Randomized coverings of a convex body with its homothetic copies, and illumination

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Abstract

We present a probabilistic model of illuminating a convex body by independently distributed light sources. In addition to recovering C.A. Rogers' upper bounds for the illumination number, we improve previous estimates of J. Januszewski and M. Naszódi for a generalized version of the illumination parameter.

1 Introduction

Given a convex body (i.e. a compact convex set with non-empty interior) K in $\mathbb{R}^n$ and points $p_1, p_2, \ldots, p_m \in \mathbb{R}^n \setminus K$, we say that the collection $\{p_1, p_2, \ldots, p_m\}$ *illuminates* K if for any point x on the boundary of K there is a point p_i such that the line passing through x and p_i intersects the interior of K at a point not between p_i and x. The *illumination number* $\mathcal{I}(K)$ is the cardinality of the smallest collection of points illuminating K.

The well known conjecture of H. Hadwider [10], independently formulated by I. Gohberg and A. Markus, asserts that $\mathcal{I}(K) \leq 2^n$ for any n-dimensional convex body, with the equality attained for parallelotopes. The problem is known to be equivalent to the question whether every convex body can be covered by at most 2^n smaller homothetic copies of itself (see, for example, V. Boltyanski, H. Martini, P. S. Soltan, [9, Theorem 34.3]). For a detailed discussion of the problem and a survey of partial results, we refer to [9, Chapter VI], K. Bezdek [4, Chapter 3] and a recent survey by K. Bezdek and M. A. Khan [6].

An upper bound for the illumination number, which follows from a classical covering argument of C.A. Rogers [15], is

$$\mathcal{I}(K) \leq (n \log n + n \log \log n + 5n) \frac{\mathrm{Vol}_n(K-K)}{\mathrm{Vol}_n(K)} \tag{1}$$

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(see, for example, K. Bezdek [4, Theorem 3.4.1]). Here, $\mathrm{Vol}_n(\cdot)$ is the Lebesgue measure in $\mathbb{R}^n$, and $K-K$ is the Minkowski sum of K and $-K$. Using the estimate of $\mathrm{Vol}_n(K-K)$ due to C.A. Rogers and G.C. Shephard [16], we get $\mathcal{I}(K) \leq (1+o(1))\binom{2n}{n} n \log n$. Moreover, for a centrally-symmetric K we clearly have $\mathrm{Vol}_n(K-K) = 2^n \mathrm{Vol}_n(K)$, whence $\mathcal{I}(K) \leq (1+o(1)) 2^n n \log n$.

The proof of (1) based on C.A. Rogers' covering of $\mathbb{R}^n$, combines probabilistic and deterministic arguments, and does not give much information about the arrangement of points illuminating K. The main motivation for this work is to present a simple probabilistic model for the illumination, which provides more data about the collection of the light sources. In fact, we consider a more general question of covering a given convex body with its positive homothetic copies of different sizes. We prove the following:

Proposition 1. *Let n be a sufficiently large positive integer, K be a convex body in $\mathbb{R}^n$ with the origin in its interior and let numbers $(\lambda_i)_{i=1}^m$ satisfy $\lambda_i \in (e^{-n}, 1)$ $(i = 1, 2, \ldots, m)$ and*

$$\sum_{i=1}^m {\lambda_i}^n \geq (n \log n + n \log\log n + 4n) \frac{\mathrm{Vol}_n(K-K)}{\mathrm{Vol}_n(K)}.$$

For each i, let X_i be a random vector uniformly distributed inside the set $K - \lambda_i K$, so that $X_1, X_2, \ldots, X_m$ are jointly independent. Then the random collection of translates $\{X_i + \lambda_i K\}_{i=1}^m$ covers K with probability at least $1 - e^{-0.3n}$.

