



Happy and sad thoughts: An exploration of children's integer reasoning[☆]

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ABSTRACT

The purpose of this study was to investigate elementary children's conceptions that might serve as foundations for integer reasoning. Working from an abstract algebraic perspective and using an opposite-magnitudes context that is relevant to children, we analyzed the reasoning of 33 children in grades K–5. We focus our report on three prominent ways of reasoning. We do this by describing and analyzing the responses of three particular children (in Grades 1, 3, and 5) who exemplify these ways of reasoning. We view each of the three ways of reasoning as rich and interesting, and we see relationships of each to formal integer reasoning. At the same time, we view these ways of reasoning in terms of increasing levels of sophistication, potentially belonging to a single learning trajectory. Thus, we see the roots of more sophisticated integer reasoning in children's early intuitions about opposite magnitudes.

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As mathematically literate adults, we sometimes forget how strange the notion of a negative number can be. We may not remember our own introductions to negatives. We have the luxury of taking them for granted. At the same time, however, negatives have special status. For example, it would seem disingenuous to say that you arrived early for an appointment, only to explain when pressed that you arrived -15 min early.

The extension of the natural numbers poses challenges for both teachers and students. Children are expected to expand their mathematical worlds to include negative integers, as well as (nonnegative) rational numbers, on the way to $\mathbf{Q}$ and $\mathbf{R}$ (Bruno & Martínón, 1999). These extensions challenge students' previous conceptions, which often involve overgeneralizations of their experiences with the natural numbers (e.g., that addition makes bigger and subtraction makes smaller). There is a wealth of research concerning teaching and learning in the rational-number domain (e.g., Behr, Harel, Post, & Lesh, 1992; Empson & Levi, 2011; Fosnot & Dolk, 2002; Lamon, 1999; Sowder, 1995). However, few reports exist that address children's reasoning about integers (Kilpatrick, Swafford, & Findell, 2001).

Integers and integer operations present conceptual difficulties for students (Janvier, 1983; Vlassis, 2004). These difficulties can be appreciated in light of the history of mathematics, wherein mathematicians struggled with counterintuitive notions associated with negative numbers (Gallardo, 2002; Henley, 1999; Thomaidis & Tzanakis, 2007). How can -7 be considered less than 3 when clearly its magnitude is greater? Why should a negative times a negative equal a positive? These questions so troubled Western mathematicians that many adamantly resisted the use of negative numbers. Others used negatives for certain purposes but refused to recognize them as legitimate numbers. This resistance continued through the 18th century, despite the fact that the rules for arithmetic of signed numbers were well known—having been established more than a millennium earlier—and negative numbers had proven useful for solving algebraic equations (Henley, 1999).

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In light of the struggles that great mathematicians faced in coming to terms with negative numbers, it is no wonder that children face difficulties in making sense of integers and operations involving them. On the other hand, researchers have found that children, even in the lower elementary grades, are capable of reasoning about integers in relatively sophisticated ways (Behrend & Mohs, 2006; Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011; Hativa & Cohen, 1995; Wilcox, 2008). For example, Bishop et al. found that first graders who had not previously been introduced to negative numbers began to invent and reason productively about them in the contexts of playing a number-line game and solving open number sentences.

In the interest of contributing to the research concerning children's reasoning about integers and integer operations, we have used a variety of tasks in interviews with children. In this report, we present analyses of children's ways of reasoning about one set of tasks.

1. Background

We lay the groundwork for the study by discussing the integers in terms of what we will call magnitudes and formalisms. We then discuss the context and specific tasks that were used in the interviews, and we relate these to the integers formally.

1.1. Magnitudes and formalisms

We distinguish two realms of integer reasoning: magnitudes and formalisms.¹ We use the term *opposite magnitudes* to refer to contexts such as lending and owing money, traveling forward and backward, elevation (above and below some relative zero), temperature (above and below zero degrees), and so on. Such contexts consist of a quantitative element (e.g., an amount of money) and a qualitative element, which is dichotomous (e.g., who owes the money to whom). For brevity, we sometimes just use the term *magnitudes*.

By the *formal realm*, we mean integers as represented numerically, together with the mathematical properties of this set of numbers. These numbers exist as abstract entities. They need not be related to quantities in the world. For example, it is a fact in the formal realm that the sum of 5 and -3 is 2. We may justify this fact by appeal to the properties of the integers. We need not relate the numbers to contexts such as money. This formal interpretation of integers applies from basic to advanced mathematics. In abstract algebraic terms, the set of integers $\mathbf{Z}$ is a ring. For the purposes of this paper, only the addition operation is relevant, so we will talk about $\mathbf{Z}$ as an abelian additive group.

