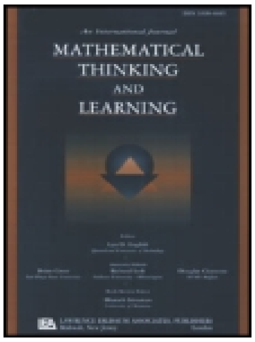

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Leveraging Structure: Logical Necessity in the Context of Integer Arithmetic

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ABSTRACT

Looking for, recognizing, and using underlying mathematical structure is an important aspect of mathematical reasoning. We explore the use of mathematical structure in children's integer strategies by developing and exemplifying the construct of logical necessity. Students in our study used logical necessity to approach and use numbers in a formal, algebraic way, leveraging key mathematical ideas about inverses, the structure of our number system, and fundamental properties. We identified the use of carefully chosen comparisons as a key feature of logical necessity and documented three types of comparisons students made when solving integer tasks. We believe that logical necessity can be applied in various mathematical domains to support students to successfully engage with mathematical structure across the K–12 curriculum.

Looking for, recognizing, and using underlying mathematical structure is an important aspect of mathematical reasoning. For example, when students think about 13×8 as the sum of the products 10×8 and 3×8, they leverage a fundamental, structuring property of our number system—the distributive property. The same property supports students to later see the equivalence of $13x$ and $10x + 3x$. Following the work of both Usiskin (1988) and Kaput (1998), we see the use of mathematical structure as a key component of algebraic reasoning. Moreover, the use of structure is consistent with research on children's mathematical thinking and the idea of relational thinking (Carpenter, Franke, & Levi, 2003; Empson & Levi, 2011; Jacobs, Franke, Carpenter, Levi, & Battey, 2007). We do not think that structure falls only within the domain of traditional school algebra; like others (Koestler, Felton, Bieda, & Otten, 2013; Mason, Stephens, & Watson, 2009), we posit that students should have opportunities to see and use structure throughout the K–12 curriculum (Common Core State Standards for Mathematics [CCSSM], 2010). In our work studying students' reasoning about integers and integer arithmetic, we have seen examples of students using mathematical structure in their approaches to integer tasks. In this article, we further develop and exemplify the use of structure by providing examples from our work investigating students' conceptions of integers. In particular, we discuss evidence of the use of mathematical structure embedded in students' strategies used for solving integer tasks—strategies we term *logical necessity*. In this way of reasoning, one leverages fundamental principles and underlying mathematical structures.

The use of structure and systems—A survey of the literature

Our view of structure is grounded in the literature on algebra and algebraic reasoning (Kaput, 1998; Kieran, 1992; Usiskin, 1988), Cuoco and colleagues' notion of mathematical habits of mind (Cuoco, Goldenberg, & Mark, 1996), and existing empirical research on structure, structure sense (Hoch &

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Dreyfus, 2004, 2005, 2006; Linchevski & Livneh, 1999), and relational thinking (Carpenter et al., 2003; Empson & Levi, 2011; Jacobs et al., 2007). After sharing an overview of research related to mathematical structure drawing on these scholars' work, we synthesize ideas related to structure from across these areas to situate our study.

What is structure?

Structure and algebra

Although mathematical structure cuts across various topics within mathematics, much of the literature related to structure is in the domain of algebra. We see the use of mathematical structure within the realm of algebra as embodying two interrelated forms of algebraic reasoning: (a) algebra as generalized arithmetic and formalizing patterns and (b) algebra as the study of structures and systems (Kaput, 1998; Usiskin, 1988). In the first view of algebra, as generalized arithmetic, arithmetic and algebraic reasoning are linked by generalizing relationships used to carry out computations with specific numbers. Arithmetic can be a context for algebraic reasoning when statements that represent rules for computation, including fundamental properties such as commutativity, additive and multiplicative identity elements, and the distributive property, are generalized from patterns students notice while computing. In fact, the first essential understanding listed in the National Council of Teachers of Mathematics publication *Developing Essential Understanding of Algebraic Thinking* is "Addition, subtraction, multiplication, and division operate under the same properties in algebra as they do in arithmetic" (Blanton, Levi, Crites, & Dougherty, 2011, p. 15). Regularities in calculations, when appropriately leveraged, can lead students to make and explore generalizations about number and operation. For example, when solving 237 – 105, a student might use an easier computation like 237 – 100 as part of his or her solution. The difference 237 – 105 might be thought of as 237 – (100 + 5), or (237 – 100) – 5 (by the distributive and associative properties). Students can then consider how the difference in the easier computation, 237 – 100, is affected when the subtrahend changes from 100 to 105. Why is the difference 5 *less* when subtracting 237 – 105 when 105 is 5 *more* than 100? Or, more generally, if the subtrahend increases (and the minuend remains constant), how does the difference change?

Algebra can be seen as a generalization of arithmetic, because observed regularities can be transformed into explicit, formal statements—generalizations of statements about numbers and arithmetic such as $n - m > n - [m + x]$ where $x > 0$, as opposed to statements about particular numbers $237 - 100 > 237 - 105$.[1] Mason identified generalized arithmetic as one context for "seeing generality through particularity" (1996, p. 69), a critical component of generalizing in a broader sense across various domains. Krutetskii (1976) found that when encountering new problems, mathematically capable students "found the generality hidden behind various particular details, saw the deep inner essence of phenomena … [and] found elements of the familiar in the new" (p. 240). These students could apply general principles that were based on "properties of various mathematical objects, schemes, or problems" (p. 306). In generalized arithmetic, students abstract, generalize, and then formalize the structures, principles, and properties that guide computation with specific numbers; they use those ideas to determine how to operate on classes of numbers. We view this type of mathematical activity as leveraging underlying mathematical structure.

In the second form of algebraic reasoning, the study of structures and systems, one makes use of the previously mentioned general properties that are attributed to operations on integers, real numbers, and polynomials. In addition to being descriptions of how numbers operate, these properties are also *objects* in their own right that, in turn, structure new sets of objects and operations. This view of algebra—the study of systems of abstract structures and the rules governing the behavior of elements in those systems—is the view we encounter in abstract

[1] This example was inspired by Schifter and colleagues' work (2008).

algebra (Cuoco et al., 1996; Usiskin, 1988). Kieran (1992) and Sfard (1991) have described the treatment of fundamental properties and symbolic forms as objects (as opposed to procedures or processes) as a *structural perspective* of algebra. In a structural perspective, mathematical objects are more general and include entities like fields, groups, rings, and polynomials. Expressions no longer refer to specific numbers, and operations are no longer performed solely on numbers. Operations are defined on sets of mathematical objects and consist of more than computations (e.g., factoring or using a property of equality to transform an equation). Moreover, variables are used as tools to represent and communicate deeper structures and relationships in a given system (Usiskin, 1988; see also Mason, 1996).

Structure and habits of mind

The use of structure is a key element in several of Cuoco and colleagues' (1996) descriptions of various mathematical "habits of mind." Two of their general habits of mind—searching for regularity and patterns (or being "pattern sniffers") and making conjectures—are aspects of structure highlighted in the above discussion of generalized arithmetic. They also identified habits of mind specific to algebra, such as the use of abstraction and extending systems. When students move through school mathematics, they extend number systems and learn how operations function in the larger systems. In this process, "new insights come when you see how a calculation or theorem behaves when you put a given system inside a larger one" (p. 399). Mason (1996), too, identified extension as an important aspect of generalization because extensions broaden "the scope of reference and application of a result, thus placing it in ever broader contexts by removing particular restrictions" (p. 69). The key question for students to ask themselves is "Which properties remain true under extension?" We view this as a question about structure related to the second form of algebraic reasoning discussed earlier.

The inclusion of structure in policy and standards documents

Additionally, the idea of mathematical structure appears both explicitly and implicitly in numerous policy and standards documents. For example, one of the Standards for Mathematical Practice identified in the CCSSM in the United States is "look for and make use of structure" (CCSSM, 2010). In the book *Connecting the NCTM Process Standards and the CCSSM Practices* (Koestler et al., 2013), this standard is described as using patterns, conjectures, and generalizations that help students recognize relationships that hold for large numbers of cases to generate new knowledge. In the United Kingdom, national mathematics curriculum documents address structure in Key Stages 3 and 4 (secondary students aged 11–16 years), stating that students should "use algebra to generalise the structure of arithmetic, including to formulate mathematical relationships" and "make and test conjectures about the generalisations that underlie patterns and relationships" (Department for Education, 2013, Key Stages 3 and 4). And in Japan, a core mathematical activity for lower secondary students (12–15 year olds) is to "discover and then extend properties of numbers and geometrical figures based on previously learned mathematics" (Japanese Ministry of Education, Culture, Sports, Science and Technology, 2010, p. 78). The stated aim of this mathematical activity is not only to discover mathematical facts or procedures but also to increase the quality of and to further refine mathematical reasoning such as induction, analogy (to develop predictions and conjectures), and deduction (to verify and justify). For this purpose, it may be necessary to re-examine the discovered properties of numbers and geometrical figures from different perspectives, such as changing the conditions and thinking about the converse, so that students can further extend their discoveries. (Japanese Ministry of Education, Culture, Sports, Science and Technology, 2010, p. 96)

In terms of the CCSSM (2010), mathematical structure is explicitly addressed under algebra standards for secondary students with the suggested guidelines of "seeing structure in expressions" and "using the structure of an expression to identify ways to rewrite it" (p. 64). Seeing and using structure requires choosing appropriate, equivalent forms of expressions to solve problems and being able to interpret complex expressions by viewing parts as a single term or entity (e.g., recognizing the expression $x^2y^4 - [7+z]^6$ as the difference of two squares; see also Hoch & Dreyfus, 2005, 2006). Transforming algebraic expressions to create equivalent expressions is also addressed in United Kingdom and Japanese curriculum documents, although these standards are not directly linked to the idea of structure.

