Advances in Mathematics 356 (2019) 106803

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

An extension of Minkowski's theorem and its applications to questions about projections for measures

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A R T I C L E I N F O

Article history:
Received 12 December 2016
Received in revised form 18 August 2017
Accepted 23 August 2017
Available online 18 September 2019
Communicated by Michael Collver

MSC:
primary 52

Keywords:
Convex bodies
Log-concave
Brunn-Minkowski
Cone-measure

A B S T R A C T

Minkowski's Theorem asserts that every centered measure on the sphere which is not concentrated on a great subsphere is the surface area measure of some convex body, and the surface area measure determines a convex body uniquely up to a shift. In this manuscript we prove an extension of Minkowski's theorem. Consider a measure μ on $\mathbb{R}^n$ with positive degree of concavity and positive degree of homogeneity. We show that a surface area measure of a convex set K, weighted with respect to μ, determines a convex body uniquely up to μ-measure zero. We also establish an existence result under natural conditions including symmetry.

We apply this result to extend the solution to classical Shephard's problem. To do this, we introduce a new notion which relates projections of convex bodies to a given measure μ, and is a generalization of the Lebesgue volume of a projection.

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https://doi.org/10.1016/j.aim.2019.106803

1. Introduction

We shall work in an n-dimensional vector space $\mathbb{R}^n$ with standard orthonormal basis $e_1, ..., e_n$ and a scalar product $\langle \cdot, \cdot \rangle$. The standard Euclidean length is denoted by $|\cdot|$.

A set K in $\mathbb{R}^n$ is said to be convex if together with every pair of points it contains the interval connecting them. Compact convex sets with non-empty interior are called convex bodies.

The standard Lebesgue measure of a set A in $\mathbb{R}^n$ shall be denoted by $|A|$ or, sometimes, $|A|_n$. When the standard Lebesgue measure on a subspace of dimension k is considered, it shall be denoted by $|\cdot|_k$. We shall denote the unit ball centered at the origin in $\mathbb{R}^n$ by B_2^n, and the unit sphere by $\mathbb{S}^{n-1}$.

Given a convex body K in $\mathbb{R}^n$, its Gauss map $\nu_K : \partial K \to \mathbb{S}^{n-1}$ is the map that corresponds to every $y \in \partial K$ the set of normal vectors at y with respect to K. The surface area measure of K is the measure on the unit sphere defined as the push forward to the sphere of the $(n-1)$-dimensional Hausdorff measure on ∂K via the map ν_K. It is denoted by σ_K.

Minkowski's existence theorem guarantees that every barycentered measure on $\mathbb{S}^{n-1}$ which is not supported on any great subsphere is a surface area measure for some convex body; moreover, a convex body is determined by its surface area measure uniquely up to a shift.

For $p \in \mathbb{R}$, the L_p surface area measure of a convex body with the support function h_K is the measure on the sphere given by $d\sigma_{p,K}(u) = h_K^{1-p}(u)d\sigma_K(u)$. It was introduced by Lutwak. The normalized L_p surface area is given by $d\bar{\sigma}_{p,K}(u) = \frac{1}{|K|}d\sigma_{p,K}(u)$. An extension of Minkowski's Theorem, called *L_p-Minkowski problem* is open in general. It asks which conditions should be required in order for a measure on the sphere to be an L_p-surface area measure, as well as whether L_p-surface area measure determines a convex body uniquely. Lutwak, Yang, Zhang have solved the normalized L_p-Minkowski problem with even data for the case $p \le 0$, and showed the uniqueness of the solution when $p < 0$. Böröczky, Lutwak, Yang, Zhang [8], [9], [10] have studied the case $p = 0$ and have, in particular, obtained the uniqueness in the case of symmetric convex bodies on the plane. Stancu [53], [54] has treated this problem for polytopes on the plane. Huang, Liu, Xu [21] have established uniqueness in $\mathbb{R}^3$ in the case when the L_p surface area is constant. The L_p-Minkowski problem is one of the main questions in the rapidly developing Brunn-Minkowski-Firey theory (see more in Ludwig [35], Lutwak [37], [38], Lutwak, Yang, Zhang [40], [41], [42], Lutwak, Oliker [39], Meyer, Werner [44], Ryabogin, Zvavitch [49], Zhu [56], [57], and the references therein).

In this manuscript, we prove an analogue of Minkowski's theorem in a different setting. Let μ be an absolutely continuous measure on $\mathbb{R}^n$, with a continuous density. Throughout the manuscript, continuity is understood in the sense of continuity on closed support. We study the surface area measure of convex bodies with respect to μ.

Definition 1.1. Let K be a convex body and ν_K be its Gauss map. Let μ be a measure on $\mathbb{R}^n$ with continuous density $g(x)$. Define $\sigma_{\mu,K}$ on $\mathbb{S}^{n-1}$, a surface area measure of K with respect to μ, as follows: for every Borel set $\Omega \subset \mathbb{S}^{n-1}$, let

$$\sigma_{\mu,K}(\Omega) = \int\limits_{\nu_K^{-1}(\Omega)} g(x)dH_{n-1}(x),$$

where H_{n-1} stands for the $(n-1)$-dimensional Hausdorff measure on ∂K, and $\nu_K^{-1}(\Omega)$ stands for the full pre-image of Ω under ν_K.

When μ is the standard Lebesgue measure, the measure $\sigma_{\mu,K}$ coincides with σ_K, the classical surface area measure.

Let $p \in (0, +\infty)$. We say that a function $f : \mathbb{R}^n \to [0, \infty]$ is p-concave if $f^p(x)$ is a concave function on its support. That is, for every $x, y \in supp(f)$ and for every $\lambda \in [0, 1]$ we have

$$f^p(\lambda x + (1-\lambda)y) \geq \lambda f^p(x) + (1-\lambda)f^p(y).$$

Let $r \in (-\infty, +\infty)$. We say that a function $f : \mathbb{R}^n \to [0, \infty]$ is r- homogenous if for every $a > 0$ and for every $x \in \mathbb{R}^n$ we have $f(ax) = a^r f(x)$.

We shall consider the class of measures on $\mathbb{R}^n$ with densities that have a positive degree of homogeneity and a positive degree of concavity. In fact, all such densities are p-concave and $\frac{1}{p}$-homogenous for the same $p \geq 0$ (see the Proposition A.2 from the Appendix). This class of measures was considered by E. Milman and L. Rotem [45], where they studied their isoperimetric properties. We remark that such measures are necessarily supported on convex cones. An example of a density function with said properties is $f(x) = 1_{\{\langle x,\theta\rangle > 0\}} |\langle x, \theta\rangle|^{\frac{1}{p}}$, where θ is a vector.

We prove an extension of Minkowski's existence theorem to the class of surface area measures with respect to measures with positive degree of concavity and positive degree of homogeneity.

Theorem 1.2. *Let μ on $\mathbb{R}^n$ be a measure and $g(x)$ be its even r-homogenous density for some $r \geq 0$, and the restriction of g to some half space is p-concave for a $p \geq 0$. Let $\varphi(u)$ be an arbitrary even measure on $\mathbb{S}^{n-1}$, not supported on any great subsphere, such that $supp(\varphi) \subset int(supp(g)) \cap \mathbb{S}^{n-1}$. Then there exists a symmetric convex body K in $\mathbb{R}^n$ such that*

$$d\sigma_{K,\mu}(u) = d\varphi(u).$$

Moreover, such convex body is determined uniquely up to a set of μ-measure zero.

In Theorem 1.2, and throughout the paper, uniqueness up to μ-measure zero means that for every pair of K and L, symmetric convex bodies with $\sigma_{K,\mu} = \sigma_{L,\mu}$, the measure of their symmetric difference $\mu(K\Delta L) = 0$.

