# On the Multicast Capacity of Full-Duplex 1-2-1 Networks

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***Abstract*—This paper studies the multicast capacity of full-duplex 1-2-1 networks. In this model, two nodes can communicate only if they point "beams" at each other; otherwise, no signal can be exchanged. The main result of this paper is that the approximate multicast capacity can be computed by solving a linear program in the activation times of links connecting pairs of nodes. This linear program has two appealing features: (i) it can be solved in polynomial-time in the number of nodes; (ii) it allows to efficiently find a network schedule optimal for the approximate capacity. Additionally, the relation between the approximate multicast capacity and the minimum approximate unicast capacity is studied. It is shown that the ratio between these two values is not universally equal to one, but it depends on the number of destinations in the network, as well as graph-theoretic properties of the network.**

## I. INTRODUCTION

With the commercial deployment of Fifth Generation (5G) cellular technology, expected as early as 2020, highly directive millimeter wave (mmWave) communication is expected to play a central role in a number of 5G applications/services. These include ultra-high resolution video streaming, vehicle-to-vehicle communication and massive machine type communication. Multicasting is foreseen to be of critical importance to enable a number of these applications, for example in multimedia broadcast and vehicular communication (for fleet management and assisted driving) [1]. In this paper, we expand on our recent investigation in [2] of unicast communication in networks with mmWave nodes and study multicast traffic and its relation to rates achieved in the unicast case.

In [2], we recently introduced Gaussian 1-2-1 networks, a model that abstracts the directivity of mmWave communication. We used this model to study the Shannon capacity for unicast traffic in arbitrary network topologies that consist of Full-Duplex (FD) mmWave nodes, that is, nodes that can receive and transmit at the same time using two highly directive beams. In particular, in [2] we proved that the unicast capacity of a Gaussian 1-2-1 network with FD mmWave nodes can be approximated to within a universal constant gap[1]. We use *approximate capacity* to refer to such an approximation in the remainder of the paper. The key differentiating point between FD 1-2-1 networks and regular FD wireless networks is that in the former, two nodes can communicate only if they point beams at each other, otherwise no signal can be exchanged, i.e., we do not have broadcast and multiple access channel opportunities. This directivity of communication inherently introduces a *scheduling* factor to the problem: in order to operate the network close to its Shannon capacity, it is necessary to understand which links should be activated (i.e., how the beams should be steered) and for how long. In other words, this leads to an optimization problem whose solution provides the network approximate capacity. Because of this feature, the 1-2-1 model shares similiarities with link scheduling [3] and hyperarc scheduling [4] in graphical models, particularly when nodes are half-duplex, i.e., nodes can either receive or transmit with a highly directive beam, but not both simultaneously.

The focus of this paper is on Gaussian FD 1-2-1 networks with *multicast* traffic where a source wishes to communicate a common message to a number of destinations in the network. In particular, we are interested in characterizing the approximate multicast capacity for Gaussian FD 1-2-1 networks. For classical wired networks (i.e., networks with orthogonal links), the multicast capacity is given by the main theorem of network coding [5]. For regular (i.e., without 1-2-1 constraints) FD wireless networks, the approximate multicast capacity can be achieved by the quantize-map-and-forward scheme [6].

Our first main result in this paper is to show that the approximate multicast capacity of Gaussian FD 1-2-1 networks can be computed using a Linear Program (LP) that is a function of the amount of time a link is active in the network. This formulation, which parallels the one in [7] for unicast traffic, enables an efficient computation (i.e., that can be performed in polynomial-time in the number of nodes in the network) of two quantities: (i) the approximate multicast capacity, and (ii) an operating schedule optimal for the approximate capacity.

Our second result in this paper is to characterize the worst-case ratio between the approximate multicast capacity and the minimum unicast capacity of the network. Note that in wired networks, by the definition of multicast capacity, the ratio between multicast and unicast capacities is always unity. Similarly, in regular FD wireless networks, the ratio between the approximate multicast and unicast capacities is always equal to one, independently of the network topology or number of destinations that share the multicast message. We here show that such a result does not hold with 1-2-1 constraints, i.e., the ratio depends on the number of destinations, as well as on graph theoretic properties of 1-2-1 networks. This is due to the fact that in 1-2-1 networks, the scheduling naturally

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[1]Constant gap refers to a quantity that is independent of the channel coefficients and operating SNR, and solely depends on the number of nodes.

