represents a set of numerable objects, he was able to productively reason about integer subtraction conjecturing that subtraction functions with −7 and −2 as it does with 7 and 2. We describe this strategy as cardinal because the child appeared to see negative numbers as related to countable sets of objects.

However, this type of reasoning might be difficult to apply to a problem such as $-9+5=\square$ because it contains a negative number and a "real" number. We have seen children use, instead, an ordinal understanding of number to evaluate $-9+5$ with a counting strategy. Using counting strategies, children typically make use of a numerical ordering by arranging adjacent negative numbers into a sequence; this ordering is then leveraged in counting up or down to solve problems:

> Negative 9 plus 1 [child raises one finger] would be negative 8. Plus 2 is negative 7 [child raises a 2nd finger], plus 3 is negative …, negative 6 [child raises a 3rd finger], plus 4 is negative 5 [child raises a 4th finger], and plus 5 is negative 4 [child raises a 5th finger].

Using this strategy, the child does not appear to treat −9 as a set of objects or as having magnitude, but instead as a number in a sequence with a distinct position—namely, between −10 and −8. Our point is that reasoning about numbers as cardinal or ordinal can render integer problems more or less difficult. The two examples provide evidence of how these views of number can support different ways of reasoning.

In this paper we focus on the affordances and limitations of an order-based way of reasoning about integers. We do so through the presentation of a case study of Violet. We chose to focus on Violet and her order-based reasoning about integers, as opposed to a case study of one using formal or cardinal views for several reasons. First, order-based strategies can be viewed as an extension of children's intuitive counting strategies used to solve whole-number problems. Second, Peled and her colleagues' research (Peled et al., 1989; see also

Peled, 1991) with third- and fifth-graders showed that young children are capable of engaging with and ordering numbers less than zero. Third, number-line diagrams (the use of which is order-based) are included in U.S. national curriculum documents, including the Common Core State Standards (CCSS, 2010) for Mathematics beginning in Grade 2 and the National Council of Teachers of Mathematics (NCTM) *Principles and Standards* (2000) in the Grades 3–5 band. In the CCSS, the use of number lines is highlighted in Grade 6 in relation to understanding negative numbers. For example, one Grade 6 Standard is "Understand a rational number as a point on the *number line*. Extend *number line diagrams* and coordinate axes familiar from previous grades to represent points on the line and in the plane with *negative number coordinates*" (p. 43). Finally, order-based views of number might be particularly useful for helping students overcome common conceptual obstacles related to negative numbers (Bishop et al., 2013).

1.4 Contextualizing the case study of Violet

Children have various conceptions about the plausibility and existence of numbers less than zero, not unlike mathematicians had historically (Bishop et al., 2013; Glaeser, 1981; Hefendehl-Hebeker, 1991; Henley, 1999; Schubring, 2005). Though our primary focus in this paper is on data from a case study of one child, Violet, her data were part of a larger study wherein we interviewed 160 K–12 students about their integer conceptions. We briefly share some of these conceptions to situate the case study findings about order-based integer reasoning presented in this paper.

The initial tasks in our interviews—such as asking students to name progressively smaller numbers and to count down (by ones) from 5 as far as they could—enabled us to ascertain whether students' numerical domains included negative numbers. Children who did not appear to know about negative numbers or whose understanding of negative numbers appeared tenuous often played a game involving a number line during the interview (for more information see Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011; Schifter, Russell, & Bastable, 2007). Students were presented with a number line, numbered starting at 0 and extending to 5 or 6, with unlabeled tick marks to the left of 0. During the course of the game, students would eventually land on an unlabeled mark to the left of 0 and were asked what that place should be called. Responses to this task helped us to identify whether children could conceive of numbers or other kinds of entities that were less than zero, and, if so, whether and how they might order numbers less than zero. Some children maintained that there is nothing smaller than zero, whereas others entertained the possibility of entities less than zero that may not be numerical in nature (e.g., one child labeled the integers the places to the left of zero with different shapes including a triangle, circle and square). And still others could conceive of the existence of distinct *numeric* entities less than zero.

Consider for example, a first grader, Nola, who invented new notation for "numbers below zero," calling them the "somethings." When asked if she could keep counting down past zero, she explained, "There is some [numbers less than zero], but I forgot the word for them. Like something 1, something 2, something 3, something 4." When asked how she would write these numbers, she wrote, S1, S2, S3, S4, and so on. Despite her nontraditional representation of negative numbers, Nola's "somethings" have a numeric component that enables her to leverage the positional and sequential nature of the "somethings" when reasoning about integer tasks. Many children, even if they could conceive of numeric entities less than zero, struggled with problems such as $3-5=\square$. Niki said, "Three minus 5 doesn't make sense because 3 is less than 5." One second grader, Rachel, provided insight into these responses in explaining, "Zero is nothing, but negative is more nothing." In a cardinal world of countable objects, numbers are used to describe “how many” of something; one meaning of zero, then, is the absence of (or none of) a given object (Lakoff & Núñez, 2000; Seife, 2000; Wilcox, 2008). If zero means *nothing*, taking something from nothing or even having an amount that is less than nothing is paradoxical, just as Niki claimed. The challenge for Niki and Rachel may stem from a difficulty in viewing numbers in new and different ways—as an ordered sequence of quantities that extend beyond zero *as well as* a conception of a number as a countable quantity.

Our focus in this paper is on how one child leveraged an order-based view of number to successfully engage with integers and integer arithmetic. By better understanding her reasoning, we hope to learn how to better support other children to extend their understandings of numbers as ordered quantities that can, in fact, exist “below” zero. In the next section we provide background on the methodology we used, participant selection, data collection, interview tasks, and data analysis.