The realms of magnitudes and formalisms are relevant to the history of integers and to discussions of the integers from an expert perspective. In addition, in the study of children's reasoning about integers, these realms provide lenses through which children's reasoning may be investigated. From the formal side, we can study children's reasoning about symbols such as -5 and expressions such as $-5-7$, including the meanings that children associate with these symbols. Alternatively, we can study children's reasoning about opposite magnitudes, such as owing and lending money, including the symbols that they associate with these.

We believe that part of understanding integers is the ability to coordinate magnitudes and formalisms, as in associating a debt of \$5 with the integer -5 . Thus, an investigation from either side may lead to a focus on this coordination. However, the two approaches are distinct, and each has its advantages and disadvantages. Our group's research into children's reasoning about integers has involved both approaches. For example, we took the formal route in reporting on high school students' reasoning about open number sentences involving integers (Lamb et al., in press). In this article, we take the magnitude approach. We discuss elementary children's responses to a task situated in the realm of magnitudes. In our analysis, we relate the children's reasoning to the formal realm.

1.2. The formal realm: integers and integer addition

In discussing the mathematics of integers for the purposes of this paper, we focus on the notions of additive inverses and of integers as representatives of equivalence classes of sums.

1.2.1. Inverses

An essential property of the integers is that of inverses. In the natural numbers, addition can never make smaller (i.e., when two addends are combined, neither addend can exceed the sum). This constraint does not apply to the integers because of the existence of additive inverses. Every integer a has an additive inverse $-a$ such that $a + -a = 0$. For example, the inverse of -3 is 3, and vice versa. We use this property when adding integers. Given any two integers with different signs, we can determine their sum by reasoning about the difference between the absolute values. For example, in $5 + -3$, the difference between the absolute values is 2. Because 5 is greater than 3, the sum is $+2$. To see this, we can think of forming zero pairs. We have $+1 + 1 + 1 + 1 + 1 + -1 + -1 + -1$, which can be regrouped as $(+1 + -1) + (+1 + -1) + (+1 + -1) + 1 + 1 = 0 + 0 + 0 + 1 + 1 = +2$. A shorthand version of this process would be to say that $5 + -3 = (2 + 3) + -3 = 2 + (3 + -3) = 2 + 0 = 2$.

¹ Note that in making this distinction, we do not intend to favor magnitudes over formalisms, or vice versa. We merely find the distinction useful.

1.2.2. Equivalence classes

An integer can be regarded as representing an equivalence class of sums. For example, -3 can be regarded as the sum of any pair of integers a and b such that $a + b = -3$. In this way, -3 stands for the equivalence class of all such pairs of integers (a, b) . For example, $-3 = 0 + -3 = 1 + -4 = 2 + -5$, and so on. We can think of equivalence classes of sums in the natural numbers as well. However, because in the natural numbers addition cannot make smaller, a given natural number is representative of only a finite number of natural-number sums. A given integer, on the other hand, is representative of infinitely many integer sums.

1.3. The realm of magnitudes: the context of happy and sad thoughts

We now turn to the interview tasks to be discussed. These tasks were situated in a context of happy thoughts and sad thoughts. The description of the context was the following:

Everyday, Jessica has happy thoughts and sad thoughts. If she has one happy thought and one sad thought, then she just feels normal—not happy or sad.

Initial tasks were like the following:

On Monday, Jessica had 2 happy thoughts and 7 sad thoughts. What kind of day was Monday?

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Children were given a sheet of paper with smiling faces and frowning faces representing the happy and sad thoughts. They were asked what Jessica's day would be like if she had particular numbers of happy and sad thoughts, as in the example above. They were given several days to consider one at a time, and then were asked to make comparisons between days. For example, if a child identified two days as sad days, she or he was asked to say whether one day was sadder than the other and to explain the answer. In some cases, additional tasks and questions were posed in this context, but the above description is common to all interviews.

1.4. Coordinating the two realms: the mathematical properties of the context

Reflecting on this task, we see two reasonable interpretations of the premise that one happy thought and one sad thought would result in Jessica's feeling "normal": (a) that a *difference* of zero in the numbers of happy and sad thoughts causes Jessica to feel normal, or (b) that a 1:1 *ratio* of happy to sad thoughts causes Jessica to feel normal. In either case, any equal numbers of happy and sad thoughts would result in Jessica's feeling normal. However, the equivalence classes of happy/sad days are influenced by the interpretation. In the additive case, a day with two happy thoughts and seven sad thoughts would belong to the same equivalence class as a day with one happy thought and six sad thoughts. That is, those sad days with a common difference in the numbers of happy and sad thoughts would be regarded as equally sad. In the multiplicative case, days with the same ratio, such as twice as many sad thoughts as happy thoughts, would belong to the same equivalence class. Thus, those sad days with a common ratio of sad to happy thoughts would be regarded as equally sad. We have seen children interpret the context in either of these ways, with the additive interpretation being much more common. In our analysis, we focus on the reasoning of those children who interpreted the context additively.