As an easy corollary of the above statement, we obtain:

Corollary 2. *Let n be a large positive integer, and K be a convex body in $\mathbb{R}^n$ with the origin in its interior. Then there is a number $R > 0$ depending only on n with the following property: Let X be a random vector uniformly distributed over $K - K$, and let*

$$m := \left\lceil (n \log n + n \log\log n + 5n) \frac{\mathrm{Vol}_n(K-K)}{\mathrm{Vol}_n(K)} \right\rceil.$$

Let $X_1, X_2, \ldots, X_m$ be independent copies of X. Then with probability at least $1 - e^{-0.3n}$ the collection $\{RX_1, RX_2, \ldots, RX_m\}$ illuminates K.

Let us note that illumination of convex sets by independent random light sources was previously considered in literature. Namely, O. Schramm [19] used such a model to estimate the illumination number for bodies of constant width; later, this approach was generalized by K. Bezdek to so-called fat spindle bodies [5].

Proposition 1 allows us to study the following notion, closely related to the illumination number. For a convex body K in $\mathbb{R}^n$, define $f_n(K)$ to be the least positive number such that for any sequence (λ_i) $(\lambda_i \in [0, 1))$ with $\sum {\lambda_i}^n > f_n(K)$ there are points $x_i \in \mathbb{R}^n$ such that the collection of homothets $\{\lambda_i K + x_i\}$ covers K. It was shown by A. Meir and L. Moser [13] that $f_n([0, 1]^n) = 2^n - 1$. For an arbitrary convex body K, J. Januszewski [11] showed that $f_n(K) \leq (n + 1)^n - 1$. Further, M. Naszódi [14] showed that for any K with its center of mass at the origin,

$$f_n(K) \leq 2^n \frac{\mathrm{Vol}_n(K + \frac{1}{2} K \cap (-K))}{\mathrm{Vol}_n(K \cap (-K))} \leq \begin{cases} 3^n, & \text{if } K = -K, \\ 6^n, & \text{otherwise.} \end{cases}$$

We refer to P. Brass, W. Moser, J. Pach [8, p. 131] for a more extensive discussion of this quantity.

Our Proposition 1, together with a Rogers–type argument, gives the following:

Corollary 3. *Let n be a (large enough) positive integer and K be a convex body in $\mathbb{R}^n$. Then, with the quantity $f_n(K)$ defined above, we have*

$$f_n(K) \leq \left\lceil (n \log n + n \log\log n + 5n) \frac{\mathrm{Vol}_n(K-K)}{\mathrm{Vol}_n(K)} \right\rceil.$$

We remark, that together with the Rogers–Shephard bound on the volume of the difference body from [16], Corollary 3 implies that

$$f_n(K) \leq \begin{cases} 2^n n \log n(1+o(1)), & \text{if } K = -K, \\ \frac{1}{\sqrt{\pi n}} 4^n n \log n(1+o(1)), & \text{otherwise.} \end{cases}$$

We note that several other illumination-related quantities, different from $f_n(K)$, were considered in literature. We refer, in particular, to [3, 7, 20].

Let us emphasize that proofs of all the above statements are very simple. The purpose of this note is to put forward a randomized model for studying the illumination number and its generalizations. We believe that such viewpoint to the Illumination Problem will prove useful. Let us note that, using a randomized model for illumination (however, different from the one considered in this note), the second author recently verified the Illumination conjecture for the class of unit balls of 1-symmetric norms in $\mathbb{R}^n$ in high dimensions [21].

We give a proof of Proposition 1 in Section 3, whereas the corollaries are derived in Section 4.

2 Notation and preliminaries

The standard vector basis in $\mathbb{R}^n$ is denoted by $\{e_1, e_2, \ldots, e_n\}$. For a non-zero vector $v \in \mathbb{R}^n$, $v^\perp$ is the hyperplane orthogonal to v. By B_∞^n we denote the cube $[-1,1]^n$.

Given two sets $A, B \subset \mathbb{R}^n$, the Minkowski sum $A + B$ is defined as

$$A + B := \{x + y \,:\, x \in A,\, y \in B\}.$$

Let K be a convex body, and let $\varepsilon > 0$. Then an ε-net $\mathcal{N}$ on K is a set of points $\{x_i\} \subset \mathbb{R}^n$ such that the collection of convex sets $\{x_i + \varepsilon K\}$ covers K.