We apply Theorem 1.2 to extend the study of volume comparison and unique determination of convex bodies related to projections.

Given a unit vector $u \in \mathbb{S}^{n-1}$, we consider an $(n-1)$-dimensional hyperplane orthogonal to it:

$$u^{\perp} = \{x \in \mathbb{R}^n : \langle x, u\rangle = 0\}.$$

An orthogonal projection of a convex body K to a subspace $u^{\perp}$ shall be denoted by $K|u^{\perp}$; that is,

$$K|u^{\perp} = \{x \in u^{\perp} : \exists t \in \mathbb{R}\, \text{s.t.}\, x + tu \in K\}.$$

Let K be an origin symmetric convex body in $\mathbb{R}^n$ with curvature function f_K. The projection body ΠK of K is defined as the origin symmetric convex body in $\mathbb{R}^n$ whose support function in every direction is equal to the volume of the hyperplane projection of K in this direction.

The Shephard problem (see Shephard [52]) is the following question: *given symmetric convex bodies K and L such that for every $u \in \mathbb{S}^{n-1}$*

$$|K|u^{\perp}|_{n-1} \leq |L|u^{\perp}|_{n-1},$$

does it follow that $|K|_n \leq |L|_n$? The problem was solved independently by Petty [47] and Schneider [50]. They showed that the answer is affirmative if $n \leq 2$ and negative if $n \geq 3$. More precisely, the answer to Shephard's problem is affirmative if and only if L is a projection body. As for general symmetric convex bodies, Ball [3] proved that if the volumes of projections of K are less than or equal to the volumes of projections of L in every direction, then $|K| \leq \sqrt{n}|L|$, for every dimension n. Goodey and Zhang [18] obtained a generalization of the Shephard problem for lower dimensional projections. A Fourier analytic approach to Shephard's problem was presented by Koldobsky, Ryabogin and Zvavitch [31]. Ryabogin and Zvavitch [49] solved the generalization of Shephard's problem for Firey projections.

The Busemann-Petty problem is in a sense dual to the Shephard problem. It asks whether symmetric convex bodies with larger central hyperplane sections necessarily have greater volume. The Busemann-Petty problem has been solved affirmatively for $n \leq 4$ and negatively for $n \geq 5$ (see Gardner, Koldobsky, Schlumprecht [17] and Zhang [55]). The answer to Busemann-Petty problem is affirmative if and only if the body with larger sections is an intersection body (see Lutwak [36] for the definition and properties of intersection bodies, and Koldobsky [24] for Fourier analytic approach to intersection bodies). Zvavitch solved an isomorphic version of Busemann-Petty problem for Gaussian measures [58], and completely generalized the solution of Busemann-Petty problem to arbitrary measures with positive density [59]. Koldobsky [28], and further Koldobsky and Zvavitch [30] obtained estimates for the isomorphic version of Busemann-Petty problem for arbitrary measures; a discrete analog of those estimates was very recently obtained by Alexander, Zvavitch, Henk [2].

We refer the reader to the books by Koldobsky [26] and Koldobsky, Yaskin [29] for a deep, yet accessible study of the Fourier-analytic approach to the Busemann-Petty and Shephard problems, as well as a general introduction to Fourier analysis in Convex geometry.

Aleksandrov in [1] proved that any symmetric convex body in $\mathbb{R}^n$ is determined uniquely by the $(n-1)$-dimensional volumes of its projections. See Zhang [55] for the discrete version of that statement under natural assumptions. In Section 5 we generalize Aleksandrov's theorem to measures with positive degree of concavity and positive degree of homogeneity.

First, we find a natural analogue of the Lebesgue measure of projection of a convex body to other measures.

Definition 1.3. Let μ be a measure on $\mathbb{R}^n$ with continuous density g, and let K be a convex body. Consider a unit vector $\theta \in \mathbb{S}^{n-1}$. Define the following function on the cylinder $\mathbb{S}^{n-1} \times [0,1]$:

$$p_{\mu,K}(\theta,t) := \frac{n}{2} \int\limits_{\mathbb{S}^{n-1}} |\langle \theta, u\rangle| d\sigma_{\mu,tK}(u). \tag{1}$$

We also consider $\mu - projection$ function on the unit sphere:

$$P_{\mu,K}(\theta) := \int\limits_0^1 p_{\mu,K}(\theta,t)dt. \tag{2}$$

In the particular case of Lebesgue measure λ we have

$$P_{\lambda,K}(\theta) = |K|\theta^{\perp}|_{n-1}.$$

The Definition 1.3 is natural since it is a generalization of Cauchy's projection formula (see below (10)). For even g, the notion of $p_{\mu,K}(\theta,t)$ can be understood geometrically as the projected weight of the boundary of tK, $t \in [0,1]$. More specifically, we define a measure μ_{tK} on $\theta^{\perp}$ to be the marginal measure of $1_{\partial(tK)}(x)g(x)dx$. In other words, for a measurable set $\Omega \subset \theta^{\perp}$, let $\mu_{tK}(\Omega) = \int_{\Omega} g(\pi^{-1}_{tK}(w))dw$, where $\pi^{-1}_{tK}(w)$ is the full pre-image of w under the projection of tK onto $\theta^{\perp}$. Then

$$p_{\mu,K}(\theta,t) = \mu_{tK}(tK|\theta^{\perp}) = \mu_{tK}(K|\theta^{\perp}),$$

where the last equality holds since $tK \subset K$. Hence,

$$P_{\mu,K}(\theta)=\int_0^1 \mu_{tK}(K|\theta^{\perp})dt.$$

We prove the following result.

Theorem 1.4. *Fix $n\geq 1$; let μ on $\mathbb{R}^n$ be a measure and $g(x)$ be its even r-homogenous density for some $r\geq 0$, and the restriction of g to some half space is p-concave for a $p\geq 0$.*

Let K and L be symmetric convex bodies, and let L additionally be a projection body. Assume that for every $\theta\in\mathbb{S}^{n-1}$ we have

$$P_{\mu,K}(\theta)\leq P_{\mu,L}(\theta).$$

Then $\mu(K)\leq\mu(L)$.

To compliment Theorem 1.4 we prove the following.

Theorem 1.5. *Fix $n\geq 1$; let μ on $\mathbb{R}^n$ be a measure and $g(x)$ be its even r-homogenous density for some $r\geq 0$, and the restriction of g to some half space is p-concave for a $p\geq 0$. Assume further that the closure of the support of μ is the whole space.*

Let L be a symmetric convex body which is not a projection body. Then there exists a symmetric convex body K such that for every $\theta\in\mathbb{S}^{n-1}$ we have

$$P_{\mu,K}(\theta)\leq P_{\mu,L}(\theta),$$

but $\mu(K)>\mu(L)$.

We remark that in the case of Lebesgue measure Theorems 1.4 and 1.5 are generalizations of the well-known solution to the classical Shephard problem (see Koldobsky [26], Chapter 8).

This paper is organized as follows. In Section 2 we present the preliminaries on the subject. In Section 3 we introduce and study the notion of mixed measure and prove an analogue of Minkowski's first inequality for measures. In Section 4 we prove Theorem 1.2. In Section 5 we prove two types of uniqueness results: one is the extension of Aleksandrov's theorem, and the other is related to the uniqueness of the solution of certain PDE in the class of support functions. In Section 6 we prove Theorems 1.4 and 1.5. In Section 7 we discuss stability and separation results for Theorem 1.4, and their corollaries.

Acknowledgment. The author would like to thank Alex Koldobsky, Artem Zvavitch, Liran Rotem and Benjamin Jaye for very fruitful discussions and encouragement.

