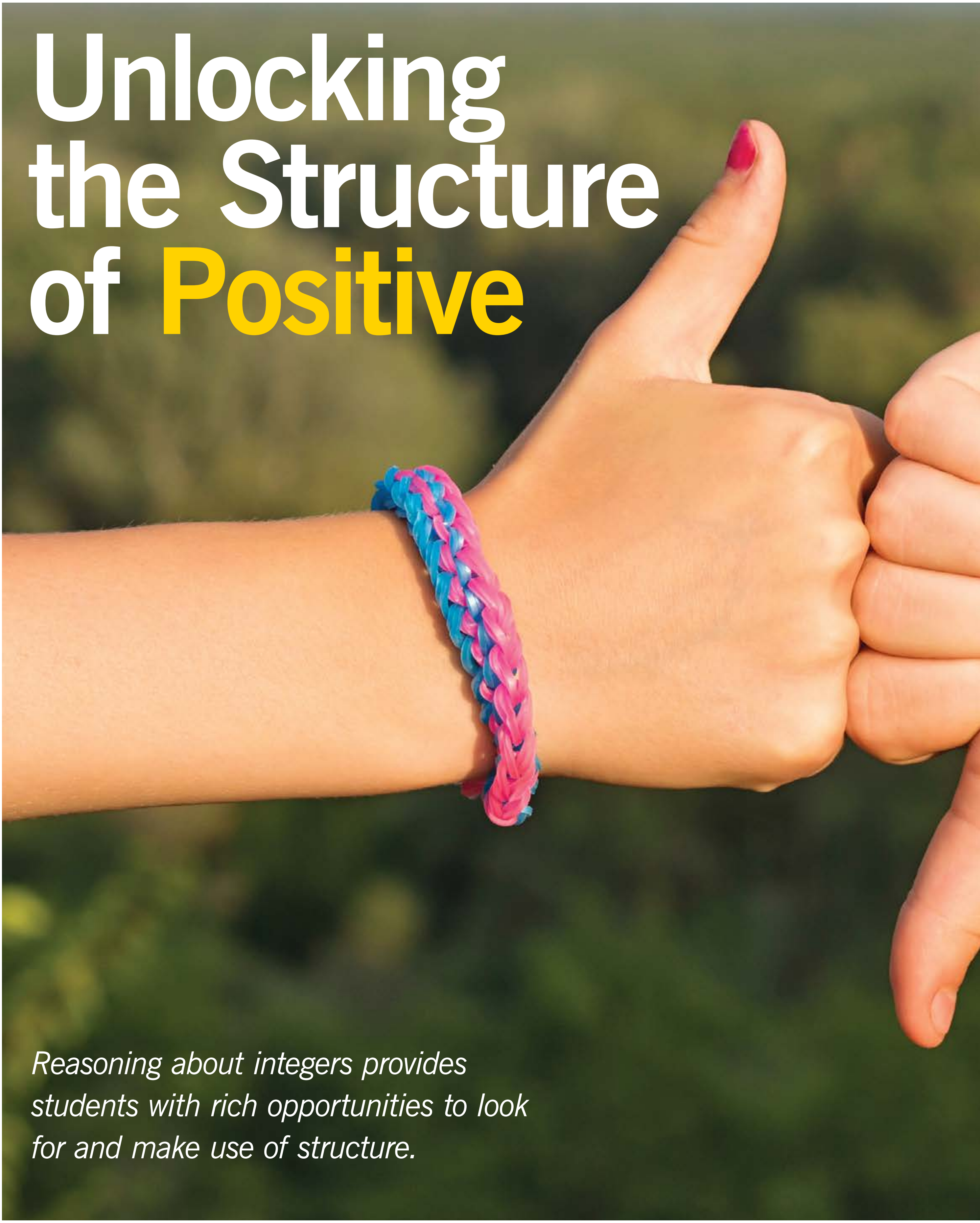

BALAKLEYPB/THINKSTOCK

Unlocking the Structure of Positive

Reasoning about integers provides students with rich opportunities to look for and make use of structure.