In discussing foundational conceptualizations of number and operation for secondary students, the CCSSM authors address structure in a second, slightly different, way:

> With each extension of number, the meanings of addition, subtraction, multiplication, and division are extended. In each new number system—integers, rational numbers, real numbers, and complex numbers—the four operations stay the same in two important ways: They have the commutative, associative, and distributive properties, and their new meanings are consistent with their previous meanings. (CCSSM, 2010, p. 58)

When students extend their understanding of numbers to new domains (from whole numbers to integers or from whole numbers to rational numbers), they have opportunities to look for and make use of underlying structures and generalizations of arithmetic by deciding how calculations *should* function (or reflecting on how calculations *do* function) within expanded number systems. It is this application of structure that is our focus in this article.

Empirical research on structure

Much of the research related to *structure* centers around the idea of structure sense. According to Linchevski and Livneh, *structure sense* is the ability to use "equivalent structures of an expression flexibly and creatively" (1999, p. 191). Students applying structure sense are able to identify and generate equivalent forms of expressions as well as identify which forms are most appropriate for the given task or goal. Structure sense includes flexibly decomposing, recomposing, and manipulating expressions in ways that are sensible to students (Linchevski & Livneh, 1999). Linchevski and Livneh operationalized this construct by considering how students used order of operations to simplify numerical expressions. They wondered whether students, desiring to simplify calculations, would overlook underlying structures to incorrectly group terms and simplify, for example, $50 - 10 + 10 + 10$ as $50 - (10 + 10 + 10)$ or $104 \div 8 \div 4$ as $104 \div (8 \div 4)$? They found that more than 50% of students in Grade 6 did just that on these tasks.

Hoch and Dreyfus's (2004, 2005, 2006) work on structure sense built on Linchevski and Livneh's notion of equivalent expressions as a key component of structure sense but also included algebraic expressions. They defined structure sense as comprised of multiple abilities including the following:

> The ability to see an algebraic expression or sentence as an entity, recognise an algebraic expression or sentence as a previously met structure, divide an entity into sub-structures, recognise mutual connections between structures, recognise which manipulations it is possible to perform, and recognise which manipulations it is useful to perform. (Hoch & Dreyfus, 2004, p. 51)

They further explained that all expressions represent some type of structure that is determined by "the relationships between the quantities and operations that are the component parts of the structure" (p. 50). Hoch and Dreyfus (2005, 2006) explored high school students' abilities to recognize simple and complex forms of expressions and equations as equivalent. For example, do students see $(x + 3)^4 - (x - 3)^4$ as the difference of squares $a^2 - b^2$? Do students see the equivalence between the expressions $30x^2 - 28x + 6$ and $(5x - 3)(6x - 2)$? The first example makes use of the *substitution principle* or *substitutional equivalence*: replacing a variable, a, with a compound term $(x + 3)^2$ (Novotna & Hoch, 2008, p. 95; see also Musgrave, Hatfield, & Thompson, 2015). The second example illustrates *transformational equivalence* and the importance of recognizing the affordances of different equivalent forms of expressions. The first expression in the second example, $30x^2 - 28x + 6$, is more easily recognized as quadratic and might prove useful for recognizing it as such; the second expression is the product of linear factors, which might prove useful when finding roots. Hoch and Dreyfus found that many high school students did not have high levels of structure sense; they struggled to recognize and apply underlying structures when solving routine algebra tasks. This finding is consistent with one Kieran noted in her handbook chapter (1992): “The majority of students do not acquire any real sense of the structural sense of algebra” (p. 412).

Some researching structure have defined and illustrated the construct to clarify it. In their definition of *mathematical structure*, Mason and colleagues (2009) argued that structural thinking is critical to deep and productive mathematical activity; they emphasized relationships, generalizations, and the importance of properties in structural thinking. For them, *mathematical structure* means “the identification of general properties which are instantiated in particular situations as relationships between elements” (p. 10). Other researchers have attempted to identify different forms of structural reasoning. For example, Gates, Cuoco, and Kang (2015) categorized more than 400 Algebra I students’ responses to problems like “Simplify $8(98^2 - 4) + 3(98^2 - 4) - 11(98^2 - 4)$” in a pencil-and-paper assessment. Many students used a purely computational approach for this task: squaring 98, subtracting 4, multiplying, and adding from left to right using the order of operations. However, Gates and colleagues identified two forms of structural reasoning some students used in their solutions—*chunking* and *hidden meaning*. Similar to employing a structural view of algebra is the use of a chunking approach, in which students treated the expression $(98^2 - 4)$ as a *single object* to obtain the new expression $(8 + 3 - 11)(98^2 - 4)$. Some students, though, transformed $98^2 - 4$ into an equivalent form, $98^2 - 2^2$, using a second type of structural reasoning Gates and associates termed “hidden meaning.” These students used meaningful transformations to write $98^2 - 4$ as $(98 + 2)(98 - 2)$ or 100×96, revealing a hidden form that was useful for evaluating the expression. Overall, researchers exploring structure and structure sense have emphasized the importance of recognizing equivalent expressions and equations, transformations, properties, and flexibility. These researchers have also found that students’ structure sense and structural reasoning may not be as common or as robust as we educators might hope.

Relational thinking

We also see the construct of *relational thinking* as closely connected to mathematical structure. *Relational thinking* has been conceptualized as both a form of formal algebraic reasoning and as a type of student thinking (Jacobs et al., 2007). It emerged from the Cognitively Guided Instruction literature and was first identified and discussed in the context of algebra with problems that addressed the meaning of equality and one’s use of field properties when performing arithmetic calculations (Carpenter, Franke, & Levi, 2003; Jacobs et al., 2007). Since then, it has been applied to students’ strategies in the context of rational number (Empson & Levi, 2011).

Relational thinking is characterized by the comparison of expressions (or equations) and the subsequent identification of relationships between those expressions that can be used to simplify the given problem without having to perform the calculations. For example, in the problem $28 + 46 = \square + 47$, students are thinking relationally if they identify a relationship between 46 and 47 (i.e., the difference is 1) and appropriately compensate for the unknown (it is 1 less than 28) to preserve the equality relation (see Carpenter et al., 2003). In the realm of fractions, relational thinking might involve using a multiplicative identity relationship to simplify the computation $20 \times ¾$ to $10 \times 1½$ to 5×3 by “halving and doubling” multiple times (Empson & Levi, 2011, p. 85). Jacobs et al. (2007) defined relational thinking as “a flexible approach to calculation in which expressions are transformed on the basis of at least implicit use of fundamental properties of number operations” and an approach that involves looking at expressions “in their entirety” to notice relationships between expressions (p. 260). Empson and Levi (2011) further extended the definition of relational thinking, emphasizing the importance of seeing and using mathematical relationships to express numbers (and algebraic expressions) in terms of other numbers, consistent with the idea of transformational equivalence, a key component of structure sense.

Our conceptual framework for structural reasoning

In looking across empirical research, theoretical writings, and policy documents related to structure, we identified several common ideas that informed our work on structure in the context of integers including (a) the importance of the field properties in mathematical structure (Blanton et al., 2011; Carpenter et al., 2003; Empson & Levi, 2011; Jacobs et al., 2007; Mason et al., 2009); (b) looking at expressions holistically and being able to treat them as mathematical objects (Empson & Levi, 2011; Gates et al., 2015; Hoch & Dreyfus, 2004, 2005, 2006; Kieran, 1992; Mason, 1996; Sfard, 1991; Usiskin, 1988); (c) transformational equivalence (CCSSM, 2010; Gates et al., 2015; Hoch & Dreyfus, 2004, 2005, 2006; Kieran, 1992; Linchevski & Livneh, 1999); (d) extensions of our number system (or other mathematical systems) and the development of, or justification for, the rules for operating in these systems (CCSSM, 2010; Cuoco et al., 1996; Mason, 1996; Usiskin, 1988); and (e) the idea of comparison or relationships in supporting students to reason structurally (Carpenter et al., 2003; Empson & Levi, 2011; Jacobs et al., 2007; Mason et al., 2009). Because not all the literature we reviewed explicitly addressed mathematical structure, we combine these five themes to highlight two related, but different, aspects of mathematical structure.[2] First, mathematical structure involves the use of fundamental properties to transform both numeric and algebraic expressions into equivalent expressions (or equations) with an awareness of the underlying relationships between the terms. And second, mathematical structure involves the use of fundamental properties in conjunction with conjecturing and deduction to reason about extensions to the number system (or other more abstract systems). Much of the research on structure, especially research on structure sense, addresses the first aspect; the second aspect of mathematical structure is largely unaddressed in the literature base. It is this aspect of structure on which we focus in this study.