We make the following observation.

Lemma 4. *Let $n \geq 2$ be an integer. For every convex body K in $\mathbb{R}^n$ and for every $\lambda \in [0,1]$ we have*

$$\mathrm{Vol}_n(K - \lambda K) \leq (1+\lambda)^n \frac{\mathrm{Vol}_n(K-K)}{2^n}.$$

Proof. Observe that

$$\mathrm{Vol}_n(K - \lambda K) = (1+\lambda)^n \mathrm{Vol}_n\left(\mu K - (1-\mu)K\right),$$

where $\mu = \frac{1}{1+\lambda} \in [0,1]$. The lemma follows from the fact that

$$\max_{\nu\in[0,1]} \mathrm{Vol}_n\left(\nu K - (1-\nu)K\right) = \mathrm{Vol}_n\left(\frac{K}{2} - \frac{K}{2}\right) = \frac{\mathrm{Vol}_n(K-K)}{2^n}. \tag{2}$$

To establish (2), let us consider an auxiliary $(n+1)$-dimensional convex set $\mathcal{C}$ given by

$$\mathcal{C} := \mathrm{conv}\left(K \times \{0\} \cup (-K) \times \{1\}\right)$$

(let us remark that the use of such auxiliary sets is rather standard and goes back at least to C.A. Rogers and G.C. Shephard [17]; also, see S. Artstein-Avidan [1]). Observe that for any $\nu \in [0,1]$ we have

$$\mathcal{C} \cap (e_{n+1}^{\perp} + \nu e_{n+1}) = (\nu K - (1-\nu)K) \times \{\nu\}.$$

The set $\mathcal{C}$ is convex and symmetric with respect to $\frac{1}{2}e_{n+1}$. Hence, the n-dimensional section of $\mathcal{C}$ given by the hyperplane $e_{n+1}^{\perp} + \frac{1}{2}e_{n+1}$, has maximal n-dimensional volume among all other sections of $\mathcal{C}$ parallel to it. □

Next, for the reader's convenience we provide a standard estimate of the covering number.

Lemma 5. *Let n be a sufficiently large positive integer, and let K be a convex body in $\mathbb{R}^n$ with the origin in its interior. Then for any $\varepsilon \in (0,1]$ there exists an ε-net on K of cardinality at most $\left(\frac{5}{\varepsilon}\right)^n$.*

Proof. By the Rogers–Zong lemma [18], there exists an ε-net of cardinality at most

$$\frac{\mathrm{Vol}_n(K - \varepsilon K)}{\mathrm{Vol}_n(\varepsilon K)}\big(n\log n + n\log\log n + 5n\big).$$

By Lemma 4, along with the Rogers–Shephard lemma [16], we estimate

$$\frac{\mathrm{Vol}_n(K - \varepsilon K)}{\mathrm{Vol}_n(\varepsilon K)}\big(n\log n + n\log\log n + 5n\big) \le (1+o(1))\frac{4^n(1+\varepsilon)^n n\log n}{\sqrt{\pi n}2^n\varepsilon^n} \le \left(\frac{5}{\varepsilon}\right)^n.$$

□

3 Proof of Proposition 1

Let $(\lambda_i)_{i=1}^m$ satisfy the assumptions of the proposition, and let $X_1, X_2, \ldots, X_m$ be jointly independent random vectors, where each X_i is uniformly distributed in $K - \lambda_i K$. Assume without loss of generality that $\mathrm{Vol}_n(K) = 1$.