2 Methods

2.1 Study background and participants

This paper is part of a larger, cross-sectional research study designed to (a) identify and categorize students’ conceptions of integers and operations on integers and (b) identify possible developmental trajectories of students’ integer understandings. In the larger project, 160 K–12 students were interviewed, but in this paper our analyses are from a series of interviews with one 2nd-grade student, Violet.

Our interviews with Violet occurred early in the project; in fact, our initial interview with Violet was one of the first interviews we conducted. At the time of the interviews, Violet was in second grade; she had received no classroom instruction on negative numbers, but we believed that her strategies had the potential to reveal natural intuitions young children have about negative numbers. We have chosen to focus on this child as an illustrative case study (Yin, 2009) for reasons we describe below.

We chose Violet because during her initial interview (a) she knew that negative numbers exist and could reason with and about them; (b) she consistently leveraged an underlying view of *numbers as ordered* in her strategies, as reflected in her use of the number line as well as her ability to extend counting strategies into negative integers; (c) she easily solved some problems while finding other seemingly similar problems quite challenging, initially puzzling us; and (d) she provided responses that appeared, on the surface, to be similar to those of other students. On the basis of the initial interview, we suspected that Violet’s conceptions of negative numbers could provide insight into how other students might reason using the underlying principles of order and ordering relations.

The methodology of teaching experiments To better understand Violet’s ways of reasoning and the related affordances and limitations, we used the methodology of a teaching experiment. Cobb and Steffe (1983) described the purpose of a teaching experiment as “formulating explanations of children’s mathematical behavior” (p. 83); those explanations usually take the form of a model of the child’s understandings and the constraints on those understandings. A model is considered viable if it *explains* individual children’s actions and responses around a given mathematical topic (Steffe & Thompson, 2000, p. 302). The model-building aspect as well as the focus on individual children differentiates teaching experiments from other methodologies. Steffe and Thompson distinguished teaching experiments from clinical interviews, explaining that a teaching experiment is aimed at understanding progress children make in reasoning over a period of time, whereas clinical interviews are used to obtain a more static picture of one's current understanding (p. 274). Though we understand this difference, we do not draw the same distinction in this paper. Because the interview tasks presented mathematical ideas that were new to Violet (namely integers), the act of reasoning and then explaining her thinking in these interactive interviews influenced the nature of her mathematical knowledge. Thus, we view the entire series of interviews as a teaching experiment, even though the first two interviews, according to Steffe and Thompson's definition, more closely resembled clinical interviews.

2.2 Data collection and interview tasks

We interviewed Violet three times over 3 months. The first two interviews occurred within 1 week in February 2010. The purpose of these initial interviews was to (a) determine whether Violet had any familiarity with negative numbers, (b) describe how she reasoned about arithmetic involving positive and negative numbers, (c) identify which problems were more and less challenging to her, and (d) use this information to build a model for her way of reasoning about integers and integer addition and subtraction. We posed a set of integer-related tasks during these interviews; the majority of the tasks were part of the interview protocol used by the larger project, but some tasks were developed in the moment to seek confirming or disconfirming evidence for the interviewer's evolving model of Violet's reasoning. The third and last interview occurred in May 2010, approximately 2.5 months after the second interview. Unlike for the first two interviews, the tasks and problems for the third interview were designed specifically (a) to test our initial model of Violet's integer reasoning after having had opportunities to analyze the first two interviews in detail and (b) to support her efforts to possibly extend that model and develop an even more sophisticated understanding of integers.

The 45–60-min videotaped interviews with Violet were conducted at her school and were comprised of four types of integer-related tasks: She was asked to (a) count up and down, name large and small numbers, and solve simple arithmetic problems in introductory tasks; (b) solve open number sentences (decontextualized problems involving negative integers such as $\square+7=-2$); (c) complete comparison tasks; and (d) solve context problems that experts would recognize as being related to integer arithmetic (e.g., situations involving owing; gains and losses in terms of people getting on and off the Polar Express [a train in a well-known children's book]; and so on.). In this paper, we focus on a subset of the interview tasks—open number sentences—because Violet's responses to these tasks enabled us to develop a model for Violet's knowledge of integers and integer arithmetic. (See the Results section for examples of the open number sentences posed to Violet.)

2.3 Analysis and model building

Teaching experiments involve generating and testing hypotheses of children's thinking to build a model of the child's mathematical reality; this construction occurs during the interview itself and in retrospective analyses of the interview (Steffe & Thompson, 2000). Thompson (1982) defined a *model* as a conceptual system with sufficient power to explain a child's interactions with a particular mathematical topic in a *systematic* way; inherent in this definition is the assumption that an underlying logic guides the child's responses and actions.

During our interviews with Violet, we began to formulate conjectures about how Violet's order-based understanding of integers supported and constrained her ability to engage with and solve problems with specific features. Though we had a series of predetermined tasks, we often deviated from the original plan to pose problems to test hypotheses and revise our conjectures.

After the first interview, we had ideas about how Violet reasoned with signed numbers, but we needed additional confirming or disconfirming evidence for these conjectures. In fact, we had not posed all the original tasks planned for the first interview, so we conducted a second interview a week later.

After the second interview, we transcribed both interviews and used video and transcript data to retrospectively analyze Violet's responses more carefully and systematically to identify the underlying conceptions of her integer reality. We were particularly interested in understanding where and why Violet's model[3] did not result in correct solutions to the open number tasks we posed. After developing an initial model of Violet's integer reasoning, we conducted a third interview to test our assumptions and to try to extend Violet's understandings so that she could engage with previously unsolvable problems. After conducting the third interview, we again analyzed video and transcripts of the last interview and revised Violet's model of integer conceptions to reflect her current understandings and our new insights in relation to her understandings.