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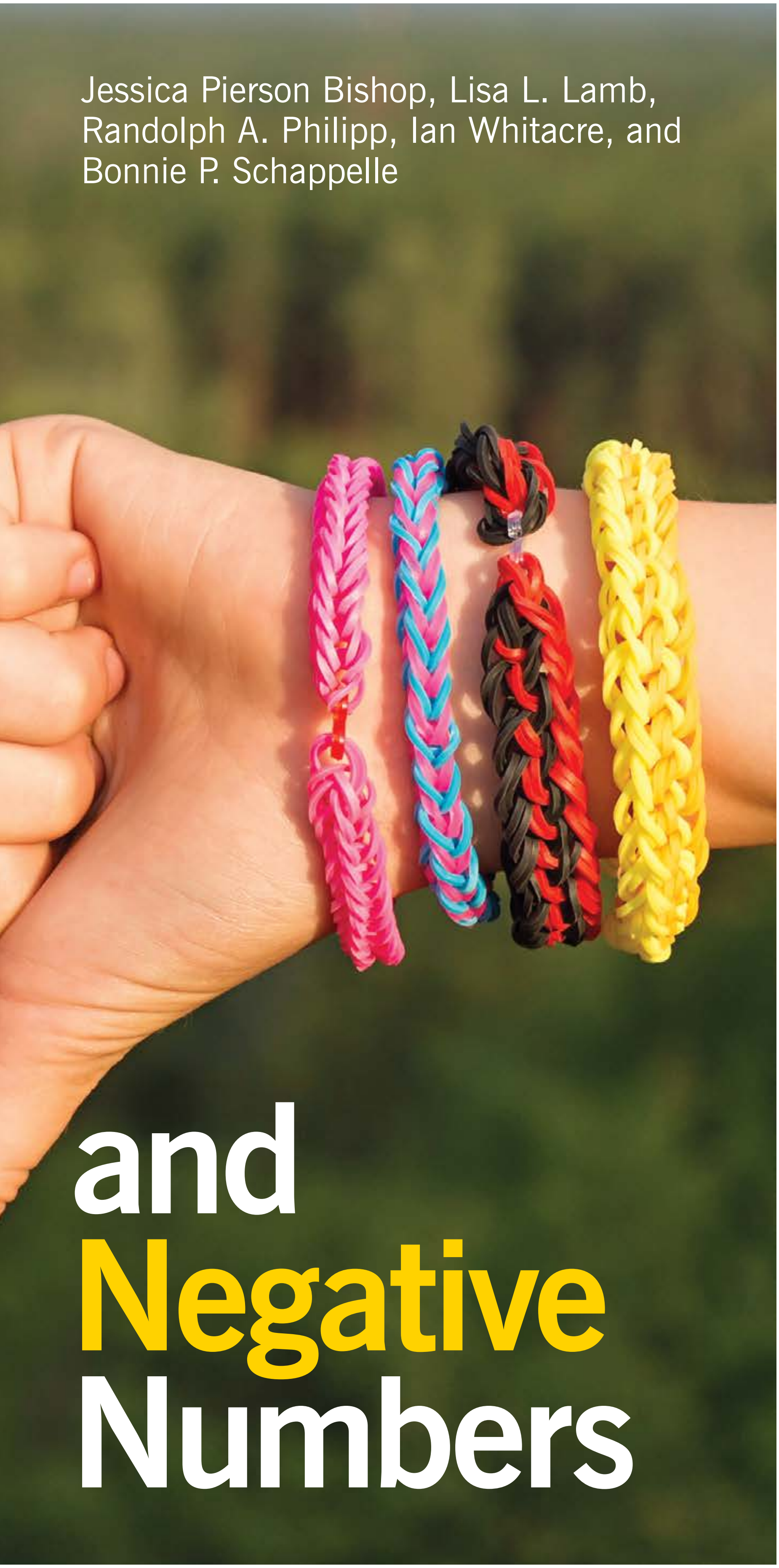

and Negative Numbers

The following problem involves basic integer arithmetic. How would you solve it? How do you think your students would solve it?

> Solve for the unknown:
> $6 + -3 = \square$

When we viewed students' responses to this problem, we were surprised to see an important type of reasoning emerge. Some students were able to use fundamental properties and the underlying structure of our number system to reason about problems similar to this one in ways that did not involve rules or procedures. For example, when solving $6 + -3 = \square$, seventh-grader Michaela answered,

> Three. I was thinking about the opposite. . . . I took away the negative sign [from −3] and said, "What if I added 6 and 3? That would be 9, so you're going forward." Then I knew if you added a negative [as in 6 + −3], it would go backward.

Strategies like Michaela's exemplify the Common Core Standard for Mathematical Practice of looking for and using mathematical structure (CCSSI 2010). Michaela leveraged the underlying structure of the number system—namely, the fact that additive inverses exist within the set of integers and are opposites—to justify her solution.