We claim that children who interpret the context additively reasoned about the tasks in a mathematical environment (Greeno, 1991) that, formally speaking, is isomorphic to the integers as an additive group. Thus, we regard these children as reasoning about integer addition, up to isomorphism. Clearly, we are making this statement from an expert perspective. We do not claim that the children thought about the context in abstract algebraic terms. Rather, the properties of the mathematical environment in which they were working were analogous to those of the integers with addition.

We briefly discuss the correspondence between the context of happy and sad thoughts, interpreted additively, and the integers as an additive group. Consider the group $G = \{ \dots, sss, ss, s, n, h, hh, hhh, \dots \}$. The group operation is concatenation, which maps the pair of elements (x, y) to the element xy . This operation is commutative, so that $xy = yx$ for all x, y in G . The identity element is n , which is to say that $xn = nx = x$ for all x in G . For every element x , there exists in G an element x^{-1} such that $xx^{-1} = n$. For example, $(sss)(hhh) = n$. In general, an element formed by the concatenation of multiple elements can be simplified via the properties of commutativity and associativity, together with the cancellation rule $sh = hs = n$. For example, $sshshs = (sh)(sh)(sh)s = s$.

Now observe that G is isomorphic to $\mathbf{Z}$. The isomorphism is a mapping that takes any negative integer a to the corresponding number of s 's and any positive integer b to the corresponding number of h 's. For example, -3 maps to sss , and $+2$ maps to hh . Zero maps to n . The set of happy and sad thoughts with the concatenation operation, as described above, behaves just as does the set of integers with the addition operation. Given this isomorphism, we are interested in children's additive reasoning in tasks involving happy and sad thoughts as a way of investigating the ways of reasoning that they might bring to the integers in the formal realm.

1.5. Limitations of contexts

We acknowledge that the context of happy and sad thoughts has limitations. Realistically, happy and sad thoughts do not necessarily cancel each other; they may coexist. Thus, the additive interpretation may not be true to life. Furthermore, happy and sad thoughts do not necessarily have equal weights. We find limitations to any context that one uses in relation to integer addition. In fact, mathematicians struggled for centuries to make sense of positive and negative numbers by relating them to real-world contexts (e.g., [Colebrooke, 1817](#), pp. 216–217 [Bhāscara]; [Fibonacci, 1202/2002](#), pp. 458–459; [Flegg, Hay, & Moss, 1484/1985](#), p. 206 [Chuquet]). Ultimately, this search for clarifying models was abandoned in favor of a formal approach and a theoretical interpretation of number ([Henley, 1999](#)). The advantage to the happy and sad context is that it is accessible to anyone who is familiar with happiness and sadness and is capable of comparing small numbers of things. Although children in the lower elementary grades tend to be unfamiliar with positives and negatives, they are familiar with the notion of opposites, and this may be a productive source of intuitions for them.

In presenting the tasks, we invited children to suspend any disbelief and to entertain a world in which happy and sad thoughts were of equal weight. We did not force an additive interpretation during the interviews, but most children (30 of 33) did interpret the behavior of happy and sad thoughts additively. The context of happy and sad thoughts was originally suggested to us by a child in an early pilot interview. We have found this context to be accessible to children in lower elementary grades, as well as to older students, and we have found some children's responses to be fascinating.

Also note that our purposes were not pedagogical in nature. We are not suggesting the use of this context as an approach to teaching children about integers. On the contrary, we are interested in what we can learn about children's reasoning from their responses to the tasks. Previous research has indicated that children are capable of reasoning about integers prior to formal instruction. We used tasks involving opposite magnitudes to investigate the ways of reasoning that children might bring with them to the study of the integers.

2. Methods

We now state our research questions and describe the research participants and the methods of data collection and analysis.

2.1. Research questions

Other researchers have used context problems to investigate children's mathematical reasoning in the whole-number domain (e.g., [Carpenter, Fennema, Franke, Levi, & Empson, 1999](#); [Empson & Levi, 2011](#)). In this study, we took a similar approach to the investigation of children's reasoning about integers. The research questions that we address are the following: How do children reason about tasks involving opposite magnitudes, and how do their ways of reasoning relate to formal integer reasoning?

2.2. Participants

A total of 55 children in grades K-5 participated in interviews during the spring of 2010. The interviews were conducted at elementary schools in both California and Texas. Various tasks in both the magnitude and formalism realms were posed to the children. Happy and Sad Thoughts tasks were posed to 33 of these children. We focus on the responses of three children—one from each of Grades 1, 3, and 5—who exemplify three distinct ways of reasoning about the tasks.