In the conceptual framework we used to guide our investigation of structural reasoning, we integrate key ideas from existing research including the application of field properties and the importance of underlying relationships and comparisons students can use in making sense of integer operations in extensions of number systems. Our goal is to investigate students' conjectures and justifications about how operations ought to (or do) behave within different sets (in this case, the integers) to determine the structure of the system under consideration. By addressing the role that structure plays in reasoning about extensions to number systems, we begin to bridge a gap in the literature. Further, by broadening the inclusion of structure to other mathematical topics such as integers, we hope to better understand the opportunities students may have to see and use structure *throughout* the K–12 curriculum. The following research questions guided our study:

1. In what ways do students use mathematical structure to reason about and justify extensions to the number system when solving integer-arithmetic tasks?
2. How often does this kind of reasoning occur, is there variation across grade levels, and what are its key features in the domain of integers?

Methods

Background and participants

The findings reported in this article are part of a larger study in which our goal was to understand K–12 students' conceptions of integers and integer arithmetic and how those conceptions change over time. In this article, we focus on one aspect of integer understanding related to structural reasoning that emerged during our analysis. Data for the larger study include clinical interviews with 160 students from 11 schools (3 elementary schools, 3 middle schools, 1 K–8 school, and 4 high schools) in a single state in the Western United States. During the 2010–2011 school year, these students participated in individual interviews about integer arithmetic. Because our focus was on students' integer conceptions and ways of reasoning, we did not gather data regarding the types of instruction students received or curricular materials used in their classrooms.

[2]There are likely other aspects of mathematical structure that exist in addition to the two we highlight here.

We conducted clinical interviews with students in grades 2, 4, 7, and 11 (40 children from each grade level). Students in grades 2 and 4 had yet to receive school-based integer instruction. We chose these grade levels to uncover intuitive and informal ways of reasoning about integer arithmetic because, in our previous research, 7–10 year olds had shown evidence of some familiarity with negative numbers. Students in grade 7 had completed instruction on integers (state standards at that time addressed integer instruction in grades 5–7). We selected this grade level to identify the kinds of integer understandings we might reasonably expect students to hold after completing school-based instruction. Students in grade 11 were enrolled in either a precalculus or calculus course, and because they would have completed a calculus course by graduation, they were deemed to be successful high school mathematics students. (Sixteen percent of high school graduates in the United States in 2009 completed calculus.[3]) This group of eleventh graders, by virtue of their course taking, presumably had more sophisticated mathematical knowledge and, consequently, provided an endpoint to a continuum of integer understanding in the K–12 context.

We selected participating schools on the basis of demographic information (such as the percentage of children receiving free or reduced-cost lunch and the ethnic composition of the school), variation in mean achievement scores on standardized state assessments, and existing relationships with teachers and staff. For each grade level we selected schools close to or at the state-achievement mean, below the state mean, and above the mean. We purposefully selected schools to include varied demographics and achievement data to ensure a range of participants. At each participating school, two teachers at each targeted grade level were identified by the school principal or volunteered to participate. All students in the two teachers' classes were invited to participate. Participating students were randomly selected from among all who returned signed consent forms (247 total students returned signed consent forms: 65 second graders, 63 fourth graders, 73 seventh graders, and 46 eleventh graders). At each school 9–11 students per grade level were selected to participate in the interviews.

Clinical interview

The 60–90-minute clinical interviews (Ginsburg, 1997) were conducted at the students' school sites during the school day and were videotaped. Although we sought to understand and follow the child's thinking during the interviews, the interviews were standardized. All children were asked the same set of 47 questions except for those students who did not have negative integers in their numeric domains.[4] However, we often posed follow-up questions to better understand and respond to the students' emerging ideas. The interview had four categories of tasks: introductory questions (asking children to name large/small numbers and to count backward), open number sentences (problems of the form $-3 + \square = 6$ and $\square + 6 = 4$, with the location of the unknown varying), contextualized problems that could be solved using negative integers (we selected two contexts, debt and elevation, common in state-adopted textbooks), and comparison problems (for a given pair of numbers like -3 and – -3, "Indicate which is larger, write an equal sign if they are equal, or write a question mark if the information given is insufficient to compare the numbers"). The introductory questions, open number sentences, and context problems are included in Appendix A in the order they were posed in the interview.

At the beginning of the interview, students were given the following tools and were told they could use them to solve problems: 20 unifix cubes, a number line labeled from 0 to 10 with 10 unlabeled tick marks to the left of zero (i.e., no negative integers were written on the number line), blank paper, and a written copy of the problems on which to record their answers. Findings shared

[3]According to the National Center for Education Statistics analysis of high school course-taking (http://nces.ed.gov/programs/coe/indicator_cod.asp, retrieved November 3, 2015).

[4]In other words, these students indicated during the early part of the interview that numbers did not extend below (or less than) zero.

in this article are from responses to only the open number sentences and focus on structural reasoning as expressed in the construct of logical necessity (which we define in the Findings section). (For analyses of other types of tasks, see Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014; Bishop et al., 2014; Lamb et al., 2012; Whitacre et al., 2012; Whitacre, Bishop, Philipp, Lamb, & Schappelle, 2014)

Pilot data

During the 2009–2010 school year we conducted a pilot study to (a) develop and test integer tasks, (b) help us determine which grade levels to select for the larger study, and (c) begin to describe the varied approaches and reasoning students used to solve integer arithmetic problems. We describe the pilot study here because it was this study that initially led us to the construct of logical necessity and because two illustrative excerpts shared later in the article are from interviews conducted during the pilot study.

We interviewed 74 students as part of the pilot study including 60 elementary students (Kindergarten through grade 5), 6 middle school students (grades 6 – 8), and 8 high school students (grades 9–12). The pilot interviews themselves were similar to the standardized interview described previously containing the same four categories of questions—introductory questions, open number sentences, contextualized problems, and comparison problems. However, in the pilot study we often tested different sequencing of problems and different number choices, tried different story problem contexts, asked only a subset of the questions from the standardized interview, or posed follow-up questions in-the-moment that were based on the child's ideas. As a result, we did not pose the same tasks to every child in the pilot study. In some cases, students were interviewed multiple times over several days or even months to test different tasks and to determine whether and how integer reasoning might change over time. The strength of the pilot interviews was that they enabled us to engage with a child to promote understanding and make visible his or her ways of reasoning. It was because of this flexibility during pilot study interviews we first noticed that some students leveraged structural reasoning in their responses.

Coding and analysis

The interviews were coded at the problem level for both correctness and the underlying way of reasoning the child used. The Ways of Reasoning coding scheme was developed and refined iteratively over a period of 2 years. We began by analyzing the 74 pilot interviews (before conducting the 160 interviews described previously), describing children's strategies with progressively more detail while the codes were developed. We identified five broad categories we call *Ways of Reasoning* and subcategories within each way of reasoning that provide more detail as to the child's specific strategy or strategies. Codes were refined and new codes were added when we began coding a subset of the 160 interviews. The final coding scheme contained five ways of reasoning: Order-based, Computational, Analogy-based, Formal, and Developmental (see Appendix B for definitions) and a total of 39 subcodes embedded within these five broader Ways of Reasoning. Each problem was assigned a way of reasoning code (some responses involved more than one way of reasoning and, thus, received multiple codes). For example, some students solved $-3 + 6 = \square$ counting up from -3 to 6 by ones. This type of solution would be coded as an Order-based way of reasoning because counting leverages the ordered and sequential nature of numbers. Another student might answer the same problem using Analogy-based reasoning explaining, "It's like I borrowed 3 dollars from my friend. It's like I owe him; that's minus 3. And I give him 3 from the 6 my mom gave me, and now I have 3." We would code this response as analogy-based reasoning about integers because the student is comparing negative integers to money and owing to reason about the problem.

The two primary coders who developed the coding framework trained five additional coders. Each additional coder coded four interviews independently and discussed all coding decisions with the primary coders (who had double-coded the same interviews). Upon completion of coder training, the set of 160 interviews was independently coded by 7 coders. Twenty percent of the interviews (32 interviews, 8 per grade level) were randomly selected for double coding to check the reliability of the coding; coders were blind as to which interviews would be double coded. Additionally, coders identified interviews that were challenging to code, and those interviews were double-coded with one of the primary coders. All coding disagreements were resolved, and final codes for double-coded interviews were generated. A total of 42 (or 26.25%) of the 160 interviews were double coded, and interrater agreement was 92% at the Ways of Reasoning level and 83% at the subcode level.

Although it was not our purpose in the larger study to investigate structure, we found that some students leveraged underlying mathematical structure in their strategies. Thus, in this article we focus on one of the strategies observed in the larger study, a subcode in the Formal Way of Reasoning category—logical necessity. *Logical necessity* exemplifies a way in which students engaged successfully with mathematical structures, fundamental principles, and invariant transformations of equations when solving integer tasks.