We shall estimate the probability

$$\mathbb{P}\Big\{K \subset \bigcup_{i=1}^{m} (X_i + \lambda_i K)\Big\}.$$

Let $\varepsilon \in (0,1]$ be chosen later, and consider an ε-net $\mathcal{N}$ on K of cardinality at most $\left(\frac{5}{\varepsilon}\right)^n$ (which exists according to Lemma 5). We can safely assume that $\mathcal{N} \subset K - \varepsilon K$. Observe that

$$\mathbb{P}\Big\{\exists x \in K \,:\, x \notin \bigcup_{i=1}^{m} (X_i + \lambda_i K)\Big\} \leq \mathbb{P}\Big\{\exists y \in \mathcal{N} \,:\, y \notin \bigcup_{i=1}^{m} (X_i + (\lambda_i - \varepsilon)_+ K)\Big\},$$

where $(\lambda_i - \varepsilon)_+ := \max(0, \lambda_i - \varepsilon)$. Hence, by the union bound,

$$\mathbb{P}\Big\{K \subset \bigcup_{i=1}^{m} (X_i + \lambda_i K)\Big\} \geq 1 - \left(\frac{5}{\varepsilon}\right)^n \max_{y \in \mathcal{N}} \mathbb{P}\Big\{y \notin \bigcup_{i=1}^{m} (X_i + (\lambda_i - \varepsilon)_+ K)\Big\}.$$

Fix any $y \in \mathcal{N}$ and note that

$$\mathbb{P}\Big\{y \notin \bigcup_{i=1}^{m} (X_i + (\lambda_i - \varepsilon)_+ K)\Big\} = \prod_{i=1}^{m} \left(1 - \mathbb{P}\big\{y \in X_i + (\lambda_i - \varepsilon)_+ K\big\}\right).$$

Further, observe that

$$\mathbb{P}\big\{y \in X_i + (\lambda_i - \varepsilon)_+ K\big\} = \mathbb{P}\big\{X_i \in y - (\lambda_i - \varepsilon)_+ K\big\} = \frac{{(\lambda_i - \varepsilon)_+}^n}{\mathrm{Vol}_n(K - \lambda_i K)},$$

where the last equality is due to the fact that $\mathrm{Vol}_n(K) = 1$ and that

$$y - (\lambda_i - \varepsilon) K \subset K - \lambda_i K$$

whenever $\lambda_i > \varepsilon$. Combining the above relations, we obtain

$$\mathbb{P}\Big\{K \subset \bigcup_{i=1}^{m} (X_i + \lambda_i K)\Big\} \geq 1 - \left(\frac{5}{\varepsilon}\right)^n \prod_{i=1}^{m} \left(1 - \frac{{(\lambda_i - \varepsilon)_+}^n}{\mathrm{Vol}_n(K - \lambda_i K)}\right).$$

Now, our aim is to show that under the assumptions of the proposition there exists an ε such that

$$\left(\frac{5}{\varepsilon}\right)^n \prod_{i=1}^{m} \left(1 - \frac{{(\lambda_i - \varepsilon)_+}^n}{\mathrm{Vol}_n(K - \lambda_i K)}\right) \leq e^{-0.3n},$$

or, equivalently,

$$\sum_{i=1}^{m} \log\left(1 - \frac{{(\lambda_i - \varepsilon)_+}^n}{\mathrm{Vol}_n(K - \lambda_i K)}\right) \leq n \log\Big(\frac{\varepsilon}{5}\Big) - 0.3n.$$

In view of Lemma 4, and the relation $\log(1-t) \le -t$ valid for all $t \ge 0$, we have

$$\sum_{i=1}^{m} \log\left(1 - \frac{(\lambda_i - \varepsilon)_+{}^n}{\mathrm{Vol}_n(K - \lambda_i K)}\right) \le \sum_{i=1}^{m} \log\left(1 - \frac{2^n (\lambda_i - \varepsilon)_+{}^n}{(1+\lambda_i)^n \mathrm{Vol}_n(K-K)}\right)$$
$$\le -\sum_{i=1}^{m} \frac{2^n (\lambda_i - \varepsilon)_+{}^n}{(1+\lambda_i)^n \mathrm{Vol}_n(K-K)}.$$

Thus, in order to prove the proposition, it is sufficient to show that for some $\varepsilon \in (0,1]$ we have

$$n \log 5 + n \log \frac{1}{\varepsilon} - \sum_{i=1}^{m} \frac{2^n (\lambda_i - \varepsilon)_+{}^n}{(1+\lambda_i)^n \mathrm{Vol}_n(K-K)} \le -0.3n. \tag{3}$$

Let $A_n := 1 - \frac{4 \log n}{n}$. We consider two complimentary subsets of $\{1, 2, \ldots, m\}$:

$$L_1 = \big\{ i \le m : \, \lambda_i \ge A_n \big\}, \quad L_2 = \big\{ i \le m : \, \lambda_i < A_n \big\}.$$

The rest of the proof splits into two cases.