2.3. Interview methods

We conducted semistructured interviews, which were designed to investigate children's conceptions, rather than to promote learning. The interviewers made an effort to use the children's language and so did not introduce terms such as *positive* and *negative*. Nor did the interviewers suggest solution strategies. Instead, they presented the tasks and then investigated the children's reasoning through the use of follow-up questions and probes ([Rubin & Rubin, 1995](#)).

2.4. Data analysis

We reviewed both video and written records of children's responses. Beginning with a subset of the data, we initiated a process of open coding of categories of responses. Using principles of grounded theory, we refined our understanding of distinct responses through a process of constant comparative analysis ([Strauss & Corbin, 1998](#)). We generated categories of ways of reasoning and then tested these against the remaining data. We found that the ways of reasoning that we identified initially were typical across the data set. That is, a small number of categories proved sufficient to characterize the reasoning of almost all the children. Having generated and refined categories of children's ways of reasoning, we then related features of these to the mathematical properties of the context. In this way, an abstract-algebraic lens also informed our analysis.

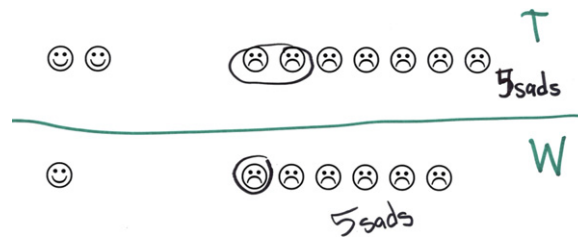


Fig. 1. Chelsea compares (2, 7) and (1, 6).

3. Results

We present three children's responses, followed by our analyses. For convenience, we use the notation (a, b) to denote the numbers of happy and sad thoughts in a given day. For example, a day with two happy thoughts and seven sad thoughts would be represented as $(2, 7)$. This notation is used here in the interest of brevity. It was not used with the children, unless they introduced such a notation.

3.1. The children's responses

We present the responses of three children and discuss their reasoning.

3.1.1. Stephen's responses

Stephen, a first grader, was unfamiliar with the notion of negative numbers. He correctly stated whether a day was happy, sad, or neither, on the basis of comparison of the numbers of happy and sad thoughts. Stephen identified $(2, 7)$, $(1, 6)$, and $(4, 8)$ as sad says. He said that $(4, 8)$ was the saddest day because it had the "most sads." Stephen also identified two happy days as equally happy because they had the same number of happy thoughts, even though the numbers of sad thoughts were unequal. At no point did he use the difference between the numbers of happy and sad thoughts as a way of quantifying the happiness of a day.

Stephen seemed to recognize the significance of having equal numbers of happy and sad thoughts. Initially, he was shown a $(1, 0)$ day, which he identified as happy, and a $(0, 1)$ day, which he identified as sad. When he was shown a $(1, 1)$ day, he said that the person would feel "happy and sad." Stephen was told that if Jessica had one happy thought and one sad thought, she would just feel *normal*—not happy or sad. He suggested that this normal state should be represented by a "confused" face with the mouth drawn straight across, rather than smiling or frowning. A few minutes later, after Stephen had made his comparisons across sad days, the $(4, 8)$ day was modified to $(5, 8)$, and then $(6, 8)$, and finally $(8, 8)$. As the happy thoughts were added, Stephen continued to refer to the day as sad—in fact, very sad. However, when he recognized that there were eight happy thoughts, he exclaimed, "It's equal!" and called this a "happy-and-sad day."

3.1.2. Stephen's reasoning

Stephen was able to decide whether a given day was happy or sad by comparing the numbers of happy and sad thoughts. He also treated equal numbers of happy and sad thoughts as resulting in a state of being neither happy nor sad. He seemed to recognize the significance of having equal numbers of happy and sad thoughts. He even suggested a special notation for feeling "confused."

Stephen did not quantify the happiness or sadness of a day on the basis of the balance of happy/sad thoughts. He simply identified days as happy or sad. Without a measure of the balance of happy and sad thoughts, he was unable to correctly compare days. To compare the sadness of two sad days, he simply compared the numbers of sad thoughts, ignoring the happy thoughts completely. For Stephen, a given day had three possible descriptions: happy, sad, and neither ("confused"). At the same time, however, not all happy/sad days were equal. Stephen compared happy days by comparing the numbers of happy thoughts in each, ignoring the numbers of sad thoughts. Likewise, he compared sad days by comparing only the numbers of sad thoughts. Thus, to compare days, he used a metric for either happy thoughts or sad thoughts, rather than a metric for the balance of happy and sad thoughts.