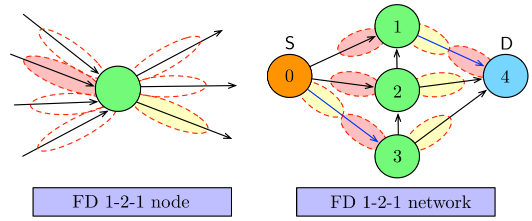


Fig. 1: Model of a 1-2-1 node (left) and an example of a 1-2-1 network with $N = 3$ relays (right).

introduces a contention for which the network seeks to best serve the collection of all destinations rather than maximizing the communication towards a single destination.

**Paper Organization.** Section II describes the Gaussian FD 1-2-1 network and presents known unicast capacity results. Section III derives the approximate multicast capacity of Gaussian FD 1-2-1 networks as the solution of an LP. Finally, Section IV proves the worst case ratio between multicast and unicast approximate capacities.

## II. System Model and Unicast Capacity

We use $[n_1 : n_2]$ to denote the set of integers from $n_1$ to $n_2 \geq n_1$; $\emptyset$ is the empty set; $\mathbb{1}_P$ is the indicator function; $0^N$ is the all-zero vector of length $N$; $|A|$ is the absolute value of $A$ when $A$ is a scalar, and the cardinality when $A$ is a set.

We consider a 1-2-1 Gaussian network denoted by $\mathcal{N}$ with $N+1$ nodes and multicast traffic. Node 0 is the source node and it wishes to communicate a common message to a set of destinations indexed by the set $\mathcal{D} \subseteq [1 : N]$. The remaining nodes $[1 : N]\backslash\mathcal{D}$ are relays that assist the communication between the source and the set of destinations. The 1-2-1 network model, introduced in [2], describes conditions on the communication between nodes in the network. In particular, at any time instant, a node in the network can only direct (beamform) its transmission towards at most one other node. Similarly, a node can only receive transmission from at most one other node by directing its receiving beam towards the node in question. A network example is depicted in Fig. 1. At node $i \in [0 : N]$, the aforementioned functionality is characterized by two states $S_{i,t}$ and $S_{i,r}$ that index the node (if any) to which node $i$ is transmitting to and receiving from, respectively. Thus, for all $i \in [0 : N]$, we have that

$$S_{i,t} \subseteq [1 : N]\backslash\{i\},\ |S_{i,t}| \leq 1, \tag{1a}$$
$$S_{i,r} \subseteq [0 : N]\backslash\{i\},\ |S_{i,r}| \leq 1. \tag{1b}$$

For the source (node 0), we additionally have that $S_{0,r} = \emptyset$ since the source does not intend to receive communication. Note that, throughout this paper, we assume that the nodes are operating in FD, and hence it is possible $\forall i \in [1 : N]$ that both $S_{i,t}$ and $S_{i,r}$ are simultaneously non-empty. In [2], it was shown that the memoryless channel model for the Gaussian 1-2-1 network model can be written as

$$Y_j = \begin{cases} h_{jS_{j,r}} \overline{X}_{S_{j,r}}(j) + Z_j & \text{if } |S_{j,r}| = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where: (i) $S_{j,t}$ and $S_{j,r}$ are defined in (1); (ii) $Y_j \in \mathbb{C}$ is the channel output at node $j$; (iii) $h_{ji} \in \mathbb{C}$ represents the channel coefficient from node $i$ to node $j$; the channel coefficients are assumed to remain constant for the entire transmission duration and hence they are known by all nodes in the network; (iv) $Z_j$ is the additive white Gaussian noise at the $j$-th node; noises across nodes in the network are assumed to be independent and identically Gaussian distributed as $\mathcal{CN}(0, 1)$; (v) $\overline{X}_i \in \mathbb{C}^{N+1}$ has elements $\overline{X}_i(k)$ defined as $\overline{X}_i(k) = X_i \mathbb{1}_{\{k \in S_{i,t}\}}$, where $X_i \in \mathbb{C}$ is the channel input at node $i$; the channel inputs are subject to an individual power constraint, i.e., $\mathbb{E}[|X_i|^2] \leq P, \forall i \in [0 : N]$; note that, if node $i$ is not transmitting, i.e., $S_{i,t} = \emptyset$, then $\overline{X}_i = 0^{N+1}$.