3 Results and discussion

In this section we present the results of our analysis of the three interviews with Violet. First, we report Violet's responses to tasks during the first two interviews. On the basis of her responses, we developed a model of Violet's reasoning to help us understand her approaches to particular problems and identify why various problems were more and less challenging for her. Then, we report findings from the third interview and revise our model of Violet's integer reasoning on the basis of these new data. We end the results section by using Violet as a particular case to help us hypothesize a more general model and consider a possible set of developmental milestones of integer reasoning grounded in an order-based understanding of number.

3.1 Constructing an initial model of Violet's integer reasoning

During our interviews with Violet, we posed a series of open number sentences. Table 1 lists a subset of the problems posed during the first two interviews along with Violet's related responses. We have organized the table on the basis of the correctness of Violet's answers; thus, the problems are not listed in the order posed to Violet. (The first three problems posed that involved negative numbers in Interview 1 were $3-5=\square$, followed by $5+\square=3$, and then $5-\square=8$.)

First, consider the problems that Violet solved correctly. Violet solved the first problem involving negative numbers, $3-5=\square$, by *jumping to zero*, what we consider to be a sophisticated version of a counting-back strategy. Much as children invoke place-value understanding to use decade numbers in incrementing strategies to solve problems involving regrouping (e.g., jumping to 30 to solve $33-5$), Violet used the friendly number of zero. By decomposing 5 into 3 and 2, she was able to jump to zero by subtracting 3 from 3. Violet successfully solved this problem by using the ordinal nature of numbers; that is, numbers appear to be unique locations or positions that exist in a sequence for Violet.[4] It is because

[3] Note that although we refer to *Violet's model*, we do not know what Violet's model is. We use the phrase *Violet's model* throughout the rest of the paper to mean our model of Violet's conceptual model for integer addition and subtraction.

[4] We do not view her use of the fact $3-3=0$ as an instance of a formal view of number because she did not indicate that that this was a generalized principle that held for all numbers. Her intuitive use of $3-3=0$ was a number fact that she used to subtract 5 from 3 and that leveraged the unique position of 0.

Table 1 Violet's Interview 1 and 2 responses

Problem	Correct Responses
3 – 5 = □	"My brain was thinking, 'Uh, 3 minus 3 is zero and then 0 minus 2'—because 3 plus 5 is 2; 3 plus 2 is 5—um, that must be negative 2."
-9 + 5 = □	Violet uses a counting strategy, saying, "Negative 9 plus 1 [puts up a finger] would be negative 8. Plus 2 is negative 7 [puts up 2nd finger], plus 3 is negative ..., negative 6 [3rd finger], plus 4 is negative 5 [4th finger], and plus 5 is negative 4 [5th finger]."
□ + 5 = 3	Violet uses trial and error on the number line, first starting at -5 and counting up five places to end at 0, then starting at -4 and ending at 1. After starting at -3 and ending in the wrong place, she answers, "Negative 2," and counts up five places to end at 3.
□ – 2 = -5	Violet looks at the number line and says something under her breath. "Three minus 2 equals negative—*negative* 3 minus 2 is negative 5. … Negative 3 minus 1 [movement from -3 to -4 on number line] is negative 4, and minus 2 [movement from -4 to -5 on number line] is negative 5."
□ + 5 = -2	Violet puts her pen point on the number line at -2 and moves it to the left. She answers, "Negative 6." She pauses then moves her pen again on the number line. "It might be negative 7" [She then writes -7 in the box]. When asked to explain her thinking, she says, "Negative 2 minus 1 [moves pen from -2 to -3 on number line] is negative 3, minus 4—minus 2 [self-correction as she moves pen from -3 to -4 on the number line] is negative 4, minus 3 [moves from -4 to -5] is negative 5, minus 4 [moves from -5 to -6] is negative 6, minus five [moves from -6 to -7] is negative 7."
Problem	**Incorrect Responses or No Response**
5 + □ = 3	"There's no way to do that. … Because you have to do 5 minus something equals 2. If you add to that, say 5 plus 2, that would be 7 not 3."
5 – □ = 8	Violet changes the minus sign to a plus sign and writes 3 in the box. When asked, "Can you answer that [the original] problem?", she says, "I don't think so."
5 + □ = 2	Violet: I'm just thinking this [pointing to the addition sign] has to be a minus for that to be possible. … Interviewer: Is there any number that could go in there [the box] that would make it possible? Violet: Not that I know. (Pause) … Well if that [points to 5 in 5 + □ = 2] were negative 5, that would, that could, be possible. Violet then used a number line to solve her new problem, -5 + □ = 2
True/False: 6 + -2 = 4 True/False: 6 + -2 = 6 – 2	Violet explains that 6 + -2 = 4 is false because, "You can't add negative 2 unless they're negative numbers. … If you have 6 and you want to plus negative, you would end up in the negatives. Like this [points to 6 in problem statement] would end up negative 6 and this [points to 4 in problem statement] would end up as negative 4." When asked, "Do you mean that when you're trying to add a negative number do all of the numbers have to be negative?" Violet responds, "Yes if you want that to work." She is then asked whether adding negative numbers is impossible or not. Violet answers, "The way that I think, yes." Violet confirms her reasoning with her response that 6 + -2 = 6 – 2 is false. "Because that's a negative 2 [pointing to -2 in 6 + -2], and you can't add negative numbers. And that's [pointing to the 2 in 6 – 2] a 2."
True/False: 5 – 3 = 5 – -3	Violet: False. 'Cause this is a negative 3 [circles the negative sign in –3]. Interviewer: Why does that make it false? Violet: Because this [the negative sign in –3] needs to be a positive. … If you try to add negative 3 of this [referring to a Unifix cube], there is no negative 3 block. So there is no way you could add. Interviewer: But this is a subtraction problem. Violet: Yeah, like there is no negative 3. Like this (holds three cubes in her hand) isn't negative 3. It's just three.

numbers are ordered that Violet could extend a counting strategy to values less than zero, as seen in her response to $3-5=\square$ as well as to $-9+5=\square$. The idea of negative numbers as ordered and positional is also evident in her responses to $\square+5=3$, $\square+5=-2$, and $\square-2=-5$; in each of these she made use of the number line and her understanding of addition as moving right (or forward) on a conventionally drawn number line and subtraction as moving left (or backward). Moreover, she appropriately transformed the start-unknown problem $\square+5=-2$ to the related subtraction problem $-2-5=\square$, implicitly drawing on the fact that addition and subtraction are inverse operations and connecting those operations to movements on the number line. We conjecture that her viewing numbers as positional (i.e., locations on a number line) and sequential was critical for Violet to successfully solve these problems.