Recognizing and using mathematical structure are key components of mathematical reasoning. We believe that one productive way to support students' use of structure is by identifying opportunities to address structure in the context of what teachers are already doing (for example, when solving typical integer tasks such as Michaela's), rather than developing additional tasks or new curriculum materials. We have found that the topic of reasoning about number systems in general, and integers in particular, provides rich opportunities for students to look for and make use of structure. When students extend their understanding of numbers to new domains (e.g., from whole numbers to integers), they have opportunities to identify and use underlying structures and generalizations of arithmetic by deciding how operations should function (or reflecting on how operations do function) and which properties remain true within expanded number systems. This broad view of structure within the context of integer arithmetic is our focus in this article.

STRUCTURAL REASONING AND LOGICAL NECESSITY

To illustrate the potential of structural reasoning in the realm of integers, we consider three vignettes and explore the students' responses through the lens of mathematical structure. We use the vignettes to help identify and illustrate characteristics of one type of reasoning about structure, called *logical necessity*—a way of reasoning in the context of integers that we have identified from our work with middle-grades students. The vignettes are representative cases from a larger study of students' ways of reasoning about integers. In our larger study, we conducted individual problem-solving interviews focused on integer addition and subtraction with 160 students in grades 2, 4, 7, and 11 in the southwestern United States. The majority of questions in the standardized interviews were open number sentences similar to the problem that Michaela solved. (For more information about the results of the study in relation to students' use of structure, see Bishop et al. 2016.)

Vignette 1

Armando was a fourth grader who had heard of negative numbers and could use counting strategies to correctly solve problems like $2 - 6 = \square$ and $-7 + \square = 4$. For example, Armando described his strategy for solving $-3 + 6 = \square$ as "counting up 6." He explained,

> I started from negative 3, and went negative 2 [raised one finger], negative 1 [raised a second finger], 0 [raised a third finger], 1 [raised a fourth finger], 2 [raised a fifth finger], 3 [raised a sixth finger].

When the next question, $-8 - 3 = \square$, was posed, Armando sat silently, and then with rising intonation answered, "Negative 11?" When asked how he thought about that problem he replied,

> I looked back up at this problem, negative 3 plus 6. I got 3 because I counted up, negative 3, negative 2. And then I thought, "Well, minus [as indicated in the problem −8 − 3] must be going down." And I got negative 11. Negative 8, negative 9 [raises first finger], negative 10 [raises second finger], negative 11 [raises third finger].

In this vignette, Armando built on what he knew, using the solution from the previous problem and the inverse relationship between addition and subtraction to help him make a reasonable conjecture about how subtraction might function when the starting value was a negative number. Armando reasoned that for addition, one counts up; and for subtraction, one "must be going down." He extended the inverse relationship that exists between addition and subtraction when operating with whole numbers to the entire set of integers. By comparing the given problem with another, known problem, Armando broadened his meaning for subtraction so that operations within his expanded number system remained consistent (i.e., when adding a positive number, count up, and when subtracting a positive number, count down, regardless of the starting value). We view Armando's use of logical necessity as engaging with important mathematical structures like inverse operations; in so doing, he extended his number system


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and the corresponding operations to include signed numbers.

Vignette 2

For this vignette, refer to Michaela's strategy for solving $6 + -3 = \square$. Michaela, a seventh grader, justified representing $6 + -3$ as starting at 6 and moving back 3 units using negation and the relationship between additive inverses. Michaela argued that because 3 and -3 are opposites (that is, $3 + -3 = 0$ or $-(-3) = 3$, more formally), adding a negative number would involve an action (going backward) that was the opposite of adding a positive number (going forward). She started with what she knew about operations with whole numbers and then leveraged the idea of additive inverses to justify her answer of 3. When reasoning about this problem and similar problems that involved adding a negative number, other students invoked commutativity; they thought about changing $6 + -3 = \square$ to $-3 + 6 = \square$. This use of logical necessity is highlighted in the third vignette.

Vignette 3

Ryan, a first grader we interviewed in our pilot study, knew about negative numbers and successfully solved problems like $3 - 5 = \square$, $-4 + 7 = \square$, and $\square - 5 = -8$ using counting strategies that he extended below zero. Ryan was troubled, however, by $5 + \square = 3$, asking, "How do you get to 3 if it's plus? . . . If you add them, how do they get to 3?" Ryan explained why this problem was difficult, saying, "If you add something, how does it get to 3? If it's 5 plus, then it's [the sum is] always past 3."