The focus in [2] was on the unicast capacity for the Gaussian FD 1-2-1 network, i.e., where the destination set is a singleton set, $\mathcal{D} = \{N\}$. Although the Shannon capacity $\mathsf{C}$ of the Gaussian 1-2-1 unicast network is not known, we have shown in [2, Theorem 1] that it can be approximated by $\mathsf{C}_{\text{cs,iid}}$ as

$$\mathsf{C}_{\text{cs,iid}} \leq \mathsf{C} \leq \mathsf{C}_{\text{cs,iid}} + O(N \log N), \tag{3a}$$

$$\mathsf{C}_{\text{cs,iid}} = \max_{\substack{\lambda_s : \lambda_s \geq 0 \\ \sum_s \lambda_s = 1}} \min_{\substack{\Omega : \Omega \subseteq [0:N-1], \\ 0 \in \Omega}} \sum_{\substack{(i,j): i \in \Omega, \\ j \in \Omega^c}} \left( \sum_{\substack{s: \\ j \in s_{i,t}, \\ i \in s_{j,r}}} \lambda_s \right) \ell_{j,i}, \tag{3b}$$

$$\ell_{j,i} = \log\left(1 + P\,|h_{ji}|^2\right), \tag{3c}$$

where: (i) $\Omega$ enumerates all possible cuts in the graph representing the network, such that the source belongs to $\Omega$; (ii) $\Omega^c = [0 : N]\backslash\Omega$; (iii) $s$ enumerates all possible network states of the 1-2-1 network in FD, where each network state corresponds to specific values for the variables in (1) for each network node; (iv) $\lambda_s$, i.e., the optimization variable, is the fraction of time for which state $s$ is active; we refer to a schedule as the collection of $\lambda_s$ for all feasible states, such that they sum up to at most 1; (v) $s_{i,t}$ and $s_{i,r}$ denote the transmitting and receiving states for node $i$ in the network state $s$ as defined in (1). In other words, for Gaussian FD 1-2-1 networks, $\mathsf{C}_{\text{cs,iid}}$ in (3) is the approximate unicast capacity, i.e., a constant gap approximation of the unicast capacity $\mathsf{C}$.

## III. Constant Gap Approximation of the Multicast Capacity

The cut-set bound for the FD 1-2-1 multicast network is

$$\mathsf{C}^{\text{multi}}_{\text{cs}} = \max_{\mathbb{P}_{\{\overline{X}_i, S_i\}}(\cdot)} \min_{d \in \mathcal{D}} \min_{\substack{\Omega \subseteq [0:N] : 0 \in \Omega, \\ d \in \Omega^c}} I(\widehat{X}_\Omega; Y_{\Omega^c} | \widehat{X}_{\Omega^c}), \tag{4}$$

where: (i) $\widehat{X}_\Omega = \{(X_i, S_i) : i \in \Omega\}$ and $X_i$ is the channel input at the $i$-th node and $S_i = (S_{i,t}, S_{i,r})$ combines the switching variables of node $i$; (ii) $Y_\Omega = \{Y_i : i \in \Omega\}$ and $Y_i$ is the signal received at node $i$.

Using arguments similar to the ones used in [7] to bound the unicast capacity, we can upper and lower bound the multicast capacity $\mathsf{C}^{\text{multi}}$ of the network as

$$\mathsf{C}^{\text{multi}}_{\text{cs,iid}} \leq \mathsf{C}^{\text{multi}} \leq \mathsf{C}^{\text{multi}}_{\text{cs,iid}} + \mathsf{GAP}, \tag{5}$$

where $\mathsf{GAP} = \log(N+1) + N\log(N^2+N)$, and $\mathsf{C}^{\text{multi}}_{\text{cs,iid}}$ is defined in (6), at the top of the next page. Note that the expression of $\mathsf{C}_{\text{cs,iid}}(d,\lambda)$ in (6) is exactly the minimum cut that we get for the single unicast case [7] when we consider node $d$ as the destination and the fixed schedule $\lambda$. Similar to the single unicast case, we can evaluate the mutual information term in $\mathsf{C}^{\text{multi}}_{\text{cs,iid}}$ in (6) for the Gaussian noise case and express it as a function of the point-to-point link capacities as

$$\mathsf{C}^{\text{multi}}_{\text{cs,iid}} = \max_{\substack{\lambda_s:\lambda_s\geq 0\\ \sum_s \lambda_s \leq 1}} \min_{d\in\mathcal{D}} \overbrace{\min_{\substack{\Omega\subseteq[0:N]:0\in\Omega,\\ d\in\Omega^c}} \sum_{\substack{(i,j):i\in\Omega,\\ j\in\Omega^c}} \ell^{(s)}_{j,i}}^{\mathsf{C}_{\text{cs,iid}}(d,\lambda)}, \quad (7)$$

$$\text{where} \quad \ell^{(s)}_{j,i} = \left( \sum_{\substack{s:\\ j\in s_{i,t},\\ i\in s_{j,r}}} \lambda_s \right) \ell_{j,i}.$$