3.1.3. Chelsea's responses

Chelsea was in Grade 3. She had heard of negative numbers but had not received any formal instruction concerning them. Chelsea said that Monday $(5, 11)$ was a sad day. She circled five of the sad thoughts, saying, "It takes away those five" (referring to the effect of the happy thoughts on the five sad thoughts). She said that there were six more sad thoughts than happy thoughts on Monday, so Monday was a sad day. Chelsea quantified the sadness of Monday as "six sad." For Wednesday $(1, 6)$, she quickly said, "Bad day. Five sads." When asked to compare Tuesday $(2, 7)$ and Wednesday $(1, 6)$, she responded, "They're both the same." When asked if the numbers of happy thoughts mattered, she said, "No 'cause they took away 'em." (See Fig. 1 for Chelsea's work.)

3.1.4. Chelsea's reasoning

Like Stephen, Chelsea was able to evaluate a given day as being happy or sad. However, unlike Stephen, Chelsea quantified the degree of happiness/sadness of a day, using the balance of happy and sad thoughts directly to label days, for example as “5 sads.” Chelsea is also distinguished from Stephen in that she was able to correctly compare days. She used her explicit quantification of the happiness/sadness of individual days to make comparisons. For example, a “5 sads” day was sadder than a “4 sads” day. She also explicitly canceled pairs of happy and sad thoughts.

3.1.5. Tania's responses

Tania, a 5th grader, had been introduced to integers in school and had received some instruction concerning integer arithmetic. Tania was able to assess whether a day was happy or sad and to make comparisons between days. She said that having one happy thought and one sad thought would lead to feeling “neutral.” Tania spontaneously identified the happy and sad thoughts as corresponding to positive and negative numbers, respectively, noting that she associated the word *positive* with being happy and *negative* with being sad. Tania crossed out happy/sad pairs and used integer notation to record the net sadness for each day. For example, she wrote “–5” next to a day with five more sad thoughts than happy thoughts. She talked about happy thoughts and sad thoughts as “balancing” one another out. Tania also wrote number sentences to represent the happiness/sadness of days:

For Monday (2, 7), she wrote $(+2) + (-7) = -5$.

For Tuesday (1, 6), she wrote $(+1) + (-6) = -5$.

For Wednesday (4, 8), she wrote $(+4) + (-8) = -4$.

She said that using the parentheses helped her keep the signs and operations separate.

3.1.6. Tania's reasoning

Tania is distinguished from both Stephen and Chelsea in that she explicitly related happy and sad thoughts to positive and negative numbers and wrote number sentences involving these. Somehow, the context resonated with her as being related to integers, which she had learned about in school. In speaking about the context here, we include the presentation of numbers of happy and sad thoughts as smiling and frowning faces, together with the premise that one happy and one sad would result in feeling “normal” or “neutral.” Evidently, these characteristics of the context led Tania to decide that these tasks were related to integer addition.

In terms of a willing suspension of disbelief, the younger children did not seem to have any reservations about whether the context was realistic. Tania, on the other hand, was attuned to this issue. Although she correctly identified two days as “the same” based on equal balances of happy/sad thoughts, she expressed reservations about whether Jessica would, in fact, feel the same on those days: “I don’t know because mathematically it would go that way maybe, but sad thoughts and happy thoughts aren’t always exactly what they work out on paper.” Thus, although Tania adopted an additive interpretation of the context and recognized a connection with the integers, she also perceived realistic limitations to that correspondence. She seemed aware that happy and sad thoughts do not necessarily behave additively.

3.2. Children's ways of reasoning

We present categories of reasoning that are exemplified by the children's responses. In doing so, we take a step away from the particular children's thinking and describe these ways of reasoning more generally. We also shift to foregrounding relationships between magnitudes and formalisms by framing our analysis in more mathematical terms.

We present these findings in two groups. First, we describe children's ways of reasoning in evaluating the happiness/sadness of a single day. Second, we describe the ways of reasoning involved in making comparisons between days.

3.2.1. Children's ways of evaluating the happiness or sadness of a day

The children's responses presented in this paper exemplify three distinct ways of evaluating the happiness or sadness of an isolated day:

1. *Sign function.* The child compares the numbers of happy and sad thoughts in the day. The child identifies the day as happy if it consists of more happy thoughts than sad thoughts and as sad if it consists of more sad thoughts than happy thoughts. A day with equal numbers of happy and sad thoughts is identified as neither happy nor sad. This way of reasoning is similar to the sign function in algebra, which takes only three possible values, -1 , $+1$, or 0 .
2. *Balance metric.* The child treats happy and sad thoughts as canceling in pairs, and after pairs are canceled (whether mentally or in writing), the balance is revealed. If only happy thoughts remain, then the day was happy, and the degree of happiness is denoted by the number of happy thoughts remaining. Likewise for sad days. If no thoughts remain, then the day was neither happy nor sad.

Table 1

Children's ways of evaluating the happiness or sadness of a day.