2. Preliminaries

2.1. Brunn-Minkowski theory

Below we present classical concepts and results of Convex geometry and Brunn-Minkowski theory. We refer the reader to books by Ball [4], Milman, Schechtman [46], Schneider [51] for a detailed introduction to the subject.

Standard Minkowski's addition for sets $A, B \subset \mathbb{R}^n$ is defined as

$$A + B := \{a + b \,:\, a \in A, b \in B\}.$$

Scalar multiplication for $\alpha \in \mathbb{R}$ and a set $A \subset \mathbb{R}^n$ is defined as

$$\alpha A := \{\alpha a \,:\, a \in A\}.$$

For Borel sets A, B in $\mathbb{R}^n$ and for arbitrary $\lambda \in [0,1]$, Brunn-Minkowski inequality states that

$$|\lambda A + (1-\lambda)B|^{\frac{1}{n}} \geq \lambda |A|^{\frac{1}{n}} + (1-\lambda)|B|^{\frac{1}{n}}.$$

See Gardner [15] for an exhaustive survey on the subject. We remark that for convex bodies the equality in the Brunn-Minkowski inequality is attained if and only if the sets A and B are closed, convex dilates of each other.

First mixed volume of convex bodies K and L in $\mathbb{R}^n$ is defined as follows:

$$V_1(K,L) := \frac{1}{n} \liminf_{\epsilon \to 0} \frac{|K + \epsilon L| - |K|}{\epsilon}.$$

Note that for any convex body K one has

$$V_1(K,K) = |K|. \tag{3}$$

Brunn-Minkowski inequality implies Minkowski's first inequality:

$$V_1(K,L) \geq |K|^{\frac{n-1}{n}} |L|^{\frac{1}{n}}. \tag{4}$$

There is equality in Minkowski's first inequality if and only if K and L are closed convex dilates of each other (see Schneider [51] for more details).

A particular case of mixed volume, is the surface area of a convex set K in $\mathbb{R}^n$:

$$|\partial K|^+ := nV_1(K, B_2^n) = \liminf_{\epsilon \to 0} \frac{|K + \epsilon B_2^n| - |K|}{\epsilon}.$$

Therefore, (4) implies classical isoperimetric inequality:

$$\frac{|\partial K|^+}{|K|^{\frac{n-1}{n}}} \geq \frac{|\partial B_2^n|^+}{|B_2^n|^{\frac{n-1}{n}}}.$$

Next, we shall discuss Brunn-Minkowski inequality for p-concave measures (see Gardner [15] for more details). For $p \in \mathbb{R}$ and for $a, b \geq 0$, $\lambda \in [0,1]$ we define a p-average as follows:

$$M_p(a, b, \lambda) = (\lambda a^p + (1-\lambda) b^p)^{\frac{1}{p}}. \tag{5}$$

In the special cases $p = 0$, $p = +\infty$ and $p = -\infty$ we have

$$M_0(a, b, \lambda) = a^\lambda b^{1-\lambda},$$

$$M_{-\infty}(a, b, \lambda) = \min(a, b),$$

$$M_{+\infty}(a, b, \lambda) = \max(a, b).$$

We say that a function $g : \mathbb{R}^n \to \mathbb{R}^+$ is p-concave if for every $x, y \in \mathbb{R}^n$ such that $g(x)g(y) > 0$, and for every $\lambda \in [0,1]$ one has

$$g(\lambda x + (1-\lambda)y) \geq M_p(g(x), g(y), \lambda).$$

We remark that 0-concave functions are also called log-concave.

The following generalized Brunn-Minkowski inequality is well known (see e.g. Borell [6], Gardner [15]). Let $p \in [-\frac{1}{n}, +\infty]$, and let μ be a measure on $\mathbb{R}^n$ with p-concave density g. Let

$$q = \frac{p}{np+1}.$$

Then the measure μ is q-concave on $\mathbb{R}^n$. That is, for every pair of Borel sets A and B and for every $\lambda \in [0,1]$ one has

$$\mu(\lambda A + (1-\lambda)B) \geq M_q(\mu(A), \mu(B), \lambda). \tag{6}$$

2.2. The surface area measure, its properties and applications

Support hyperplane of a convex body K at a point $y \in \partial K$ is a hyperplane which contains y and does not contain any of the interior points of K. By convexity, such hyperplane exists at every point $y \in \partial K$, and is unique almost everywhere with respect to the $(n-1)$-dimensional Hausdorff measure on ∂K. The vector orthogonal to a support hyperplane at $y \in \partial K$ is called *normal vector* at y; if such vector is unique it shall be denoted n_y. The Gauss map $\nu_K : \partial K \to \mathbb{S}^{n-1}$ corresponds $y \in \partial K$ to the set of its normal vectors.

The push forward of the $(n-1)$-dimensional Hausdorff measure on ∂K under the Gauss map ν_K to the sphere is called surface area measure of K and is denoted by σ_K. In particular, $|\partial K|^+$ (the surface area of K) can be found as

$$|\partial K|^+ = \int\limits_{\mathbb{S}^{n-1}} d\sigma_K(u).$$

A class of strictly convex bodies whose support function is twice continuously differentiable we shall denote by $C^{2,+}$ (strict convexity means that the interior of every interval connecting a pair of points in the body is fully contained in the interior of the body). For such bodies, the Gauss map is a bijection, and the surface area measure σ_K has a continuous density $f_K(u)$, which is called curvature function of K.

One can see via approximation by polytopes, that

$$\int\limits_{\mathbb{S}^{n-1}} u d\sigma_K(u) = 0.$$

Conversely, the following Minkowski's existence Theorem holds (see e.g. Schneider [51] or Koldobsky [26]).

Theorem 2.1 *(Minkowski). Let μ be a measure on the sphere, not supported on any subspace, and such that*

$$\int\limits_{\mathbb{S}^{n-1}} u d\mu(u) = 0.$$

Then there exists a unique convex body K so that $d\sigma_K(u) = d\mu(u)$ for all $u \in \mathbb{S}^{n-1}$.

We refer the reader to Schneider [51] for an accessible proof of Minkowski's existence theorem, and to Pogorelov [48] for a detailed survey on the differential geometric approach to Minkowski's existence theorem, its strengthening and related results.

The support function h_K of a convex body K, containing the origin, is defined on $\mathbb{R}^n$ via

$$h_K(x) = \max_{y \in K} \langle x, y \rangle.$$

Geometrically, for a unit vector θ, the value of $h_K(\theta)$ represents distance to the support hyperplane of K in the direction θ. Due to the fact that h_K is 1-homogenous, one has

$$\langle \nabla h_K(u), u \rangle = h_K(u), \tag{7}$$

for every $u \in \mathbb{S}^{n-1}$, provided that $\nabla h_K(u)$ is well-defined. In this case, $\nabla h_K(n_y) = y$ for all $y \in \partial K$.

We state a formula for a volume of a convex body K with surface area measure σ_K:

$$|K| = \frac{1}{n}\int\limits_{\mathbb{S}^{n-1}} h_K(u)d\sigma_K(u). \tag{8}$$

The validity of this formula can be seen in the case when K is a polytope and the general case follows by approximation. Moreover, for arbitrary convex bodies K and L one has the following:

$$V_1(K,L) = \frac{1}{n}\int\limits_{\mathbb{S}^{n-1}} h_L(u)d\sigma_K(u). \tag{9}$$

Another formula involving surface area measure is the so called Cauchy projection formula:

$$|K|\theta^{\perp}|_{n-1} = \frac{1}{2}\int\limits_{\mathbb{S}^{n-1}} |\langle u,\theta\rangle| d\sigma_K(u), \tag{10}$$

where θ is an arbitrary unit vector, and K is a convex symmetric body. The validity of (10), once again, can be seen for polytopes and it follows by approximation for arbitrary convex bodies. See Koldobsky [26] for more details about (8), (9) and (10).