Findings

In this study, we investigated whether, how, and with what frequency students used mathematical structure when solving integer-arithmetic tasks. We also sought to identify key features of mathematical structure that students used when reasoning about integers. To answer our first research question, we describe the construct of logical necessity and relate it to the practice of looking for and using structure. We then share interview excerpts to illustrate examples of logical necessity used by students solving integer tasks. To answer our second research question, we provide data on the frequency of logical-necessity use, trends across grade levels, and characteristics of logical necessity that emerged from our analysis.

Logical necessity defined and exemplified in the realm of integer arithmetic

One finding from our study is a particular strategy involving mathematical structure that some students used to solve integer problems. This approach, which we call *logical necessity*, is more formal in nature than the other types of reasoning we saw; it includes students' use of underlying mathematical principles to justify existing procedures or to extend their number systems from whole numbers to the entire set of integers so that students treat negative numbers in ways consistent with principles already accepted as true for whole numbers. When students in our study engaged with integer tasks, some were in the process of extending their personal numeric domains[5] from whole numbers to include negative numbers. As a result, using and identifying common underlying properties and structures is a powerful sense-making strategy that supported these students in making conjectures about how integer arithmetic should operate in their newly expanded number systems. Other students used properties and mathematical structures to help them justify rules and procedures they already knew to be true. We use the term *logical necessity* to describe a strategy wherein students treat negative numbers like formal objects that are part of a mathematical system and use the ideas of structural similarity, well-defined expressions, deduction, fundamental mathematical principles (e.g., commutativity, negation), and, sometimes, proof by contradiction in their problem-solving approaches. On the basis of the research literature, we see logical necessity as closely related to research on relational thinking (Carpenter et al., 2003; Empson & Levi, 2011; Jacobs et al., 2007). (Note that we are using the term *logical necessity* to describe a strategy that students in our

[5]A student's *personal numeric domain* is the number system within which he or she operates. It is the set of values that the student has knowledge of, considers to be legitimate numbers, and can use as input values for arithmetic operations.

study used. We are not making claims about what a logically necessary inference, in an objective sense, is.)

A key characteristic of logical necessity in the context of integers is maintaining consistency with what one knows to be true for one's personal numeric domain (for most students, that number system is whole numbers). For example, when using logical necessity, a student may reason about a problem involving negative numbers (or make a generalization about operating with negative numbers) by making a comparison to a similar problem for which an answer is known and extending some element of that reasoning. A student may know that -5 + 1 is -4 (perhaps by using a number line or counting strategy). He or she may use this fact, together with the assumption that the commutative property holds for the extended number system, to determine a plausible answer for the related problem 1 + -5. Then, the child may reflect and consider, more generally, the meaning of adding a negative number. Additionally, we found that each occurrence of logical necessity in our data set involved the use of a comparison between two problems. Sometimes, the numbers were held constant and the operations changed, and other times the operation was constant and other features (e.g., order or a second addend) were changed.

In the following sections, we share five examples of logical necessity across a range of grades to identify key features of logical necessity and to illustrate how mathematical structure can be used productively in the context of integer arithmetic. We chose these examples to (a) exemplify the three different comparisons students made, (b) for the clarity of students' explanations, and (c) to represent the different grade levels, ethnicities, and genders of students who used logical necessity. Female and male students at all grade levels in our study as well as students from different ethnic backgrounds (Latino, African American, and Caucasian) used logical necessity. Our choice of examples reflects that diversity. The first two cases we share involve comparisons in which the sign of the number is varied, the second two cases involve comparisons in which the operation is varied, and the last case involves a comparison in which the order of the addends is varied.

Comparisons in which the sign of the number is varied

In the following sections, we present cases of both an 11th-grade student and a 2nd-grade student who invoked logical necessity by comparing two expressions for which the minuend (or first addend) and operation remained constant and the sign of the subtrahend (or second addend) was varied.

The case of Beth. Beth was an 11th grader in our main study who, like many of the eleventh graders we interviewed, knew how to correctly solve integer problems and was efficient in her use of strategies, often using standard procedures. As our first example of logical necessity, we highlight Beth's response to the problem 6 – -2 = □. Initially, Beth solved 6 – -2 = □ by invoking a computational procedure she knew and rewrote 6 – -2 as 6 + $^{+}$2 to get 8.

Beth: My answer is 8. I just, I guess you can kind of think of it as 6 minus, 2 [*sic*], the negative signs just cancel each other out and turn into positives … It's 6 plus 2 …

Interviewer: You said 6 plus 2 is 8. I can understand that. But I still don't know that I understand why 6 minus negative 2 is 8.

When she was asked to revisit the original problem,[6] Beth eventually turned to the number line, comparing 6 – 2 and 6 – -2.

Beth: Um, so, maybe I can go to a number line (she picks up a provided number line). So if you're at 6, and you know that minus normally sends you to the left (moves pen on

[6]As part of the interview protocol, interviewers asked each participant about the meaning of 6 – -2.

number line to the left of 6). So 6 minus 2 would send you this way (draws arrow oriented to the left). You would have 4. And since that [4] can't be the same answer since it's a different problem (she points to the original problem of 6 – -2 = □), you have to go right 2 (she moves her pen two spaces on the number line to 8).

Interviewer: Why do you have to go right?

Beth: Well, because, okay, 'cuz 6 minus 2 is 4. So 6 minus negative 2 can't be 4. It just, it, I mean, unless there were multiple dimensions to this number line. …

Interviewer: So your rationale for going to the right is what?

Beth: It's, it's logic, I guess. Because if 6 minus 2 equals 6 minus negative 2 [she writes 6 – 2 = 6 –(-2)], um, you wouldn't write them differently. You wouldn't write the same problem [differently]. Uh, well, see my logic breaks there because you do write problems like that—that are equal but you write them different ways. It wouldn't—I don't know, just to me, it wouldn't make sense to have 6 minus 2 equal 6 minus negative 2 because you're taking away a different quantity. Positive 2 is a different quantity than negative 2. So, if you're taking [away] different numbers, you can't get the same answer.

Beth held the operation of subtraction constant and compared the results of subtracting 2 versus subtracting -2 from 6. She reasoned that if subtracting 2 represents a movement to the left, then subtracting -2 (its opposite), should represent a movement to the right, the only other possible option on a 2-dimensional number line. In our interpretation, Beth was grappling with the underlying structure of our number system, essentially arguing that subtraction should be a well-defined function that yields unique output values given unique inputs (in other words, mapping the set of integers to itself using addition (or subtraction) is a one-to-one function).

In the given problem, one could consider a function *f*, that maps integers onto integers via $f(a) = 6 - a \ \forall \ a \in Z$. Because *f* is a one-to-one function (which is necessary for additive inverses to exist) and 2 ≠ -2, we know that $f(2) \neq f(-2)$. Or, in Beth's language, "It wouldn't make sense to have 6 minus 2 equal 6 minus negative 2 because you're taking away a different quantity" (i.e., because 2 ≠ -2 then 6 – 2 ≠ 6 – -2). Beth's explanation reflects an informal understanding of structural principles in our number system in her use of sophisticated mathematical ideas to justify why 6 – -2 cannot be 4 and must be 8. In fact, she argued that this difference has to be 8 to maintain the internal consistency and mathematical structure of integers and integer arithmetic in our number system. If not, if 6 – -2 could equal 4, our number system, as we know it, would break.

The case of Violet. As another example, we present the reasoning of Violet, a second grader interviewed in our pilot study (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014). We selected Violet because of the clarity of her reasoning and her explicit use of the term "opposite." Although many other students who used logical necessity reasoned in a similar manner, their responses were not as articulate as Violet's. In her strategies, Violet consistently leveraged an underlying view of numbers as ordered, as reflected in her use of the number line as well as in her ability to extend counting strategies into negative integers. For example, Violet solved the problem -9 + 5 = □ with a counting strategy, counting up from -9 to -4 by ones, and she solved the start-unknown problem □ + 5 = 3, using trial and error on the number line.

However, Violet could not solve the problem 5 + □ = 3. Her response to 5 + □ = 3 was "There's no way to do that. … Because you have to do 5 minus something equals 2.[7] If you add to that, say 5 plus 2, that would be 7 not 3." Because this problem contradicts the widely held overgeneralization that addition makes larger when interpreting addition as joining quantities or sets, we suspected that open number sentences of this form might be particularly problematic (see Bishop et al., 2011, 2014; Bofferding, 2014; Peled & Carraher, 2008). Similarly, we expected the open number sentence 5 – □ =

[7]It is not clear whether Violet meant that she needed to subtract 2 from 5 to equal 3, or if she misspoke and meant to say "5 minus something equals 3."

8, which Violet could not solve, to be challenging because it contradicts the notion that subtraction makes smaller.