Now consider problems Violet solved incorrectly or described as unsolvable (see the lower section of Table 1). Violet was unable to solve $5+\square=3$ or $5+\square=2$.[5] Because both problems contradict the widely held overgeneralization that addition makes larger when interpreting addition as joining quantities or sets, we suspected that open number sentences of this form might be particularly problematic (see Bishop et al., 2013; Bishop et al., 2011). Similarly, we expected the open number sentence $5-\square=8$ to be challenging because it contradicts the notion that subtraction makes smaller.[6] For young children who are familiar primarily with whole numbers, such generalizations about addition and subtraction are sensible.

Given Violet's insistence that $5+\square=3$ was unsolvable, we anticipated a similar response to $\square+5=3$; in both problems the sum is less than the given addend, and the latter problem is a start-unknown problem, typically harder than a change-unknown problem (Carpenter et al., 1999). But Violet not only engaged with the start-unknown problem, she correctly solved it by interpreting adding as motion and using a trial-and-error strategy on a number line.

In the symbolic representation of an addition problem, one might treat the addends as the *same* in some sense; thus, we expected the placement of the 5 in $\mathbf{5}+\square=3$ and $\square+\mathbf{5}=3$ to be less important than the fact that the sum of 3 is less than the given addend. However, Violet approached these problems quite differently and appeared to have different interpretations of the problems themselves. Violet's differential treatment of the addends aligns with research on whole-number arithmetic showing that children attach different meanings to addends (Fuson, 1992). Upon closer inspection, we recognize that the open number sentence $\square+5=3$ is consistent with the idea of addition moves right (or makes larger) *in Violet's motion context*. Consider the location of the unknown within a motion context in terms of the CGI problem-type framework (Carpenter et al., 1999).

Starting Point	+	Change	=	Ending Point

Violet interpreted the open number sentence $\square+5=3$ as asking her to start at some unknown place, move right 5 units, and end at 3. In fact Violet said, "You're adding when you go this way (points right), and you're subtracting when you go this way (points left)." When one starts at −2

[5] In fact, none of the 40 second graders interviewed in the larger study solved $5+\square=3$ correctly; 85 % of these second graders said that this problem was not possible to solve as written. One child thought that the box might be a negative number but was unsure.

[6] For the problem $5-\square=8$, only one of the 40 second graders was able to solve it correctly; 80 % of the children said that this problem was impossible to solve, with the remaining responses including "3" or other numbers. Again, one student thought that the answer could be a negative number, but was unsure.

and adds 5, the resultant sum is, in fact, larger than the starting point of −2, thereby satisfying the addition-makes-larger idea. When operating with the number line, one typically treats the second addend not as an amount, but as a change or movement. Consequently, it need not be smaller than the sum. But the problem 5+□=3 proves difficult to interpret within a motion context. On a number line, if addition means to *move right*, how can one start at 5, move right, and end to the left of the starting point at 3?

Violet's responses indicate that a context of motion, similar to real-life contexts for whole numbers, may support differential treatment and interpretations of the two addends. However, for reasons stated above, unlike in the Cognitively Guided Instruction problem-solving framework for whole numbers, the start-unknown problem proved to be an easier problem for Violet to solve than the related change-unknown problem. Further, Violet, like many second- and fourth-grade students we interviewed, was unable to imagine a kind of number she could add but move left, or add to yield a smaller sum. To be clear, Violet *could* add a negative number but *only* as the first addend. The existence of a solution, much less a negative solution, to 5+□= 3 was beyond her frame of reference. She was unable to make sense of adding or subtracting a negative number (e.g., in 6+−2) or the corresponding problems that necessitated their use (e.g., 5+□=3). When asked explicitly to interpret and solve problems with such signs, Violet maintained, "You can't add negative numbers [pointing to −2 in 6+−2]." She responded similarly when thinking about the expression 5−−3, explaining that solving it was not possible, and that −3 needed to be 3 to make the problem solvable. She provided no interpretation and hypothesized no meaning for adding (or subtracting) a negative but instead claimed that adding negative numbers is not possible in her way of thinking. Toward the end of the second interview, we attempted to get Violet to engage with a possible meaning for subtracting a negative number by posing the following true/false tasks. Her responses follow the tasks.

T/F	$7 - 7 = 0$	"True. Any number minus that number is zero."
T/F	$-7 - -7 = 0$	"False. Negative 7 minus something goes that way" (Violet points to the left of − 7 on her number line).

Even though Violet maintained that any number minus itself was zero, she did not extend this generalization to negative numbers. The competing interpretation of subtraction as moving left took precedence over the generalization $a - a = 0$ for the second true/false task.

We now share the model we have developed for Violet's order-based understanding of integers. We begin in Table 2 with Violet's conceptions of number and the operations of addition and subtraction. Following Table 2, we explain how these conceptions of number and operation interact and combine with Violet's interpretation of open number sentences (as start + change = result) to yield model-specific affordances and limitations for Violet.