2.3. Fourier transform on $\mathbb{S}^{n-1}$ and its applications to Convex geometry

Fourier transform in Convexity plays a very important role. See books by Koldobsky [26], Koldobsky, Yaskin [29], and a survey by Koldobsky, Ryabogin, Zvavitch [32] for a detailed introduction to the subject.

The Schwartz class $\mathbf{S}$ is the space of complex valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. Every locally integrable real valued function f on $\mathbb{R}^n$ with power growth at infinity represents a distribution acting by integration:

$$\langle f,\varphi\rangle = \int\limits_{\mathbb{R}^n} f(x)\varphi(x)dx,$$

for $\varphi \in \mathbf{S}$.

The Fourier transform $\widehat{f}$ of a distribution f is defined by

$$\langle \widehat{f},\widehat{\varphi}\rangle = (2\pi)^n\langle f,\varphi\rangle,$$

for every test function $\varphi \in \mathbf{S}$.

Let μ be a finite Borel measure on the unit sphere $\mathbb{S}^{n-1}$. Let μ_e be a $-(n+1)$-homogenous extension of μ to $\mathbb{R}^n$. μ_e is called the extended measure of μ if for every $\varphi \in \mathbf{S}$,

$$\langle \mu_e, \varphi \rangle = \frac{1}{2} \int\limits_{\mathbb{S}^{n-1}} \langle r^{-2}, \varphi(ru) \rangle d\mu(u).$$

The following geometric representation of Fourier transform on the sphere was proved by Koldobsky, Ryabogin, Zvavitch [31] (see also Koldobsky [26]):

$$\widehat{\mu_e}(\theta) = -\frac{\pi}{2} \int\limits_{\mathbb{S}^{n-1}} |\langle u, \theta \rangle| d\mu(u), \tag{11}$$

for every $\theta \in \mathbb{S}^{n-1}$.

Note that (10) and (11) imply that

$$d\widehat{\sigma_K}(\theta) = -\pi |K|\theta^{\perp}| d\theta, \tag{12}$$

where σ_K is the surface area measure of a symmetric convex body K, extended to $\mathbb{R}^n$ with degree of homogeneity $-(n+1)$.

The following Parseval-type identity was proved by Koldobsky, Ryabogin, Zvavitch [31] (see also Koldobsky [25], [26]): for symmetric convex bodies K, L, so that the support function of K is infinitely smooth,

$$\int\limits_{\mathbb{S}^{n-1}} \widehat{h_K}(\theta) \widehat{f_L}(\theta) = (2\pi)^n \int\limits_{\mathbb{S}^{n-1}} h_K(\theta) f_L(\theta), \tag{13}$$

where the Fourier transform of h_K is considered with respect to its 1-homogenous extension, and the Fourier transform of f_L is considered with respect to its $-(n+1)$-homogenous extension.

By Minkowski's existence Theorem, for every symmetric convex body L and for every even density g, not supported on a great subsphere, there exists a symmetric convex body $\tilde{L}$ such that

$$\sigma_{\mu,L} = \sigma_{\tilde{L}}.$$

Therefore, for all infinitely smooth symmetric convex bodies K, L in $\mathbb{R}^n$, and for every even continuous density g, one has

$$\int\limits_{\mathbb{S}^{n-1}} \widehat{h_K}(\theta) d\widehat{\sigma_{\mu,L}}(\theta) = (2\pi)^n \int\limits_{\mathbb{S}^{n-1}} h_K(\theta) d\sigma_{\mu,L}(\theta), \tag{14}$$

where the Fourier transform of h_K is considered with respect to its 1-homogenous extension, and the Fourier transform of $\sigma_{\mu,L}$ is considered with respect to its $-(n+1)$-homogenous extension.

Another observation is that (11) implies:

$$d\widehat{\sigma_{\mu,L}}(\theta) = -\frac{\pi}{2} \int\limits_{\mathbb{S}^{n-1}} |\langle u, \theta\rangle| d\sigma_{\mu,L}(u), \tag{15}$$

where the Fourier transform of $\sigma_{\mu,L}$ is considered with respect to its $-(n+1)$-homogenous extension.

In particular, considering tL in place of L we get

$$\widehat{\sigma_{\mu,tL}}(\theta) = -\frac{\pi}{n} p_{\mu,L}(\theta, t), \tag{16}$$

and

$$\widehat{\int\limits_0^1 \sigma_{\mu,tL}(\theta) dt} = -\frac{\pi}{n} P_{\mu,L}(\theta). \tag{17}$$

Remark 2.2. The degree of homogeneity with which a function on the sphere is extended to $\mathbb{R}^n$ impacts radically its Fourier transform, and, in particular, the restriction of its Fourier transform back to the unit sphere (see more in Goodey, Yaskin, Yaskina [19]). We would like to emphasize the fact that the homogeneity properties of the measure μ on $\mathbb{R}^n$ are completely irrelevant to the study of Fourier transforms of h_K and $\sigma_{\mu,K}$. In fact, we always extend h_K and $\sigma_{\mu,K}$ in the most convenient way, after having already translated all the information about the underlying measure μ onto the sphere. The proof of Theorem 1.4, much like the classical Shephard's problem (see [31]), consists of combining Fourier analysis and Brunn-Minkowski theory; the part which involves Fourier transform works for arbitrary measures, while the Brunn-Minkowski part is what reinforces the assumptions of concavity and homogeneity on the density of μ.

2.4. Projection bodies

Let K be an origin symmetric convex body in $\mathbb{R}^n$ with curvature function f_K. The projection body ΠK of K is defined as the origin symmetric convex body in $\mathbb{R}^n$ whose support function in every direction is equal to the volume of the orthogonal projection of K in this direction. We extend $h_{\Pi K}$ to a homogeneous function of degree 1 on $\mathbb{R}^n$. By (12),

$$h_{\Pi K}(\theta) = -\frac{1}{\pi} \widehat{f_K}(\theta).$$

The curvature function of a convex body is non-negative. Therefore, $\widehat{h_{\Pi K}} \leq 0$. On the other hand, by Minkowski's existence theorem, an origin symmetric convex body K in

$\mathbb{R}^n$ is the projection body of some origin symmetric convex body if and only if there exists a measure μ on $\mathbb{S}^{n-1}$ so that

$$\widehat{h_K} = -\mu_e.$$

The condition that L is a projection body is equivalent to L being a centered zonoid (see Gardner [16]). Zonoids are characterized as polar bodies of unit balls of finite dimensional sections of L_1.

Every origin symmetric convex body on the plane is a projection body (see Herz [20], Ferguson [14], Lindenstrauss [33]). It was proved by Koldobsky [23] that p-balls in $\mathbb{R}^n$ for $n \geq 3$ and $p \in [1, 2]$ are not projection bodies.

3. Mixed measures and related results

3.1. Mixed measures

As an analogue of the classical mixed volume consider the following notion.