Violet had no model to help her make sense of adding or subtracting a negative number (as in 6 + -2) or the corresponding problems that necessitated doing so (e.g., 5 + □ = 3). When asked explicitly to interpret and solve problems with such signs, Violet said, "You can't add with negative numbers [pointing to -2 in 6 + -2]." She responded similarly when thinking about the expression 5 – -3, explaining that solving it was not possible and that -3 needed to be 3 to make the problem solvable. To be clear, Violet *could* add a negative number but *only* as the first addend. Thus, she could solve □ + 5 = 3 using a number line but could not solve 5 + □ = 3; she had not yet developed a meaning for a negative change value.

Two months later, we interviewed Violet again as part of our pilot study and were surprised at her response to the problem 5 + □ = 2:

> I'm not sure it's this, but [she writes −3 in the box]. 'Cuz negative 3 is kind of like minusing 3. … I was just thinking that negative 3, well, it has a minus sign in front, so people might think that you're minusing.

She continued her explanation, reasoning that her answer could not be positive 3 because then she would be going to 8 (not 2). Violet further clarified her thinking in another problem, 4 + □ = -3, saying, "If you add negative 7, to me, it's kind of like you're going backwards. … It goes the *opposite* direction of what the signs say it's supposed to go." Violet appeared to reason about her answer to the problem 4 + □ = -3 by thinking about the two related expressions 4 + 7 and 4 + -7. She explained, "If you go to 4 and then you go all the way to 7 more, that would be adding positive 7." When comparing 4 + $^{+}$7 to 4 + -7, the difference was that she was now adding negative 7. She continued, "If you add negative 7, to me, it's kind of like you're going backwards."

Violet's conjecture leveraged a critical component of understanding integers, namely, that of negation (Lamb et al., 2012; Thompson & Dreyfus, 1988). Although she did not use this term, Violet treated the negative sign as *negating*, or doing the opposite of what one would do with positive numbers. In solving the problem 4 + □ = -3, she essentially held the first number and the operation constant, varying the sign of the second addend. She reasoned that when one added a negative number, the result was the opposite (a movement left) of what it normally would be (a movement right). Violet implicitly recognized that the answers for $a + b$ and $a + -b$, for $b \neq 0$, *must* differ, and she determined *how* they might differ by leveraging the idea of negation. We view this example of logical necessity as leveraging mathematical structure because Violet generalized beyond a specific case by making a comparison to another, known, problem and broadened her meaning for addition so that her number system remained consistent. In particular, she compared the sum of 4 and 7 with the sum of 4 and -7 and used her initial understanding of additive inverses and negation to develop possible meanings for adding (and subtracting) negative numbers.

Comparisons in which the operation is varied

We now present cases of a fourth-grade student and a seventh-grade student who both invoked logical necessity by comparing expressions for which the operation varied.

The case of Armando. Armando was a fourth grader in our main study who, like many of our other fourth-grade participants, had heard of negative numbers and could proficiently use counting strategies to correctly solve problems like 3 − 5 = □ and -4 + □ = 2. For example, Armando described his strategy for solving -3 + 6 = □: "I counted up 6 and I got to 3. … I counted, I started from negative 3, and went negative 2 (raised on finger), negative 1 (raised a second finger), negative 0, I mean 0, (raised a third finger), 1 (raised a fourth finger), 2 (raised a fifth finger), 3 (raised a sixth finger)." After the next question, -8 – 3 = □, was posed, Armando sat silently for about 30 seconds. Then, with rising intonation and a furrowed brow, he answered, "Negative 11?" When asked how he thought about that problem he replied, "Well I looked back up at this problem [points to -3 + 6 = □], negative 3 plus 6. I got 3 because I counted up; I mean, I counted down with negative 3, negative 2. And then I thought, 'Well, minus [as indicated by the subtraction sign in -8 – 3] must be going up. So I have to go up.'" And I got 11. . . . Negative 8, negative 9 (raises one finger), negative 10 (raises second finger), negative 11 (raises third finger)."

When asked to clarify what he meant by "going up" and "going down," Armando said, "Like I, I went up, I went up in the. . . . I went down in the numbers [pointing to the problem -3 + 6]. Like from negative 3 to negative 2, I went down in the numbers. And then [pointing to the problem -8 – 3] I went up by going up from 8 to 9, negative 8 up to negative 9."

In this instance of logical necessity, Armando used what he knew about the sum of -3 and 6 to help him make a reasonable conjecture about how subtraction might function when the starting value was a negative number. Armando reasoned that if addition means to "count down," then subtraction "must be going up."[8] The inverse relationship that exists between addition and subtraction when operating with whole numbers was extended to the entire set of integers. Armando solved this subtraction problem by making a comparison to another, known problem using the mathematical idea of an inverse operation, which again, we see as an example of engaging with and using mathematical structure. This comparison enabled him to broaden his meaning for subtraction so that operations within his expanded number system remained consistent.

The cases of Armando and Violet involve students who had had no school-based instruction on integers and integer arithmetic. Logical necessity had a conjectural quality in these instances, inasmuch as students were making decisions about how operations *might* function in an expanded number system. They did not already know how those in the mathematical community *do* operate with negative numbers. Instead, much like explorers encountering new terrain, they followed hunches and intuitions to map the traits and behaviors of a new number system in a systematic, logical, and structural fashion. When students encountered negative numbers, they decided, on the bases of the rules of the existing system, how these numbers should operate; students determined which rules should be maintained (e.g., commutativity) and which could be relaxed (e.g., addition need not make the sum smaller). They were, however, quick to admit their uncertainty, often hedging conjectures with phrases like "I'm not that good at negative numbers, so I'm just gonna kind of guess." "I was just thinking," "in my way of thinking," or "I wasn't sure." These conjectural cases of logical necessity contrast with Beth's reasoning and the next case we present (the case of Alma). In these cases, Beth and Alma already knew the rules for operations with integers. Their use of logical necessity and mathematical structure was to justify a claim or result they already knew to be true.

The case of Alma. Alma was a seventh grader who participated in our main study. Her interview occurred after she had completed the integer unit in her mathematics class. Overall she used a mix of strategies, including order-based reasoning (primarily referring to number lines and motion/movement), efficient computational procedures, and, at times, more formal ways of reasoning to correctly solve integer tasks. In this example, Alma was solving the problem -5 – -3 = □. She initially answered -8 and then paused. "No, it's negative 2. Because when you add, like when I'm adding negative 5 to negative 1, it's gonna be negative 6. [The problem she had just solved was -5 + -1 = □.] So it [-5 + -1] can't be the same as this [-5 – -3]. So when you subtract negative 3 from negative 5 it's gonna be negative 2."

Unlike Beth, who varied the sign of the subtrahend when comparing 6 – -2 to 6 – 2, Alma varied the operation; she compared the sum of two negative numbers (-5 + -1) to the difference of two negative numbers (-5 – -3). She reasoned that because addition and subtraction were opposites, the results of those operations would be "the other way around." When asked to say more about her reasoning, she continued, "It [the result to -5 – -3] can't be the same because it's [circles the addition symbol in -5 + -1] a different sign you're working with. . . . It's [-5 – -3] not gonna like work the

[8]What Armando calls "counting down" from -3 to -2 to -1 is, in fact, what we in the mathematical community describe as counting up. In the counting sequence -3, -2, -1, the magnitude or absolute value is decreasing, but the numbers are getting larger. However, for many children like Armando, the idea of counting up (and counting down) with negative numbers is tied to increasing (and decreasing) magnitudes. Thus we see Armando describing his strategy for solving -3 + 6 = □ as counting down and his strategy for solving -8 – 3 = □ as counting up.

same as this one [-5 + -1]. It's gonna work the other way around, like it's gonna work different." Later, she applied the same reasoning to the open number sentence -7 – -9 = □, justifying her answer of 2 because "when you add -7 to -9 it's -16. So when you're doing the opposite, when you're subtracting -9 it's like doing the opposite. So it's like adding 9."

Alma justified her solutions to problems involving the difference of two negative numbers by comparing them to problems involving the sum of two negative numbers and using the inverse relationship of addition and subtraction. The idea of inverse operations was critical in her reasoning. If the sum of -5 and -3 was -8, then the difference between -5 and -3 (the opposite operation) could not have the same solution. In fact, it should "do the opposite" in Alma's words. Again, we view Alma's reasoning as engaging with important mathematical structures such as inverse operations and negation.

Notice that both Alma's and Armando's use of logical necessity involved comparisons that *varied the operations* of addition and subtraction, leveraging their inverse relationship. In contrast, Violet's and Beth's use of logical necessity involved comparisons that *varied the signs of a number* and leveraged the idea of negation (i.e., the negative sign negates what one would normally do when operating with positive numbers). In the following case we share a third type of comparison: one that *varied the order of the addends*. We see each of the three types of comparisons as powerful sense-making strategies for students.

Comparisons in which the order of the addends is varied

For our final case we present a first-grade student who invoked logical necessity by comparing expressions for which the order of the addends varied.