The expression in (7) looks very similar to the approximate capacity expression for the single unicast case in (3), with the difference that in multicast we also have a minimization over $d \in \mathcal{D}$, which ensures that each destination is able to reliably decode the message. Note that the inner minimization in (7) is the min-cut over a graph with edge capacities $\ell^{(s)}_{j,i}$. Thus, for fixed $d \in \mathcal{D}$ and fixed schedule $\lambda$, we can replace $\mathsf{C}_{\text{cs,iid}}(d,\lambda)$ in (7) with its equivalent max-flow formulation, and we obtain

$$\begin{aligned} \mathsf{P0}:\ \mathsf{C}^{\text{multi}}_{\text{cs,iid}} &= \max_{\substack{\lambda_s:\lambda_s\geq 0\\ \sum_s \lambda_s\leq 1}} \min_{d\in\mathcal{D}} \max_{\{F^{(d)}_{d,j}\}} \sum_{j\in[0:N]\backslash d} F^{(d)}_{d,j} \\ & 0 \leq F^{(d)}_{j,i} \leq \ell^{(s)}_{j,i} \quad \forall (i,j)\in[0:N]\times[1:N], d\in\mathcal{D} \quad (8) \\ & \sum_{j\in[1:N]\backslash\{i\}} F^{(d)}_{j,i} = \sum_{k\in[0:N]\backslash\{i\}} F^{(d)}_{i,k} \quad \forall i\in[1:N]\backslash\{d\}, d\in\mathcal{D}, \end{aligned}$$

where $F^{(d)}_{j,i}$ is the information flow from node $i$ to node $j$ when the destination is node $d$.

We observe that in the optimization problem P0 in (8), if we can exchange the inner min-max with max-min, then we can write $\mathsf{C}^{\text{multi}}_{\text{cs,iid}}$ in (8) as the following LP

$$\begin{aligned} \mathsf{P1}:\ \mathsf{C}^{\text{multi}}_{\text{cs,iid}} &= \max_{\lambda_s} \max_{\mathbf{F}} \min_{d\in\mathcal{D}} \left\{ \sum_{j\in[0:N]\backslash d} F^{(d)}_{d,j} \right\} \\ & 0 \leq F^{(d)}_{j,i} \leq \ell^{(s)}_{j,i} \quad \forall (i,j)\in[0:N]\times[1:N], d\in\mathcal{D} \\ & \sum_{j\in[1:N]\backslash\{i\}} F^{(d)}_{j,i} = \sum_{k\in[0:N]\backslash\{i\}} F^{(d)}_{i,k} \quad \forall i\in[1{:}N]\backslash\{d\}, d\in\mathcal{D} \\ & \sum_s \lambda_s \leq 1, \quad (9) \\ & \lambda_s \geq 0 \quad \forall s, \end{aligned}$$

where $\mathbf{F} = \bigcup_{d\in\mathcal{D}} \{F^{(d)}_{d,j}\}$. We next prove that, without loss of generality, we can indeed exchange the min-max with the max-min. Towards this end, we start by noting that, by the max-min inequality, we have

$$\min_{d\in\mathcal{D}} \max_{\mathbf{F}} \sum_{j\in[0:N]\backslash d} F^{(d)}_{d,j} \geq \max_{\mathbf{F}} \min_{d\in\mathcal{D}} \sum_{j\in[0:N]\backslash d} F^{(d)}_{d,j}. \quad (10)$$

Thus, to prove equality between the two sides of (10) (and consequentially the equivalence between P0 and P1), we need to prove that

$$\min_{d\in\mathcal{D}} \max_{\mathbf{F}} \sum_{j\in[0:N]\backslash d} F^{(d)}_{d,j} \leq \max_{\mathbf{F}} \min_{d\in\mathcal{D}} \sum_{j\in[0:N]\backslash d} F^{(d)}_{d,j}. \quad (11)$$

We prove the inequality in (11) by leveraging the result in the following lemma.

