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Anomaly cancellation in effective supergravity from the heterotic string with an anomalous $U(1)$ ☆

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Abstract

We show that a choice of Pauli-Villars regulators allows the cancellation of all the conformal and chiral anomalies in an effective field theory from $\mathbb{Z}_3$ compactification of the heterotic string with two Wilson lines and an anomalous $U(1)$.

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1. Introduction

Starting with the determination of the full anomaly structure of Pauli-Villars (PV) regularized supergravity [1], we recently showed [2] that an appropriate choice of PV regulator fields allows for cancellation of all the T-duality (hereafter referred to as "modular") anomalies by the four-dimensional version of the Green-Schwarz term in $\mathbb{Z}_3$ and $\mathbb{Z}_7$ compactifications of the heterotic string without Wilson lines.[1] We further matched our results to a string calculation [3] of the chiral anomaly in those theories. Here we extend our results to a specific $\mathbb{Z}_3$ compactification [4] (hereafter referred to as FIQS) with two Wilson lines and therefore an anomalous $U(1)$, hereafter referred to as $U(1)_X$. In the following section we briefly describe the orbifold model we are studying. In Section 3 we outline the four-dimensional Green-Schwarz mechanism and the structure of the anomaly when an anomalous $U(1)$ is present. In Section 4 we discuss some aspects of the cancellation of ultra-violet (UV) divergences and anomaly matching that are specific to the case with an anomalous $U(1)$, as well as some simplifications with respect to the $\mathbb{Z}_7$ case studied in [2]. We summarize our results in Section 5. The full set of conditions for cancellation of UV divergences and anomaly matching are given in Appendix A, a sample solution to these constraints is presented in Appendix B, and the full spectrum for the FIQS model is displayed in Appendix C. The determination of the correct Pauli-Villars (PV) masses can have implications for soft supersymmetry breaking terms [5].

2. The FIQS model

Here we will give a brief review of the orbifold model we will consider for the rest of the paper. The FIQS model [4] is a $\mathbb{Z}_3$ orbifold compactification of the 10d $E_8 \otimes E_8$ heterotic string compactified to T^6 with two Wilson lines and a nonstandard embedding for the shift vector. The embeddings of the shift vector and Wilson lines are given by

$$V = \frac{1}{3}(1,1,1,1,2,0,0,0)(2,0,0,0,0,0,0,0)' \tag{2.1}$$

$$a_1 = \frac{1}{3}(0,0,0,0,0,0,0,2)(0,1,1,0,0,0,0,0)' \tag{2.2}$$

$$a_3 = \frac{1}{3}(1,1,1,2,1,0,0,1,1)(1,1,0,0,0,0,0,0)' \tag{2.3}$$

Where the prime indicates that the last 8 elements of the above vectors correspond to the second factor of E_8. With these specifications, the massless spectrum of the FIQS model can be worked out following the standard recipes [6]. The 4D gauge group is $SU(3) \otimes SU(2) \otimes SO(10) \otimes U(1)^8$. The generators of the eight $U(1)$ factors can be written as linear combinations of the $E_8 \otimes E_8$ Cartan subalgebra generators H^I as

$$Q_a = \sum_{I=1}^{16} q_a^I H^I \tag{2.4}$$

The constants q_a^I are determined by requiring that $q_a \cdot q_b = 0$ and $q_a \cdot \alpha_{bj} = 0$, where the α_{bj} are the sixteen dimensional simple root vectors of the nonabelian gauge group factors. Thus the

[1] Corrections to this paper are given in Appendix D.

index b corresponds to $SU(3)$, $SU(2)$, or $SO(10)$ and j runs over the rank of each group. One choice of q_a's is [7]:

$$\vec{q}_1 = 6(1, 1, 1, 0, 0, 0, 0, 0)(0, 0, 0, 0, 0, 0, 0, 0)' \tag{2.5}$$

$$\vec{q}_2 = 6(0, 0, 0, 1, -1, 0, 0, 0)(0, 0, 0, 0, 0, 0, 0, 0)' \tag{2.6}$$

$$\vec{q}_3 = 6(0, 0, 0, 0, 0, 1, 0, 0)(0, 0, 0, 0, 0, 0, 0, 0)' \tag{2.7}$$

$$\vec{q}_4 = 6(0, 0, 0, 0, 0, 0, 1, 0)(0, 0, 0, 0, 0, 0, 0, 0)' \tag{2.8}$$

$$\vec{q}_5 = 6(0, 0, 0, 0, 0, 0, 0, 1)(0, 0, 0, 0, 0, 0, 0, 0)' \tag{2.9}$$

$$\vec{q}_6 = 6(0, 0, 0, 0, 0, 0, 0, 0)(1, 0, 0, 0, 0, 0, 0, 0)' \tag{2.10}$$

$$\vec{q}_7 = 6(0, 0, 0, 0, 0, 0, 0, 0)(0, 1, 0, 0, 0, 0, 0, 0)' \tag{2.11}$$

$$\vec{q}_8 = 6(0, 0, 0, 0, 0, 0, 0, 0)(0, 0, 1, 0, 0, 0, 0, 0)' \tag{2.12}$$

To get the charges of the matter fields, one normalizes the $U(1)_a$ generators as

$$Q_a \to \frac{1}{\sqrt{2}\,|q_a|} Q_a, \tag{2.13}$$

where the $\sqrt{2}$ is inserted to adhere to the standard phenomenological normalization. For this choice, one finds that the traces of Q_6, Q_7, and Q_8 are all nonzero. One can perform a re-definition of the generators so that only one factor of U(1) has a nonzero trace. In [4], the following re-definition was made:

$$q_6^{(FIQS)} = q_6 + q_7 \tag{2.14}$$

$$q_7^{(FIQS)} = q_7 + q_8 \tag{2.15}$$

$$q_X = q_6 - q_7 + q_8 \tag{2.16}$$

While $\mathrm{Tr}\left[Q_6^{(FIQS)}\right] = \mathrm{Tr}\left[Q_7^{(FIQS)}\right] = 0$ in this basis, one also has $\mathrm{Tr}\left[Q_6^{(FIQS)} Q_7^{(FIQS)} Q_X\right] \neq 0$, which is rather undesirable. Therefore, we will use a different choice such that the above mixed anomaly does not appear. In particular, we define

$$q_6^{(N)} = q_6 - q_8 = q_6^{(FIQS)} - q_7^{(FIQS)} \tag{2.17}$$

$$q_7^{(N)} = q_6 + 2q_7 + q_8 = q_6^{(FIQS)} + q_7^{(FIQS)} \tag{2.18}$$

In what follows, we will simply drop the superscript N and use these as the definition of the $U(1)_6$ and $U(1)_7$ generators. As a final note, the charges defined above are generally not orthogonal to one another, i.e. $\mathrm{Tr}\,[Q_a Q_b] \neq 0$ for some $a \neq b$. It is possible to define a new set of charges that are mostly orthogonal to one another, but we will not need to do so for our purposes.

We close this section with some relations among the gauge charges q_a^p and modular weights q_n^p of the chiral superfields Φ^p of the model. These will be useful in the analysis that follows. These include the universality conditions

$$8\pi^2 b = C_a + \sum_p \left(2q_i^p - 1\right) C_a^p = \frac{1}{24}\left(2\sum_p q_n^p - N + N_G - 21\right) \quad \forall \quad i, a,$$
$$-2\pi^2 \delta_X = \frac{1}{24}\mathrm{Tr} T_X = \frac{1}{3}\mathrm{Tr} T_X^3 = \mathrm{Tr}(T_a^2 T_X) \quad \forall \quad a \neq X. \tag{2.19}$$

Here C_a is the quadratic Casimir in the adjoint representation of the gauge group factor $\mathcal{G}_a$ and C_a^p is the Casimir for the representation of the chiral supermultiplet Φ^p, T_a is a generator of $\mathcal{G}_a$, and N, N_G are the number of chiral and gauge supermultiplets respectively, with, in the FIQS model,

$$N = 415, \qquad N_G = 64, \qquad 8\pi^2 b = 6, \qquad -4\pi^2\delta_X = 3\sqrt{6}. \tag{2.20}$$

In addition we will use the sum rules

$$\sum_p q_n^p = A_1, \qquad \sum_p q_m^p q_n^p = A_2 + B_2\delta_{mn},$$
$$\sum_p q^l q_m^p q_n^p = A_3 + B_3\left(\delta_{lm} + \delta_{mn} + \delta_{nl}\right) + C_3\delta_{lm}\delta_{mn},$$
$$\sum_b q_a^b q_n^b = Q_{1a}, \qquad \sum_b q_a^b q_m^b q_n^b = Q_{2a} + P_{2a}\delta_{mn}, \tag{2.21}$$

with, in particular,

$$B_2 = 42, \qquad P_{2X} = 5\sqrt{6}. \tag{2.22}$$

3. Anomalies and anomaly cancellation with an anomalous $U(1)$

The effective supergravity theory from generic orbifold compactifications with Wilson lines is anomalous under both $U(1)_X$ and T-duality:

$$T'^i = \frac{a_i - ib_iT^i}{ic_iT^i + d_i}, \qquad a_ib_i - c_id_i = 1, \qquad a_i, b_i, c_i, d_i \in \mathbf{Z}, \qquad i = 1, 2, 3,$$
$$\Phi'^a = e^{-\sum_i q_i^a F^i(T^i)}\Phi^a, \qquad F^i(T^i) = \ln(ic_iT^i + d_i), \tag{3.1}$$

where Φ^a is any chiral supermultiplet other than a diagonal Kähler modulus T^i, and q_i^a are its modular weights.

We are working in the covariant superspace formalism of ref. [8] in which the chiral multiplets $Z^p = T^i, S, \Phi^a$, with S the dilaton superfield, are *covariantly* chiral:

$$\mathcal{D}^{\dot\beta} Z^p = 0, \tag{3.2}$$

with $\mathcal{D}_A,\ A = a, \alpha$ a fully covariant superspace derivative. In particular, under a $U(1)$ gauge transformation

$$Z'^p = g^{q_a^p} Z^p, \qquad \bar{Z}'^p = g^{-q_a^p}\bar{Z}^p, \qquad A'^a_A = A^a_A - g^{-1}\mathcal{D}_A g, \tag{3.3}$$

where g is a hermitian superfield, and A_A is the gauge potential in superspace. Gauge invariance assures that holomorphy of the superpotential is maintained under (3.3). If gauge invariance is unbroken, the gauge potential A_A does not appear explicitly in the superspace Lagrangian. Instead the usual Yang-Mills superfield strength W_α is obtained as a component of the two-form superfield strength F_{AB}. One can still introduce [8] a superfield superpotential V_a such that

$$W_\alpha = -\frac{1}{8}(\bar{\mathcal{D}}^2 - 8R)\mathcal{D}_\alpha V_a, \qquad V_a' = V_a + \Lambda_a + \bar\Lambda_a, \tag{3.4}$$

but V_a never appears in the Lagrangian and the chiral superfield Λ_a is independent of g in (3.3).

However in the presence of an anomalous $U(1)$, gauge invariance is broken. It is easy to see that the UV divergences cannot be regulated by PV fields that all have $U(1)_X$ invariant masses. There is a quadratically divergent term proportional to $D_X\mathrm{Tr}T_X$, where D_X is the auxiliary field of the $U(1)_X$ supermultiplet, which must be canceled by the analogous term from the PV sector. Invariant masses require the coupling of PV fields with equal and opposite charges that do not contribute to $(\mathrm{Tr}T_X)_{PV}$. Noninvariant masses arise from the superpotential for PV fields Φ^C:

$$W(\Phi^C, \Phi'^C) = \mu_C \Phi^C \Phi'^C, \tag{3.5}$$

with μ_C constant (in the absence of threshold corrections, as for the cases considered here). If $Q_X^C + Q_X'^C \neq 0$, holomorphy of (3.5) is not respected under (3.3) for $a = X$. For this reason we do not include the $U(1)_X$ connection in the covariant derivative (3.2). Instead of (3.3) we require

$$\Phi'^C = e^{-Q_X^C \Lambda}\Phi^C, \qquad \bar{\Phi}'^C = e^{-Q_X^C \bar{\Lambda}}\bar{\Phi}^C \tag{3.6}$$

under a $U(1)_X$ transformation, and the Kähler potential depends on $U(1)_X$-charged fields through the invariant operators $\bar{\Phi}e^{Q_X V_X}\Phi$.

It was shown in [1] that modular noninvariant masses can be restricted to a subset of PV chiral supermultiplets Φ^C with diagonal Kähler metric:

$$K(\Phi^C, \bar{\Phi}^C) = \exp[f^C(Z, \bar{Z})]|\Phi^C|^2 \tag{3.7}$$

and superpotential (3.5).