The case of Ryan. The last example of logical necessity we present occurred during an interview with Ryan, a first grader who participated in our pilot study. Although this third type of comparison was not as common as the other two types, other students who varied the order of the addends gave explanations similar to Ryan's. We chose this example because of the clarity of Ryan's reasoning and his explicit use of the commutative property (although he did not name it as such). Ryan knew of negative numbers, recognized them symbolically, could correctly order them, and solved problems like 3 − 5 = □, -4 + 7 = □, and □ − 5 = -8 successfully by using counting strategies that he extended below zero. Ryan was troubled, however, by 5 + □ = 3, asking, "How do you get to 3 if it's plus? … If you add them, how do they get to 3?" Ryan explained, "If you add something, how does it get to 3? If it's 5 plus, then it's [the sum's] always past 3." He did not think that this problem was solvable. Toward the close of the interview, we posed the problem -2 + 5 = □ to Ryan. His answer was 3, which he obtained using a counting strategy, counting up five from -2. "It was negative 2, negative 1 (puts up one finger), 0 (puts up second finger), 1 (puts up third finger), 2 (puts up fourth finger), and then 3 (puts up fifth finger)." We then asked him to consider the problem 5 + -2 = □. This was the first time we had posed a problem that involved adding a negative number as the second addend, and we did so immediately after asking him to solve -2 + 5, hoping that he would notice that the addends were the same. Ryan's answer to 5 + -2 was 3, which he explained as follows:

> Because it's pretty much the same thing (points to -2 + 5). Five plus negative 2 and negative 2 plus 5. If you add the same things, and you just say 5 first and [negative] 2 second, it's still the same thing. … You always add the same things together.

In this example of logical necessity, Ryan invoked a fundamental principle of mathematics, the commutative property of addition, to reason about a possible meaning for adding a negative number. He assumed that negative numbers obey the commutative property of addition; consequently, his answer had to be 3. For Ryan's newly expanded numeric system to be consistent, 3 was the necessary answer. We next asked Ryan to solve 6 + □ = 4, which Ryan compared to the problem he had previously solved, 5 + -2 = 3, saying, "It's kinda like that one [he pointed to 5 + -2 = 3] because there's plus a negative. Six plus negative 2 goes two back." When reminded that he had earlier said that this type of problem was "impossible," he smiled and said, "Now it's plus a negative. So if it's a negative number, you have to fill it back in. … It's like minus." Ryan extended the fundamental property of commutativity for the addition of whole numbers to this *new* type of number, which enabled him to develop a possible meaning for adding a negative number and then to solve a previously counterintuitive problem wherein the sum was smaller than one of the addends.

Trends across the data

In the previous sections, we documented the use of logical necessity in students' approaches to integer arithmetic and showed how this strategy supports students to engage with mathematical structure. We now share our second finding by looking across the interview data from our main study to determine how often this type of reasoning occurred and to consider trends across grade levels. Of the 160 students we interviewed in grades 2, 4, 7, and 11 in our main study, 119 had negative numbers in their personal numeric domains (i.e., could entertain the idea of numbers less than zero), and 41 students showed no evidence of knowing of negative numbers. Looking across the interviews, we identified the number of student responses that included logical necessity *anywhere* during a student's explanation. If, at any time in a student's response to a given question he or she invoked this strategy, that response was considered an instance of logical necessity. The relative frequency of logical-necessity use across the grade levels is shown in Table 1.

Across the 119 students and the 25 open number sentences posed, the logical-necessity code was invoked 24 times across 12 different students on 11 of the 25 open number sentences. We found that for all students, regardless of grade level, logical-necessity use was rare, with only 10% of the students who knew of negative numbers (12 of 119) using this strategy.[9] The number of instances did increase slightly from seventh to eleventh grade, from 7 to 10 occurrences, but we view this as a nominal increase. Eight of the 12 students who used logical necessity when solving open number sentences did so on more than one problem.

Additionally, this form of reasoning appeared in solutions for multiple open number sentences (11 of the 25), indicating to us that logical necessity was not tied to particular problems. However, it was invoked most frequently on the following tasks: $-5 - -3 = \square$ (4 instances), $6 - -2 = \square$ (4 instances), $6 + -3 = \square$ (4 instances), $-8 - 3 = \square$ (3 instances), and $-7 - -9 = \square$ (3 instances). All but one of these 11 problems involves the explicit addition or subtraction of a negative number.

Features of logical necessity

As mentioned earlier, each of the 24 instances of logical necessity in our data set involved the use of a comparison between two problems. Thus, our third finding is the identification of comparisons as the key feature of logical necessity along with the three types of comparisons students made when operating

Table 1. Occurrence of logical necessity by grade.

	Logical-Necessity Use		
	Number of students	Number of occurrences	Number of distinct problems
Grades 2 & 4* (*n* = 39, 17 males and 22 females)	4 (10.3%)	7	6
Grade 7 (*n* = 40, 17 males and 23 females)	4 (10%)	7	6
Grade 11 (*n* = 40, 23 males and 17 females)	4 (10%)	10	7
Total Across Grades	12 (10%)	24	11

*Only the 39 grades 2 and 4 students who had negative numbers in their personal numeric domains are included in this table. There were no instances of logical-necessity use for those 41 grades 2 and 4 students who showed no evidence of knowing of negative numbers.

[9]This result is consistent with findings Kieran reported in her synthesis of research on algebra. She reported that 7%–10% of students demonstrated a structural conception of expressions and equations (1992, p. 408).

Table 2. Types of logical necessity comparisons.

Type of Comparison	Example	Frequency*
Comparisons in which the sign of the number was varied	6 − 2 = □ and 6 − -2 = □ Students use the result of 6 − 2 to help them reason about 6 − -2	16 instances
Comparisons in which the operation was varied	-8 + 3 = □ and -8 − 3 = □ Students use the result of -8 + 3 to help them reason about -8 − 3	10 instances
Comparisons in which other features, such as the order of addends, were varied	-2 + 5 = □ and 5 + -2 = □ Students use the result of -2 + 5 to help them reason about 5 + -2	2 instances

*In 4 of the 24 occurrences of logical necessity, students made more than one kind of comparison.

with integers. Those three types of comparisons are (a) varying the sign of the number (e.g., 6 – -2 compared to 6 – 2), (b) varying the operation (-8 – 3 compared to -8 + 3), and (c) varying other features such as the order of addends (e.g., -2 + 5 compared to 5 + -2). Moreover, in all but 1 of the 24 instances of logical necessity, students generated the comparison problem on his or her own, as part of the response (the exception being one grade 2 student, whose attention was drawn to a previously solved addition problem by the interviewer after she struggled to solve a related subtraction problem). Table 2 briefly presents explanations and examples of the three types of comparisons students in our study made.

Two thirds of the instances of logical necessity (16 of 24) involved comparisons in which the sign of the number was varied (e.g., see Violet's and Beth's responses). This was the most common type of comparison students made. Within this group, four students (all in fourth or seventh grade) reasoned about negatives and positives as separate, but related, *classes* of numbers; we describe this subset of comparisons as Negative Land and Positive Land. For example, Morgan, a fourth grader, reasoned about -5 – -3 by comparing it to 5 – 3. She reasoned that because 5 – 3 moves one closer to zero (or to the left of 5), then -5 – -3 should also move one closer to zero (than -5), even though the result of the subtraction (-2) is to the right of the starting value (-5). She rationalized, "It's [the negatives are] like the opposite with these (she gestured to right side of number line)." Other students echoed this idea, explaining that because negatives "are on the wrong side of zero," they, in general, behave oppositely. Note that using logical necessity to reason about the *classes* of positive and negative numbers occurred with only the following open number sentences, all of which involved the sum or difference of two negative numbers: -5 – -3 = □, -7 – -9 = □, and -5 + -1 = □.

In the second most common type of comparison (10 of 24 instances of logical necessity), the operation was varied, as in the examples of Alma's and Armando's reasoning. The following pairs of expressions were compared in this way: -8 – 3 and -8 + 3, -5 – -3 and -5 + -3, and -5 + 5 and -5 – 5. Finally, two instances of logical necessity involved comparisons varying other problem features. We refer the reader to Ryan as an example of this type of comparison. Ryan used the commutative property to vary the location of the addends, comparing -2 + 5 (to which he already knew the answer) to 5 + -2 to surmise the meaning of adding a negative number.

We also observed that logical necessity emerged in cases in which students were *justifying* a result they knew to be true and in cases in which students were *conjecturing* about how an operation within **Z** could function. Of those students in our study who used logical necessity, all the eleventh graders in our study used logical necessity to justify a procedure or rule, whereas all the second and fourth graders who used logical necessity conjectured about how negative numbers might behave by applying and relaxing properties and generalizations they had made about whole numbers. And, despite the fact that formal integer instruction was completed by the time seventh graders were interviewed, four of the seven instances of logical necessity for this grade level were conjectural in nature (judged by the tentative language used and the presence of hedges), indicating that some of these students had not yet memorized computational rules and procedures.

In summary, we found that logical necessity is a powerful way of approaching some integer tasks. Of the 119 students who knew about negative numbers, 12 (or 10%) used logical necessity on 24 occasions. Logical-necessity use did not emerge as prevalently as we might hope; however, it emerged with multiple students at each grade level, and we believe that it is something that can and should be encouraged in instruction. Moreover, we have identified the use of carefully chosen comparisons and contrasting cases as a key feature of logical necessity that we believe can be applied both within and outside the domain of integers to support students to successfully engage with mathematical structures.