**Lemma 1.** *Let $\mathcal{I}$ be a discrete set and $\mathcal{X} = \prod_{i\in\mathcal{I}} \mathcal{X}_i$. Consider the set of functions $\{f_i(\cdot)\}_{i\in\mathcal{I}}$, $f_i : \mathcal{X} \to \mathbb{R}$ such that $f_i(\cdot)$ depends only on $\mathcal{X}_i$. Then, we have*

$$\min_{i\in\mathcal{I}} \max_{x\in\mathcal{X}} f_i(x) \leq \max_{x\in\mathcal{X}} \min_{i\in\mathcal{I}} f_i(x). \quad (12)$$

*Proof.* For any $x \in \mathcal{X}$, we can write it as $x=[x_1, x_2, \cdots, x_{|\mathcal{I}|}]$, such that $x_i \in \mathcal{X}_i$. Since the value of $f_i(\cdot)$ depends only on $x_i$, then we can define $x^\star_i$ as the value of $x_i$ that maximizes $f_i$. Thus, for $x^\star = [x^\star_1, x^\star_2, \cdots, x^\star_{|\mathcal{I}|}]\in\mathcal{X}$, we have that

$$\max_{x\in\mathcal{X}} f_i(x) = f_i(x^\star).$$

Furthermore, we have that

$$\max_{x\in\mathcal{X}} \min_{i\in\mathcal{I}} f_i(x) \overset{(a)}{\geq} \min_{i\in\mathcal{I}} f_i(x^\star) = \min_{i\in\mathcal{I}} \max_{x\in\mathcal{X}} f_i(x), \quad (13)$$

where $(a)$ follows from the fact that we are considering a particular $x \in \mathcal{X}$. This proves Lemma 1. □

**Remark 1.** *In Lemma 1, consider $\mathcal{I} = \mathcal{D}$, $x = \mathbf{F}$, $x_i = \{F^{(i)}_{i,j}\}$ and $f_i(x) = \sum_{j\in[0:N]\backslash i} F^{(i)}_{i,j}$. Then, we have the intended relation in* (11).

We now would like to rewrite the LP P1 such that the schedule variables $\lambda_s$ are combined into variables $\lambda_{ji}$ representing the fraction of time the link of capacity $\ell_{j,i}$ is active. With this, we can obtain a significant reduction in the number of optimization variables for scheduling, i.e., from exponential in $N$ in P1 to polynomial in $N$. It is not difficult to show that, by using the change of variables

$$\lambda_{ji} = \sum_{\substack{s:\\ j\in s_{i,t},\\ i\in s_{j,r}}} \lambda_s,$$

the LP P1 is equivalent to the LP P2 below

$$\begin{aligned} \mathsf{P2}:\ \mathsf{C}^{\text{multi}}_{\text{cs,iid}} &= \max_{\mathbf{F},\ \lambda} \min_{d\in\mathcal{D}} \left\{ \sum_{j\in[0:N]\backslash d} F^{(d)}_{d,j} \right\} \\ (2a)\ & 0 \leq F^{(d)}_{j,i} \leq \lambda_{ji}\ell_{j,i} \quad \forall (i,j)\in[0{:}N]\times[1{:}N],\ d\in\mathcal{D} \\ (2b)\ & \sum_{j\in[1:N]\backslash\{i\}} F^{(d)}_{j,i} = \sum_{k\in[0:N]\backslash\{i\}} F^{(d)}_{i,k} \quad \forall i\in[1{:}N],\ d\in\mathcal{D} \\ (2c)\ & \sum_{j\in[1:N]\backslash\{i\}} \lambda_{ji} \leq 1, \quad \forall i\in[0:N] \quad (14) \\ (2d)\ & \sum_{k\in[0:N]\backslash\{i\}} \lambda_{ik} \leq 1, \quad \forall i\in[1:N] \\ (2e)\ & \lambda_{ji} \geq 0, \quad \forall (i,j)\in[0{:}N]\times[1{:}N]. \end{aligned}$$

$$\mathsf{C}_{\text{cs,iid}}^{\text{multi}} = \max_{\mathbb{P}_{\{S_i\}}(\cdot)} \min_{d\in\mathcal{D}} \overbrace{\min_{\substack{\Omega\subseteq[0:N]:0\in\Omega,\\ d\in\Omega^c}} \max_{\mathbb{P}_{\{\overline{X}_i\}|\{S_i\}}(\cdot)} \sum_s \lambda_s\, I(\overline{X}_\Omega; Y_{\Omega^c}|S_{[0:N+1]}{=}s, \overline{X}_{\Omega^c})}^{\mathsf{C}_{\text{cs,iid}}(d,\lambda)}, \tag{6}$$

In P2 the number of variables is $O(|\mathcal{D}|N^2)$. This, together with the fact that P2 has a polynomial in $N$ number of constraints, ensure that the approximate multicast capacity $\mathsf{C}_{\text{cs,iid}}^{\text{multi}}$ can be computed efficiently, i.e., in polynomial-time in $N$.