Case 1:

$$\sum_{i \in L_1} \lambda_i{}^n \ge \left(1 - \frac{1}{\log n}\right) \sum_{i=1}^{m} \lambda_i{}^n. \tag{4}$$

Choose $\varepsilon := \frac{A_n}{n \log n}$. Then

$$(\lambda_i - \varepsilon)^n \ge \lambda_i{}^n \left(1 - \frac{1}{\log n}\right) \quad \text{for all } i \in L_1,$$

whence, using the condition $\lambda_i \le 1$ together with (4) and the condition on the sum of $\lambda_i{}^n$, we get

$$\sum_{i \in L_1} \frac{2^n (\lambda_i - \varepsilon)^n}{(1+\lambda_i)^n \mathrm{Vol}_n(K-K)} \ge \left(1 - \frac{1}{\log n}\right) \sum_{i \in L_1} \frac{\lambda_i{}^n}{\mathrm{Vol}_n(K-K)}$$
$$\ge \left(1 - \frac{1}{\log n}\right)^2 \sum_{i=1}^{m} \frac{\lambda_i{}^n}{\mathrm{Vol}_n(K-K)}$$
$$\ge \left(1 - \frac{1}{\log n}\right)^2 (n \log n + n \log\log n + 4n).$$

It is easy to check, using the above inequality, that (3) is satisfied, and Case 1 is settled.

Case 2:

$$\sum_{i \in L_2} \lambda_i{}^n > \frac{1}{\log n} \sum_{i=1}^{m} \lambda_i{}^n.$$

Set $\varepsilon := \frac{e^{-n}}{n \log n}$. By the assumption of the proposition, $\lambda_i \ge e^{-n}$ for all i. Hence,

$$(\lambda_i - \varepsilon)_+{}^n \ge \left(1 - \frac{1}{\log n}\right) \lambda_i{}^n, \quad i \le m,$$

and the left hand side of (3) is less than

$$n\log 5+n^2+n\log n+n\log\log n-\left(1-\frac{1}{\log n}\right)\frac{2^n}{\mathrm{Vol}_n(K-K)}\sum_{\lambda_i\in L_2}\frac{{\lambda_i}^n}{(1+\lambda_i)^n}$$

$$n\log 5+n^2+n\log n+n\log\log n-\frac{1}{\log n}\left(1-\frac{1}{\log n}\right)\frac{2^n}{\mathrm{Vol}_n(K-K)}\sum_{i=1}^{m}\frac{{\lambda_i}^n}{(1+\lambda_i)^n}$$

$$\leq n\log 5+n^2+n\log n+n\log\log n-\left(1-\frac{1}{\log n}\right)\frac{2^n}{(1+A_n)^n}\frac{n\log n+n\log\log n+4n}{\log n}$$

$$\ll -0.3n.$$

Here, we used the definition of A_n, and the assumption on the sum of ${\lambda_i}^n$. Thus, Case 2 is settled, and the proof of Proposition 1 is complete.

4 Proof of the Corollaries

4.1 Proof of Corollary 2

First, we recall that if K is a convex body and $K\subset\cup_{i=1}^m \mathrm{int}(K)+x_i$ for some non-zero vectors $x_1,x_2,\dots,x_m$, then there exists $R>0$ such that points $Rx_1,Rx_2,\dots,Rx_m$ illuminate K (see, for example, [9, proof of Theorem 34.3]). It is not difficult to verify the following quantitative version of above observation: If $K\subset\cup_{i=1}^m(1-\varepsilon)K+x_i$ for some $\varepsilon\in(0,1)$, then $Rx_1,Rx_2,\dots,Rx_m$ illuminate K whenever $R>\frac{1}{\varepsilon}$.