Perhaps unsurprisingly, we note that Violet's ways of understanding negative numbers are rooted in an ordinal approach, which is reflected in her responses in Table 1 as well as the order-based conceptions of number identified in Table 2. But we saw evidence that Violet understood numbers as cardinal, or as representing a countable set. During the first interview when Violet was asked what a number is, she replied, "It's how you know how much something is. (Violet grabbed a handful of cubes.) Like this is, I don't know. (She dropped the cubes and then picked up two cubes.) Okay, this is two." When asked if there were numbers smaller than zero, Violet replied,

> I don't think so. Negative numbers aren't really numbers. … They're just acting like other numbers except there is a minus in front of them. … We don't really count them [negative numbers] in school, and there's no negative 1 cube and stuff (She holds up one cube).

Table 2 Violet's conceptions of number and addition/subtraction—initial model

Conceptions of Number

Whole numbers are conceived as a *countable set of objects* (related to a cardinal view of number). Negative numbers are **not** conceived as a countable set of objects.

Both whole numbers and negative numbers are conceived as *part of the same ordered set or sequence* of numbers (an ordinal view of number).

Each whole number and negative number is conceived as a *unique location on the number line* (an ordinal view of number).

Conceptions of Addition and Subtraction

Addition is conceived as having a result greater than the starting value; subtraction is conceived as having a result that is less than the starting value.

On a conventionally drawn number line, addition is represented by movement to the right whereas subtraction is represented by movement to the left, except for problems involving negative change values, such as $6+-3=\square$, which are unsolvable.

Notions of the inverse relationship between addition and subtraction are emerging, as evidenced by the occasional use of subtraction to solve addition problems and addition to solve subtraction problems.

Conceptions of the meaning of "double signs" (adding or subtracting a negative number as the change term, such as $5+-1$ or $5--1$) in open number sentences are not developed.

One way that Violet seemed to understand numbers was, in her own words, as "how you know how much something is"; we describe this as a cardinal understanding of number. However, Violet did not appear to understand negative numbers in the same manner; she objected to their status as numbers because, after all, one could not have a negative number of cubes.

Additionally, for Violet, the operation of addition results in a sum larger than the addends (the first addend, in particular), which corresponds to moving right on the number line. And subtraction indicates a movement left or to count down when she uses a counting sequence. Violet's conceptions of number and operation, in combination with her order-based interpretation of open number sentences as starting and change values, supported her general sense-making model for integer addition and subtraction: namely, to identify the starting value, a change to the starting value, a resultant ending value, and then to use the two given quantities to determine the unknown value.

Violet's order-based model had both affordances and constraints that we conjecture explain which integer problems were easy, difficult, and impossible for Violet to solve. In terms of affordances,

- Her order-based model supported the existence of negative numbers. Interpreting a negative number as existing in a sequence of numbers or as a position on a number line presents an alternative to an interpretation of a negative number as a number less than nothing. Conceiving of a number of objects less than nothing is challenging, whereas a location to the left of zero, for many children, is neither different from nor less plausible than any other location on a number line.
- Her order-based model supports the successful resolution of the paradoxical notion of removing something from nothing, or more than one has, as in the problem $3-5$.
- Her order-based model supports strategies that enable her to reason about tasks in which start and result values can be negative quantities as long as her overgeneralizations that addition does not move left/make-smaller and subtraction does not move right/make-bigger are not violated. For example, Violet can appropriately reason about $-4+\square=3$ or $\square-3=-10$ but not $-4+\square=-7$.

The primary constraint of Violet's model lies in developing meaning for negative "change" values; specifically, the meaning of adding a negative number (e.g., −5+−1) and subtracting a negative number (e.g., 6−−2). Additionally, Violet's model does not explicitly challenge the overgeneralizations she appears to have formed that addition cannot move left/make-smaller or subtraction cannot move right/make-larger.

4 Revising and expanding the model of Violet's integer reasoning

Two months after the second interview, we interviewed Violet again. We posed another series of open number sentences, designed and sequenced to test our hypotheses about the model we had developed for her integer understanding and to create opportunities for her to form conjectures about what *adding a negative number* might mean. We planned multiple possible sequences of problems that might support Violet to reason about adding a negative number. For example, in one sequence of open number sentences we leveraged the commutative property by asking Violet to see if she could use what she knew about her solution to −3+5=□ to help her think about 5+−3=□ and then about 6+□=4. She proved to not need these carefully sequenced tasks when she readily engaged with problems involving the addition of a negative number using the number line. This facility surprised us inasmuch as the number line does not provide an easily interpretable meaning for subtracting or adding a negative number. But Violet used her order-based reasoning *in conjunction with* a key conjecture she made: that negatives involve doing the opposite. All the tasks in Table 3 were posed during this third interview and in the order listed in the table. Because of space constraints, not all problems posed during the interview are listed (see also Violet's number line in Fig. 1).

Violet's responses during the third interview surprised us because we did not anticipate that she would so readily engage with the problem 5+□=2, the second problem posed during the third interview. Because Violet was insistent in both the first and second interviews that these kinds of problems did not have answers, we posed this question early in the interview simply to verify her understanding that addition could not move left/make smaller. However, counterintuitive problems and notions of what adding a negative number could mean had been on Violet's mind for some time. She explained to us that after the second interview she kept thinking about the problems we posed. "I've been kind of thinking about it. I've just been thinking about it." And the result of her continued intellectual engagement was considering what adding a negative number on a number line might mean.[7] When explaining her interpretation of 5+−3, Violet said, "I was just thinking that negative 3, well, it has a minus sign in front, so people might think that you're minusing." Instead of maintaining that this operation was not possible to perform, Violet began to entertain possibilities of what adding a negative number might mean. She later clarified her thinking in another problem, saying, "If you add negative 7, to me, it's kind of like you're going backwards …; it goes the opposite direction of what the signs say it's supposed to go." Violet appeared to reason about the problem 4+□=−3 by thinking about the related problem 4+7=11. She explained, "If you go to 4 and then you go all the way to 7 more, that would be adding positive 7." Notice that Violet herself introduced the directional language, referring to "adding *positive* 7." When comparing 4 + $^{+}$7 to 4+−7, she recognized that the difference was that she was now adding negative 7. She continued, "If you add negative 7, to me, it's kind of like you're going backwards."