Definition 3.1. Given sets K and L, we define their **mixed μ-measure** as follows.

$$\mu_1(K, L) = \liminf_{\epsilon \to 0} \frac{\mu(K + \epsilon L) - \mu(K)}{\epsilon}.$$

We observe that in the absence of homogeneity of μ, the mixed measure $\mu_1(K, L)$ is not homogenous in the first argument. However, it is necessarily homogenous in the second argument:

$$\mu_1(K, sL) = s\mu_1(K, L).$$

If, additionally, the measure μ is α-homogenous, i.e.

$$\mu(tA) = t^\alpha \mu(A)$$

for all $t \in \mathbb{R}^+$ and Borel sets A, then

$$\mu_1(tK, L) = t^{\alpha-1}\mu_1(K, L).$$

Definition 3.2. We also introduce the following analogue of mixed volume:

$$V_{\mu,1}(K, L) = \int_0^1 \mu_1(tK, L)dt.$$

Note that in the case of the Lebesgue measure λ we have

$$V_{\lambda,1}(K,L)=V_1(K,L).$$

Definition 3.1 implies that for $t\in(0,\infty)$,

$$\mu_1(tK,K)=\mu(tK)'_t; \tag{18}$$

this derivative exists by monotonicity. Therefore,

$$V_{\mu,1}(K,K)=\int\limits_0^1 \mu_1(tK,K)dt=\int\limits_0^1 \mu(tK)'dt=\mu(tK)|_0^1=\mu(K). \tag{19}$$

Recall that we use the notation $\sigma_{\mu,K}$ for a surface area measure of a convex body K with respect to a measure μ on $\mathbb{R}^n$. That is, for a Borel set $A\subset\mathbb{S}^{n-1}$,

$$\sigma_{\mu,K}(A)=\int\limits_{\nu_K^{-1}(A)} g(x)dH_{n-1}(x),$$

where $dH_{n-1}(x)$ stands for the $(n-1)$-dimensional Hausdorff measure on ∂K. Following the idea from the appendix of [34], we prove the following representation for $\mu_1(K,L)$.

Lemma 3.3. *Given convex bodies K and L containing the origin, and a measure μ with continuous density g on $\mathbb{R}^n$, we have*

$$\mu_1(K,L)=\int\limits_{\mathbb{S}^{n-1}} h_L(u)d\sigma_{\mu,K}(u).$$

Here h_K and h_L are support functions of K and L and $\sigma_{\mu,K}$ is the surface area measure of K.

The proof is outlined in the Appendix (see Lemma A.1).

In order to provide some intuition about $\sigma_{\mu,K}$, we describe it explicitly in a couple of partial cases.

Proposition 3.4. *If a body K is C^2-smooth and strictly convex then its surface area measure has representation*

$$d\sigma_{\mu,K}(u)=f_K(u)g(\nabla h_K(u))du.$$

Proposition 3.5. *The surface area measure of a convex polytope P with respect to a measure μ has representation*

$$d\sigma_{\mu,P}(u) = \sum_{i=1}^{N} \delta_{u_i} \mu_{n-1}(F_i) du,$$

where u_i, $i = 1, ..., N$ are the normals to the faces of the polytope, F_i are the corresponding faces, and $\mu_{n-1}(F_i)$ stands for $\int_{F_i} g(x)dx$.

See the Appendix for the proofs of Propositions 3.4 (Proposition A.5) and 3.5 (Proposition A.4).

We remark that Lemma 3.3, Proposition 3.4, along with (16) and (13) imply for all symmetric convex infinitely smooth bodies K and L:

$$\begin{aligned} \mu_1(tK, L) &= (2\pi)^{-n} \int_{\mathbb{S}^{n-1}} \widehat{h_L}(u) \widehat{d\sigma_{\mu,tK}}(u) du \\ &= -\frac{\pi}{n}(2\pi)^{-n} \int_{\mathbb{S}^{n-1}} \widehat{h_L}(u) p_{\mu,K}(t,u) du. \end{aligned} \tag{20}$$

As an immediate corollary of Lemma 3.3 and (19) we derive the following expression of the measure of a $C^{2,+}$ convex body (see also [11]).

Lemma 3.6. *Let μ be a measure with continuous density g. Let K be a $C^{2,+}$ convex body with support function h_K and curvature function f_K. Then*

$$\mu(K) = \int_{\mathbb{S}^{n-1}} h_K(u) f_K(u) \int_0^1 t^{n-1} g\left(t \nabla h_K(u)\right) dt du. \tag{21}$$

We outline that if the density of a measure μ on $\mathbb{R}^n$ is r-homogenous, then

$$\mu(K) = \int_0^1 \mu_1(tK, K) dt = \mu_1(K, K) \int_0^1 t^{n+r-1} dt = \frac{1}{n+r} \mu_1(K, K). \tag{22}$$

In view of (22), Lemma 3.3 and Proposition 3.5 imply the following.

Proposition 3.7. *Let μ be a measure with r-homogenous density $g(x)$ on $\mathbb{R}^n$, and consider a polytope with N faces:*

$$P = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq \alpha_i\},$$

where $u_i \in \mathbb{S}^{n-1}$ and $\alpha_i > 0$, $i = 1, ..., N$. Let F_i be faces of P orthogonal to u_i, $i = 1, ..., N$. Then

$$\mu(P) = \frac{1}{n+r} \sum_{i=1}^{N} \alpha_i \mu_{n-1}(F_i),$$

where $\mu_{n-1}(F_i)$ *stands for* $\int_{F_i} g(x)dx$.

3.2. Minkowski's first inequality generalized

The main result of this subsection is the following theorem.

Theorem 3.8. *Let* μ *on* $\mathbb{R}^n$ *be an absolutely continuous measure. Assume that* μ *is* $F(t)$*-concave, i.e. there exists a differentiable invertible function* $F : \mathbb{R}^+ \to \mathbb{R}$ *such that for every* $\lambda \in [0,1]$ *and for every pair of Borel sets* K *and* L *in a certain class, we have*

$$\mu(\lambda K + (1-\lambda)L) \geq F^{-1}\left(\lambda F(\mu(K)) + (1-\lambda)F(\mu(L))\right). \tag{23}$$

Then the following holds:

$$\mu_1(K,L) \geq \mu_1(K,K) + \frac{F(\mu(L)) - F(\mu(K))}{F'(\mu(K))}, \tag{24}$$

for all K, L *in that class.*

Proof. We write

$$\begin{aligned}\mu(K+\epsilon L) &= \mu\left((1-\epsilon)\frac{K}{1-\epsilon} + \epsilon L\right)\\ &\geq F^{-1}\left((1-\epsilon)F\left(\mu(\frac{K}{1-\epsilon})\right) + \epsilon F\left(\mu(L)\right)\right) =: G_{K,L,\mu,F}(\epsilon).\end{aligned}$$

The function $G_{K,L,\mu,F}$ is differentiable since μ is absolutely continuous and F is differentiable; in the case $F'(t) = 0$ the expression from (24) is understood as a limit. Note that $G_{K,L,\mu,F}(0) = \mu(K)$. Therefore,

$$\mu_1(K,L) \geq G'_{K,L,\mu,F}(0).$$

We note that

$$\mu\left(\frac{K}{1-\epsilon}\right)'|_{\epsilon=0} = \mu_1(K,K).$$

Using the above along with standard rules of differentiation, such as

$$(F^{-1}(a))' = \frac{1}{F'(F^{-1}(a))},$$

we get the statement of the Theorem. $\square$

A standard argument implies that the equality cases of the inequality (24) coincide with equality cases of (23). We shall formulate a few corollaries of Theorem 3.8 in some special cases.

Corollary 3.9. *Let $p \geq 0$. Let $g : \mathbb{R}^n \to \mathbb{R}^+$ be a p-concave density of measure μ. Let $q = \frac{1}{n+\frac{1}{p}}$. Then for every pair of Borel sets K and L we have*

$$\mu_1(K,L) \geq \mu_1(K,K) + \frac{\mu(L)^q - \mu(K)^q}{q\mu(K)^{q-1}}.$$

The Corollary 3.9 follows from Theorem 3.8 via considering $F(t) = t^q$. We also obtain the following nicer-looking corollary for measures with p-concave and $\frac{1}{p}$-homogenous densities. It was originally proved by E. Milman and L. Rotem [45].