**Remark 2.** The formulation in P2 for the approximate multicast capacity of Gaussian FD 1-2-1 networks differs from the formulation in unicast only in the number of flow variables needed for the multicast case. However, with similar feasibility constraints on the link activation times $\{\lambda_{ji}\}$ both in multicast and unicast, we can use the algorithm in [7, Appendix F] to compute an optimal schedule $\{\lambda_s\}$ for the approximate capacity in polynomial-time by using the solution of P2.

In the following section, we focus on characterizing a lower bound on the ratio between the multicast approximate capacity in P2 and the minimum unicast approximate capacity in FD.

## IV. Multicast vs Minimum unicast approximate capacity

In this section, we focus on characterizing the ratio between the approximate multicast and unicast capacities of Gaussian FD 1-2-1 networks. For brevity, we denote $\mathsf{C}_{\text{cs,iid}}^{\text{multi}}$ with $\mathsf{C}_{\text{multicast}}$. In particular, the approximate multicast capacity is given by the LP P2 in (14). For destination $d \in \mathcal{D}$, the approximate unicast capacity $\mathsf{C}_d$ is given by [7]

$$\begin{aligned}
\text{P3}_d\ :\ \mathsf{C}_d = \max_{\mathbf{F},\ \lambda} & \left\{ \sum_{j\in[0:N]\setminus d} F_{d,j}^{(d)} \right\} \\
& 0 \le F_{j,i}^{(d)} \le \lambda_{ji}^{(d)} \ell_{j,i} \qquad \forall (i,j) \in [0:N]\times[1:N] \\
& \sum_{j\in[1:N]\setminus\{i\}} F_{j,i}^{(d)} = \sum_{k\in[0:N]\setminus\{i\}} F_{i,k}^{(d)} \qquad \forall i \in [1:N] \\
& \sum_{j\in[1:N]\setminus\{i\}} \lambda_{ji}^{(d)} \le 1, \qquad \forall i \in [0:N] \\
& \sum_{k\in[0:N]\setminus\{i\}} \lambda_{ik}^{(d)} \le 1, \qquad \forall i \in [1:N] \\
& \lambda_{ji}^{(d)} \ge 0 \qquad \forall (i,j) \in [0:N]\times[1:N].
\end{aligned} \tag{15}$$

Note that the superscript $(d)$ is fixed throughout the LP and is included to specialize the variables used in computing $\mathsf{C}_d$.

We now define the following parameters for our network:

- $\mathsf{C}_{\text{mu}}$: minimum unicast approximate capacity from the source to the destinations, i.e., $\mathsf{C}_{\text{mu}} = \min_{d\in\mathcal{D}} \mathsf{C}_d$;
- $\Delta^+$: maximum number of incoming links (with non-zero capacity) to a node in the 1-2-1 network;
- $\Delta^-$: maximum number of outgoing links (with non-zero capacity) from a node in the 1-2-1 network.

With these definitions, we can now state the following theorem that relates $\mathsf{C}_{\text{mu}}$ and $\mathsf{C}_{\text{multicast}}$.

**Theorem 2.** *For a Gaussian FD 1-2-1 network with destination set $\mathcal{D}$, we have that*

$$\mathsf{C}_{\text{multicast}} \ge \frac{1}{\min\{|\mathcal{D}|,\ \max\{\Delta^+,\Delta^-\}\}} \mathsf{C}_{\text{mu}}. \tag{16}$$

*Furthermore, there exists a class of networks for which this ratio is tight.*

*Proof.* Without loss of generality, the destination nodes are indexed by $\{1, 2, \cdots, D\}$, with $D = |\mathcal{D}|$. The key intuition behind the worst-case ratio in Theorem 2 is that, when the destinations are spread out in different places in the network (e.g. in Fig. 2 and Fig. 3), the network scheduling needs to balance the amount of traffic to be delivered to each destination. Thus, because of this, the approximate multicast capacity decreases. In what follows, we formalize this notion by considering two different cases, namely $|\mathcal{D}| \le \max\{\Delta^+,\Delta^-\}$ and $|\mathcal{D}| > \max\{\Delta^+,\Delta^-\}$, respectively. In each of the two cases, we show that there exists a feasible schedule (in terms of link activation times) in (14) that achieves the bound in (16). Moreover, we also present network examples for which the ratio guarantee in (16) is indeed tight.