Let n be a sufficiently large integer, denote

$$m:=\left\lceil(n\log n+n\log\log n+5n)\frac{\mathrm{Vol}_n(K-K)}{\mathrm{Vol}_n(K)}\right\rceil,$$

and select $\varepsilon=\varepsilon(n)>0$ small enough so that

$$m(1-\varepsilon)^n\geq(n\log n+n\log\log n+4n)\frac{\mathrm{Vol}_n(K-K)}{\mathrm{Vol}_n(K)}.$$

Let $X_1,X_2,\dots,X_m$ be i.i.d. uniformly distributed in $K-K$. Then, by Proposition 1, with probability at least $1-e^{-0.3n}$ the collection $\{X_i+(1-\varepsilon)K\}_{i=1}^m$ forms a covering of K; hence, for any fixed $R>\frac{1}{\varepsilon}$, the vectors $RX_1,RX_2,\dots,RX_m$ illuminate K with probability at least $1-e^{-0.3n}$.

4.2 Proof of Corollary 3

The next lemma is a variation of the well known theorem of C.A. Rogers [15] on economical coverings of $\mathbb{R}^n$ with convex bodies.

Lemma 6. *Let n be a sufficiently large positive integer and K be a convex body in $\mathbb{R}^n$ such that $-\frac{1}{n}K\subset K\subset B_\infty^n$. Further, let $L\geq n^2$ and let $(\lambda_i)_{i=1}^M$ be a sequence of numbers in $[1/2,1]$ with*

$$\sum_{i=1}^{M}\mathrm{Vol}_n(\lambda_iK)\geq(n\log n+n\log\log n+5n)\mathrm{Vol}_n(LB_\infty^n).$$

Then there exists a translative covering of LB_∞^n by $\{\lambda_i K\}_{i=1}^M$.

Proof. The proof to a large extent follows [15]; we provide it only for reader's convenience. Let $M' \le M$ be the least number such that

$$\sum_{i=1}^{M'} \mathrm{Vol}_n(\lambda_i K) \ge (n\log n + n\log\log n + 4n)\mathrm{Vol}_n(LB_\infty^n),$$

and let Y_i $(1 \le i \le M')$ be independent random vectors uniformly distributed in $LB_\infty^n - 2K$. For any point $x \in LB_\infty^n - K$, using the condition $K \subset B_\infty^n$, we have

$$\begin{aligned}
&\mathbb{P}\Big\{x \notin \Big(\lambda_i - \frac{1}{2n\log n}\Big)K + Y_i \ \text{ for all } \ i \le M'\Big\}\\
&= \prod_{i=1}^{M'}\Big(1 - \mathbb{P}\Big\{Y_i \in x - \Big(\lambda_i - \frac{1}{2n\log n}\Big)K\Big\}\Big)\\
&\le \prod_{i=1}^{M'}\left(1 - \Big(1 - \frac{1}{n\log n}\Big)^n \frac{\mathrm{Vol}_n\big(\lambda_i K\big)}{\mathrm{Vol}_n(LB_\infty^n - 2K)}\right),
\end{aligned}$$

where the last inequality is due to the fact that $x - \big(\lambda_i - \frac{1}{2n\log n}\big)K \subset LB_\infty^n - 2K$. Further, we have