As in [2], we define a superfield

$$\mathcal{M}_C^2 = \mathcal{M}_{C'}^2 = \exp(K - f^C - f'^C) = \exp(K - 2\bar{f}^C), \qquad \bar{f}^C = \frac{1}{2}(f^C + f'^C), \tag{3.8}$$

whose lowest component $m_C^2 = \mathcal{M}_C^2\big|$ is the Φ^C, Φ'^C squared mass. Then the anomalous part of the one-loop corrected supergravity Lagrangian takes the form [1]

$$\mathcal{L}_{\rm anom} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_r = \int d^4\theta E\,(L_0 + L_1 + L_r) \equiv \int d^4\theta E\Omega, \tag{3.9}$$

where E is the superdeterminant of the supervielbein, and

$$L_0 = \frac{1}{8\pi^2}\left[\mathrm{Tr}\eta \ln \mathcal{M}^2 \Omega_0 + K\,(\Omega_{GB} + \Omega_D)\right], \tag{3.10}$$

with $\eta = \pm 1$ the PV signature. The operators in (3.10) are given explicitly in [1,2], except that now

$$\Omega_0 = \Omega_{\rm YM}^0 + \Omega_0', \tag{3.11}$$

where Ω_0' contains the Gauss-Bonnet Chern-Simons superfield and operators composed of auxiliary superfields of the gravity supermultiplet, and

$$\Omega_{\rm YM}^0 = \sum_{a \neq X} \Omega_{\rm YM}^a = \Omega_{\rm YM} - \Omega_{\rm YM}^X, \tag{3.12}$$

is the Yang-Mills Chern-Simons superfield without the $U(1)_X$ term, and $\Omega_{\rm YM}^a$ is defined by its chiral projection:

$$(\bar{\mathcal{D}}^2 - 8R)\Omega_{\rm YM}^a = W_a^\alpha W_\alpha^a. \tag{3.13}$$

Ω_r is composed of terms linear and higher order in $\ln\mathcal{M}$, and Ω_D represents a "D-term" anomaly [1,2] that, together with a contribution to the Gauss-Bonnet term $\Omega_{\rm GB}$, arises from uncanceled total derivatives with logarithmically divergent coefficients, requiring the introduction of a field-dependent cut-off:

$$\partial_\mu \Lambda = \frac{1}{4}\partial_\mu K. \tag{3.14}$$

L_1 is defined by its variation:

$$\Delta L_1 = \frac{1}{8\pi^2}\frac{1}{192}{\rm Tr}\eta\Delta\ln\mathcal{M}^2\Omega'_L = \frac{1}{8\pi^2}\frac{1}{192}{\rm Tr}\eta H\Omega'_L + {\rm h.c.}, \tag{3.15}$$

where under (3.1) and (3.6) $\ln\mathcal{M}^2$ transforms as

$$\Delta\ln\mathcal{M}^2 = H + \bar{H}, \tag{3.16}$$

with H holomorphic. Defining

$$(\bar{\mathcal{D}}^2 - 8R)\Omega_f = f^\alpha f_\alpha, \qquad (\bar{\mathcal{D}}^2 - 8R)\Omega_{\bar{f}} = \bar{f}^\alpha\bar{f}_\alpha, \qquad (\bar{\mathcal{D}}^2 - 8R)\Omega_{\bar{f}X} = \bar{f}^\alpha X_\alpha,$$
$$f_\alpha = -\frac{1}{8}(\bar{\mathcal{D}}^2 - 8R)\mathcal{D}_\alpha f, \qquad \bar{f}_\alpha = -\frac{1}{8}(\bar{\mathcal{D}}^2 - 8R)\mathcal{D}_\alpha\bar{f}, \tag{3.17}$$

we have

$$\Omega'_L = 192\Omega_f - 128\Omega_{\bar{f}} - 64\Omega_{\bar{f}X},$$
$$\Delta L_1 = \frac{1}{8\pi^2}{\rm Tr}\eta H\left(\Omega_f - \frac{2}{3}\Omega_{\bar{f}} - \frac{1}{3}\Omega_{\bar{f}X}\right) + {\rm h.c.} \tag{3.18}$$

In the presence of an anomalous $U(1)_X$ the form of f^C is taken to be

$$f^C = \alpha^C K(Z,\bar{Z}) + \beta^C g(T,\bar{T}) + \delta^C k(S,\bar{S}) + \sum_n q_n^C g^n(T^n,\bar{T}^n) + Q_X^C V_X,$$
$$\bar{f}^C = \bar{\alpha}^C K + \bar{\beta}^C g + \bar{\delta}^C k + \sum_n \bar{q}_n^C g^n + \bar{Q}_X^C V_X,$$
$$H^C = \left(1 - 2\bar{\gamma}^C\right)F(T) - 2\sum \bar{q}_n^C F^n(T^n) - 2\bar{Q}_X^C\Lambda, \qquad \bar{\gamma}^C = \bar{\alpha}^C + \bar{\beta}^C, \tag{3.19}$$

where k is the dilaton Kähler potential, and g is defined in (3.31) below. The traces in $\Delta\mathcal{L}_{\rm anom}$ can be evaluated using only PV fields with noninvariant masses or using the full set of PV fields, since those with invariant masses, $H^C = 0$, drop out. The contribution ΔL_0 to the anomaly is linear in the parameters $\alpha^C, \beta^C, q_n^C, Q_X^C$, and the trace of the coefficient of Ω'_0 is completely determined by the sum rules [9]

$$N' = \sum_C \eta^C = -N - 29, \qquad N'_G = \sum_\gamma \eta_\gamma^V = -12 - N_G,$$
$$\sum_C \eta^C f^C = -10K - \sum_p q_n^p g^n - \sum_a q_X^a V_X, \tag{3.20}$$

that are required to assure the cancellation of quadratic and logarithmic divergences. In (3.20) the index C denotes any chiral PV field, the index γ runs over the Abelian gauge PV superfields that are needed to cancel some gravitational and dilaton-gauge couplings, and the sum over p includes all the light chiral multiplet modular weights with $q_n^S = 0$, $q_n^{T^i} = 2\delta_n^i$. All PV fields with noninvariant masses have $\delta = 0$, and most[2] with $\delta \neq 0$ have $\alpha = \beta = q_n = 0 = Q^C_X$. For the purposes of the present analysis we can largely ignore the latter. Similarly, the cancellation of linear divergences that give rise to the chiral anomaly proportional to

$$\mathrm{ImTr}\phi G\cdot\widetilde{G} \ni \mathrm{Im}\frac{1}{2}\sum_{a\neq X}\left\{F(t)C_a - \sum_p\left[F(t) - 2\sum_n q^p_n F^n(t^n) - 2q^p_X\lambda\right](T^p_a)^2\right\}F^a\cdot\tilde{F}_a \tag{3.21}$$

fixes the coefficient of Ω^0_{YM}. Here $G_{\mu\nu} \ni -iT_a F^a_{\mu\nu}$ is the field strength associated with the fermion connection, $t^i = T^i\big|$, $\lambda = \Lambda|$ are the lowest components of the chiral supermultiplets T^i, Λ, and a left-handed fermion f transforms as

$$f \to e^\phi f \tag{3.22}$$

under modular and $U(1)_X$ transformations; $\phi = -\frac{i}{2}\mathrm{Im}F$ for gauginos, and

$$\phi = \frac{i}{2}\mathrm{Im}F - \sum_n q^p_n F^n(t^n) - q^p_X\lambda \tag{3.23}$$

for chiral fermions χ^p. The compensating PV contribution

$$\mathrm{Im}\left(\mathrm{Tr}\eta\phi G\cdot\widetilde{G}\right)_{PV} \ni \mathrm{Im}\sum_C \eta^C\left(\phi^C + \phi'^C\right)(T^C_a)^2 F_a\tilde{F}^a = -\mathrm{ImTr}\phi G\cdot\widetilde{G} \tag{3.24}$$

that cancels (3.21) determines the anomaly coefficient of Ω^0_{YM}, since for each pair Φ^C, Φ'^C the sum of fermion phases $\phi^C + \phi'^C = H^C$ is just the holomorphic part of the variation (3.16), (3.19) of the PV mass term $\Delta\ln\mathcal{M}^2_C$.

In the chiral formulation for the dilaton, the anomaly is canceled by the variation of the superspace Lagrangian

$$\mathcal{L} = \int d^4\theta E\left(S + \bar{S}\right)\Omega, \tag{3.25}$$

where Ω is the real superfield introduced in (3.9). The quantum Lagrangian varies according to

$$\Delta\mathcal{L}_{\mathrm{anom}} = \int d^4\theta\left\{b\left[F(T) + \bar{F}(\bar{T})\right] - \frac{\delta_X}{2}\left(\Lambda + \bar{\Lambda}\right)\right\}\Omega, \tag{3.26}$$

so the full Lagrangian is invariant provided

$$\Delta S = -bF(T) + \frac{\delta_X}{2}\Lambda, \qquad F = \sum_i F^i. \tag{3.27}$$

However the classical Kähler potential for the dilaton is no longer invariant and must be modified:

$$k_{\mathrm{class}}(S,\bar{S}) = -\ln(S + \bar{S}) \to k(S,\bar{S}) = -\ln(S + \bar{S} + V_{GS}), \tag{3.28}$$

[2] There is a set of chiral multiplets in the adjoint representation of the gauge group that has $f = K - k$; these get modular invariant masses though their coupling in the superpotential to a second set with $f = k$. These cancel renormalizable gauge interactions and gauge-gravity interactions, respectively. Together with a third set, that has $f = 0$ and contributes to the anomaly, they cancel the Yang-Mills contribution to the beta-function..

where V_{GS} is a real function of V_X and of the chiral supermultiplets; it transforms under (3.1) and (3.4), (3.6) as

$$\Delta V_{GS} = b\left(F + \bar{F}\right) - \frac{\delta_X}{2}\left(\Lambda + \bar{\Lambda}\right). \tag{3.29}$$

A simple solution consistent with string calculation results [10,11] is

$$V_{GS} = bg(T, \bar{T}) - \frac{\delta_X}{2} V_X, \tag{3.30}$$

where

$$g(T, \bar{T}) = \sum_i g^i(T^i, \bar{T}^i), \qquad g^i = -\ln(T^i + \bar{T}^i) \tag{3.31}$$

is the Kähler potential for the moduli. The modification (3.28) is the 4d Green-Schwarz (GS) term in the chiral formulation. As discussed in [2], the 4d GS mechanism is more simply formulated in the linear multiplet formalism [8] for the dilaton. In this case the linear dilaton superfield L remains invariant, its Kähler potential is unchanged, and instead one adds a term to the Lagrangian:

$$\mathcal{L}_{GS} = -\int d^4\theta E L V_{GS}, \qquad \Delta\mathcal{L}_{GS} = -\Delta\mathcal{L}_{\text{anom}} \tag{3.32}$$

Only terms in the anomaly that are linear in the combination $\tilde{H}$, where

$$\tilde{H} = bF(T) - \frac{\delta_X}{2}\Lambda, \tag{3.33}$$

can be canceled by the Green-Schwarz term. The values of b and δ_X are fixed by the conditions (3.20), (3.24) for the cancellation of divergences, together with the universality conditions (2.19), that hold for all $\mathbb{Z}_3$ and $\mathbb{Z}_7$ orbifold compactifications.

In contrast to $\mathcal{L}_0$, the contributions to the anomaly from $\mathcal{L}_1$ and $\mathcal{L}_r$ are nonlinear in the parameters α, β, q_n, Q_X, and depend on the details of the PV sector. In particular $\mathcal{L}_r$ has no terms linear in $\ln\mathcal{M}$ and must vanish. To ensure that the anomaly coefficient depends on the T-moduli only through $F(T)$ we impose [2]

$$\bar{q}_n^C = 0 \tag{3.34}$$

for (almost[3]) all PV fields with noninvariant masses.

4. The anomaly and cancellation of UV divergences in the FIQS model

The full set of conditions for cancellation of the divergences and for obtaining an anomaly linear in $\tilde{H}$, Eq. (3.33), that matches the string result [3] is given in the Appendix A. In this section we outline some features of the case of $\mathbb{Z}_3$ with an anomalous $U(1)_X$. We will be primarily concerned with the contribution of ΔL_1, Eq. (3.18), to the anomaly. This expression is nonlinear in the parameters q_n^C, Q_X^C of the PV fields, and therefore model dependent, as noted above. This was illustrated in [2] where it was shown that cancellation of the modular anomaly requires (3.34). However, the contribution cubic in Q_X^C is model independent. It is given by

[3] The exception is for some PV fields, introduced in Appendix B.6, needed to cancel divergences from light fields with Abelian gauge charges.

$$\Delta L_1(Q_X^3) = -\frac{2(\Lambda+\bar\Lambda)}{24\pi^2}\mathrm{Tr}\eta\bar Q_X\left(3Q_X^2 - 2\bar Q_X^2\right)\Omega_{YM}^X = -\frac{2(\Lambda+\bar\Lambda)}{24\pi^2}\mathrm{Tr}\eta Q_X^3\Omega_{YM}^X, \tag{4.1}$$

where the sum is over all PV fields, and we used the definition (3.6), (3.19) of $\bar Q^X$ and the fact that

$$\sum_C \eta^C (Q_X^C)^p (Q_X'^C)^{p'} = \sum_C \eta^C (Q_X^C)^{p'} (Q_X'^C)^p, \tag{4.2}$$

for any powers p, p'. Cancellation of the term in $\mathrm{Tr}\phi G\cdot\widetilde G$ that is cubic in Q_X^3 requires

$$-\frac{2(\Lambda+\bar\Lambda)}{24\pi^2}\mathrm{Tr}\left(\eta Q_X^3\right)\Omega_{YM}^X = \frac{2(\Lambda+\bar\Lambda)}{24\pi^2}\mathrm{Tr}\left(q_X^3\right)\Omega_{YM}^X = -\frac{\delta_X}{2}(\Lambda+\bar\Lambda)\Omega_{YM}^X, \tag{4.3}$$

from (2.19), so the anomaly (4.1) is consistent with the requirement for anomaly cancellation.