**Case 1:** $|\mathcal{D}| \le \max\{\Delta^+,\Delta^-\}$.

In this particular case, $\forall d \in \mathcal{D}$, let $\{\lambda_{ji}^{(d)\star}\}$ be an optimal schedule in P3$_d$ for the approximate unicast capacity from the source to destination $d$. We can define a feasible schedule for the LP P2 in (14) as

$$\lambda'_{ji} = \frac{1}{|\mathcal{D}|} \sum_{d\in\mathcal{D}} \lambda_{ji}^{(d)\star} \qquad \forall (i,j) \in [0:N]\times[1:N].$$

In other words, for multicast traffic, we timeshare the network with the optimal schedule for each of the destinations $d \in \mathcal{D}$. Let $\{F_{j,i}^{\prime(d)}\}$ be the optimal flow variables that maximize the objective function for the fixed schedule $\{\lambda'_{ji}\}$ in P2. By the timesharing argument, it is not difficult to see that for all destinations, we have that the evaluation of the objective function in P2 using this timesharing schedule gives that

$$\sum_{j\in[0:N]\setminus d} F_{d,j}^{\prime(d)} \ge \frac{1}{|\mathcal{D}|} \sum_{j\in[0:N]\setminus d} F_{d,j}^{(d)\star} = \frac{1}{|\mathcal{D}|}\mathsf{C}_d \quad \forall d \in \mathcal{D},$$

where $\{F_{d,j}^{(d)\star}\}$ are optimal for P3$_d$. Since the computed $\{F_{j,i}^{\prime(d)}\}$ and $\{\lambda'_{ji}\}$ are feasible in the LP P2, then we have the desired ratio, i.e.,

$$\mathsf{C}_{\text{multicast}} \ge \min_{d\in\mathcal{D}} \sum_{j\in[0:N]\setminus d} F_{d,j}^{\prime(d)} \ge \min_{d\in\mathcal{D}} \frac{1}{|\mathcal{D}|}\mathsf{C}_d = \frac{1}{|\mathcal{D}|}\mathsf{C}_{\text{mu}}.$$

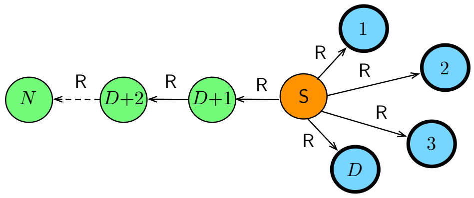


Fig. 2: Network with tight ratio for $|\mathcal{D}| \leq \max\{\Delta^+, \Delta^-\}$. The $D = |\mathcal{D}|$ destinations are shown with a bold border.

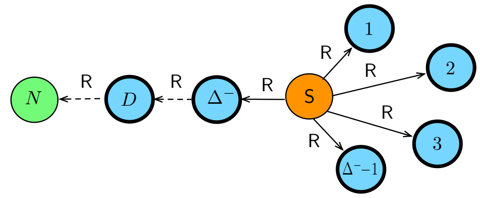


Fig. 3: Network with tight ratio for $|\mathcal{D}| > \max\{\Delta^+, \Delta^-\}$. The $D = |\mathcal{D}|$ destinations are shown with a bold border.

To show that the bound is tight, consider the network topology illustrated in Figure 2. It is not difficult to see that, for this particular network, $\mathsf{C}_{\text{mu}} = \mathsf{R}$. For the multicast approximate capacity $\mathsf{C}_{\text{multicast}}$, the source has to timeshare between the $|\mathcal{D}|$ destinations to achieve a rate of $\mathsf{R}/|\mathcal{D}|$. Thus,

$$\mathsf{C}_{\text{multicast}} = \frac{1}{|\mathcal{D}|}\mathsf{C}_{\text{mu}}.$$

This concludes the proof for the first case.
**Case 2:** $|\mathcal{D}| > \max\{\Delta^+, \Delta^-\}$.
In this particular case, we define the multicast schedule as

$$\lambda'_{ji} = \frac{1}{\max\{\Delta^+, \Delta^-\}} \mathbb{1}_{\{\ell_{ji}>0\}} \qquad \forall (i,j) \in [0:N]\times[1:N].$$