Corollary 3.10 *(E. Milman, L. Rotem). Let $p \geq 0$. Let $g : \mathbb{R}^n \to \mathbb{R}^+$ be a p-concave $\frac{1}{p}$-homogenous density of measure μ. Let $q = \frac{1}{n+\frac{1}{p}}$. Then for every pair of Borel sets K and L we have*

$$\mu_1(K,L) \geq \frac{1}{q}\mu(K)^{1-q}\mu(L)^q, \tag{25}$$

and

$$V_{\mu,1}(K,L) \geq \mu(K)^{1-q}\mu(L)^q. \tag{26}$$

Proof. Note that if g is $\frac{1}{p}$-homogenous then μ is an $(n+\frac{1}{p}) = \frac{1}{q}$-homogenous measure. Therefore,

$$V_{\mu,1}(K,L) = \int_0^1 \mu_1(tK,L)dt = \mu_1(K,L)\int_0^1 t^{\frac{1}{q}-1}dt = q\mu_1(K,L), \tag{27}$$

and in particular

$$\mu(K) = q\mu_1(K,K) \tag{28}$$

Corollary 3.9 together with (28) implies (25). Also, (25) together with (27) implies (26). $\square$

Recall that a measure μ is called log-concave if for all Borel sets K and L,

$$\mu(\lambda K + (1-\lambda)L) \geq \mu(K)^\lambda \mu(L)^{1-\lambda}.$$

Applying Theorem 3.8 with $F(t) = \log t$ (as $\log t$ is an increasing function), we get the following corollary.

Corollary 3.11. *Let measure μ be log-concave. Then for every pair of Borel sets K and L we have*

$$\mu_1(K,L) \geq \mu_1(K,K) + \mu(K)\log\frac{\mu(L)}{\mu(K)}.$$

In particular, the following isoperimetric-type result follows from Theorem 3.8.

Proposition 3.12. *Let a measure μ be log-concave. Then for every pair of Borel sets K and L such that $\mu(K) = \mu(L)$, one has*

$$\mu_1(K,L) \geq \mu_1(K,K).$$

For example, if γ is the standard Gaussian measure γ (that is, the measure with density $\frac{1}{\sqrt{2\pi}^n} e^{-\frac{|x|^2}{2}}$), and K is a convex set containing the origin, then the expression

$$\int_{\partial K} \langle y, \nu_L(y)\rangle e^{-\frac{|y|^2}{2}} d\sigma(y)$$

is minimized when $L = K$, where L is such convex region that $\gamma(K) = \gamma(L)$, and ν_L is it Gauss map.

Another strengthening of Corollary 3.11 in the case of the standard Gaussian measure is possible to obtain using Ehrhard's inequality (see Ehrhard [13], Borell [7]). Recall the notation

$$\Psi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{t^2}{2}} dt.$$

It was shown by Ehrhard (for convex sets), and further extended by Borell, that for every pair of Borel sets K and L and for every $\lambda \in [0,1]$ we have

$$\Psi^{-1}\left(\gamma(\lambda K + (1-\lambda)L)\right) \geq \lambda \Psi^{-1}(\gamma(K)) + (1-\lambda)\Psi^{-1}(\gamma(L)).$$

Hence the next Corollary follows.

Corollary 3.13. *For the standard Gaussian measure γ and for every pair of convex sets K and L we have*

$$\gamma_1(K,L) \geq \gamma_1(K,K) + \sqrt{2\pi} e^{-\frac{\Psi^{-1}(\gamma(K))^2}{2}} \left(\Psi^{-1}(\gamma(L)) - \Psi^{-1}(\gamma(K))\right).$$

To obtain this corollary we use the fact that Ψ is an increasing function and the relation

$$\Psi^{-1}(a)' = \sqrt{2\pi} e^{\frac{\Psi^{-1}(a)^2}{2}}.$$

4. Extension of the Minkowski's existence theorem

This section is dedicated to proving an extension of Minkowski's existence theorem. We use ideas from the proof of the classical Minkowski's existence theorem (see Schneider [51]).

First, we state a definition.

Definition 4.1. For a measure μ on $\mathbb{R}^n$, we say that a convex body K in $\mathbb{R}^n$ with particular properties is **μ-unique** if every pair of convex bodies with said properties coincides up to a set of μ-measure zero.

Theorem 4.2. *Let μ on $\mathbb{R}^n$ be a measure and $g(x)$ be its even r-homogenous density for some $r \geq 0$, such that a restriction of g on some half space is p-concave for $p \geq 0$. Let φ be an even measure on $\mathbb{S}^{n-1}$, not supported on any great subsphere, such that supp(φ) is spherically convex and supp$(\varphi) \subset$ int(supp$(g)) \cap \mathbb{S}^{n-1}$. Then there exists a μ-unique convex body K in $\mathbb{R}^n$ such that*

$$d\sigma_{K,\mu}(u) = d\varphi(u).$$

The existence part of Theorem 4.2 follows by approximation from the lemma below. We remark, that for an $(n-1)$-dimensional surface F, the notation $\mu_{n-1}(F)$ stands for

$$\mu_{n-1}(F) = \int_F g(x)dx,$$

where $g(x)$ is the density of μ, and dx is the area element on F.

Lemma 4.3. *Let μ on $\mathbb{R}^n$ be a measure and $g(x)$ be its even r-homogenous density for some $r > -n$. Let $N \geq 2n$ be an even integer. Let $u_1, ..., u_N$ be unit vectors spanning the $\mathbb{R}^n$, $u_i \in$ int(supp(g)), such that $u_i = -u_{\frac{N}{2}+i}$. Let $f_1, ..., f_N$ be arbitrary positive numbers such that $f_i = f_{\frac{N}{2}+i}$.*

Then there exist positive $\alpha_1, ..., \alpha_N$ such that the convex polytope

$$P = \cap_{i=1}^N \{|\langle x, u_i \rangle| \leq \alpha_i\}$$

with faces $F(u_1), ..., F(u_N)$ satisfies

$$\mu_{n-1}(F(u_i)) = f_i.$$

Moreover, if restriction of g on a half space is p-concave for $p \geq 0$ then such polytope P is μ-unique.

Proof. For a vector $A = (\alpha_1, ..., \alpha_N) \in \mathbb{R}^N$ we shall consider a polytope

$$P(A) = \cap_{i=1}^N \{x \in \mathbb{R}^n \,:\, |\langle x, u_i\rangle| \leq \alpha_i\}.$$

Consider a set $M \subset \mathbb{R}^N$ defined as follows:

$$M := \{A \in \mathbb{R}^N \,:\, \mu(P(A)) \geq 1\}.$$

Note that $M \subset \{A \,:\, \alpha_i \geq 0 \; \forall i = 1, ..., N\}$. It is nonempty since the measure is unbounded. As the set M is closed, and $f_i > 0$, the linear functional

$$\varphi(A) = \frac{1}{n+r} \sum_{i=1}^N f_i \alpha_i$$

attains its minimum on M. Let $A^* = (\alpha_1^*, ..., \alpha_N^*)$ be the minimizing point, $P^* = P(A^*)$, and let F_i^* stand for the facet of P^* orthogonal to u_i. Denote the value of the minimum $\varphi(A^*) = m^{n+r-1}$.