Toward the close of the interview, we posed the problem $-2 + 5 = \square$ to Ryan. His answer was 3, which he obtained using a counting strategy, counting up 5 from -2.

> It was negative 2, negative 1 [puts up one finger], 0 [puts up second finger], 1 [puts up third finger], 2 [puts up fourth finger], and then 3 [puts up fifth finger].

We then asked him to consider the problem $5 + -2 = \square$. This was the first time we had posed a problem that involved adding a negative number as the second addend. He answered,

> Three, because it's pretty much the same thing [points to $-2 + 5$]. Five plus negative 2 and negative 2 plus 5. If you add the same things, and you just say 5 first and [negative] 2 second, it's still the same thing. . . . You always add the same things together.

Ryan's reasoning used the commutative property of addition to reason about a possible meaning for adding a negative number. He assumed that all numbers, even negative numbers, obey the commutative property of addition; consequently, his answer had to be 3. For Ryan's newly expanded number system to be consistent, his choice of 3 was the necessary answer. The use of the commutative property of addition was particularly useful for students who, prior to school-based instruction, conjectured that commutativity did, in fact, extend to this new kind of number. This conjecture enabled Ryan and other students to develop a possible meaning for adding a negative number and to reconsider the counterintuitive notion that a sum can be smaller than either addend (Bishop et al. 2011; Bishop et al. 2014; and Karp, Bush, and Dougherty 2014).

Table 1 These examples show types of logical-necessity comparisons.

Type of Comparison	Example
Comparisons in which the sign of the number was varied	$6 - -2 = \square$ and $6 - 2 = \square$ Students use the result of 6 – 2 to help them reason about 6 – –2.
Comparisons in which the operation was varied	$-8 - 3 = \square$ and $-8 + 3 = \square$ Students use the result of -8 + 3 to help them reason about -8 – 3.
Comparisons in which other features, such as the order of addends, were varied	$-2 + 5 = \square$ and $5 + -2 = \square$ Students use the result of -2 + 5 to help them reason about 5 + –2.

FEATURES OF LOGICAL NECESSITY

When students in our study engaged with integer tasks, some were in the process of extending their number systems from whole numbers to include negative numbers. Identifying and using underlying properties and structures was a powerful sense-making strategy that supported students to make conjectures about which properties should remain true (e.g., does addition of integers result in a larger sum) and how integer arithmetic should operate in their newly expanded number systems. We describe this type of structural reasoning about number systems as logical necessity. As noted earlier, logical necessity is a type of structural reasoning wherein students use familiar mathematical principles and ideas (e.g., commutativity, inverses, and, sometimes, proof by contradiction) to make logical inferences and deductions in their problem-solving approaches to integer arithmetic (Bishop et al. 2016).

Table 2 Integer tasks encourage structural reasoning.

Goal	Integer Tasks*	Feature Compared and Key Understandings Leveraged
a. Make sense of or justify adding and subtracting a positive integer when the starting value is a negative number.	$-8 + 3$ and $-8 - 3$ $-5 + 5$ and $-5 - 5$	Vary operation Leverage inverse operations and knowledge of zero (in second pairing)
b. Make sense of or justify adding a negative number	$7 + 5$ and $7 + -5$ $-5 + 1$ and $-5 + -1$ $5 + 1$ and $-5 + -1$	Vary sign of number Leverage negative/additive inverses (In the third pairing, the signs of both addends are varied.)
	$-3 + 6$ and $6 + -3$ $-9 + 5$ and $5 + -9$	Vary order of addends Leverage commutative property
c. Make sense of or justify subtracting a negative number	$-5 - 3$ and $-5 - -3$ $6 - 2$ and $6 - -2$ $10 - 4$ and $-10 - -4$	Vary sign of number Leverage negative/additive inverses (In the third pairing, the signs of the minuend and subtrahend are varied.)
	$-5 + -3$ and $-5 - -3$ $-7 + -9$ and $-7 - -9$	Vary operation Leverage inverse operations
	$-5 - -5$ and $-5 - -1$ $-7 - -7$ and $-7 - -8$	Vary subtrahend Leverage knowledge of zero (These pairings are useful if students can productively engage with $-5 - -5$ and/or $-7 - -7$. If not, consider posing $5 - 5$ before $-5 - -5$.)

* The problem pairs in **table 2** could also be rewritten as True/False statements (e.g., True/False $-8 + 3 = -8 - 3$).

In each use of logical necessity that we identified in our study, students compared two related problems. Like Armando, some students held the numbers constant and varied the operation. At other times, students held the operation constant and varied other features.