In contrast, anomaly terms quadratic in Q_X^2 are model dependent. For example, in [1] it was assumed that $\bar f^C = f^C$ for all PV fields with noninvariant masses, giving a contribution

$$\Delta L_1(FQ_X^2) = \frac{F+\bar F}{24\pi^2}\mathrm{Tr}\eta\,(1-2\bar\gamma)\left(3Q_X^2 - 2\bar Q_X^2\right)\Omega_{YM}^X \tag{4.4}$$

$$= \frac{F+\bar F}{24\pi^2}\mathrm{Tr}\eta\,(1-2\bar\gamma)\,Q_X^2\Omega_{YM}^X = \frac{F+\bar F}{24\pi^2}\mathrm{Tr}q_X^2\Omega_{YM}^X = \frac{b}{3}(F+\bar F)\Omega_{YM}^X, \tag{4.5}$$

from (3.21) and (3.24) with $a = X$, and (2.19). Here we instead assume, in addition to (3.34), that $\bar Q_X^C = 0$ if $1-\bar\gamma \neq 0$, that is PV masses can be noninvariant under either T-duality or $U(1)_X$, but not both. In this case the last term in (4.4) drops out and we recover a factor three, in agreement with the requirement for anomaly cancellation.

The full set of PV fields sufficient to regulate light field couplings is described in Section 3 of [1]. These include a set $\dot Z^P = \dot Z^I, \dot Z^A$, with negative signature, $\eta^{\dot Z} = -1$, that regulates most of the couplings, including all renormalizable couplings, of the light chiral supermultiplets $Z^p = T^i, \Phi^a$. The $\dot Z$ get invariant masses through a superpotential coupling to PV fields $\dot Y_P$ with the same signature, opposite gauge charges and the inverse Kähler metric:

$$(T_a)_{\dot Y} = -(T_a^T)_{\dot Z} = -(T_a^T)_Z. \tag{4.6}$$

It remains to cancel the divergences introduced by the fields $\dot Y$. To this end we take the following set:

$$\begin{aligned}
&\psi^{Pn}: \ f^{Pn} = \alpha_\psi^P K + \beta_\psi^P g + q_\psi^P g^n + Q_\psi^P V_X, \qquad \alpha_\psi^P + \beta_\psi^P = \gamma_\psi^P, \qquad \bar q_\psi^P = 0,\\
&T^P: \ f_T^P = \alpha_T^P K + \beta_T^P g + Q_T^P V_X, \qquad \alpha_T^P + \beta_T^P = \gamma_T^P,\\
&\phi^C: \ f^{\phi^C} = \alpha^C K.
\end{aligned} \tag{4.7}$$

In the solution to the constraints given in Appendix B, the ψ^C and T^C are further subdivided, together with additional fields, into sets S_a, $a = 1,\ldots,12$, some of which are charged under the nonanomalous gauge group. The ϕ^C regulate certain gravity supermultiplet loops and nonrenormalizable coupling of chiral multiplets. These must be included together with the other PV fields introduced above in implementing the sum rules (3.20). Their contributions will be included in all the finiteness and anomaly conditions that involve only the parameters α in (4.7); otherwise they play no role in the analysis below. In the expressions given in the remainder of this section, we drop terms that contain only X_α or $X_{\mu\nu}$ since their contributions are included in the sums (3.20) and the additional sum rule [9]

$$\sum_C \eta^C \alpha_C^2 = -4. \tag{4.8}$$

In [2] we also introduced pairs Φ^P, Φ'^P with modular invariant masses that did not contribute to the anomaly, but played an important role in canceling certain divergences. However, because the $\mathbb{Z}_3$ sum rules (2.21) are much simpler than the analogous sum rules for the $\mathbb{Z}_7$ case studied in [2], here we need only the set in (4.7).

The quadratic and logarithmic divergences we are concerned with here involve the superfield strengths $-i(T_a)W_\alpha^a$,

$$\Gamma^C_{D\alpha} = -\frac{1}{8}(\bar{\mathcal{D}}^2 - 8R)\mathcal{D}_\alpha Z^p \Gamma^C_{Dp}, \tag{4.9}$$

and

$$X_\alpha = -\frac{1}{8}(\bar{\mathcal{D}}^2 - 8R)\mathcal{D}_\alpha K, \tag{4.10}$$

associated with the Yang-Mills, reparameterization and Kähler connections, $i(T_a)^C_D A_\mu$, $\partial_\mu Z^p \Gamma^C_{pD}$ and $\delta^C_D \Gamma_\mu$, respectively, where

$$\Gamma_\mu = \frac{i}{4}\left(\mathcal{D}_\mu z^i K_i - \mathcal{D}_\mu \bar{z}^{\bar{m}} K_{\bar{m}}\right). \tag{4.11}$$

Cancellation of quadratic divergences requires

$$\mathrm{Tr}\eta\Gamma_\alpha = \mathrm{Tr}\eta T_X = 0, \tag{4.12}$$

and cancellation of logarithmic divergences requires

$$\mathrm{Tr}\eta\Gamma_\alpha\Gamma_\beta = \mathrm{Tr}\eta\Gamma_\alpha T^a = \mathrm{Tr}\eta(T^a)^2 = 0, \tag{4.13}$$

where $\eta = +1$ for light fields. Cancellation of all contributions linear and quadratic in X_α is assured by the conditions in (3.20) and (4.8). The Yang-Mills contribution to the term quadratic in W_α is canceled by chiral fields in the adjoint (see footnote on page 7) that we need not consider here. Finally, cancellation of linear divergences requires cancellation of the imaginary part of

$$\mathrm{Tr}\eta X_\chi = \mathrm{Tr}\eta\phi G \cdot \widetilde{G}, \qquad \widetilde{G}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}, \tag{4.14}$$

where $G_{\mu\nu}$ is the field strength associated with the fermion connection[4]; for left-handed fermions:

$$G_{\mu\nu} = -\Gamma^C_{D\mu\nu} + iF^a_{\mu\nu}(T_a)^C_D + \frac{1}{2}X_{\mu\nu}\delta^C_D, \tag{4.15}$$

where

$$\begin{aligned} X_{\mu\nu} &= \left(\mathcal{D}_\mu z^i \mathcal{D}_\nu \bar{z}^{\bar{m}} - \mathcal{D}_\nu z^i \mathcal{D}_\mu \bar{z}^{\bar{m}}\right) K_{i\bar{m}} - iF^a_{\mu\nu}(T_a z^i) K_i \\ &= 2i\left(\partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu\right), \qquad i = p, s, \end{aligned} \tag{4.16}$$

[4] Here we neglect the spin connection whose contribution was discussed in [2].

is the field strength associated with the Kähler connection (4.11). For a generic PV superfield Φ^C with diagonal metric, its fermion component χ^C transforms under (3.1) and (3.6) as

$$\chi'^C = e^{\phi^C}\chi^C, \qquad \phi^C = \left(\frac{1}{2} - \alpha^C - \beta^C\right)F - \sum_i F^i(t^i)q_i^C - \lambda Q_X. \tag{4.17}$$

In evaluating (4.14) we will use the fact that the expression[5]

$$\epsilon^{\mu\nu\rho\sigma} g^i_{\mu\nu} g^i_{\rho\sigma} = 0, \tag{4.18}$$

vanishes identically, and the expressions

$$\begin{aligned}
X^{ij} &= \epsilon^{\mu\nu\rho\sigma}\mathrm{Im}F^i g^i_{\mu\nu} g^{j\neq i}_{\rho\sigma} = 4\epsilon^{\mu\nu\rho\sigma}\mathrm{Im}F^i \partial_\mu g^i_\nu \partial_\rho g^j_\sigma = 4\partial_\rho\left(\epsilon^{\mu\nu\rho\sigma}\mathrm{Im}F^i\partial_\mu g^i_\nu g^j_\sigma\right),\\
X^i &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Im}F^i g^i_{\mu\nu} X_{\rho\sigma} = 4i\partial_\rho\left(\epsilon^{\mu\nu\rho\sigma}\mathrm{Im}F^i\partial_\mu g^i_\nu \Gamma_\sigma\right),\\
X^{ia} &= \epsilon^{\mu\nu\rho\sigma}\mathrm{Im}F^i g^i_{\mu\nu} F^a_{\rho\sigma} = 4\partial_\rho\left(\epsilon^{\mu\nu\rho\sigma}\mathrm{Im}F^i\partial_\mu g^i_\nu A^a_\sigma\right),
\end{aligned} \tag{4.19}$$

are total derivatives, where A^a_μ is an Abelian gauge field, and

$$g^i = -\ln(t^i + \bar{t}^{\bar{i}}), \qquad g^i_\mu = -\frac{\partial_\mu t^i - \partial_\mu \bar{t}^{\bar{i}}}{t^i + \bar{t}^{\bar{i}}}, \qquad g^i_{\mu\nu} = \partial_\mu g^i_\nu - \partial_\nu g^i_\mu. \tag{4.20}$$

The full Kähler potential for $\dot{Y}$, with no anomalous $U(1)_X$, is given in [1,2]; here it takes the form

$$\begin{aligned}
K(\dot{Y}) &= e^{\dot{G}}\left(\sum_A e^{-g^a - q^a V_X}|\dot{Y}_A|^2 + \sum_I e^{-2g^i}|\dot{Y}_I|^2 + \sum_N |\dot{Y}_N|^2\right) + \dots,\\
g^a &= \sum_n q^a_n g^n, \qquad \dot{G} = \dot{\alpha}K + \dot{\beta}g, \qquad \dot{\alpha} + \dot{\beta} = 1,
\end{aligned} \tag{4.21}$$

where $\dot{Y}_{N=1,2,3}$ (and their counterparts $\dot{Z}^N$) are gauge singlet PV fields needed [9] to make the Kähler potential and superpotential terms for $\dot{Z}, \dot{Y}$ fully invariant, and the ellipsis represents terms that make no contribution to the expressions given below. Using the sum rules in (2.21) and (3.20) we obtain:

$$\begin{aligned}
\mathrm{Tr}\dot{\eta}\Gamma^{\dot{Y}}_\alpha &= -\left[(N+2)\dot{\beta} - A_1\right]g_\alpha, \qquad \mathrm{Tr}\dot{\eta}T^{\dot{Y}}_X = \mathrm{Tr}T_X,\\
\mathrm{Tr}\dot{\eta}\Gamma^{\dot{Y}}_\alpha\Gamma^{\dot{Y}}_\beta &= -2\dot{\alpha}\left[\dot{\beta}(N+2) - A_1\right]X_\alpha g_\beta - \left[\dot{\beta}^2(N+2) - \dot{\beta}A_1 + A_2\right]g_\beta g_\alpha\\
&\quad - B_2\sum_n g^n_\alpha g^n_\beta\\
\mathrm{Tr}\dot{\eta}\Gamma^{\dot{Y}}_\alpha T_a &= \delta_{aX}\mathrm{Tr}T^{\dot{Y}}_X \dot{G}_\alpha - Q_{1a}g_\alpha, \qquad \dot{G}_\alpha = \dot{\alpha}X_\alpha + \dot{\beta}g_\alpha.
\end{aligned} \tag{4.22}$$

[5] It was noted in [2] that the expression (4.18), which is in fact the T-dependent part of the chiral anomaly found in [3], vanishes. The authors of [3] attribute [12] this to their approximation that neglects higher order corrections. However if these corrections take the form $g^i(T^i, \bar{T}^i) \to g^i(T^i, \bar{T}^i) + \Delta^i(T^i, \bar{T}^i)$, our results our unchanged. Note that the functional form of Δ^i is severely restricted by the fact that it has to be invariant under T-duality.

Using (4.19) and (2.21), the part of $X^{\dot{Y}}$ that is independent of gauge charges takes the form:

$$X^{\dot{Y}}_{\chi} \ni \frac{1}{2}\left[(N+2)-2A_1\right] F\dot{G}\cdot\tilde{\dot{G}} - (A_1-2A_2)\,F\dot{G}\cdot\tilde{g} - A_3 Fg\cdot\tilde{g}$$
$$+ \text{total derivative}, \qquad \dot{G}_{\mu\nu} = \dot{\alpha}X_{\mu\nu} + \dot{\beta}g_{\mu\nu}. \tag{4.23}$$

The modular weights for the ψ satisfy

$$\sum_{m,n} g^n q_n^{P_m} = g q_\psi^P, \qquad \sum_P \eta_\psi^P q_l^{P_k} q_n^{P_k} q_n^{P_k} = 0,$$
$$\sum_{l,m,n} g^m g^n q_m^{P_l} q_n^{P_l} = (q_\psi^P)^2 \sum_n g^n g^n. \tag{4.24}$$

Like $X^{\dot{Y}}_{\chi}$, X^{ψ}_{χ} depends only on F, $g_{\mu\nu}$ and $X_{\mu\nu}$, and (4.22) and (4.23) can be canceled by some combination of the fields in (4.7), with the condition

$$\sum_P \eta_\psi^P (q_\psi^P)^2 = B_2. \tag{4.25}$$

The pure T-moduli anomaly is given by

$$\Delta L_1(Fg^2) = \frac{F}{8\pi^2}\mathrm{Tr}\eta_\psi\left(1-2\bar{\gamma}_\psi\right) q_\psi^2 \Omega_g, \qquad (\bar{\mathcal{D}}^2 - 8R)\Omega_g = \sum_n g_n^\alpha g_\alpha^n. \tag{4.26}$$

Consistency with string results [13] requires

$$\mathrm{Tr}\eta_\psi\left(1-2\bar{\gamma}_\psi\right) q_\psi^2 = -8\pi^2 b \tag{4.27}$$

Finally, we require

$$\Delta L_1(Q_X g^2) = -\frac{2\Lambda}{8\pi^2}\mathrm{Tr}\eta\bar{Q}_X\Omega_f = \frac{1}{2}\Lambda\delta_X\Omega_g. \tag{4.28}$$

Using (4.24), the condition (4.28) requires

$$\sum_P \eta_\psi^P \bar{Q}_\psi^P (q_\psi^P)^2 = -4\pi^2\delta_X. \tag{4.29}$$

All other contributions to ΔL_1 are required to vanish.