We show that mP^* is the polytope which solves the problem. Indeed, consider hyperplanes

$$H_1 = \{A \in \mathbb{R}^N \,:\, \frac{1}{n+r} \sum_{i=1}^N f_i \alpha_i = m^{n+r-1}\},$$

$$H_2 = \{A \in \mathbb{R}^N \,:\, \frac{1}{n+r} \sum_{i=1}^N \mu_{n-1}(F_i^*) \alpha_i = 1\}.$$

Note that all $\alpha_i^* > 0$. Thus, by Proposition 3.7,

$$\mu(P^*) = \frac{1}{n+r} \sum_{i=1}^N \alpha_i^* \mu_{n-1}(F_i^*).$$

On the other hand, the linear functional φ attains its minimum on the boundary of M, and hence

$$\mu(P^*) = 1. \tag{29}$$

We conclude that $A^* \in H_1 \cap H_2$.

Observe that $H_1 \cap int(M) = \varnothing$, as otherwise A^* would not be the minimum. Consider a vector $A \in H_1$ different from A^*. For any $\lambda \in [0, 1]$, the vector $\lambda A^* + (1 - \lambda)A \in H_1$, and hence

$$\mu(P(\lambda A^* + (1 - \lambda)A)) \leq 1.$$

Note also that

$$\lambda P(A^*) + (1-\lambda)P(A) \subset P(\lambda A^* + (1-\lambda)A),$$

and thus

$$\mu(\lambda P^* + (1-\lambda)P(A)) \leq 1. \tag{30}$$

Therefore, by homogeneity of μ, (29) and (30),

$$\begin{aligned}
\mu_1(P^*, P(A)) &= \liminf_{\epsilon\to 0} \frac{\mu(P^* + \epsilon P(A)) - \mu(P^*)}{\epsilon} \\
&= \liminf_{\epsilon\to 0} \frac{(1+\epsilon)^{n+r}\mu(\frac{1}{1+\epsilon}P^* + \frac{\epsilon}{1+\epsilon}P(A)) - 1}{\epsilon} \leq \liminf_{\epsilon\to 0} \frac{(1+\epsilon)^{n+r} - 1}{\epsilon} \\
&= n + r.
\end{aligned}$$

On the other hand, if $\alpha_i > 0$, by Proposition 3.5 and Lemma 3.3 we have

$$\mu_1(P^*, P(A)) = \sum_{i=1}^{N} \alpha_i \mu_{n-1}(F_i^*),$$

and hence

$$\frac{1}{n+r}\sum_{i=1}^{N} \alpha_i \mu_{n-1}(F_i^*) \leq 1. \tag{31}$$

Therefore, there exists an open subset of H_1,

$$U := H_1 \cap \{A \in \mathbb{R}^N \, : \, \alpha_i > 0\},$$

which is fully contained in the half space

$$H_2^- = \{A \in \mathbb{R}^N \, : \, \frac{1}{n+r}\sum_{i=1}^{N} \mu_{n-1}(F_i^*)\alpha_i \leq 1\},$$

and, in addition, the interior of U contains $A^* \in H_1 \cap H_2$. This implies that $H_1 = H_2$.

Therefore,

$$\mu_{n-1}(F_i^*)m^{n+r-1} = f_i.$$

Using homogeneity of g once again, we conclude that the polytope

$$mP^* = \cap_{i=1}^{N}\{x \in \mathbb{R}^n \, : \, \langle x, u_i\rangle \leq \beta_i\},$$

with $\beta_i = m\alpha_i^*$, satisfies the conclusion of the Lemma.

The uniqueness part follows in the same manner as in subsection 4.1 for all convex bodies, therefore we skip the argument here. $\square$

We remark that no concavity was necessary to prove the existence part for polytopes; however, it is used in the proof for uniqueness, and it is used in the approximation argument below.

4.1. Proof of the uniqueness part of Theorem 4.2

Proof. Let $\tilde{\mu}$ be measure with density $\tilde{g}(u) = g(u)1_{\{\langle u,v\rangle>0\}}$, for some unit vector v, such that $\tilde{g}$ is p-concave and $\frac{1}{p}$-homogenous on its support for some $p \geq 0$ (assumptions of the Theorem along with Proposition A.2 of the appendix allow us to select such vector). Fix $q = \frac{p}{np+1}$. Assume that there exist two symmetric convex bodies K and L such that

$$d\sigma_{\mu,K}(u) = d\sigma_{\mu,L}(u) \tag{32}$$

for all $u \in \mathbb{S}^{n-1}$. Observe that

$$\begin{aligned}\mu_1(K, L) &= \int\limits_{\mathbb{S}^{n-1}} h_K(u) d\sigma_{\mu,L}(u) \\ &= \int\limits_{\mathbb{S}^{n-1}} h_K(u) d\sigma_{\mu,K}(u) = \mu_1(K, K) = \frac{1}{q}\mu(K).\end{aligned}$$

By symmetry of K and L, it implies that

$$\tilde{\mu}_1(K, L) = \frac{1}{q}\tilde{\mu}(K).$$

By Corollary 3.10,

$$\frac{1}{q}\tilde{\mu}(K) = \tilde{\mu}_1(K, L) \geq \frac{1}{q}\tilde{\mu}(K)^{1-q}\tilde{\mu}(L)^q, \tag{33}$$

and hence $\tilde{\mu}(K) \geq \tilde{\mu}(L)$. Analogously, by considering $\tilde{\mu}_1(L, K)$, we get that $\tilde{\mu}(K) \leq \tilde{\mu}(L)$. Hence $\tilde{\mu}(K) = \tilde{\mu}(L)$, and hence there is equality in (33). Milman and Rotem ([45] Corollary 2.17) proved, using the results from Dubuc [12], that in this case K and L have to coincide up to a dilation and a shift on the support of $\tilde{\mu}$. As we assume that K and L are symmetric, we get that $K = aL$ for some $a > 0$ almost everywhere with respect to $\tilde{\mu}$. But as g is $\frac{1}{p}$-homogenous, we have

$$d\sigma_{\mu,K}(u) = d\sigma_{\mu,aL}(u) = a^{n+\frac{1}{p}-1} d\sigma_{\mu,L}(u),$$

and hence by (32), $a = 1$. Which means that $K = L$ μ-almost everywhere. $\square$

4.2. Proof of the existence part of Theorem 4.2

Proof. We shall use Lemma 4.3 and argue by approximation. Let $d\varphi(u)$ be an even measure on $\mathbb{S}^{n-1}$. For a positive integer k, consider a symmetric partition of $\mathbb{S}^{n-1} \cap supp(\varphi)$ into disjoint sets A_i, $i = 1, ..., 2N$ with spherically convex closures of diameters at most $\frac{1}{k}$ (recall that a subset of the sphere is called spherically convex if the geodesic interval connecting any pair of points in the set is fully contained in this set). This is possible to do since $\mathbb{S}^{n-1} \cap supp(\varphi)$ is spherically convex by the assumption. Consider the vector

$$c_i = \frac{1}{\varphi(A_i)} \int\limits_{A_i} u d\varphi(u).$$

Note that $c_i \neq 0$. Select $u_i \in \mathbb{S}^{n-1}$ and $f_i \in \mathbb{R}^+$ to be such that $c_i = f_i u_i$. Note that $u_i \in int(A_i)$. Therefore, for every $u \in A_i$, $|u - u_i| \leq \frac{1}{k}$, and hence

$$1 - \frac{1}{k} \leq f_i \leq 1. \tag{34}$$

According to Lemma 4.3, there exists a polytope

$$P_k = \{x \in \mathbb{R}^n : |\langle x, u_i \rangle| \leq \alpha_i\}$$

with faces F_{P_K}, such that

$$\mu_{n-1}(F_{P_K}(u_i)) = \int\limits_{A_i} \varphi(u) du.$$

Consider a measure φ_k on $\mathbb{S}^{n-1}$ such that for every Borel set $\Omega \subset \mathbb{S}^{n-1}$,

$$\varphi_k(\Omega) = \sum_{u_i \in \Omega} \mu_{n-1}(F_{P_K}(u_i)).$$

Consider a bounded Lipschitz function $a(u)$ on $\mathbb{S}^{n-1}$. Observe that

$$\left| \int\limits_{\mathbb{S}^{n-1}} a(u) d\varphi(u) - \int\limits_{\mathbb{S}^{n-1}} a(u) d\varphi_k(u) \right| \leq \sum \int\limits_{A_i} |a(u) - a(u_i) f_i| d\varphi(u).$$

Observe as well, that by (34),

$$\begin{aligned} |a(u) - f_i a(u_i)| &\leq |a(u_i) - f_i a(u_i)| + |a(u) - a(u_i)| \\ &\leq \frac{1}{k} ||a||_{Lip} + ||a||_\infty |1 - f_i| \leq \frac{1}{k}(||a||_{Lip} + ||a||_\infty) \to_{k \to \infty} 0. \end{aligned}$$

Thus $\varphi_k \to \varphi$ weakly, as k tends to infinity.