Discussion

In this article, we demonstrated how integers and integer arithmetic can provide fertile ground for exploration of mathematical structure through a particular approach some students used when solving integer problems—logical necessity. In particular, we found that some students leveraged fundamental principles, invariant transformations, and underlying mathematical structures in their problem-solving strategies. In addition to describing how students use mathematical structure when solving integer tasks through logical necessity, we identified the use of comparisons as a key feature of the use of structure in the context of integers. Each instance of logical necessity in our data set involved a comparison between two related problems in which some key mathematical attribute of the problem statement was purposefully exploited to justify a proposed solution. We documented three types of comparisons students made when solving arithmetic problems: varying the sign of the number, varying the operation, or varying another feature (such as the order of the addends). Additionally, every instance of logical necessity involved either justification of an already known answer, procedure, or rule or a conjecture about how an operation on integers *should* behave based on previous knowledge about how the operation behaves with whole numbers.

In this study we have documented a new form of structural reasoning, logical necessity, in a content area not typically associated with structural reasoning. Much of the existing empirical research to define, exemplify, or categorize different types of structural reasoning has been focused on equivalent expressions and transformations of those expressions that reflect underlying and structuring relationships between quantities (CCSSM, 2010; Hoch & Dreyfus, 2004, 2005, 2006; Kieran, 1992; Linchevski & Livneh, 1999). In contrast, we have shown that the topic of reasoning about number systems in general, and integers in particular, provides rich opportunities for students to look for and make use of structure. Specifically, our data indicate the importance of comparing, conjecturing, and generalizing as key components of logical necessity as students consider the effects of broader contexts on mathematical objects (in our case, extensions of number systems). Although some scholars have linked number systems to structure on the basis of mathematical or theoretical arguments (Cuoco et al., 1996; Mason, 1996; Usiskin, 1988), we have presented empirical evidence from student interviews to document how students can engage with mathematical structure in the context of number systems through the use of logical necessity. Additionally, our data illustrate ways to practically enact the vision advocated for in policy documents that call for the use of structure *across* the curriculum (CCSSM, 2010; Japanese Ministry of Education, Culture, Sports, Science and Technology, 2010). We suggest that one way to support students' use of structure is by identifying opportunities to address structure in the context of what teachers are already doing—such as solving typical integer tasks like those shared in this article.

As described in the Findings section, we found that making comparisons that are based on underlying mathematical relationships is a key feature of logical necessity. The use of contrasting cases (6 – -2 compared to 6 – 2 in the case of Beth), known information (6 – 2 = 4), and some form of deductive reasoning (applying the meaning of additive inverses) were combined in each instance of logical necessity. Thus, we contribute to the growing literature on structural reasoning by extending existing research to identify *relationships* as a key element of structural reasoning.[10] For example, looking holistically at expressions to identify relationships between them is a key component of relational thinking (Carpenter et al., 2003; Empson & Levi, 2011; Jacobs et al., 2007). Similarly, Mason and colleagues (2009) defined *mathematical structure* as being able to see "relationships between elements" as instantiations of general structuring properties (p. 10). Students use relationships in logical necessity when they vary some attribute of a given problem to create a comparison problem and then exploit the underlying relationship between the compared attributes. Often when people compare objects (or strategies or equations), they notice distinguishing features and variations that they may not have noticed previously. Through the comparison of alternatives, people can identify key differences and the potential consequences of those differences. Not only have we found *that* relationships are an important element in structural reasoning, but we have identified *how* these relationships function in the context of integer arithmetic through the comparison of problem attributes. Moreover, we have begun to identify key problem attributes in the realm of integer addition and subtraction, including (a) the sign of the number (along with a corresponding interpretation of "–" as negation—see Lamb et al., 2012; and Thompson & Dreyfus, 1988), (b) the operations of addition and subtraction (and the inverse relationship between them), and (c) the order of addends (and the appropriate use of the commutative property). The list of attributes would likely change if, for example, we had been studying integer multiplication or even fraction division. We suspect, however, that the act of making comparisons between carefully chosen contrasting cases would be constant across content domains and, therefore, may be a more general feature of instruction and curriculum design that will support students to look for, recognize, and use mathematical structure.

Limitations and future research

Because the purpose of our interviews was to investigate students' integer-related reasoning, not to provide opportunities for them to make use of logical necessity, we suspect that we are likely underrepresenting the prevalence of this type of reasoning by students. The fact that students *did not* use logical necessity does not imply that they *cannot* engage in this type of reasoning. One reason logical necessity may not have been as prevalent in our data as we would have hoped is the lack of opportunities for students to engage in reasoning of this type in school settings. Another reason is the nature of our interviews. Our focus in the interviews was on understanding the strategies students used *without* support rather than trying to support the use of any particular strategy; consequently, in all but one instance of logical necessity that occurred in the main study, students themselves generated the second comparison problem. We believe that with active support for the use of logical necessity, a greater percentage of students will use it. Consequently, one area for future research is to investigate features that increase the prevalence of logical necessity. For example, we wonder about the potential effects of introducing explicit comparisons in interview and teaching situations rather than waiting for students to generate their own problems. With additional instructional supports in place, will more students engage in logical necessity? We also encourage future work to validate pairs of tasks we have identified as productive for students' use of logical necessity, to explore new problem pairs and their corresponding attributes, and to investigate other contextual features (e.g., interventions in the form of probing questions) that support logical necessity use. We also see potential for researchers to investigate how logical necessity can be supported in classrooms during integer instruction (as opposed to interview settings) by making use of our suggested problem pairings.

Another limitation of this study is our narrow focus on integers and integer operations. By restricting our study to the context of integer arithmetic and the use of open number sentences, we

[10]We also find noteworthy that contrasting cases and their connecting feature(s) have been identified as features that support and refine thinking in general (see Bransford, Franks, Vye, & Sherwood, 1989; National Research Council, 2000) as well as in domain-specific thinking within mathematics.

may have overlooked or failed to identify other key features of logical necessity that could emerge in other content domains and with other types of problems. In the future, researchers might focus explicitly on structural reasoning and explore the use of logical necessity and other forms of structural reasoning in other number systems (such as rational and irrational numbers). By broadening the contexts for investigation, researchers may identify new comparisons students use that are productive for logical necessity. However, one important outcome of our study is to highlight an often overlooked and under-researched aspect of mathematical structure: the use of fundamental properties in conjunction with conjecturing and deduction to reason about extensions to the number system (or other more abstract systems). Without a focus on integers, we might not have been able to identify and exemplify this aspect of mathematical structure. We encourage researchers to continue to expand the notion of structure as a researchable construct because it is a productive area for research.

Implications

More than 20 years ago, Kieran (1992) identified the development of structural conceptions in students as an important next step in research related to algebra. Our results indicate instructional changes that could provide students with increased opportunities to develop understanding of mathematical structure, both within the context of integer instruction and in other mathematical topics.

Incorporate paired comparison tasks into instruction

At the beginning of the study, our research team did not expect students as young as six and seven years to engage with logical necessity to reason about integer arithmetic. Likewise, classroom teachers may not expect (a) that these simple integer tasks can engender this type of rich reasoning or (b) that their students are capable of producing these explanations and strategies. Thus, our first instructional implication is to incorporate purposefully sequenced integer tasks that students in our study used to reason more formally about underlying structural aspects of mathematical systems (Bishop, Lamb, Philipp, Whitacre, & Schappelle, in press). Purposefully sequencing related problems such as -8 + 3 and -8 – 3, as seen in Armando's case, may encourage students to grapple with which way to count when starting with a negative number. This practice may lead to conversations about comparing and ordering negative numbers.

The suggested problem combination of -8 + 3 and -8 – 3 varies the operations of addition and subtraction, whereas other combinations might instead vary the sign of the number. Problem combinations that vary the sign of the number provide a different type of contrast that supports students to justify or make sense of adding and subtracting a negative number. For example, consider comparing 6 + -3 and 6 + 3 or comparing -5 + -1 and -5 + 1. In these pairs, the sign of the second addend is varied whereas the operation is held constant. We conjecture that because the idea of negation is tacitly indicated in the problem statements themselves, students may respond similarly to Violet and Beth, especially if students are familiar with an interpretation of the minus sign as "the opposite of" (Lamb et al., 2012). In the case of Ryan, the interviewer posed a contrasting pair of problems that supported logical necessity (in this example, -2 + 5 followed by 5 + -2). We believe that the exchange with Ryan illustrates the potential pedagogical benefit of introducing an explicit comparison, and we suggest that one way to enhance the use of logical necessity is to introduce an explicit comparison by posing a carefully chosen follow-up problem or asking students to compare two problems instead of waiting for them to generate their own comparison problems.

Table 3 shows tasks that include different comparisons in service of particular goals related to integer instruction. We envision teachers posing the first task in the given pair (e.g., 7 + 5 under the second goal) and then using the second task (7 + -5) to build on what students already know about a more familiar task to

Table 3. Integer tasks to encourage use of logical necessity.