The work of Michaela, Armando, and Ryan illustrate the three kinds of comparisons that students made when using logical necessity. They involved comparisons in which—

1. the sign of the number was varied (e.g., Michaela's comparison of $6 + -3$ to $6 + 3$);
2. the operation was varied (e.g., Armando's comparison of $-8 - 3$ to $-8 + 3$ or, more accurately, $-3 + 6$); and
3. other features, such as the order of the addends, were varied (e.g., Ryan's comparison of $-2 + 5$ to $5 + -2$ and invoking the commutative property).

See **table 1** for an example of each type of comparison.

Logical necessity involves both choosing an appropriate contrasting case and an ability to identify the logical consequence of changing a particular problem feature (e.g., the operation). The use of comparisons reminds us of Fosnot and Dolk's (2001) work with number strings. Number strings are a deliberately structured sequence of related problems (e.g., 6×4, 6×40, 60×40) that highlight specific relationships (e.g., the associative property; factorization; and, in this example, multiplying by powers of ten) used to promote number sense. Similarly, the comparisons students used in our work—contrasting problems that highlighted key structural relationships—promoted logical necessity.

Additionally, the use of logical necessity can occur when students are either justifying an already known answer, procedure, or rule, or conjecturing how an operation on integers might behave on the basis of previous knowledge. Michaela, the seventh-grader whose strategy was shared at the beginning of this article and again in vignette 2, had completed school-based integer instruction. Thus, she used logical necessity to justify a claim or result she already knew to be true. In contrast, Armando, the fourth grader highlighted in vignette 1, did not know how to operate with negative numbers but instead made a reasoned decision about how operations might function in an expanded number system. His use of logical necessity was conjectural in nature.

TEACHER MOVES TO ENCOURAGE THE USE OF LOGICAL NECESSITY

About 10 percent of the students we interviewed invoked logical necessity (similar to ways illustrated in the three vignettes above) at some point during the interview. We believe that teachers can encourage the use of logical necessity at much higher rates to not only support students' abilities to solve these problems correctly but also expand their abilities to look for and make use of structure. In the sections below, we share two teaching moves to enhance students' use of logical necessity: strategically selecting pairs of problems for students to solve and posing specific questions after students solve the problem pairs.

Strategically Selecting Pairs of Problems

Initially, we did not expect students to reason about integer arithmetic by using logical necessity. Likewise, classroom teachers may not expect that these kinds of seemingly simple integer tasks can engender this type of rich reasoning or that their students will produce these kinds of strategies or justifications. Consequently, we encourage teachers to incorporate in their integer instruction pairs of tasks that invite students to reason more formally about underlying structural aspects of mathematical systems.

For example, problems such as –3 + 6 and –3 – 6 may help students grapple with which way to count when adding to or subtracting from a negative number (see type (a) comparisons in **table 2**). This sequencing may lead to conversations about comparing and ordering negative numbers and making explicit the meanings of adding and subtracting positive numbers. In **table 2**, we present tasks that leverage different comparisons in service of particular goals related to integer instruction. We envision teachers posing the first task in the given pair (e.g., 7 + 5 in type (b) comparisons) and then using the second task (7 + –5 in type (b) comparisons) to build on what students already know about a more familiar task to successfully engage with (or justify the result of) the second task. Similar to the pair that evoked Michaela's strategy, the type (b) pairings support students to use negation or additive inverses. In the example of comparing 7 + 5 to 7 + –5, students can leverage the fact that the opposite of 5 is –5, and, because of this inverse relationship, the result of operating with a negative number is the opposite of operating with a positive number. Thus, if one moves 5 units to the right on the number line to add 5 to 7, one must do the opposite and move 5 units to the left on the number line to add –5 to 7.

These tasks can support classroom discussions about underlying mathematical structures, including fundamental properties and their importance as well as equivalent transformations. However, students may need support in making the structural aspects