$$\begin{aligned}
&\prod_{i=1}^{M'}\left(1 - \Big(1 - \frac{1}{n\log n}\Big)^n \frac{\mathrm{Vol}_n\big(\lambda_i K\big)}{\mathrm{Vol}_n(LB_\infty^n - 2K)}\right)\\
&\le \exp\left(-\Big(1 - \frac{1}{n\log n}\Big)^n \sum_{i=1}^{M'} \frac{\mathrm{Vol}_n\big(\lambda_i K\big)}{\mathrm{Vol}_n(LB_\infty^n - 2K)}\right)\\
&\le \exp\left(-\frac{L^n}{(L+2)^n}\Big(1 - \frac{1}{n\log n}\Big)^n \sum_{i=1}^{M'} \frac{\mathrm{Vol}_n\big(\lambda_i K\big)}{\mathrm{Vol}_n(LB_\infty^n)}\right)\\
&\le \exp\left(-\frac{1}{(1+2n^{-2})^n}\Big(1 - \frac{1}{\log n}\Big)\big(n\log n + n\log\log n + 4n\big)\right)\\
&\le \exp(-n\log n - n\log\log n - 2n),
\end{aligned}$$

where in the last inequality we used the assumption that n is large. Hence, there is a non-random collection of vectors $\{y_i\}_{i=1}^{M'}$ such that

$$\begin{aligned}
&\mathrm{Vol}_n\Big((LB_\infty^n - K) \setminus \bigcup_{i=1}^{M'}\Big(\Big(\lambda_i - \frac{1}{2n\log n}\Big)K + y_i\Big)\Big)\\
&\quad \le \exp(-n\log n - n\log\log n - 2n)\mathrm{Vol}_n(LB_\infty^n - K).
\end{aligned}$$

Suppose that $S := LB_\infty^n \setminus \bigcup_{i=1}^{M'}\big(\lambda_i K + y_i\big)$ is non-empty. Let $\mathcal{N}$ be a maximal discrete subset of S such that

$$\Big(y - \frac{1}{2n\log n}K\Big) \cap \Big(y' - \frac{1}{2n\log n}K\Big) = \emptyset \ \text{ for all } \ y \ne y' \in \mathcal{N}.$$

Note that $\mathcal{N}-\frac{1}{2n\log n}K\subset(LB_\infty^n-K)\setminus\bigcup_{i=1}^{M'}\big((\lambda_i-\frac{1}{2n\log n})K+y_i\big)$, whence

$$|\mathcal{N}|\leq\frac{\exp(-n\log n-n\log\log n-2n)\mathrm{Vol}_n(LB_\infty^n-K)}{(2n\log n)^{-n}\mathrm{Vol}_n(K)}\leq\frac{\mathrm{Vol}_n(LB_\infty^n-K)}{e^n\mathrm{Vol}_n(K)}. \tag{5}$$

On the other hand, by the choice of $\mathcal{N}$ and in view of the inclusion $-\frac{1}{n}K\subset K$, we have

$$S\subset\mathcal{N}+\frac{1}{2n\log n}(K-K)\subset\mathcal{N}+\frac{1}{2}K.$$

Finally, from the choice of M' it follows that

$$\sum_{i=M'+1}^{M}\mathrm{Vol}_n(\lambda_iK)\geq\mathrm{Vol}_n(LB_\infty^n)-\mathrm{Vol}_n(K),$$

whence, using (5), $M-M'\geq\frac{\mathrm{Vol}_n(LB_\infty^n)-\mathrm{Vol}_n(K)}{\mathrm{Vol}_n(K)}\geq|\mathcal{N}|$. It remains to define the points $\{y_i\}_{i=M'+1}^M$ so that $\{y_i\}_{i=M'+1}^M=\mathcal{N}$; then the collection $\{y_i+\lambda_iK\}_{i=1}^M$ covers LB_∞^n. □

Proof of Corollary 3. Let $(\lambda_i)_{i=1}^\infty$ be a sequence of numbers in $(0,1)$ such that

$$\sum_{i=1}^{\infty}\mathrm{Vol}_n(\lambda_iK)>(n\log n+n\log\log n+5n)\mathrm{Vol}_n(K-K).$$

We need to show that in this case there exists a covering of K of the form $\{y_i+\lambda_iK\}_{i=1}^\infty$. We will assume that $\sum\limits_{i=1}^{\infty}\mathrm{Vol}_n(\lambda_iK)<\infty$. First, suppose that

$$\sum_{i:\lambda_i\geq n^{-5}}\mathrm{Vol}_n(\lambda_iK)\geq(n\log n+n\log\log n+4n)\mathrm{Vol}_n(K-K).$$

Then the result immediately follows from Proposition 1.