We conclude this section by noting that cancellation of divergences linear in the $U(1)_a$ field strengths is much simpler than for the $\mathbb{Z}_7$ case considered in [2], as outlined below.

The gauge charges for the FIQS [4] model are listed[6] in Appendix C. The universality of the anomaly term quadratic in Yang-Mills fields strengths is guaranteed by the universality condition (2.19), as discussed in Section 3. Since gauge transformations commute with modular transformations, a set of chiral multiplets Φ^b that transform according to a nontrivial irreducible representation R of a nonabelian gauge group factor $\mathcal{G}_a$ have the same modular weights q_n^R such that

$$\sum_{b\in R} q_n^b (T_a)_b^b = q_n^R (\mathrm{Tr}T_a)_R = 0. \tag{4.30}$$

[6] We have made some corrections to the $U(1)_a$ charges given in (2).

Therefore terms linear in Yang-Mills field strengths occur only for Abelian gauge group factors. We need to cancel the $\dot{Y}$-loop contribution to logarithmic divergences

$$\left(\mathrm{Tr}\eta\sum_n q_n g^n_\alpha T_a\right)_{\dot{Y}} = -\sum_{b,n} q^b_n Q^b_a g^n_\alpha = -Q_{1a} g_\alpha, \tag{4.31}$$

and, dropping terms proportional to the last expression in (4.19), the relevant $\dot{Y}$ contributions to linear divergences:

$$\begin{aligned} X^{\dot{Y}}_\chi &\ni \sum_{a,b,n} Q^b_a \tilde{F}^a \cdot \left[g^n q^b_n\left(F - 2\sum_m q^b_m F^m\right) + 2q^b_n F^n\left(\dot{G} - \frac{1}{2}X\right)\right] \\ &= \sum_a \tilde{F}^a \cdot \left\{\left[g\left(1+2\dot{\beta}\right) + X\left(2\dot{\alpha}-1\right)\right] Q_{1a} F - 2\sum_n g^n F^n Q_{2a}\right\}, \end{aligned} \tag{4.32}$$

where we used (2.21). The last term in (4.32) is canceled by

$$X^\psi_\chi \ni -2\sum_{a,P,l,m,n} \eta^P_\psi Q^P_a q^{P_l}_m q^{P_l}_n F^m \tilde{F}^a \cdot g^n = -2\sum_{a,P} \eta^P_\psi Q^P_a (q^P)^2 \tilde{F}^a \cdot \sum_n g^n F^n, \tag{4.33}$$

provided

$$\sum_P \eta^P_\psi Q^P_a (q^P)^2 = -Q_{2a}. \tag{4.34}$$

The remaining terms in (4.32), as well as (4.31) can be canceled by a combination of the fields in (4.7). For $a = X$ there are additional terms proportional to $(\mathrm{Tr}\eta T_X)_{PV} = -\mathrm{Tr}T_X$.

5. The final anomaly in the FIQS model

In Appendix A we show that is possible to cancel all the ultraviolet divergences from the $\dot{Y}$ fields with a choice of the set (4.7) such that the fields with noninvariant masses have the properties

$$\mathrm{Tr}\eta(\ln\mathcal{M})^{n>1} = \Delta\mathrm{Tr}\eta(\ln\mathcal{M})^{n>1} = \mathrm{Tr}\eta(\Delta\ln\mathcal{M})(\bar{f}_\alpha)^{n>0} = 0. \tag{5.1}$$

Then, including the results of [2], the anomaly due to the variation of (3.9) takes the form

$$\delta\mathcal{L}_{\mathrm{anom}} = \int d^4\theta E\left(bF - \frac{1}{2}\delta_X\Lambda\right)\Omega + \int d^4\theta E bF\Omega', \tag{5.2}$$

where

$$\begin{aligned} \Omega &= \Omega_{\mathrm{YM}} - \Omega_{\mathrm{GB}} + \Omega_g, \\ \Omega' &= -\frac{b_{\mathrm{spin}}}{48b}\left(4G_{\dot{\beta}\alpha}G^{\alpha\dot{\beta}} - 16R\bar{R} + \mathcal{D}^2R + \bar{\mathcal{D}}^2\bar{R}\right) - \frac{1}{8\pi^2 b}\Omega_D, \end{aligned} \tag{5.3}$$

where Ω_g is defined in (4.26), and b_{spin} governs the contributions from PV masses, as opposed to those arising from uncanceled divergences:

$$8\pi^2 b_{\mathrm{spin}} = 8\pi^2 b + 1, \tag{5.4}$$

with $8\pi^2 b = 6$ in the FIQS model. In the absence of an anomalous $U(1)$, $\Lambda = 0$, the anomaly can be canceled by the four dimensional GS mechanism as described in [2]. However with $\Lambda \neq 0$, the anomaly as written in (5.3) is no longer universal and cannot be canceled by the GS term alone. However all of the "D-terms", in other words the full expression Ω', can be removed [14] by adding counterterms to the Lagrangian, giving a universal anomaly which can now be canceled by the GS term.[7]

The results for the Gauss-Bonnet and Yang-Mills terms are well-established [10] and result from the universality conditions (2.19).

6. Conclusions

We have shown that a suitable choice of Pauli-Villars regulator fields allows for a full cancellation of the chiral and conformal anomalies associated, respectively, with the linear and logarithmic divergences in the effective supergravity theory from a $\mathbb{Z}_3$ orbifold compactification with Wilson lines and an anomalous $U(1)$.

A future work [13] will compare this result with that obtained directly from string theory.

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Appendix A. Conditions for the cancellation of ultraviolet divergences and the evaluation of the anomaly

A.1. Notation

We pair PV fields according to their mass terms. A pair of PV fields (Φ^P, Φ'^P) has a superpotential coupling

$$W_{PV} = \sum_P \mu_P \Phi'^P \Phi^P \tag{A.1}$$

and a Kähler potential

$$K_{PV} = \sum_P e^{f^P} \Phi^{*P} \Phi^P + \sum_P e^{f'^P} \Phi'^{*P} \Phi'^P, \tag{A.2}$$

where

$$f^P = \alpha^P K + \beta^P g + \sum_n q_n^P g^n \tag{A.3}$$

with an identical definition holding for f'^P but with primes on the constants $\{\alpha^P, \beta^P, q_n^P\}$. While we will not use it often, summing over the index C means summing over PV fields and then their

[7] The elimination of Ω_D further obviates the need for a modification of the linear-chiral duality transformation, a possibility considered in Appendix B of [2] and Appendix E of [1].

primed partners whereas summing over P means summing over only the unprimed or primed fields, depending on the quantity being summed. For example,

$$\sum_C \eta^C \alpha^C = \sum_P \eta_P \alpha^P + \sum_P \eta_P \alpha'^P. \tag{A.4}$$

However, to reduce clutter, we will abbreviate the above. When summing over primed and unprimed fields, we will use "Tr". When summing over only primed or unprimed ones, we will use "Sum". Thus the above would be written as

$$\mathrm{Tr}[\eta\alpha] = \mathrm{Sum}[\eta\alpha] + \mathrm{Sum}[\eta\alpha']. \tag{A.5}$$

We will also encounter sums over various combinations of U(1) charges, $\mathrm{U}(1)_X$ charges, and modular weights. To abbreviate these, especially when dealing with the quantum numbers of the light fields, we will define

$$Q_{1a} = \mathrm{Sum}[\eta Q_a q_n] \tag{A.6}$$

$$Q_{2a} + P_{2a}\delta_{nm} = \mathrm{Sum}[\eta Q_a q_n q_m] \tag{A.7}$$

$$R_a = \mathrm{Sum}[\eta Q_a Q_X q_n] \tag{A.8}$$

$$R_{ab} = \mathrm{Sum}[\eta Q_a Q_b q_n] \tag{A.9}$$

$$S_a = \mathrm{Sum}[\eta Q_a Q_X] \tag{A.10}$$

$$S_{ab} = \mathrm{Sum}[\eta Q_a Q_b]. \tag{A.11}$$

A.2. Conditions for regularization

The terms we must cancel come from linear, logarithmic, and quadratic divergences. It is helpful to organize these terms by forming subsets based on whether terms depend on nonabelian gauge interactions, nonanomalous Abelian gauge interactions, anomalous Abelian gauge interactions, or none of the above. We will refer to these groupings as nonabelian divergences, $\mathrm{U}(1)_a$ divergences, $\mathrm{U}(1)_X$ divergences, and modular divergences, respectively. As an overview, the divergences come from the terms

$$\mathrm{Tr}[\eta\Gamma_\alpha] \tag{A.12}$$

$$\mathrm{Tr}[\eta\Gamma_\alpha\Gamma_\beta] \tag{A.13}$$

$$\mathrm{Tr}[\eta\Gamma_\alpha T_a] \tag{A.14}$$

$$\mathrm{Tr}[\eta T_a T_b] \tag{A.15}$$

$$\mathrm{Tr}[\eta Q_a], \tag{A.16}$$

where

$$\Gamma^C_{D\alpha} = -\frac{1}{8}\left(\bar{\mathcal{D}}^2 - 8R\right)\mathcal{D}_\alpha Z^i \Gamma^C_{Di} \tag{A.17}$$

$$\phi^C = \left(\frac{1}{2} - \alpha^C - \beta^C\right) F - \sum_i F^i q^C_i - q^C_X \Lambda \tag{A.18}$$

$$G_{\mu\nu} = \Gamma^C_{C\mu\nu} - \frac{1}{2} X_{\mu\nu}\delta^C_D - i F^a_{\mu\nu}(T_a)^C_D - i F^X_{\mu\nu}(Q_X)^C_D \tag{A.19}$$

for our PV fields defined above.

The PV fields involved in this procedure are numerous. We take all of the PV fields described in sections 3 and 4 of [1] and supplement them with further fields. However, to satisfy the divergences above, we need only focus on the $\dot{Y}$ and $\hat{\phi}$ fields of [1]. We now group all the terms in the above expressions with our organizational scheme.

Modular Divergences

To cancel all the modular divergences, we require

$$0 = -\mathrm{Tr}\left[\eta\beta\left(\frac{1}{2}-\alpha\right)^2\right] - \mathrm{Tr}\left[\eta q_n\left(\frac{1}{2}-\alpha\right)^2\right] \tag{A.20}$$

$$\begin{aligned} 0 = &-\frac{1}{2}\mathrm{Tr}\left[\eta(1-2\alpha)\beta(1-2\gamma)\right] + \mathrm{Tr}\left[\eta\beta q_n(1-2\alpha)\right] \\ &-\frac{1}{2}\mathrm{Tr}\left[\eta(1-2\alpha)(1-2\gamma)q_n\right] + \mathrm{Tr}\left[\eta q_n q_m(1-2\alpha)\right] \end{aligned} \tag{A.21}$$

$$\begin{aligned} 0 = &\frac{1}{2}\mathrm{Tr}\left[\eta\beta^2(1-2\gamma)\right] - \mathrm{Tr}\left[\eta\beta^2 q_n\right] + \mathrm{Tr}\left[\eta\beta(1-2\gamma)q_n\right] - 2\mathrm{Tr}\left[\eta\beta q_n q_m\right] \\ &+\frac{1}{2}\mathrm{Tr}\left[\eta(1-2\gamma)q_n q_m\right] - \mathrm{Tr}\left[\eta q_n q_m q_k\right]. \end{aligned} \tag{A.22}$$

U(1)$_X$ Divergences

To cancel all the U(1)$_X$ divergences, we need

$$0 = \mathrm{Tr}[\eta Q_X] \tag{A.23}$$

$$0 = \mathrm{Tr}[\eta Q_X\beta] + \mathrm{Tr}[\eta Q_X q_m] \tag{A.24}$$

$$0 = \mathrm{Tr}[\eta Q_X\alpha] \tag{A.25}$$

$$0 = \mathrm{Tr}\left(\eta Q_X\left(\alpha-\frac{1}{2}\right)^2\right) \tag{A.26}$$

$$0 = -\mathrm{Tr}\left(\eta Q_X\beta\left(\alpha-\frac{1}{2}\right)\right) + \mathrm{Tr}\left(\eta Q_X q_n\left(\alpha-\frac{1}{2}\right)\right) \tag{A.27}$$

$$0 = \mathrm{Tr}\left(\eta Q_X\beta^2\right) + 2\mathrm{Tr}\left(\eta Q_X q_n\beta\right) + \mathrm{Tr}\left(\eta Q_X q_n q_m\right) \tag{A.28}$$

$$0 = \mathrm{Tr}\left(\eta Q_X^3\right) \tag{A.29}$$

$$0 = \mathrm{Tr}\left(\eta Q_X^2\left(\frac{1}{2}-\gamma\right)\right) - \mathrm{Tr}\left(\eta Q_X^2 q_n\right) \tag{A.30}$$

$$0 = \mathrm{Tr}\left(\eta Q_X^2\left(\alpha-\frac{1}{2}\right)\right) \tag{A.31}$$

$$0 = 2\mathrm{Tr}\left(\eta Q_X\left(\alpha-\frac{1}{2}\right)\left(\frac{1}{2}-\gamma\right)\right) - 2\mathrm{Tr}\left(\eta Q_X q_n\left(\alpha-\frac{1}{2}\right)\right) \tag{A.32}$$

$$0 = -2\mathrm{Tr}\left(\eta Q_X^2\beta\right) - 2\mathrm{Tr}\left(\eta Q_X^2 q_n\right) \tag{A.33}$$

$$\begin{aligned} 0 = &2\mathrm{Tr}\left(\eta Q_X\beta\left(\frac{1}{2}-\gamma\right)\right) - 2\mathrm{Tr}\left(\eta Q_X q_n\beta\right) + 2\mathrm{Tr}\left(\eta Q_X q_n\left(\frac{1}{2}-\gamma\right)\right) \\ &- 2\mathrm{Tr}\left(\eta Q_X q_n q_m\right). \end{aligned} \tag{A.34}$$

Note that only fields that have $\bar{Q}_X \neq 0$ will contribute to Eq. (A.29).