To show that this schedule is feasible in the LP P2, we note that for every node $i$ in the network, we have that

$$\begin{aligned}\sum_{k\in[0:N]\backslash\{i\}} \lambda'_{ik} &= \sum_{k\in[0:N]\backslash\{i\}} \frac{1}{\max\{\Delta^+, \Delta^-\}} \mathbb{1}_{\{\ell_{ik}>0\}} \\ &= \frac{1}{\max\{\Delta^+, \Delta^-\}} \sum_{k\in[0:N]\backslash\{i\}} \mathbb{1}_{\{\ell_{ik}>0\}} \\ &= \frac{1}{\max\{\Delta^+, \Delta^-\}} \sum_{k\in N^+(i)} 1 = \frac{|N^+(i)|}{\max\{\Delta^+, \Delta^-\}} \stackrel{(a)}{\leq} 1, \end{aligned} \quad (17)$$

where: (i) $N^+(i) = \{k \in [0:N] | \ell_{ik} > 0\}$ is the set of neighboring nodes to $i$ that have incoming edges into $i$ with non-zero point-to-point link capacity; (ii) the inequality in $(a)$ follows from the definition of $\Delta^+$ that ensures that $|N^+(i)| \leq \Delta^+$, $\forall i \in [0{:}N]$. Using similar arguments, we can also show that

$$\sum_{j\in[1:N]\backslash\{i\}} \lambda'_{ji} \leq 1.$$

The analysis above proves that the constructed schedule $\lambda'_{ji}$ is feasible, i.e., it satisfies the constraints in $(2c)-(2e)$ in the LP P2. By fixing and substituting $\{\lambda'_{ji}\}$ in P2, we can now compute the achievable multicast rate through this LP

$$\begin{aligned}&\mathsf{R}'_{\text{multicast}} = \max_{\mathbf{F}} \min_{d\in\mathcal{D}} \left\{ \sum_{j\in[0:N]\backslash d} F^{(d)}_{d,j} \right\} \\ &0 \leq F^{(d)}_{j,i} \leq \frac{1}{\max\{\Delta^+,\Delta^-\}} \ell_{j,i} \quad \forall (i,j) \in [0{:}N]\times[1{:}N], d\in\mathcal{D} \\ &\sum_{j\in[1:N]\backslash\{i\}} F^{(d)}_{j,i} = \sum_{k\in[0:N]\backslash\{i\}} F^{(d)}_{i,k} \quad \forall i \in [1{:}N], d\in\mathcal{D}. \end{aligned} \quad (18)$$

Note that the only variables in the LP in (18) are the flow variables $\{F^{(d)}_{j,i}\}$. Thus, in (18), we are effectively computing the multicast capacity in a wired network with link capacities

$$\ell'_{j,i} = \frac{1}{\max\{\Delta^+, \Delta^-\}} \ell_{j,i}.$$

$$\mathsf{R}'_{\text{multicast}} = \frac{1}{\max\{\Delta^+, \Delta^-\}} \min_{d\in\mathcal{D}} \left\{ \mathsf{C}^{(\text{wired})}_d \right\}, \quad (19)$$

where $\mathsf{C}^{(\text{wired})}_d$ is the unicast capacity to destination $d$ when we consider a wired network with the same link capacities as our network. In other words, the network has orthogonal links that can be activated for $100\%$ of the time. Thus, it is not difficult to see that $\mathsf{C}_d \leq \mathsf{C}^{(\text{wired})}_d$, $\forall d \in \mathcal{D}$ and we have that

$$\begin{aligned}\mathsf{C}_{\text{multicast}} \geq \mathsf{R}'_{\text{multicast}} &= \frac{1}{\max\{\Delta^+, \Delta^-\}} \min_{d\in\mathcal{D}} \left\{ \mathsf{C}^{(\text{wired})}_d \right\} \\ &\geq \frac{1}{\max\{\Delta^+, \Delta^-\}} \min_{d\in\mathcal{D}} \{\mathsf{C}_d\} = \frac{1}{\max\{\Delta^+,\Delta^-\}} \mathsf{C}_{\text{mu}},\end{aligned}$$

which proves the lower bound in the second case. To show that the bound is indeed tight in this case, consider the network shown in Figure 3. For this particular case, it is not difficult to see that the unicast approximate capacity to each of the destinations is $\mathsf{R}$. Furthermore, in multicast, the source needs to switch (equally) between the $\Delta^-$ different paths connected to it to satisfy different destinations. Thus, we have

$$\mathsf{C}_{\text{multicast}} = \frac{1}{\max\{\Delta^+, \Delta^-\}} \mathsf{C}_{\text{mu}}.$$

This concludes the proof of Theorem 2. □

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