Goal	Integer Tasks	Feature Compared and Key Understandings Leveraged
Make sense of or justify adding and subtracting a positive integer when the starting value is a negative number	-8 + 3 and -8 – 3 -5 + 5 and -5 – 5	Vary operation Leverage inverse operations and knowledge of zero (in second pairing)
Make sense of or justify adding a negative number	7 + 5 and 7 + -5 -5 + 1 and -5 + -1 5 + 1 and -5 + -1 -3 + 6 and 6 + -3 -9 + 5 and 5 + -9	Vary sign of number Leverage negation/additive inverses (In the third pairing, the signs of both addends are varied.) Vary order of addends Leverage commutative property
Make sense of or justify subtracting a negative number	-5 – 3 and -5 – -3 6 – 2 and 6 – -2 10 – 4 and -10 – -4 -5 + -3 and -5 – -3 - 7 + -9 and -7 – -9 -5 – -5 and -5 – -1 -7 – -7 and -7 – -8	Vary sign of number Leverage negation/additive inverses (In the third pairing, the signs of the minuend and subtrahend are varied) Vary operation Leverage inverse operations Vary subtrahend Leverage knowledge of zero (These pairings are useful if students can productively engage with -5 – -5, -7 – -7, or both. If not, consider posing 5 – 5 before -5 – -5.)

successfully engage with (or justify the result of) the second task. The problem pairs in Table 3 could also be rewritten as True/False statements instead of used as stand-alone problems (e.g., True/False -8 + 3 = -8 – 3). Additionally, these tasks may support classroom discussions about logical necessity, underlying mathematical structures including fundamental properties and their importance, and equivalent transformations.

Use probing questions to support logical necessity

Even with the use of paired comparison tasks, students may need additional support in making the structural aspects of their reasoning explicit. In some instances of logical necessity, the interviewer posed additional questions that supported student reasoning; they may have asked for justification, pushed for alternative models/explanations of integer arithmetic, asked for clarification, or problematized some aspect of the student's response. Thus, certain contextual conditions, like the presence of probing questions, may support students to engage in logical necessity. Our second instructional implication, then, is to incorporate probing questions to support students to engage in logical necessity and make their reasoning explicit (see Bishop et al., in press for a more detailed discussion of probing questions). For example, if students used procedures or rules to transform problems like -5 – -3 = □, 6 – -2 = □, and 6 + -3 = □, the following types of follow-up questions were often posed: "Why can you change the problem like that (e.g., -5 – -3 to -5 + 3)?" or "Does changing the signs always work? Why do you think it always works?" Additionally, students were sometimes asked to generate an explanation for solving the *original* open number sentence (i.e., before they changed the problem): "I noticed that when you solved this problem (6 – -2 = □), you changed it to 6 + 2 = □. I understand how you solved 6 + 2, but do you have a way of thinking about what it means to subtract negative 2 from 6?" In other instances of logical-necessity-use, follow-up prompts may have problematized contradictions in student reasoning to provoke cognitive dissonance.

Probing questions sometimes, but not always, appeared to positively support students to make logical necessity explicit. At other times, students spontaneously engaged in logical necessity without any interviewer support. Because we did not code the larger data set for interviewer moves, we do not know how probing questions, in general, influenced the emergence of logical necessity. However, to provide students with the best opportunity to engage in logical necessity, we encourage the judicious use of probing questions along with carefully chosen contrasting pairs of problems.

Provide opportunities to conjecture

Our final instructional implication is based on the importance of conjectures as related to logical necessity. Instead of showing or telling students rules, we suggest that providing them more opportunities to hypothesize about how operations could or should behave would encourage them to think about underlying mathematical structure. Conjecturing also provides students opportunities to engage in argumentation, particularly if multiple, contrasting conjectures are advanced. As mentioned earlier, we believe that possibilities for logical necessity exist beyond the realm of integers and can be leveraged when students begin to reason about any new type of number—fractions, imaginary numbers, irrational numbers, and so on. Given the use of mathematical structure we have seen students use in integer arithmetic, we encourage teachers to allow students to take a conjectural stance toward developing the meaning of operations with extensions of the number system, be it with integers, rational numbers, or other types of numbers. We believe that instruction can be designed so that progress and understanding are powered by logical necessity.

Summary

In closing, we have shown that logical necessity can be used when students engage in integer tasks. About 10% of students in our study who had heard of negative numbers used logical necessity to approach and use numbers in a formal, algebraic way, leveraging key mathematical ideas about inverses, the structure of our number system, and fundamental algebraic properties to solve problems previously unsolvable for them. This kind of thinking, which is integral to modern mathematics, helped students to conjecture about meanings for adding and subtracting negative numbers and then to accept or reject those conjectures on the basis of whether the structure and logic of the system were preserved. We see this as a powerful way of thinking and an important form of structural reasoning for students to develop. Although the majority of the research on structure is situated in algebra or prealgebra contexts, our work provides evidence that with the emergence of logical necessity, structure can be addressed within other mathematical topics. Just as this article moves beyond transformational equivalence and clear links to algebra into the idea of number systems and extensions of those systems, we wonder what other aspects of mathematical structure exist and can be fruitfully explored. A broader understanding of where and how students use mathematical structure across more mathematical topics will help us in our efforts to broaden students' opportunities to see and use structure across the K–12 curriculum.

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Appendix A: Integers problem solving interview

Introductory Questions

1) Name a big number. Can you name a bigger number?
2) Name a small number. Can you name a smaller number? If the child responds, "Zero," ask, Is there a number smaller than zero?
3) Can you count backward, starting at 5? If child stops at 0 or 1, ask, Can you keep counting back?
4) What can you tell me about negative numbers? (Ask only if the student has previously mentioned the term "negative.")

Open Number Sentences

5) 5 + 6 = □ 6) 4 + □ = 9 7) □ − 4 = 6
8) 8 − □ = 4 9) 3 − 5 = □ 10) 6 + □ = 4
11) 5 − □ = 8 12) □ + 6 = 2 13) -3 + 6 = □
14) 8 − 3 = □
15) Yesterday you borrowed $8 from your friend to buy a school t-shirt. Today you borrowed another $5 from the same friend to buy lunch. What's the situation now? Can you write an equation or number sentence that describes this story? Explain how this number sentence (equation) relates to the story?
16) -2 + □ = 4 17) □ − 5 = -1 18) -9 + □ = -4
19) -2 − □ = -8 20) -5 + □ = -8 21) -3 − □ = 2
22) -8 − □ = -2 23) -8 + □ = 0 24) -5 + -1 = □
25) -5 − -3 = □ 26) 6 − -2 = □ 27) 6 + -3 = □
28) 3 + □ = 0
29) Imagine that you are standing on the beach. There is a bird flying 20 feet above the surface of the water and a fish swimming 5 feet below the surface of the water. How many feet higher is the bird than the fish? Can you write an equation or number sentence that describes this story? Explain how this number sentence (equation) relates to the story?
30) -5 − -5 = □ 31) -7 − -9 = □ 32) □ + -7 = -3
33) □ + -2 = −10 34) 3 − □ = -6 35) -2 − 7 = □

Appendix B: Integers ways-of-reasoning coding framework

Ways of reasoning categories	Definitions
Order-based	In this way of reasoning, one leverages the sequential and ordered nature of numbers to reason about a problem. Strategies include use of the number line with motion as well as counting forward or backward by 1s or another incrementing amount.
Analogy-based	This way of reasoning is characterized by relating numbers and, in particular, signed numbers, to another idea, concept, or object and reasoning about negative numbers on the basis of behaviors observed in this other concept. At times, signed numbers may be related to contexts (e.g., debt or digging holes). Analogy-based reasoning is often tied to ideas about cardinality and understanding a number as having magnitude.
Formal	In this way of reasoning, signed numbers are treated as formal objects that exist in a system and are subject to mathematical principles that govern behavior. Students may leverage the ideas of structural similarity, well-defined expressions, the structure of our number system, and fundamental principles (such as the field properties). This way of reasoning includes generalizing beyond a specific case by making a comparison to another, known, problem and appropriately adjusting one's heuristic so that the logic of the approach remains consistent or generalizing beyond a specific case to apply properties of classes of numbers, such as generalizations about zero.
Computational	In this way of reasoning, one uses a procedure, rule, or calculation to arrive at an answer. For example, some students used a rule to change the operation of a given problem along with the corresponding sign of the subtrahend or second addend (i.e., changing 6 − -2 to 6 + 2 or 5 + -7 to 5 − 7). Students explained these changes by referring to rules like "Keep Change Change" (*keep* the sign of the first quantity, *change* the operation, and *change* the sign of the second quantity).
Developmental	The category of reasoning often reflects preliminary attempts to compute with signed numbers. For many strategies in this category, the domain of possible solutions is locally restricted to non-negatives. For example, a child may overgeneralize that addition always makes larger, and, as a result, claim that a problem for which the sum is less than one of the addends (6 + □ = 4) has no answer. The domain of possible solutions appears to be restricted to whole numbers and the effect (or possible effect) of adding a negative number is not considered.