[7] We do not know whether Violet had conversations with others about negative numbers between interviews 2 and 3, and, if she did, what the nature of those conversations was.

Table 3 Violet's Interview 3 responses

Problem	Response
5 + □ = 2	"I'm not sure it's this, but (she writes -3 in the box). 'Cuz negative 3 is kind of like minusing 3. … I was just thinking that negative 3, well, it has a minus sign in front, so people might think that you're minusing. I get confused a lot with negative numbers." She continues, reasoning that her answer could not be positive 3 because then she would be going to 8 [not 2].
4 + □ = -3	"Negative 7 … because positive 7 gets you to 11. … If you go to 4 and then you go all the way to 7 more, that would be adding positive 7, and if you add negative 7, to me, it's kind of like you're going backwards."
5 + -3 = □	"Well you start at 5 and, like I said, to me adding negative numbers is like going backwards, so I went 1 (draws line from 5 to 4 on a number line), 2 (draws line from 4 to 3), 3 (draws line from 3 to 2), and I got to 2."
-5 + -3 = □	"Start at negative 5 and go backwards 3. I got to negative 8. … Adding negatives is like going back, 1, 2, 3" (moving back from -5 on the number line three places).
□ – 4 = -6	Violet started at -6 on the number line and counted to the right four places, ending at -2. "Negative 2, because you said, 'What minus 4 equals that?' (points to -6). So I started at negative 6 and went up four." When pressed further, Violet offers, "It would take longer to go like, 'Zero minus 4, oh, that's negative 4. And then negative 1 minus 4.' It's just easier (she points to -6 on the number line) to go 1 (she moves to -5 on the number line), 2 (she moves to -4 on the number line), 3 (she moves to -3 on the number line), 4 (she moves to -2) and find out that number."
-5 – □ = -2	"I don't know. Because if you um subtract anything, it will become these numbers (points to numbers on the number line that are to the left of -5). It's hard." Her final answer, at the time, was "I don't know."
2 – □ = 6	She pauses and then moves her pen on the number line between 2 and 6, pauses again, and writes -4. Then she laughs, seemingly quite amused at her answer. She explains herself, saying, "Because when it goes .... A negative number to me, when you're adding **or subtracting it**, it goes the other way that you want it that it usually does with a positive number. When you're adding a negative number, you're going that way (points left), so I thought when you're subtracting [a negative number], you went that way (points right)."
-3 – □ = 2	Violet answers, "Negative 5," and explains that when working with negative numbers, "When you're doing it, uh, the opposite, it goes the opposite direction of what the signs say it's supposed to go. I just counted up from negative 3: 1, (moves from -3 to -2), 2, 3, 4, 5 (she continues to move on the number line while she counts from -3 to 2), and I got to 2. Then I added a negative sign. Negative 5."

In her conjecture, Violet leveraged a critical aspect of understanding integers, namely, that of negation (Lamb et al., 2012; Thompson & Dreyfus, 1988). Although she did not use this language, Violet treated the negative sign as *negating*, or doing the opposite of what one would do with positive numbers. In other words, Violet essentially conjectured that for positive change values a, $-a$ represents a movement in the direction opposite of a. Once Violet had a sensible interpretation for adding a positive number, she leveraged the meaning of a negative number as doing the opposite of a positive number to make sense of adding a negative number. In solving the problem $4+\square=-3$, she essentially held the first number and the operation

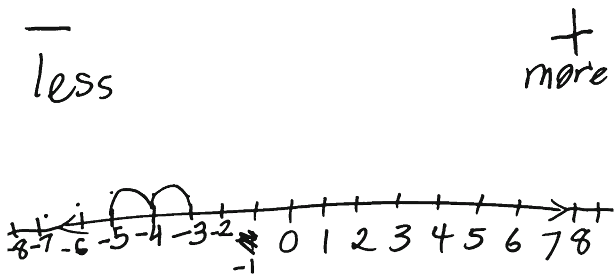

Fig. 1 Violet's number line

constant, varying the sign of the second addend. She reasoned that to add a negative number, one did the opposite (move left) of what one normally did (move right). Violet generalized beyond a specific case by making a comparison to another, known, problem and adjusted her heuristic so that the logic of her approach remained consistent. In particular, she compared the sum of 4 and 7 with the sum of 4 and −7 in combination with her beginning understanding of additive inverses and negation to develop possible meanings for adding a negative number.[8]

Now consider Violet's response to the problem $-5-\square=-2$. In this task she was unable to successfully extend her earlier generalization about the behavior of negative numbers to the operation of subtraction. Violet responded that she "didn't know" the answer to this counterintuitive problem: How could she start at −5, move left (because it was a subtraction problem), and end to the right of −5? As Violet explained, "Because if you, um, subtract anything, it will become these numbers (she points to numbers on the number line that are less than, or to the left of, −5). It's hard."

But the very next problem, $2-\square=6$, which also exemplifies the idea of subtraction's making larger, Violet successfully solved! She considered the problem and then laughed while she wrote her answer of −4, saying, "I don't really know what I'm doing." On the basis of the tentative nature of her response and the consistency of her previous "not possible" answers, we think that Violet developed a new conjecture *in that moment.* We do not know whether posing a second, similar, task supported the formation of this conjecture or if using the nonnegative numbers of 2 and 6 in the problem statement might explain why Violet was now able to entertain the possibility of subtracting a negative number. Regardless, she was able to engage with a previously unsolvable task during this interview.