Table 3 Types of probing questions to support structural reasoning

Supporting Questions	Examples
Questions that support justification or an alternative explanation	I noticed that when you solved this problem (6 – –2 = $\square$), you changed it to 6 + 2 = $\square$. I understand how you solved 6 + 2, but do you have a way of thinking about what it means to subtract negative 2 from 6 in the original problem? If you were explaining to a younger student why you can change the problem like that, what would you say?
Questions that support an orientation to notice similarities and differences in problem pairs	Can someone tell me how these two problems are similar and how they are different? Is there a way that this difference might help you consider how to solve the second problem? How?
Questions that support an explicit rationale or claim for students' reasoning	You mentioned 6 + 3 = 9 in your response. How does knowing that 6 plus 3 equals 9 help you think about 6 + –3 = $\square$?
Questions that support clarity or verification in response to the specifics of a student's strategy	If, when solving 6 – –2, a student explains that a "negative is like the opposite," a teacher could ask, "What do you mean by opposite? The opposite of what?" You initially answered 5 (to the problem –3 – $\square$ = 2), and then you changed your answer. How did you decide the answer wasn't 5? I need you to explain that one more time please.
Questions that problematize contradictions in reasoning (as an attempt to promote cognitive dissonance)	If a student incorrectly answers –5 for –8 – 3 = $\square$, explaining that she counted 3 places from –8 to reach –5, the teacher can respond, "I noticed that when you solved –6 + 3 that you **added** (with emphasis) 3, and you said –5, –4, –3. And this problem, [–8 – 3] is **minus** (with emphasis])."

of their reasoning explicit, for which appropriate and timely use of probing questions becomes important. We call special attention to type (c) problems in **table 2** wherein students grapple with subtracting a negative number. In our study, an average of only 56 percent of the seventh-grade students correctly answered the five items that involved subtracting a negative number. This low percentage speaks to a need to support students' understanding of these types of problems. Although one common approach is to have students memorize the sign rules (subtracting a negative is equivalent to adding a positive), we found that students often misapplied the rule and invoked it at inappropriate times. Supporting students' use of logical necessity gives teachers opportunities to have students reason, rather than simply compute in relation to these problem types.

Posing Probing Questions after Students Solve Pairs of Problems

During our interviews, the interviewer sometimes posed questions that supported student thinking or made explicit the underlying argument related to the use of logical necessity. We believe that teachers can use these questions to productively support their students' use of logical necessity. In general, these questions push students for justification or alternative models/explanations for integer arithmetic. For example, if students use procedures or rules to transform problems that involve adding or subtracting negative numbers, such as $-5 - -3 = \square$ and $6 + -3 = \square$, teachers can pose the following questions (similar to those posed by the interviewers):

- "Why can you change the problem like that [i.e., a change from –5 – –3 to –5 + 3]?" or
- "Does changing the signs always give a correct result? Why do you think so?"

Additionally, teachers can press students to generate explanations for their solutions for the original open number sentence. For example, teachers can ask,

- "I noticed that when you solved this problem [e.g., $6 - -2 = \square$], you changed it to $6 + 2 = \square$. I understand how you solved 6 + 2, but do you have a way of thinking about what it means to subtract negative 2 from 6?"

We describe these kinds of questions as ways to support justification or an alternative explanation (see **table 3**, row 1).

A second category of supporting questions consists of those that help students notice differences in problem pairs and the attribute being compared, such as, "Can someone tell me how these two problems are similar and how they are different?" A third category of probing questions includes responses that support students to make rationales, claims, arguments, and connections more explicit. This category of follow-up questions includes the following types of responses:

- "You mentioned 6 + 3 = 9 in your response. How does knowing that 6 plus 3 is 9 help you think about 6 + –3?" and
- "So, what is your reason for moving to the right?"

A fourth category includes specific questions about a student's response. Some questions and statements can be used to clarify or verify the teacher's understanding of the student's strategy. Examples include these:

- "I need you to explain that one more time please."
- "How did you decide the answer wasn't 5?"
- "What do you mean by opposite? The opposite of what?"
- "It sounds as though you were comparing this problem to a different problem. Were you?"

The final category of probing questions includes prompts that problematize contradictions in student reasoning to provoke cognitive dissonance. Some follow-ups in this category might involve posing a new problem that is based on the teacher's conjecture about how a child was reasoning. For example, in response to the interaction shared in the last row of **table 3**, the teacher might pose the problem $-8 + 3 = \square$ rather than refer to a previously solved problem, to highlight the contradiction.

In closing, we have found that purposeful problem combinations and the use of probing questions can encourage students to engage with mathematical structure. We suspect that the act of making comparisons between carefully chosen, contrasting cases is not restricted to integer tasks but is a more general feature of instruction and curriculum design that will support students to look for, recognize, and use mathematical structure across the K–grade 12 curriculum. We encourage teachers and mathematics educators to think about structure more broadly as including the use of fundamental properties in conjunction with conjecturing and deduction to reason about extensions to the number system—both within the context of integer instruction and in other mathematical topics.

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