	More sad than happy thoughts (e.g., (2, 7))	More happy than sad thoughts (e.g., (3, 2))	Equal numbers of thoughts (e.g., (8, 8))
Sign function	Sad	Happy	Neither
Balance metric	5 sads	1 happy	Neither
Explicit integer sum	Sad ($2 + -7 = -5$)	Happy ($3 + -2 = 1$)	Neutral ($8 + -8 = 0$)

3. *Explicit integer sum.* The number of happy thoughts is translated to a positive integer, and the number of sad thoughts is translated to a negative integer. The net happiness or sadness of a day is given by the sum of the two integers. The result is interpreted in terms of happiness/sadness.

These ways of reasoning are exemplified by the cases that were presented. Stephen used the sign-function approach. Chelsea's reasoning exemplifies the balance-metric approach. Tania used explicit integer sums. Table 1 presents examples of each of these ways of reasoning.

The use of happy and sad thoughts as inverses was clear in the balance-metric and explicit-integer-sum approaches. Although integer sums can be computed in various ways, Tania talked about the happy and sad thoughts as canceling one another in pairs. Children using the sign-function approach may implicitly treat happy and sad thoughts as inverses; that is, they identify days with more happy thoughts than sad thoughts as simply being happy, as opposed to more happy than sad. (By contrast, if Jessica had four blue balloons and seven red balloons, to say that her set of balloons was red would not be sensible.) Thus, these children may have extended the given premise that one happy thought and one sad thought resulted in Jessica's feeling "normal" to a more general way of resolving sets of happy and sad thoughts to a single result (happy or sad).

For days with equal numbers of happy and sad thoughts, each way of reasoning enabled the children to recognize that these days would be neither happy nor sad. From the sign-function approach, the number of happy thoughts did not exceed the number of sad thoughts, or vice versa, so the day was neither happy nor sad.² Using a balance metric, all of the happy and sad thoughts canceled in pairs, and nothing was left. In terms of explicit integer sums, the sum of the values of the day's thoughts was 0. In each case, the children extended from the premise that one happy thought and one sad thought resulted in neutrality to the apparent generalization that any equal number of happy and sad thoughts would also result in neutrality.

3.2.2. Children's ways of comparing happy or sad days

Although the task of evaluating the happiness or sadness of a single day did not require the children to quantify the balance of happy and sad thoughts, in making comparisons across days, one needs to attend to the specific differences in some way. The ability to correctly compare across days was a major distinguishing characteristic in children's responses. We present three distinct ways of reasoning for making these comparisons:

1. *Uncoordinated metric.* The child compares days by comparing only the numbers of happy (sad) thoughts. The day with more happy (sad) thoughts is identified as happier (sadder).
2. *Coordinated metric.* The child compares days by first applying a balance metric to each day. The metric is then used to compare the two days.
3. *Comparing integer sums.* The child translates the happy and sad thoughts of each day to integer sums. The sums are determined and compared, and the result of the comparison is interpreted in terms of happiness/sadness.

Stephen used the uncoordinated-metric approach. Chelsea used a coordinated metric, which built naturally on her balance-metric approach. Tania built on her use of explicit integer sums to compare happy/sad days by comparing integer sums. Table 2 provides examples of each way of reasoning in cases in which the days belong to different equivalence classes or to the same equivalence class. For comparison, we also include a formal algebraic treatment, situated purely in the integers.

The uncoordinated metric does not allow for a valid comparison between days, except in the special case of neutral days. For example, Stephen referred to both (1, 1) and (8, 8) as "happy-and-sad" days. Neutral days were the only equivalence class that Stephen recognized. Chelsea and Tania, by contrast, recognized various cases in which days belonged to the same equivalence class, as well as cases in which days belonged to distinct equivalence classes.

Chelsea and Tania explicitly quantified the happiness/sadness of a day to make comparisons between days. Chelsea identified days with scores such as "5 sads." Tania used integer labels, such as "-5" for the same purpose. By quantifying happiness/sadness in these ways, the children gave names to the equivalence classes of days. For example, a day identified by Chelsea as "5 sads" could be readily recognized as "the same" as any other "5 sads" day.

² It may be that Stephen saw such days as being both happy and sad, rather than neutral. In any case, he recognized days with equal numbers of happy and sad thoughts as belonging to their own special class.

Table 2
Ways of comparing happy or sad days.