Note the subtle differences in her reasoning here. When considering a meaning for adding a negative number, Violet used the idea of opposites and contrasted adding a negative number with adding a positive number. When conjecturing about a possible meaning for subtracting a negative, Violet made *two* comparisons. First, she added to her earlier idea that when adding, and now subtracting, a negative number, "it goes the other way that you want it that it usually does with a positive number." This idea now holds for *both* addition and subtraction. Then, Violet offered a second comparison: "When you're adding a negative number, you're going that way (points left), so I thought when you're subtracting [a negative number] you went that way (points right)." Here Violet compared the operations of addition and subtraction, implicitly holding the numbers constant (presumably comparing $2+-4$ with $2--4$; see the

[8] We believe that Violet's intuitive use of the relationship that adding 7 moves in the opposite direction as adding negative 7 provides a solid foundation for her to develop and make sense of the more formal definition of *additive inverses* in the future, namely $7=-(-7)$ and $7+(-7)=0$. In her strategy, we see Violet's emerging ideas about additive inverses.

problem 2−□=6 in Table 3) and reasoning that if she moved left when adding a negative number, she should move right when subtracting a negative number. This second comparison leveraged the idea of additive inverses in a slightly different way, building from the idea that addition and subtraction are inverse operations (an understanding reflected earlier in her response to □+5=−2 in Table 1 as reported in Interviews 1 and 2 as well as her response to □−4=−6 as reported in Table 3).

On the basis of Violet's interactions during the third interview, we present a revised model of her order-based reasoning about integers and integer arithmetic. This new model incorporates possible explanations for Violet's letting go of the addition-makes-larger and subtraction-makes-smaller generalizations to consider, and then successfully interpret, the meaning of adding and subtracting negative numbers. Again, we begin with Violet's conceptions of number and the operations of addition and subtraction in Table 4; the changes and additional conceptions appear toward the bottom of each section and are italicized (see Table 2 for a comparison).

Again, Violet's understandings of number involve ordinal and cardinal views, with the ordinal view being invoked almost exclusively during the integer tasks. However, the new conception that emerged in Interview 3 was that negative numbers could be conceived of as doing the "opposite" of positive numbers. The first time Violet made this conjecture was in relationship to the operation of subtraction when considering the problem 5+□=2. She explained, "'Cuz negative 3 is kind of like minusing 3. … I was just thinking that negative 3, well, it has a minus sign in front, so people might think that you're minusing." Violet's connection of negative numbers to minusing seemed to provide the impetus for her to more fully reason about negatives as doing the opposite of what they normally do. Violet then extended the idea of opposites to the addition and subtraction of negative numbers. In her

Table 4 Violet's conceptions of number and addition/subtraction—revised model

Conceptions of Number
Whole numbers are conceived as a countable set of objects (related to a cardinal view of number). Negative numbers are **not** conceived as a countable set of objects.
Both whole numbers and negative numbers are conceived as part of the same ordered set or sequence of numbers (an ordinal view of number).
Each whole number and negative number is conceived as a unique location on the number line (an ordinal view of number).
A negative number can be defined in relation to a positive number; namely, a negative number does the opposite of a positive number. This conception is a precursor to a more formal understanding of additive inverses.
Conceptions of Addition and Subtraction
Addition is conceived as having a result greater than the starting value; subtraction is conceived as having a result that is less than the starting value, for positive change values.
On a conventionally-drawn number line, addition is represented by movement to the right whereas subtraction is represented by movement to the left, except for problems involving negative change values, such as 6+−3=□.
Ideas of the inverse relationship between addition and subtraction are emerging, as evidenced by the occasional use of subtraction to solve addition problems and addition to solve subtraction problems.
On a conventionally drawn number line, addition of a negative change value is conceived as doing the opposite of "normal," so that adding a negative change value corresponds to movement left on the number line. Subtraction of a negative change value is conceived as doing the opposite of "normal," so that subtracting a negative change value corresponds to movement right on the number line
Conceptions of the meaning of "double signs" (adding or subtracting a negative number as the change term, such as 5+−1 or 5−−1) in open number sentences are developed.

words, adding negative numbers is "kind of like you're going backwards," and, "When you're doing it, uh, the opposite, it goes the opposite direction of what the signs say it's supposed to go." For Violet, the operation of addition on a number line could now indicate movement right or left, *depending* on the sign of the second addend/value. What was a constraint in the previous version of Violet's model (an inability to compute with negative "changes") became an affordance because her revised model includes a possible meaning for adding and subtracting negative *change* values. This modification to her model also enabled Violet to solve problems that contradict the idea she expressed in Interviews 1 and 2—that addition could not move left/make smaller and subtraction could not move right/make larger. In Violet's revised model, adding negative numbers does, in fact, move left, and subtracting negative numbers moves right.

5 Hypothesized developmental milestones for order-based integer reasoning

We now situate Violet's responses in a broader context, using findings from our larger data set of interviews of 160 K–12 students described earlier in the Conceptual Framework; we do so to present a hypothesized set of developmental milestones of integer reasoning using order-based understandings of number. The numbered points below are major integer *milestones* that we *hypothesize* students might pass through while they develop more robust order-based understandings of integers and integer arithmetic. A milestone may include a way of reasoning or the types of problems one might be able to solve. Violet's case indicates the presence of Milestones 3 and 4 in the hypothesized, order-based set of developmental milestones of integer reasoning. Evidence for the existence of Milestones 3a and 3b comes from Violet's responses during Interviews 1 and 2, and evidence for Milestone 4 comes from Violet's responses in the last interview. Drawing from findings from other researchers cited in the initial presentation of the Milestones and from interviews in our larger data set, we suggest the presence of Milestones 0, 1 and 2 as well.