Nonabelian Divergences

To cancel the nonabelian divergences, we need

$$0 = \mathrm{Tr}[\eta T_a T_b] \tag{A.35}$$

$$0 = \mathrm{Tr}[\eta Q_X T_a T_b] \tag{A.36}$$

$$0 = \mathrm{Tr}\left[\eta T_a T_b \left(\gamma - \frac{1}{2}\right)\right], \tag{A.37}$$

where T^a is a generator of a nonabelian gauge group factor.

U(1)$_a$ Divergences

Finally, the conditions for canceling the abelian divergences are

$$0 = \mathrm{Tr}[\eta Q_a] \tag{A.38}$$

$$0 = \mathrm{Tr}[\eta Q_a \alpha] \tag{A.39}$$

$$0 = \mathrm{Tr}[\eta Q_a \beta] + \mathrm{Tr}[\eta q_n Q_a] \tag{A.40}$$

$$0 = \mathrm{Tr}[\eta Q_X Q_a Q_b] \tag{A.41}$$

$$0 = \mathrm{Tr}[\eta Q_X Q_a \beta] + \mathrm{Tr}[\eta Q_X q_n Q_a] \tag{A.42}$$

$$0 = \mathrm{Tr}\left[\eta Q_X Q_a \left(\alpha - \frac{1}{2}\right)\right] \tag{A.43}$$

$$0 = -\mathrm{Tr}\left[\eta Q_X Q_a \left(\frac{1}{2} - \gamma\right)\right] + \mathrm{Tr}[\eta Q_X Q_a q_n] \tag{A.44}$$

$$0 = \mathrm{Tr}\left[\eta Q_a \left(\alpha - \frac{1}{2}\right)\left(\left(\frac{1}{2} - \gamma\right) - q_n\right)\right] \tag{A.45}$$

$$0 = \mathrm{Tr}\left[\eta Q_a \beta \left(\left(\frac{1}{2} - \gamma\right) - q_n\right)\right] + \mathrm{Tr}\left[\eta Q_a q_n \left(\left(\frac{1}{2} - \gamma\right) - q_n\right)\right] \tag{A.46}$$

$$0 = \mathrm{Tr}\left[\eta Q_a Q_b \left(\left(\gamma - \frac{1}{2}\right) + q_n\right)\right]. \tag{A.47}$$

In all of the above sets, we have assumed that the modular weights of all PV fields satisfy sum rules reminiscent of those satisfied by the light sector, (2.21). Indeed, this will be baked directly into our choice of PV fields. We have also used the total derivative identities (4.19). In addition to the above conditions, we must enforce the sum rules of [1]:

$$-N - 29 = \mathrm{Tr}[\eta] \tag{A.48}$$

$$-10 = \mathrm{Tr}[\eta\alpha] \tag{A.49}$$

$$-4 = \mathrm{Tr}[\eta\alpha^2] \tag{A.50}$$

$$0 = \mathrm{Tr}\left[\eta\beta\right] \tag{A.51}$$

$$0 = \mathrm{Tr}\left[\eta\beta^2\right] \tag{A.52}$$

$$0 = \mathrm{Tr}\left[\eta\beta\alpha\right]. \tag{A.53}$$

A.3. Conditions for anomaly matching

By drawing an analogy with the calculation of [3], we infer that in four dimensions the anomaly polynomial for the FIQS model has the form [13]

$$I_6 = \left(-\frac{b}{4\pi}\sum_{i=1}^{3} G_i + \frac{\delta_X}{8\pi}F_X \right)\left(\mathrm{tr}(R^2) - \sum_n (F_n^{SU(3)})^2 - \sum_n (F_n^{SU(2)})^2 - \sum_n (F_n^{SO(10)})^2 - \sum_{a=1}^{7}(F_a)^2 - (F_X)^2 + 2\sum_i G_i^2 \right) \tag{A.54}$$

where

$$G_i = dZ_i \tag{A.55}$$

$$Z_i = \frac{1}{2i}\frac{d(T^i - \bar{T}^i)}{T^i + \bar{T}^i} \tag{A.56}$$

and

$$\mathrm{tr}(R^2) = R^a{}_b R^b{}_a \tag{A.57}$$

$$= \frac{1}{4}R^\tau{}_{\epsilon\mu\nu}R^\epsilon{}_{\tau\rho\sigma}dx^\mu dx^\nu dx^\rho dx^\sigma \tag{A.58}$$

$$(F_A)^2 = \frac{1}{4}F_{A\mu\nu}F_{A\rho\sigma}dx^\mu dx^\nu dx^\rho dx^\sigma \tag{A.59}$$

In the above, we have implicitly assumed wedge products in the multiplication of differential forms. To get the 4D anomaly from the 6-form anomaly polynomial, one goes through the usual descent equations:

$$2\pi I_6 = dI_5 \tag{A.60}$$

$$\delta I_5 = dI_4 \tag{A.61}$$

For example, under a modular transformation, $Z_i \to Z_i + d\mathrm{Im}(F^i)$ so that the modular-gravity-gravity anomaly has the form

$$\int I_4 \supset \int -\frac{3}{32\pi^2}\left(\sum_{i=1}^{3}\mathrm{Im}(F^i)\right) R^\tau{}_{\omega\mu\nu}R^\omega{}_{\tau\rho\sigma}\epsilon^{\mu\nu\rho\sigma}\sqrt{g}d^4x \tag{A.62}$$

which is precisely what one would expect if one considers the modular-gravity anomaly to have the same form as a U(1)-gravity anomaly. To match this anomaly, we look at the anomalous contributions of PV fields with masses that are noninvariant under modular and $\mathrm{U}(1)_X$ transformations. The general form of their contribution is

$$\mathcal{L}_{\mathrm{anom}} = \int d^4\theta E(L_0 + L_1 + L_r) \tag{A.63}$$

with

$$L_0 = \frac{1}{8\pi^2}\left(\mathrm{Tr}[\eta\ln(\mathcal{M}^2)]\Omega_0 + K(\Omega_{GB} + \Omega_D)\right) \tag{A.64}$$

$$L_r = -\frac{1}{192\pi^2}\mathrm{Tr}\left[\eta\int d\ln(\mathcal{M})\Omega_r\right]. \tag{A.65}$$

Focusing on the second term of Eq. (A.63), we again break up terms based on whether they contribute to the $U(1)_X$ related anomalies or the pure modular anomaly.

$U(1)_X$ Anomaly Conditions

To match the anomalies involving $U(1)_X$, we require

$$0 = \frac{2}{3}\mathrm{Tr}\left[\eta\bar{Q}_X\left(2\bar{\alpha}^2 + \bar{\alpha} - 3\alpha^2\right)\right] \tag{A.66}$$

$$0 = \frac{2}{3}\mathrm{Tr}\left[\eta\bar{Q}_X\left(\bar{\beta} + 4\bar{\alpha}\bar{\beta} - 6\alpha\beta\right)\right] \tag{A.67}$$

$$0 = \frac{2}{3}\mathrm{Tr}\left[\eta\bar{Q}_X\left(2\bar{\beta}^2 - 3\beta^2\right)\right] \tag{A.68}$$

$$0 = -4\mathrm{Tr}\left[\eta\left(\alpha\bar{Q}_X q_n\right)\right] \tag{A.69}$$

$$0 = -4\mathrm{Tr}\left[\eta\left(\beta\bar{Q}_X q_n\right)\right] \tag{A.70}$$

$$8\pi^2\delta_X\delta_{mn} = -2\mathrm{Tr}\left[\eta\left(\bar{Q}_X q_n q_m\right)\right] \tag{A.71}$$

$$0 = \frac{1}{3}\mathrm{Tr}\left[\eta\left(Q_X\left(-4\bar{\alpha} + 6\alpha - 1\right)\left(1 - 2\bar{\gamma}\right)\right)\right] \tag{A.72}$$

$$0 = \frac{2}{3}\mathrm{Tr}\left[\eta Q_X\left(1 - 2\bar{\gamma}\right)\left(3\beta Q_X - 2\bar{\beta}\bar{Q}_X\right)\right)\right] \tag{A.73}$$

$$0 = 2\mathrm{Tr}\left[\eta\left(Q_X q_n\left(1 - 2\bar{\gamma}\right)\right)\right] \tag{A.74}$$

$$8\pi^2 b = \frac{1}{3}\mathrm{Tr}\left[\eta\left(1 - 2\bar{\gamma}\right)\left(3Q_X^2 - 2\bar{Q}_X^2\right)\right] \tag{A.75}$$

$$0 = \frac{2}{3}\mathrm{Tr}\left[\eta\bar{Q}_X\left(4\bar{\alpha}\bar{Q}_X + \bar{Q}_X - 6\alpha Q_X\right)\right] \tag{A.76}$$

$$0 = \frac{1}{3}\mathrm{Tr}\left[\eta\left(8\bar{\beta}\bar{Q}_X^2 - 12\beta Q_X\bar{Q}_X\right)\right] \tag{A.77}$$

$$0 = -4\mathrm{Tr}\left[\eta\left(Q_X\bar{Q}_X q_n\right)\right] \tag{A.78}$$

$$-4\pi^2\delta_X = \mathrm{Tr}\left[\eta\left(\frac{4\bar{Q}_X^3}{3} - 2Q_X^2\bar{Q}_X\right)\right] = -\frac{2}{3}\mathrm{Tr}\left[\eta Q_X^3\right]. \tag{A.79}$$

Note that the last term is fixed by cancellation of the linear divergence term Eq. (A.29).

Pure Modular Anomaly Conditions

To match the pure modular anomaly, we require

$$0 = \frac{1}{3}\mathrm{Tr}\left[\eta\left(1 - 2\bar{\gamma}\right)\left(-2\bar{\alpha}^2 - \bar{\alpha} + 3\alpha^2\right)\right] \tag{A.80}$$

$$0 = \frac{1}{3}\mathrm{Tr}\left[\eta\left(1 - 2\bar{\gamma}\right)\left(3\beta^2 - 2\bar{\beta}^2\right)\right] + 2\mathrm{Tr}\left[\eta\beta\left(1 - 2\bar{\gamma}\right)q_n\right] \tag{A.81}$$

$$0 = \frac{1}{3}\mathrm{Tr}\left[\eta\left(1-2\bar{\gamma}\right)\left(6\alpha\beta-\left(4\bar{\alpha}+1\right)\bar{\beta}\right)\right] + 2\mathrm{Tr}\left[\eta\alpha\left(1-2\bar{\gamma}\right)q_n\right] \tag{A.82}$$

$$-8\pi^2 b\delta_{mn} = \mathrm{Tr}\left[\eta q_m q_n\left(1-2\bar{\gamma}\right)\right]. \tag{A.83}$$

As for the third term of Eq. (A.63), we need it to vanish identically. This can be achieved so long as the following are satisfied

$$0 = \mathrm{Tr}\left[\eta x(1-2\bar{\gamma})^2\right] \tag{A.84}$$

$$0 = \mathrm{Tr}\left[\eta x\bar{q}_X(1-2\bar{\gamma})\right] \tag{A.85}$$

$$0 = \mathrm{Tr}\left[\eta x\bar{q}_X^2\right] \tag{A.86}$$

$$0 = \mathrm{Tr}\left[\eta\bar{\alpha}\bar{\beta}(1-2\bar{\gamma})\right] \tag{A.87}$$

$$0 = \mathrm{Tr}\left[\eta\bar{\alpha}\bar{\beta}\bar{q}_X)\right] \tag{A.88}$$

$$0 = \mathrm{Tr}\left[\eta\bar{\beta}^k(1-2\bar{\gamma})\right] \tag{A.89}$$

$$0 = \mathrm{Tr}\left[\eta\bar{\beta}^k\bar{q}_X\right]\mathrm{Tr}\left[\eta\bar{\beta}^3\bar{q}_X\right], \tag{A.90}$$

where $x = 1, \bar{\alpha}, \bar{\beta}, \bar{q}_X, \bar{\alpha}^2, \bar{\beta}^2, \bar{q}_X^2, \bar{\alpha}\bar{\beta}, \bar{\alpha}\bar{q}_X, \bar{\beta}\bar{q}_X$ and $k = 1, 2, 3$.