Otherwise,

$$\sum_{i:\lambda_i<n^{-5}}\mathrm{Vol}_n(\lambda_iK)\geq\mathrm{Vol}_n(K-K)\geq2^n\mathrm{Vol}_n(K). \tag{6}$$

Further, without loss of generality (for example, by applying John's theorem [12, 2] together with an appropriate affine transformation) we can assume that $B_\infty^n\subset K\subset n^{3/2}B_\infty^n$. Let $\{x_j+n^{-3/2}B_\infty^n\}_{j=1}^N$ be a minimal covering of K by cubes with pairwise disjoint interiors. Let us define subsets $I_k\subset\mathbb{N}$ by

$$I_k:=\big\{i\in\mathbb{N}:\ \lambda_in^5\in(2^{-k},2^{-k+1}]\big\},\ \ k=1,2,\dots$$

and for every $k\in\mathbb{N}$ set

$$F(k):=\Big\lfloor\frac{\sum_{i\in I_k}\mathrm{Vol}_n(\lambda_iK)}{(n\log n+n\log\log n+6n)\mathrm{Vol}_n(2^{-k+1}n^{-3/2}B_\infty^n)}\Big\rfloor.$$

Further, for those k with $F(k) > 0$, we let I_k^ℓ ($\ell = 1, 2, \ldots, F(k)$) be a partition of I_k such that

$$\sum_{i \in I_k^\ell} \mathrm{Vol}_n(\lambda_i K) \geq (n \log n + n \log\log n + 5n) \mathrm{Vol}_n(2^{-k+1} n^{-3/2} B_\infty^n), \quad \ell = 1, 2, \ldots, F(k). \tag{7}$$

Note that such partitions can always be constructed as the volume of each $\lambda_i K$ ($i \in I_k$) is negligible compared to $\mathrm{Vol}_n(2^{-k+1} n^{-3/2} B_\infty^n)$. Further,

$$\sum_{k : F(k) < n} \sum_{i \in I_k} \mathrm{Vol}_n(\lambda_i K) \leq 4n^2 \log n \, \mathrm{Vol}_n(n^{-3/2} B_\infty^n) \ll n^{-n} \mathrm{Vol}_n(K),$$

whence, by (6)

$$\begin{aligned}
\sum_{k : F(k) \geq n} F(k) \mathrm{Vol}_n(2^{-k+1} n^{-3/2} B_\infty^n) &\geq \sum_{k : F(k) \geq n} \frac{n}{n+1} \frac{\sum_{i \in I_k} \mathrm{Vol}_n(\lambda_i K)}{(n \log n + n \log\log n + 6n)} \\
&\geq \frac{n}{n+1} \frac{2^n \mathrm{Vol}_n(K)}{n \log n + n \log\log n + 6n} - n^{-n} \mathrm{Vol}_n(K) \\
&> \mathrm{Vol}_n\Big(\bigcup_{j=1}^N (x_j + n^{-3/2} B_\infty^n)\Big),
\end{aligned}$$

where the last inequality follows from the trivial observation $\bigcup\limits_{j=1}^N (x_j + n^{-3/2} B_\infty^n) \subset (1 + 2n^{-3/2})K$ and our assumption that n is large. The last relation implies that there exists a finite collection of translates

$$\mathcal{C} := \big\{ y_k^\ell + 2^{-k+1} n^{-3/2} B_\infty^n, \ k : F(k) \geq n, \ \ell = 1, 2, \ldots, F(k) \big\} \quad (y_k^\ell \in \mathbb{R}^n)$$

such that the union of the cubes from $\mathcal{C}$ covers $\bigcup\limits_{j=1}^N (x_j + n^{-3/2} B_\infty^n)$, hence, K. For each cube from $\mathcal{C}$ we construct a translative covering by sets $\{\lambda_i K\}_{i \in I_k^\ell}$ using condition (7) and Lemma 6. This completes the proof. □

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