0. Nothing less than zero exists. Whole numbers exist. On a conventionally drawn number line, children can conceive of the operation of addition as moving right and subtraction as moving left for all problems with start, change, and result values that are in the set of whole numbers.
1. *Something* less than zero exists. The names, symbolic representations, or correct positional order may not be known, but children have ideas about the existence of something less than zero (see also Bishop et al., 2011; Peled, 1991; Peled et al., 1989; Wilcox, 2008).
2. Children can conceive of negative numbers as locations or positions in a sequence with the same ordering as the conventional ordering of negative numbers; in other words, negative numbers exist as part of an ordered, numeric set that extends less than, or *below*, zero. This understanding may be related to children's understandings of what numbers are, how to place and locate numbers on a number line, or in a counting sequence and what constitutes greater and less than (see also Behrend & Mohs, 2006; Bishop et al., 2011; Peled, 1991; Peled et al., 1989; Thomaidis & Tzanakis, 2007; Wilcox, 2008).

3a. Children can conceive of negative numbers as an ending point in an addition/subtraction open number sentence when the start and change values are positive. Thus, children can solve problems such as $4-7=\square$. This extension into negative numbers conserves the principle of moving left or counting down for subtraction when operating in the domain of natural numbers and includes the additional allowance that ending points can be negative; starting points, however, are *positive*. In contrast, when the starting value is *negative*, children who use order-based reasoning *may* have to grapple with which way to count (or move on the number line) when adding or subtracting and may be unable to solve these problems. Consider the problem $-8-1=\square$. Is the answer -7 or -9? If subtraction makes smaller, as many children claim, in what ways can -9, a number 9 units from zero, be reasonably thought of as *smaller than* -7, a number only 7 units from zero? Even though -9 is *to the left* of -7 on the number line, it has a larger absolute value than -7. Smaller and larger can be confusing to the left of zero! (See also Ball, 1993.)

3b. Children can conceive of negative numbers as two distinct types of locations in addition/subtraction open number sentences: starting *and* ending points, so long as the change value is positive (i.e., the sum is larger than the first addend for addition problems and the difference is less than the minuend for subtraction problems). Thus, students can view negative numbers as a *starting point* in problems for which (a) the negative value is explicitly given, such as in $-4-3=\square$, or (b) the negative value is the unknown, such as in $\square+5=2$. Additionally, students can view negative numbers as a result, or *ending point*, of an operation (e.g., solutions to problems such as $3-5=\square$ or $6-\square=-5$). Note that subcategories likely exist within this developmental milestone because students may struggle with other features of problems. For example, some students may be able to solve result-unknown problems with negative starting or ending points but be unable to solve start- or change-unknown problems of this type.

4. Children can conceive of negative numbers as both locations and changes in addition/subtraction number sentences. Specifically, children can account for *change* in problems such as $5+\square=2$, in which the unknown is the negative number, or in problems such as $6--2=\square$, in which the negative number appears as the subtrahend (or the second addend). Note that for negative numbers to exist as a change, students must develop meanings for adding and subtracting a negative number.

After the first two interviews, Violet appeared to have achieved all but the fourth milestone. We hypothesized that for Violet (and students who reason similarly using order-based conceptions) to reach this final milestone, she would have to either adopt a new way of reasoning about integers or expand her order-based understanding by conceiving of addition and subtraction as movement right or left as determined by the second operator in conjunction with the operation. In the last interview Violet showed evidence that she had now reached the most sophisticated stage in this order-based set of hypothesized developmental milestones; that is, she developed a meaning for a negative number as a change, a meaning that rested on her conjecture that negative and positive numbers have opposite effects. By understanding Violet's reasoning, we hope to learn how to better support other students to leverage order-based reasoning to engage successfully with integer arithmetic.

6 Conclusion and implications

Violet has shown us that students have intuitive ideas about negative numbers prior to school-based instruction. Given that, we see three instruction-related implications based on our findings. First, Violet's order-based understanding provides insight regarding how some students might approach integer addition and subtraction. Our model of her reasoning and the related possible set of developmental milestones provide a framework that may help teachers to better know how to support student learning of integers. We believe that understanding Violet's order-based reasoning can help us listen to other children.

Second, subtle differences in problem types and the location of the unknown affect whether and how a child engages with integer arithmetic. Violet can solve a range of problems, yet if we were to consider only the bottom half of Table 1, we might think Violet's understanding of negative numbers was impoverished. Tasks matter—different tasks give different views of a child's understanding because they provide children different affordances that are based on the child's underlying way of reasoning about integers.

And third, given the powerful ways of reasoning Violet (and many students in the larger study) exhibited *prior to formal instruction*, curriculum designers should be attentive to ways to draw out and build upon the ideas of students like Violet, whose intuitions can provide a starting point for curriculum design and instruction. This is not to say that number lines and the idea of order are absent from textbooks; in fact, number lines are often introduced in textbooks, but they are almost exclusively introduced in conjunction with a prescribed set of rules one should follow to get correct answers (Whitacre et al., 2011). Although Violet's use of the number line might outwardly resemble the textbook procedures for using number lines, the process by which she developed *her rules* was quite different.

In closing, attention to students' powerful and complicated mathematical ideas is a hallmark of effective instruction. The benefits of teachers' building from students' thinking have been well documented in the domain of whole numbers (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Carpenter et al., 1999; Fuson & Briars, 1990; Fuson et al., 1997; Hiebert et al., 1997). We offer the case of Violet and her order-based way of reasoning as an initial foray into children's thinking about negative numbers. This case indicates ways that teachers may be able to build on students' order-based reasoning to help them make sense of integer arithmetic.

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