Appendix B. Solution to the Pauli-Villars regularization conditions

We will now elucidate a solution to the system described above. The solution consists of sets S_a, $a = 1, 2, \ldots$ of PV fields that address each of the divergence and anomaly sets of conditions more or less separately. For example, it is possible to introduce PV fields that cancel only the nonabelian divergences and contribute to no other conditions. We will try to follow the same strategy for all the sets of conditions described above. It is not entirely possible to do so – for example, fields that solve the modular anomaly conditions will generically contribute to modular divergences. Of course, this is far from the only way to tackle the system, but it is a straightforward method to illustrate that a solution can be found. To this end, we define the notion of clone fields for PV fields. For a given pair of PV fields $\left(\Phi^P, \Phi'^P\right)$, we define clone fields $\left(\Phi^P_{cl}, \Phi'^P_{cl}\right)$ that have almost the same parameters $(\alpha, \beta, q_n, \ldots)$ and quantum numbers as the original pair but with negative signature. We say almost here because this notion is only useful if the $\left(\Phi^P, \Phi'^P\right)$ have quantum numbers different from the clones so that the two sets cancel each other's contributions to some subset of the conditions, but not all conditions. As a concrete example, which will be described below, one can introduce PV fields with nonabelian gauge interactions to eliminate divergences associated with those same interactions. One can then introduce clone PV fields without gauge interactions that exactly cancel the contributions of the gauge charged PV fields to all other terms. The primary advantage of this technique is tidiness.

B.1. PV Fields for U(1)$_X$ anomaly matching

The fields described here will satisfy Eqs. (A.66)–(A.79) and will contribute to some of the U$(1)_X$ divergence conditions (A.24)–(A.34). In particular, only PV fields with $\bar{Q}_X \neq 0$ contribute to Eq. (A.29), so this condition will be satisfied by this sector only. The sets of PV fields we need are

- S_1: A set of PV fields with modular invariant masses, $\alpha_1 = \alpha_1' = \bar{\gamma}_1 = 1/2$, and $\bar{q}_n^{(1)} = 0$ and modular weights of the form $(q^{(1)})_m^C = q_{(1)}^P \delta_m^n$ and clone fields with no U$(1)_X$.
- S_2: A set of PV fields with $\bar{\alpha}_2 = \bar{\beta}_2 = \bar{\gamma}_2 = \bar{Q}_X^{(2)} = (q^{(2)})_n^C = 0$ and clone fields with no U$(1)_X$ charge.

We then place the following conditions on the parameters of these fields:

$$\text{Sum}\left[Q_X^{(L)}\right] = -\text{Sum}\left[\eta Q_X^{(1)}\right] \tag{B.1}$$

$$\text{Sum}\left[(Q_X^L)^3\right] = -\text{Tr}\left[\eta_1 (Q_X^{(1)})^3\right] \tag{B.2}$$

$$0 = \text{Tr}\left[\eta_1 \bar{Q}_X^{(1)} \left(1 - 3\alpha_1^2\right)\right] \tag{B.3}$$

$$0 = \left[\eta_1 \bar{Q}_X^{(1)} \alpha_1 q_n^{(1)}\right] \tag{B.4}$$

$$0 = \text{Tr}\left[\eta_1 \alpha_1 \bar{Q}_X^{(1)} Q_X^{(1)}\right] \tag{B.5}$$

$$0 = \text{Tr}\left[\eta (\bar{Q}_X^{(1)})^2\right] \tag{B.6}$$

$$0 = \text{Tr}\left[\eta (\bar{Q}_X^{(1)})^3\right] \tag{B.7}$$

$$0 = \text{Tr}\left[\eta (\bar{Q}_X^{(1)})^4\right] \tag{B.8}$$

$$0 = \text{Tr}\left[\eta \bar{Q}_X^{(1)} q_n^{(1)}\right] \tag{B.9}$$

$$0 = \text{Tr}\left[\eta \bar{Q}_X^{(1)} Q_X^{(1)} q_n^{(1)}\right] \tag{B.10}$$

$$-4\pi^2 \delta_X \delta_{nm} = \text{Tr}\left[\eta \bar{Q}_X^{(1)} q_n^{(1)} q_m^{(1)}\right] \tag{B.11}$$

$$2\pi^2 \delta_X = -\frac{1}{3}\text{Sum}\left[(Q_X^{(L)})^3\right] = \text{Tr}\left[\eta \bar{Q}_X^{(1)} (Q_X^{(1)})^2\right]. \tag{B.12}$$

Once again, the first condition is a linear divergence term that can only be canceled by fields with masses that are noninvariant under U$(1)_X$. This in turn forces the correct coefficient for the pure U$(1)_X$ anomaly in the last condition. The second set must satisfy

$$0 = \text{Tr}[\eta_2] \tag{B.13}$$

$$0 = \text{Tr}\left[\eta_2 \alpha_2 Q_X^{(2)}\right] \tag{B.14}$$

$$0=\mathrm{Tr}\left[\eta_2\beta_2 Q_X^{(2)}\right] \tag{B.15}$$

$$8\pi^2 b=\mathrm{Tr}\left[\eta_2 (Q_X^{(2)})^2\right]. \tag{B.16}$$

The first condition here comes from Eq. (A.84) and can be relaxed.

B.2. PV fields for modular anomaly matching

The fields described here will satisfy conditions (A.80)–(A.83) and contribute to the modular divergence conditions (A.20)–(A.22). The sets are

- S_3: A set of pairs of PV fields with $\beta_3=\beta_3'=0$, $q_n^{(3)}=q_n^{\prime(3)}=0$.
- S_4: A set of pairs of PV fields with $\alpha_4=\alpha_4'=\beta_4=\beta_4'=\bar{q}_4^n=0$, $(q^{(4)})_m^C=(q^{(4)})^P\delta_m^n$, and clone fields with no modular weights.

These fields will contribute to the modular divergence conditions, as outlined below. We also have to consider the $\hat{\phi}$ fields of [1] here since they have noninvariant masses under modular transformations. These fields have no β or modular weight parameters, but do have $f_{\hat{\phi}}=\hat{\alpha}K$. Then the conditions the S_3, S_4, and $\hat{\phi}$ fields must satisfy are

$$0=\mathrm{Tr}\left[\hat{\eta}(1-2\bar{\hat{\alpha}})^2\right]+\mathrm{Tr}\left[\eta_3(1-2\bar{\alpha}_3)^2\right] \tag{B.17}$$

$$0=\mathrm{Tr}\left[\hat{\eta}\bar{\hat{\alpha}}(1-2\bar{\hat{\alpha}})^2\right]+\mathrm{Tr}\left[\eta_3\bar{\alpha}_3(1-2\bar{\alpha}_3)^2\right] \tag{B.18}$$

$$0=\mathrm{Tr}\left[\hat{\eta}\bar{\hat{\alpha}}^2(1-2\bar{\hat{\alpha}})^2\right]+\mathrm{Tr}\left[\eta_3\bar{\alpha}_3^2(1-2\bar{\alpha}_3)^2\right] \tag{B.19}$$

$$0=\mathrm{Tr}\left[\hat{\eta}\left(1-2\bar{\hat{\alpha}}\right)\left(-2\bar{\hat{\alpha}}^2-\bar{\hat{\alpha}}+3\hat{\alpha}^2\right)\right]+\mathrm{Tr}\left[\eta_3\left(1-2\bar{\alpha}_3\right)\left(-2\bar{\alpha}_3^2-\bar{\alpha}_3+3\alpha_3^2\right)\right] \tag{B.20}$$

and

$$-8\pi^2 b=\mathrm{Tr}\left[\eta_4 q_4^P q_4^P\right]=2\mathrm{Sum}\left[\eta_4 q_4^P q_4^P\right]. \tag{B.21}$$

B.3. PV fields for the regulation of modular divergences

Here we introduce fields that can cancel the contributions to Eqs. (A.20)–(A.22) from the $\dot{Y}$, S_3, and S_4 and contribute to the sum rules in Eqs. (3.37), (3.38) and (A.16) of [1]. The only new set we introduce here is

- S_5: A set of pairs of PV fields with $\bar{\gamma}_5=\frac{1}{2}$ and $(\bar{q}^{(5)})_n^C=0$ with $(q^{(5)})_m^C=(q^{(5)})^P\delta_m^n$.

Then the conditions we must satisfy are

$$0=(N+2)\dot{\beta}\left(\frac{1}{2}-\dot{\beta}\right)^2-A_1\left(\frac{1}{2}-\dot{\beta}\right)^2-\mathrm{Sum}\left[\eta_5\beta_5\left(\frac{1}{2}-\alpha_5\right)^2\right]$$

$$- \mathrm{Sum}\left[\eta_5\beta'_5\left(\frac{1}{2}-\alpha'_5\right)^2\right] - \mathrm{Sum}\left[\eta_5 q_5^P\left(\frac{1}{2}-\alpha_5\right)^2\right] + \mathrm{Sum}\left[\eta_5 q_5^P\left(\frac{1}{2}-\alpha'_5\right)^2\right] \tag{B.22}$$

$$\begin{aligned}0 =\ & (N+2)\dot\beta\left(\frac{1}{2}-\dot\beta\right) - A_1\left(\frac{1}{2}-\dot\beta\right) - 2A_2\dot\beta\left(\frac{1}{2}-\dot\beta\right) + 2A_2\left(\frac{1}{2}-\dot\beta\right)\\ &-\frac{1}{2}\left(\mathrm{Sum}\left[\eta_5\beta_5(1-2\alpha_5)(1-2\gamma_5)\right] + \mathrm{Sum}\left[\eta_5\beta'_5(1-2\alpha'_5)(1-2\gamma'_5)\right]\right)\\ &+\mathrm{Sum}\left[\eta_5 q_5^P\beta_5(1-2\alpha_5)\right] - \mathrm{Sum}\left[\eta_5 q_5^P\beta'_5(1-2\alpha'_5)\right]\\ &-\frac{1}{2}\mathrm{Sum}\left[\eta_5 q_5^P(1-2\alpha_5)(1-2\gamma_5)\right] - \frac{1}{2}\mathrm{Sum}\left[\eta_5 q_5^P(1-2\alpha'_5)(1-2\gamma'_5)\right]\\ &+\mathrm{Sum}\left[\eta_5 q_5^P q_5^P(1-2\alpha_5)\right] + \mathrm{Sum}\left[\eta_5 q_5^P q_5^P(1-2\alpha'_5)\right] + 2\mathrm{Sum}\left[\eta_4 q_4^P q_4^P\right]\end{aligned} \tag{B.23}$$

$$\begin{aligned}0 =\ & (N+2)\frac{\dot\beta^2}{2} - A_1\dot\beta + \frac{A_2}{2} - A_1\dot\beta^2 + 2A_2\dot\beta - A_3\\ &+\frac{1}{2}\left(\mathrm{Sum}\left[\eta_5\beta_5^2(1-2\gamma_5)\right] + \mathrm{Sum}\left[\eta_5\beta_5'^2(1-2\gamma'_5)\right]\right)\\ &-\left(\mathrm{Sum}\left[\eta_5 q_5^P\beta_5^2\right] - \mathrm{Sum}\left[\eta_5 q_5^P\beta_5'^2\right]\right) + \left(\mathrm{Sum}\left[\eta_5\beta_5 q_5^P(1-2\gamma_5)\right]\right.\\ &\left.-\mathrm{Sum}\left[\eta_5\beta'_5 q_5^P(1-2\gamma'_5)\right]\right) - 2\left(\mathrm{Sum}\left[\eta_5\beta_5 q_5^P q_5^P\right] + \mathrm{Sum}\left[\eta_5\beta'_5 q_5^P q_5^P\right]\right)\\ &+\frac{1}{2}\left(\mathrm{Sum}\left[\eta_5(1-2\gamma_5)q_5^P q_5^P\right] + \mathrm{Sum}\left[\eta_5(1-2\gamma'_5)q_5^P q_5^P\right]\right) + \mathrm{Sum}\left[\eta_4 q_4^P q_4^P\right].\end{aligned} \tag{B.24}$$

We include an explicit P in the modular weights simply to remind ourselves that we sum over the "P" index and not the "n" index since C = (P,n).

B.4. PV fields for the regulation of $U(1)_X$ divergences

Here we introduce fields that cancel the contributions to Eqs. (A.24)–(A.34) from the $\dot Y$, S_1, and S_2. Note that we will omit Eq. (A.29) since has been taken care of above. We introduce the following set:

- S_6: A set of pairs of PV fields with $Q_X^{(6)} = -Q_X'^{(6)}$ and $\bar q_n^{(6)} = 0$ and clone fields without $U(1)_X$ charge.