	Different equivalence classes (comparing (4, 8) and (2, 7))	The same equivalence class (comparing (2, 4) and (1, 3))
Uncoordinated metric	(4, 8) is sadder than (2, 7) because it has more sad faces.	(2, 4) is sadder than (1, 3) because it has more sad faces.
Coordinated metric	(2, 7) is “5 sad” and (4, 8) is “4 sad,” so (2, 7) is the sadder day.	(1, 3) and (2, 4) are both “2 sad” days, so they are the same.
Comparing integer sums	(2, 7) yields -5 , and (4, 8) yields -4 , so (2, 7) is the sadder day.	(1, 3) yields -2 , and (2, 4) yields -2 , so they are equal.
Formal algebraic	(2, 7) belongs to the equivalence class -5 , or $[(0, 5)]$, whereas (4, 8) belongs to the equivalence class -4 , or $[(0, 4)]$. Because $-5 < -4$, $\alpha < \beta$ for all α in $[(0, 5)]$ and β in $[(0, 4)]$.	(1, 3) and (2, 4) both belong to the equivalence class $[(0, 2)]$. Therefore, they are equivalent.

3.3. Paired ways of reasoning and levels of sophistication

We chose to code separately for children’s ways of reasoning within a single day and in comparing days. However, because these tasks are related, the ways of reasoning pair together in a natural way. Stephen (the 1st grader) used sign function and uncoordinated metric, Chelsea (the 3rd grader) used balance metric and coordinated metric, and Tania (the 5th grader) used explicit integer sums and comparing integer sums. These pairings were typical. In fact, among the 30 children whose reasoning was identified as additive, every instance of uncoordinated metric was paired with sign function, every instance of coordinated metric was paired with balance metric, and every instance of comparing integer sums was paired with explicit integer sums.³

We conceptualize these pairs as increasingly sophisticated ways of reasoning about opposite magnitudes. This view is based on our judgment of the mathematical sophistication of each way of reasoning. It is also corroborated by grade-level trends in the data. The sign function and uncoordinated metric pair was used by the majority (11 of 18) of the students in grades K-3, whereas it was used by few of the 4th graders (2 of 9), and none of the 5th graders (0 of 5). The balance metric and coordinated metric pair was used by the majority of the 4th graders (5 of 9) and the 5th graders (3 of 5). Explicit integer sums and comparing integer sums were only used by 5th graders (2 of 5).⁴

In addition to representing increasing levels of sophistication, we view the ways of reasoning as potentially belonging to a single learning trajectory. That is, we can imagine Stephen coming to reason like Chelsea, and Chelsea, in turn, coming to reason like Tania.

4. Discussion

We briefly discuss a few themes that arise from this work.

4.1. Potential generality of the children’s ways of reasoning

In discussing the ways of reasoning that we identified, we have related these to the formal realm. To relate them to other contexts involving opposite magnitudes also seems useful. We see no reason to expect that the ways of reasoning that we identified are specific to the context of happy and sad thoughts. Rather, we anticipate that children reason similarly in various contexts that share the same mathematical properties.

Consider events at bus stops, as discussed by Streefland (1996). At a given stop, a certain number of people get off the bus, and a certain number of people get on. The bus gains or loses a number of passengers at each stop, on the basis of the balance of the number who boarded versus exited. If we are interested in the total number of passengers on the bus at a given time, then clearly this context behaves additively. We can use the notation (a, b) for an event at a particular stop, where a represents the number of people who board the bus, and b is the number who exit the bus. The equivalence classes of these events look just like equivalence classes of happy/sad days, under the additive interpretation (e.g., $[(0, 5)] = [(1, 6)]$).

One can imagine children applying the same ways of reasoning to this context. The sign-function approach would entail identifying whether more or fewer people were on the bus after a stop than before it, but without quantifying the number gained or lost. The balance metric would be the net change in the number of people on the bus. Likewise, one could write

³ We note that some students who used the sign function approach used a different way of reasoning to compare days. This alternative approach, which we call *coordinating between* involves comparing the numbers of happy thoughts between the two days, as well as comparing the number of sad thoughts between the two days, and using these differences to determine which day was happier or sadder. This way of reasoning can enable children to correctly compare days, and we regard it as more sophisticated than the uncoordinated-metric approach. It is primarily for reasons of length that we do not elaborate further on the coordinating-between approach in this paper.

⁴ It is unsurprising that 5th graders were the only interview participants to use explicit integer sums because doing so requires familiarity with integer addition. We would expect this way of reasoning to be more prevalent amongst middle and high school students.

expressions involving integer addition to represent each event. In terms of comparisons of events, we can imagine Stephen applying an uncoordinated metric, as in saying that more people got on the bus at Stop A (7, 4) than at Stop B (5, 1), without accounting for the number who got off. We can see how a balance-metric approach would enable a child like Chelsea to correctly compare across events, and similarly how Tania might do this by comparing integer sums.