Then the conditions we must satisfy are

$$0 = 12C'_{GS}\dot\beta + 2\mathrm{Sum}\left[\eta_2 Q^X_{(2)}\beta_2\right] + \mathrm{Sum}\left[\eta_6 Q^X_{(6)}\beta_6\right] - \mathrm{Sum}\left[\eta_6 Q^X_{(6)}\beta'_6\right] \tag{B.25}$$

$$0 = 12C'_{GS}(1-\dot\beta) + 2\mathrm{Sum}\left[\eta_2 Q^X_{(2)}\alpha_2\right] + \mathrm{Sum}\left[\eta_6 Q^X_{(6)}\alpha_6\right] - \mathrm{Sum}\left[\eta_6 Q^X_{(6)}\alpha'_6\right] \tag{B.26}$$

$$0 = 12C'_{GS}\left(\frac{1}{2}-\dot{\beta}\right)^2 + \text{Sum}\left[\eta_6 Q^X_{(6)}\left(\alpha_6-\frac{1}{2}\right)\right] - \text{Sum}\left[\eta_6 Q^X_{(6)}\left(\alpha'_6-\frac{1}{2}\right)\right] \tag{B.27}$$

$$\begin{aligned}0 = {}& 12C'_{GS}\dot{\beta}\left(\frac{1}{2}-\dot{\beta}\right) + Q^{(L)}_{1X}\left(\frac{1}{2}-\dot{\beta}\right) + \text{Sum}\left[\eta_2 Q^X_{(2)} q^P_2\right]\\ &+ \text{Sum}\left[\eta_6 Q^X_{(6)}\beta_6\left(\alpha_6-\frac{1}{2}\right)\right] - \text{Sum}\left[\eta_6 Q^X_{(6)}\beta'_6\left(\alpha'_6-\frac{1}{2}\right)\right]\\ &- \text{Sum}\left[\eta_6 Q^X_{(6)} q^P_6\left(\alpha_6-\frac{1}{2}\right)\right] - \text{Sum}\left[\eta_6 Q^X_{(6)} q^P_6\left(\alpha'_6-\frac{1}{2}\right)\right]\end{aligned} \tag{B.28}$$

$$\begin{aligned}0 = {}& 12\dot{\beta}^2 C'_{GS} - 2\dot{\beta}Q^{(L)}_{1X} + Q^{(L)}_{2X} + \text{Sum}\left[\eta_1 Q^X_{(1)} q^P_1 q^P_1\right] + \text{Sum}\left[\eta_1 Q'^X_{(1)} q^P_1 q^P_1\right]\\ &+ \text{Sum}\left[\eta_6 Q^X_{(6)}\beta^2_6\right] - \text{Sum}\left[\eta_6 Q^X_{(6)}\beta'^2_6\right] + 2\text{Sum}\left[\eta_6 Q^X_{(6)}\beta_6 q^P_6\right]\\ &+ 2\text{Sum}\left[\eta_6 Q^X_{(6)}\beta'_6 q^P_6\right]\end{aligned} \tag{B.29}$$

$$\begin{aligned}0 = {}& \frac{1}{2}\text{Tr}\left[(Q^X_{(L)})^2\right] - R^{(L)}_X - \text{Sum}\left[\eta_1 (Q^X_{(1)})^2 q^P_1\right] + \text{Sum}\left[\eta_1 (Q'^X_{(1)})^2 q^P_1\right]\\ &+ \text{Sum}\left[\eta_2 (Q^X_{(2)})^2\right] + \text{Sum}\left[\eta_6 (Q^X_{(6)})^2\left(\frac{1}{2}-\gamma_6\right)\right] + \text{Sum}\left[\eta_6 (Q^X_{(6)})^2\left(\frac{1}{2}-\gamma'_6\right)\right]\end{aligned} \tag{B.30}$$

$$\begin{aligned}0 = {}& \text{Tr}\left[(Q^X_{(L)})^2\left(\frac{1}{2}-\dot{\beta}\right)\right] - \text{Sum}\left[\eta_2 (Q^X_{(2)})^2\right] + \text{Sum}\left[\eta_6 (Q^X_{(6)})^2\left(\alpha_6-\frac{1}{2}\right)\right]\\ &+ \text{Sum}\left[\eta_6 (Q'^X_{(6)})^2\left(\alpha_6-\frac{1}{2}\right)\right]\end{aligned} \tag{B.31}$$

$$\begin{aligned}0 = {}& -\frac{1}{2}Q^{(L)}_{1X}\left(\frac{1}{2}-\dot{\beta}\right) + R^{(L)}_X\left(\frac{1}{2}-\dot{\beta}\right) + \text{Sum}\left[\eta_2 Q^X_{(2)}(\alpha_2+\gamma_2)\right] + \text{Sum}\left[\eta_2 Q^X_{(2)} q^P_2\right]\\ &+ \text{Sum}\left[\eta_6 Q^X_{(6)}\left(\alpha_6-\frac{1}{2}\right)\left(\frac{1}{2}-\gamma_6\right)\right] - \text{Sum}\left[\eta_6 Q^X_{(6)}\left(\alpha'_6-\frac{1}{2}\right)\left(\frac{1}{2}-\gamma'_6\right)\right]\\ &- \text{Sum}\left[\eta_6 Q^X_{(6)} q^P_6\left(\alpha_6-\frac{1}{2}\right)\right] - \text{Sum}\left[\eta_6 Q^X_{(6)} q^P_6\left(\alpha'_6-\frac{1}{2}\right)\right]\end{aligned} \tag{B.32}$$

$$\begin{aligned}0 = {}& -\dot{\beta}\text{Tr}\left[(Q^X_{(L)})^2\right] + R^{(L)}_X + \text{Sum}\left[\eta_1 (Q^X_{(1)})^2 q^P_1\right] - \text{Sum}\left[\eta_1 (Q'^X_{(1)})^2 q^P_1\right]\\ &+ \text{Sum}\left[\eta_6 (Q^X_{(6)})^2\beta_6\right] + \text{Sum}\left[\eta_6 (Q^X_{(6)})^2\beta'_6\right]\end{aligned} \tag{B.33}$$

$$\begin{aligned}0 = {}& -6\dot{\beta}C'_{GS} + \beta Q^{(L)}_{1X} + \frac{1}{2}Q^{(L)}_{1X} - Q^{(L)}_{2X} - \text{Sum}\left[\eta_1 Q^X_{(1)} q^P_1 q^P_1\right] - \text{Sum}\left[\eta_1 Q'^X_{(1)} q^P_1 q^P_1\right]\\ &+ \text{Sum}\left[\eta_2 Q^X_{(2)}\beta_2\right] + \text{Sum}\left[\eta_2 Q^X_{(2)} q^P_2\right] + \text{Sum}\left[\eta_6 Q^X_{(6)}\beta_6\left(\frac{1}{2}-\gamma_6\right)\right]\\ &- \text{Sum}\left[\eta_6 Q^X_{(6)}\beta'_6\left(\frac{1}{2}-\gamma'_6\right)\right] - \text{Sum}\left[\eta_6 Q^X_{(6)}\beta_6 q^P_6\right] - \text{Sum}\left[\eta_6 Q^X_{(6)}\beta'_6 q^P_6\right]\end{aligned}$$

$$+ \text{Sum}\left[\eta_6 Q^X_{(6)} q^P_6 \left(\frac{1}{2} - \gamma_6\right)\right] + \text{Sum}\left[\eta_6 Q^X_{(6)} q^P_6 \left(\frac{1}{2} - \gamma'_6\right)\right]. \tag{B.34}$$

B.5. PV fields for the regulation of nonabelian divergences

Here we introduce fields to cancel Eqs. (A.35)-(A.37). We consider a separate PV set for each of the nonabelian factors of the FIQS gauge group as follows

- S_7: A set of pairs of PV fields in the fundamental of SU(3) (anti-fundamental for the primed fields) with no modular weights, uniform coefficients, and clone fields with no gauge charges. By uniform coefficients, we mean that α^C and β^C are independent of index within the set: $\alpha^C = \alpha$ and $\beta^C = \beta$.
- S_8: A set of pairs of PV fields in the fundamental of SU(2) with no modular weights, uniform constants, and clone fields with no gauge charges.
- S_9: A set of pairs of PV fields in the **16** (and $\overline{\mathbf{16}}$ for primed fields) of SO(10) and a set of pairs of PV fields in the **10** of SO(10), all with no modular weights, uniform coefficients, and clone fields with no gauge charges.
- S_{10}: A set of PV fields with $\gamma = \gamma' = 1/2$, zero modular weights, a nonzero trace $U(1)_X$ charge matrix, and charged under the nonabelian gauge groups in the same reps as the light fields and clone fields without nonabelian gauge charges.

Let us discuss this choice briefly. First we need to check the number of fields in a given representation. This is because we care about the quantity

$$C^M_{(\mathcal{G})} = C^m_{(\mathcal{G})} N_{(\mathcal{G})}, \tag{B.35}$$

which comes from the first term in the list above. The technique in [2] relies on having an even number of light fields in a given representation for all the gauge factors. Let us check if this is the case for the FIQS model. See Appendix C for a detailed breakdown of the FIQS spectrum. For the SU(3) of FIQS, the total number of triplets charged under this gauge group is

$$\begin{aligned} &N^{SU(3)}_{Q_L} + N^{SU(3)}_{u_L} + N^{SU(3)}_{u_2} + \sum_{i=1}^{2} N^{SU(3)}_{d_i} + \sum_{j=1}^{4} N^{SU(3)}_{D_j} + \sum_{j=1}^{2} N^{SU(3)}_{\bar{D}_j} \\ &\qquad = 6 + 3 + 12 + 15 = 36. \end{aligned} \tag{B.36}$$

For the SU(2) of FIQS, there are

$$\begin{aligned} &N^{SU(2)}_{Q_L} + \sum_{i=1}^{4} N^{SU(2)}_{\bar{G}_i} + \sum_{i=1}^{5} N^{SU(2)}_{G_i} + \sum_{i=1}^{4} N^{SU(2)}_{F_i} \\ &\qquad = 9 + 3 + 33 + 3 = 48 \end{aligned} \tag{B.37}$$

doublets. Note that we have used the fact that each state in the table of Appendix C has a degeneracy of 3, with the exception of the states Y_1, Y_2, and Y_3. The number of states charged under the SU(3) and SU(2) groups are indeed even, but this is not the case for SO(10), since there are only 3 **16**'s charged under this gauge factor. To resolve this, we begin by listing the Casimirs of the first few SO(10) representations:

$$\text{Fundamental } \mathbf{10}: C_{10} = 1 \tag{B.38}$$

$$\text{Spinor } \mathbf{16}: C_{16} = 2 \tag{B.39}$$

$$\text{Adjoint } \mathbf{45}: C_{45} = 8 \tag{B.40}$$

Note that these satisfy the sum rule (5.12) of [1] when considering the fields charged under SO(10):

$$C_{45} - 3C_{16} + 2C_{16} \sum_i \delta_n^i = 8 - 6 + 4 = 6 \tag{B.41}$$

The first divergence we cancel is $\mathrm{Tr}(\eta T_a T_b)$. The $\dot{Y}$ give the negative of the contribution of the light fields, so in the case of SO(10) this trace is simply $-3C_{16} = -6$. Since PV fields come in pairs, we cancel this with at least 2 fields and so we need

$$3C_{16} = 2 \sum_P \eta^P C^P \tag{B.42}$$

Thus, we have two options. We can have a PV pair in the **16** (and $\overline{\mathbf{16}}$) plus a PV pair in the **10** or we can have 3 pairs of PV fields in the **10**. The other divergence from gauge interactions we have to get rid of is the linear divergence proportional to the Casimir. We note that the $\dot{Y}$'s here give

$$(-1)\left(\frac{F}{2} - F + \sum_n q_n^a F^n\right) C_{(\mathcal{G}_a)} = \left(-\frac{1}{2}\right)(C_{GS} - C_{\mathcal{G}}) \tag{B.43}$$

since $\dot{\alpha} + \dot{\beta} = 1$. The overall sign is the sum of the signatures. Cancellation then requires

$$\frac{C_{GS} - C_{\mathcal{G}}}{2} = \sum_C \eta^C C_{\mathcal{G}_C}\left(\frac{1}{2} - \gamma^C\right) \tag{B.44}$$

$$= \sum_P \eta^P C_{\mathcal{G}_P}\left(1 - 2\bar{\gamma}^P\right) \tag{B.45}$$

provided that the PV fields have no modular weights. The first sum is over all PV fields whereas the second is over PV pairs. Both of our potential solutions can work since we have either one or two free parameters in the γ's. In the list of sets of PV fields above, we opted for the combination of PV fields in the **10** and **16** of SO(10). For the last nonabelian divergence, Eq. (A.36), we explicitly write out the contribution from the $\dot{Y}$ so that is takes the form

$$0 = \mathrm{Tr}(Q_X^L) C_{\mathcal{G}}^m + \mathrm{Tr}\left[\eta Q_X^{PV} T_a T_b\right], \tag{B.46}$$

where $C_{\mathcal{G}}^m$ is the Casimir of the representation of the matter fields. If we consider fields from the set S_{10}, this becomes

$$-\mathrm{Tr}(Q_X^L) = \mathrm{Tr}(Q_X^{PV}) = 2\mathrm{Sum}\left[\eta \bar{Q}_X^{PV}\right] \tag{B.47}$$

The fields in S_{10} contribute to Eq. (A.35) but not to Eq. (A.37) since we have restricted their γ parameters to be $\gamma = \frac{1}{2}$. Their contribution to Eq. (A.35) is not an issue since we can simply include more fields in the other sets described in this section to cancel their contribution. Finally, the clone fields ensure that none of the sets described in this section contribute to other conditions.

B.6. PV fields for the regulation of abelian divergences

Here we satisfy the conditions Eqs. (A.40)–(A.47). The $\dot{Y}$ contribute here, and to cancel them we will need to introduce fields with $\bar{q}_n \neq 0$, which is different from all other fields considered thus far. This would alter some of the expressions we have used above, but we will not consider these alterations since we will employ clone fields that cancel contributions to previously considered terms from the fields introduced here. Specifically, we consider

- S_{11}: A set of pairs of PV fields such that the unprimed fields have the same abelian gauge charges as the light fields (including $U(1)_X$), $\alpha_{11}^P = \dot{\alpha}$, $\beta_{11}^P = \dot{\beta}$, $q_n^{(11)} = -q_n^{(L)}$, $\alpha_{11}'^P = \frac{1}{2}$, $\beta_{11}'^P = Q_{11}'^X = q_n'^{(11)} = 0$, and positive signature and clone fields with no $U(1)_a$ charges.
- S_{12}: A set of pairs of PV fields with no β parameters or modular weights and with $\alpha_{12}^P = \alpha_{12}'^P = 1/2$, $Q_{(12)}'^X = 0$, $Q_{(12)}^X = 4Q_{(L)}^X$, and $U(1)_a$ charges $Q_{(12)}^a = Q_{(L)}^a/\sqrt{2}$ and negative signature and clone fields with no $U(1)_a$ charges.

These satisfy

$$0 = -S_{ab}^{(L)} + 2\text{Sum}\left[\eta_{11} Q_{(11)}^a Q_{(11)}^b\right] + 2\text{Sum}\left[\eta_{12} Q_{(12)}^a Q_{(12)}^b\right] \tag{B.48}$$

$$0 = -2\pi^2 \delta_X + \text{Sum}\left[\eta_{11} Q_{(11)}^X Q_{(11)}^a Q_{(11)}^b\right] + \text{Sum}\left[\eta_{12} Q_{(12)}^X Q_{(12)}^a Q_{(12)}^b\right] \tag{B.49}$$

$$0 = -\dot{\beta} S_a^{(L)} + R_a^{(L)} + \text{Sum}\left[\eta_{11} Q_{(11)}^a Q_{(11)}^X \beta_{11}\right] + \text{Sum}\left[\eta_{11} q_n^{(11)} Q_{(11)}^a Q_{(11)}^X\right] \tag{B.50}$$

$$0 = -\left(\frac{1}{2} - \dot{\beta}\right) S_a^{(L)} + \text{Sum}\left[\eta_{11} Q_{(11)}^X Q_{(11)}^a \left(\alpha_{11} - \frac{1}{2}\right)\right] \tag{B.51}$$

$$0 = -\frac{1}{2} S_a^{(L)} + R_a^{(L)} + \text{Sum}\left[\eta_{11} Q_{(11)}^X Q_{(11)}^a \left(\gamma_{11} - \frac{1}{2}\right)\right] + \text{Sum}\left[\eta_{11} Q_{(11)}^X Q_{(11)}^a q_n^{(11)}\right] \tag{B.52}$$

$$0 = -\frac{1}{2} S_{ab}^{(L)} + R_{ab}^{(L)} + \text{Sum}\left[\eta_{11} Q_{(11)}^a Q_{(11)}^b \left(\gamma_{11} - \frac{1}{2} + q_n^{(11)}\right)\right], \tag{B.53}$$

where again a subscript or superscript (L) implies a trace over the corresponding values of the light fields. Note that we have omitted some conditions that are automatically zero. There are also terms in the above that vanish for the choice of U(1) charges defined in this paper but do not vanish for other choices. If one substitutes the parameters of S_{11} and S_{12} as per the discussion above, one sees that all the remaining conditions above are satisfied.

Appendix C. The FIQS spectrum

The FIQS model was described in [15,16,4,17,18]. The modular weights in this model are simple: the fields in the ith untwisted sector have $q_n^i = \delta_n^i$, and the twisted sector fields have $q_n = \frac{2}{3}$, except for the Y^i which have $q_n^i = \delta_n^i + \frac{2}{3}$. Here we will focus in particular on the U(1) charges of the low-energy matter spectrum. The U(1) charge generators arising from the Cartan subalgebra of the $E_8 \times E_8$ and the corresponding charges were worked out in [17,18]. Table 2 of [16] lists the charges of the massless spectrum. However, the linear combinations of generators given in [4] have a mixed anomaly:

$$\mathrm{Tr}(Q_6 Q_7 Q_X) = 1296. \tag{C.1}$$

To avoid this, one should re-define Q_6 and Q_7. The fix is very simple:

$$Q_6' = Q_6 - Q_7, \tag{C.2}$$
$$Q_7' = Q_6 + Q_7. \tag{C.3}$$

Below we produce a table of the new charge designations.

(n_1, n_3)	Field	Rep.	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6^N	Q_7^N	X
Untwisted	Q_L	(3, 2)	−6	−6	0	0	0	0	0	0
	u_L	($\bar{3}$,1)	6	0	0	−6	0	0	0	0
	$\bar{G}_1$	(1, 2)	0	6	0	6	0	0	0	0
	$16'$	1	0	0	0	0	0	0	0	9
(0, 0)	D_1	(3, 1)	0	4	0	0	0	4	4	4
	$\bar{G}_2$	(1, 2)	6	−2	0	0	0	4	4	4
	$\bar{A}_1$	1	−3	−2	−3	−3	−3	4	4	4
	$\bar{A}_2$	1	−3	−2	3	−3	3	4	4	4
	A_1	1	−3	−2	−3	3	3	4	4	4
	A_2	1	−3	−2	3	3	−3	4	4	4
(1, 0)	S_4	1	6	4	0	0	−2	2	10	4
	S_5	1	6	4	0	0	−2	−4	−8	4
	S_6	1	6	4	0	0	−2	2	−2	−8
	$\bar{A}_3$	1	−3	−2	−3	−3	1	2	10	4
	$\bar{A}_4$	1	−3	−2	−3	−3	1	−4	−8	4
	$\bar{A}_5$	1	−3	−2	−3	−3	1	2	−2	−8
	A_3	1	−3	−2	3	3	1	2	10	4
	A_4	1	−3	−2	3	3	1	−4	−8	4
	A_5	1	−3	−2	3	3	1	2	−2	−8
(−1, 0)	S_7	1	6	4	0	0	2	6	−2	4
	S_8	1	6	4	0	0	2	0	4	−8
	S_9	1	6	4	0	0	2	−6	−2	4
	$\bar{A}_6$	1	−3	−2	3	−3	−1	6	−2	4
	$\bar{A}_7$	1	−3	−2	3	−3	−1	0	4	−8
	$\bar{A}_8$	1	−3	−2	3	−3	−1	−6	−2	4
	A_6	1	−3	−2	−3	3	−1	6	−2	4
	A_7	1	−3	−2	−3	3	−1	0	4	−8
	A_8	1	−3	−2	−3	3	−1	−6	−2	4
(0, 1)	d_1	($\bar{3}$, 1)	0	0	0	2	2	0	−8	4
	F_1	(1, 2)	3	0	−3	−1	−1	0	−8	4
	$\bar{A}_9$	1	3	6	3	−1	−1	0	−8	4
	A_9	1	3	−6	3	−1	−1	0	−8	4
	$\bar{l}_1$	1	−6	0	0	−4	2	0	−8	4
	S_{10}	1	−6	0	0	2	−4	0	−8	4
(1, 1)	D_2	(3, 1)	6	0	0	2	0	−2	−2	4
	u_2	($\bar{3}$,1)	0	0	0	−4	0	−2	−2	4
	F_2	(1,2)	3	0	3	−1	−3	−2	−2	4
	F_3	(1, 2)	3	0	−3	−1	3	−2	−2	4
	S_1	1	−6	0	0	2	0	4	4	−8
	Y_1	1	−6	0	0	2	0	−2	−2	4
	$\bar{A}_{10}$	1	3	6	−3	−1	−3	−2	−2	4
	$\bar{A}_{11}$	1	3	6	3	−1	3	−2	−2	4
	A_{10}	1	3	−6	3	−1	3	−2	−2	4
	A_{11}	1	3	−6	−3	−1	−3	−2	−2	4

(n_1, n_3)	Field	Rep.	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6^N	Q_7^N	X
$(-1, 1)$	d_2	$(\bar{3}, 1)$	0	0	0	2	−2	−4	4	4
	F_4	$(1, 2)$	3	0	3	−1	1	−4	4	4
	$\bar{A}_{12}$	1	3	6	−3	−1	1	−4	4	4
	A_{12}	1	3	−6	−3	−1	1	−4	4	4
	$\bar{l}_2$	1	−6	0	0	−4	−2	−4	4	4
	S_{11}	1	−6	0	0	2	4	−4	4	4
$(0, -1)$	$\bar{D}_1$	$(\bar{3}, 1)$	−3	2	−3	1	1	2	−2	4
	D_3	$(3, 1)$	3	2	3	1	1	2	−2	4
	$\bar{G}_3$	$(1, 2)$	0	2	0	4	−2	2	−2	4
	G_1	$(1, 2)$	0	2	0	−2	4	2	−2	4
	S_2	1	0	−4	0	−2	−2	−4	4	−8
	Y_2	1	0	−4	0	−2	−2	2	−2	4
	l_1	1	0	−4	0	4	4	2	−2	4
	$\bar{l}_3$	1	0	8	0	−2	−2	2	−2	4
	$\bar{A}_{13}$	1	−9	2	3	1	1	2	−2	4
	A_{13}	1	9	2	−3	1	1	2	−2	4
$(1, -1)$	$\bar{D}_2$	$(\bar{3}, 1)$	−3	2	3	1	−1	0	4	4
	D_4	$(3, 1)$	3	2	−3	1	−1	0	4	4
	$\bar{G}_4$	$(1, 2)$	0	2	0	4	2	0	4	4
	G_2	$(1, 2)$	0	2	0	−2	−4	0	4	4
	S_3	1	0	−4	0	−2	2	0	−8	−8
	Y_3	1	0	−4	0	−2	2	0	4	4
	l_2	1	0	−4	0	4	−4	0	4	4
	$\bar{l}_4$	1	0	8	0	−2	2	0	4	4
	$\bar{A}_{14}$	1	−9	2	−3	1	−1	0	4	4
	A_{14}	1	9	2	3	1	−1	0	4	4
$(-1, -1)$	G_3	$(1, 2)$	0	2	0	−2	0	4	−8	4
	G_4	$(1, 2)$	0	2	0	−2	0	−2	10	4
	G_5	$(1, 2)$	0	2	0	−2	0	−2	−2	−8
	l_3	1	0	−4	0	4	0	−2	10	4
	l_4	1	0	−4	0	4	0	4	−8	4
	l_5	1	0	−4	0	4	0	−2	−2	−8

Appendix D. Corrections to [2]

Equation (3.11) should read:

$$(\bar{\mathcal{D}}^2 - 8R)\Omega_W = W^{\alpha\beta\gamma} W_{\alpha\beta\gamma}, \quad (\bar{\mathcal{D}}^2 - 8R)\Omega_X = X^\alpha X_\alpha, \quad (\bar{\mathcal{D}}^2 - 8R)\Omega_{\rm YM} = W_a^\alpha W_\alpha^a$$

In Eqs. (3.6) and (5.2) the factor 1/24 in front of $\Omega_{\rm GB}$ should be removed.
Equation (5.3) and the remainder of section 5 should read

$$8\pi^2 b_{\rm spin} = 8\pi^2 b + 1 = 31, \qquad \widetilde{\Omega}_f = {\rm Tr}\eta \Delta \ln \mathcal{M}^2 \Omega_f. \tag{5.3}$$

The results for the Gauss-Bonnet and Yang-Mills terms are well-established [10] and result from the universality conditions (2.3) and (B.7), as illustrated in the appendices. The only other term in (5.2) that contains a chiral anomaly is Ω_f, which, using the set (4.11) of PV fields, is a priori a product of the chiral superfields X_α, g_α and g^n_α. We show in Appendix A that we may choose the PV parameters such that

$$(\bar{\mathcal{D}}^2 - 8R)\widetilde{\Omega}_f = 30 \sum_n g_n^\alpha g_\alpha^n, \tag{5.4}$$

in agreement with the string calculation of [4].

The anomaly is canceled provided the Lagrangian for the dilaton $S, \bar{S}$ is specified by the coupling (2.5) and the Kähler potential (2.9), or, equivalently, the linear superfield L satisfies (1.3) and the GS term (1.3) is added to the Lagrangian.

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