Our analysis of children's ways of reasoning about happy and sad thoughts becomes useful insofar as these ways of reasoning relate to (a) children's ways of reasoning in other contexts involving opposite magnitudes and (b) the ways of reasoning that children bring with them to their study of the integers. One can imagine how a child like Chelsea could bring her balance-metric reasoning to bear on integers and integer sums. We note, however, that Chelsea's future experiences with integers may or may not connect to her own ways of reasoning about opposite magnitudes. We have found that students who have been taught rules for operating with integers may apply these rules algorithmically and often incorrectly, instead of attempting to make sense of tasks such as open number sentences. Whether Chelsea grows up to resemble these students or builds on her intuitions about opposite magnitudes to develop an understanding of integers and integer operations is a question of great concern to us, and it seems to hinge on her experiences with integers in formal instruction during middle school.

4.2. Recognizing integers in context

Tania explicitly related happy and sad thoughts to positive and negative numbers and spontaneously wrote number sentences involving integers to describe happy and sad days. Instead of discussing Tania's conceptions related to integer addition, which is beyond the scope of this paper, we briefly discuss her recognition of the context as being related to integers. The ability to "see integers" in story-problem or real-world contexts is a characteristic of individuals with sophisticated understandings of integers. It involves the explicit coordination of the realm of opposite magnitudes with the formal realm. As discussed earlier, we can identify an isomorphism between the integers, regarded as an additive group, and the set of happy and sad thoughts, under the additive interpretation. We certainly do not expect that Tania saw the correspondence between happy and sad thoughts and the integers in abstract algebraic terms, so what enabled her to recognize a correspondence?

As discussed earlier, the Happy and Sad Thoughts context involves opposite magnitudes. Happy and sad are opposite emotions, and the assumption that they behave as inverses make this relationship precise. The context is discrete, like the chip model that is sometimes used in integer instruction. Further, the connotations of the words *positive* and *negative* afforded a natural mapping to happy and sad thoughts, which Tania explicitly recognized. We offer these relationships between the context and integers as the beginnings of a hypothesis concerning how Tania recognized integers in this context. Accounting for how and why children like Tania can recognize integers in contexts may be a topic of future research.

In addition to recognizing a correspondence between the happy/sad context and the integers, Tania also recognized limitations to that correspondence. She pointed out during her interview that happy and sad thoughts might not realistically behave the way they did in our story problems. When those in our research group think about contexts involving opposite magnitudes, we can choose to superimpose the integers on those contexts or not. We also recognize ways in which correspondences between opposite magnitudes and the integers test our willingness to suspend disbelief. For example, although we are capable of seeing integer arithmetic in a temperature context, we sympathize with Freudenthal (1983) who described the notion of negative temperature gains as "unworthy of belief" (p. 437). Thus, although we regard the recognition of a correspondence between opposite magnitudes and the integers as an aspect of sophisticated integer reasoning, we also view recognition of the limitations of that correspondence as part of a mature understanding of integers.

4.3. Relationships between reasoning about opposite magnitudes and reasoning about integers

We have seen early indications of relationships between children's reasoning about opposite magnitudes and their potential for reasoning formally about integers. For example, some middle and high school students reason that the answer to $2 + -7 = \square$ is negative before determining its specific value. We refer to this as *inferring the sign* of the unknown. Taking this reasoning further, they then determine the sum based on the difference between the absolute values of the two addends.⁵ Thus, in the formal realm, many of the children we have interviewed have inferred the signs of unknown numbers by reasoning about the relative sizes of given numbers, together with the results of adding or subtracting. In the realm of opposite magnitudes, we saw young children reasoning similarly in determining whether a day was happy or sad: they could determine *that* a day was happy or sad without necessarily determining *how* happy or sad the day was. We believe that the ability to infer the signs of answers is one critical aspect of understanding integers, and we have found that young children exhibit nascent ideas in this vein. Given appropriate support, these ideas could be leveraged to develop robust reasoning about integer arithmetic.

⁵ This is in contrast to a procedure in which the student notices that the signs are different, subtracts 2 from 7 because the signs are different (and the rule says "different signs, subtract"), and then assigns to the result the sign of the "larger" number. Inferring the sign entails first reasoning about the sign of the unknown and then determining its specific magnitude.

5. Conclusion

We have presented three paired ways of reasoning about tasks involving opposite magnitudes through the discussion of three children's responses. We have identified increasing levels of sophistication in this reasoning in drawing contrasts between younger and older children. We believe that these findings illustrate the potential for tasks in opposite-magnitude contexts to reveal important aspects of children's reasoning that are relevant to understanding integers. Although we used the context of happy and sad thoughts, our intention is not to argue for the use of that particular context, either for research or pedagogical purposes. We simply regard it as an example of a context involving opposite magnitudes that is accessible to children in Grades K-5. Such tasks enable us to investigate children's intuitions about opposite magnitudes prior to instruction concerning negative numbers. This study provides additional evidence that children have intuitions that can potentially support their later reasoning about integers in sophisticated ways. These findings and others lend credence to the hypothesis that children may be capable of reinventing integer arithmetic for themselves, in lieu of being taught the sign rules through direct instruction.

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