It remains to show that all the polytopes P_k are bounded on the support of μ: then, by Blaschke selection theorem (see [51], Theorem 1.8.6), applied on the support of μ, there exists a subsequence of $\{P_k\}$ which converges to some convex body P in Hausdorff metric. Then $\sigma_{\mu,P_k} \to \sigma_{\mu,P}$ weakly (see Proposition A.3 from the appendix), and hence, by the uniqueness of the weak limit, we have $d\sigma_{\mu,P}(u) = d\varphi(u)$.

To show the boundedness, observe first that $\mu^+(\partial P_k) = \int_{\mathbb{S}^{n-1}} \varphi(u)du =: \tilde{C}_\varphi$, where $\mu^+(\partial P_k)$ stands for $\mu_1(P_k, B_2^n)$.

Let $\tilde{g}$ be the restriction of g to a half space where it is *p-concave*. By Corollary 3.10,

$$\tilde{\mu}(P_k) \le \left(q\tilde{\mu}(B_2^n)^{-q}\tilde{\mu}^+(\partial P_k)\right)^{\frac{1}{1-q}},$$

and hence, by symmetry of P_k,

$$\mu(P_k) \le \left(q\mu(B_2^n)^{-q}\mu^+(\partial P_k)\right)^{\frac{1}{1-q}} \le C_{\mu,\varphi}. \tag{35}$$

Here $q = \frac{p}{np+1}$, and $C_{\mu,\varphi}$ depends only on the measures μ and φ. On the other hand, for any $x \in P_k$ we have

$$h_{P_k}(u) \ge \langle u, x\rangle^+ = |x|\langle u, v\rangle^+,$$

where $v \in \mathbb{S}^{n-1}$ is such that $x = |x|v$, and $\langle u, x\rangle^+$ stands for the positive part of $\langle u, x\rangle$. We note that for k large enough,

$$\int_{\mathbb{S}^{n-1}} \langle u, v\rangle^+ d\varphi_k(u) \ge \frac{1}{2}\int_{\mathbb{S}^{n-1}} \langle u, v\rangle^+ d\varphi(u) =: C_\varphi > 0,$$

where $C_\varphi > 0$ is a positive constant depending on φ only. Therefore,

$$\mu(P_k) = \frac{1}{n+r}\int_{\mathbb{S}^{n-1}} h_{P_K}(u)d\varphi_k(u) \ge |x|C_\varphi. \tag{36}$$

By (35) and (36), $|x| \le \frac{C_{\mu,\varphi}}{C_\varphi}$. As x was an arbitrary point from P_k, we conclude that the sequence $\{P_k\}$ is indeed uniformly bounded. □

5. Applications to the questions about uniqueness

5.1. An extension of Aleksandrov's theorem

Theorem 5.1. *Let μ be a measure with density with positive degree of concavity and positive degree of homogeneity. Let K and L be symmetric convex bodies such that in every direction θ, $P_{\mu,K}(\theta) = P_{\mu,L}(\theta)$. Then $K = L$ μ-almost everywhere.*

Proof. Given $g(x)$ on $\mathbb{R}^n$, the density of μ, let $\tilde{\mu}$ on $\mathbb{R}^n$ be the measure with density $\tilde{g}(x) = \frac{g(x)+g(-x)}{2}$. Recall that by (16),

$$d\widehat{\sigma_{\tilde{\mu},K}}(\theta) = -C(\mu)\frac{\pi}{n}P_{\tilde{\mu},K}(\theta)$$

and

$$d\widehat{\sigma_{\tilde{\mu},L}}(\theta) = -C(\mu)\frac{\pi}{n}P_{\tilde{\mu},L}(\theta),$$

where $C(\mu)$ depends only on the dimension and the degree of homogeneity of μ, and the Fourier transform is considered with respect to $-(n+1)$-homogenous extensions of $\sigma_{\tilde{\mu},K}$ and $\sigma_{\tilde{\mu},L}$.

Note that $P_{\mu,K}(\theta) = P_{\mu,L}(\theta)$ implies $P_{\tilde{\mu},K}(\theta) = P_{\tilde{\mu},L}(\theta)$ for every θ. By Fourier inversion formula, we get that $\sigma_{\tilde{\mu},K} = \sigma_{\tilde{\mu},L}$ everywhere on the sphere. By Theorem 1.2 we conclude that K and L coincide up to a set of μ-measure zero. $\square$

5.2. Uniqueness of solutions for certain PDE's in the class of support functions

Proposition 5.2. *Let K and L be two symmetric $C^{2,+}$ convex bodies in $\mathbb{R}^n$ with support functions h_K and h_L and curvature functions f_K and f_L such that*

$$\frac{\partial h_K(u)}{\partial x_1}f_K(u) = \frac{\partial h_L(u)}{\partial x_1}f_L(u)$$

for every $u \in \mathbb{S}^{n-1}$. Then $K = L$.

Proof. Let $g : \mathbb{R}^n \to \mathbb{R}^+$ be given via

$$g(x) = |x_1|.$$

Then, for every $x \in \mathbb{R}^n$,

$$g(\nabla h_K) = \left|\frac{\partial h_K(u)}{\partial x_1}\right|.$$

By the symmetry, the Proposition 3.4 and the condition of the Corollary,

$$\sigma_{\mu,K} = f_K(u)g(\nabla h_K(u)) = f_L(u)g(\nabla h_L(u)) = \sigma_{\mu,L} \tag{37}$$

for every $u \in \mathbb{S}^{n-1}$. Observe that the restriction of g onto $\{x \in \mathbb{R}^n : x_1 > 0\}$ is 1-homogenous and 1-concave. Therefore, it satisfies the condition of Theorem 4.2, and thus, by (37), $K = L$ μ-almost everywhere. In this case it means that $K = L$ coincide almost everywhere with respect to Lebesgue measure, and as they are also convex bodies, it means that $K = L$. $\square$

We remark that the curvature function f_K can be written in the Aleksandrov's form as $det(\delta_{ij}h+h_{ij})$, where h is the support function of K, h_{ij} are derivatives of it taken with respect to an orthonormal frame on $\mathbb{S}^{n-1}$, and δ_{ij} is the usual Kroneker symbol. Therefore, Proposition 5.2 implies that a PDE

$$\frac{\partial h}{\partial x_1}det(\delta_{ij}h+h_{ij})=F$$

has a unique solution in the class of even support functions of convex bodies. The existence of such solution for even continuous function F which is not supported on any great subsphere can be derived from Theorem 1.2.

Remark 5.3. Observe that

$$\frac{\partial(h_K(u)f_K(u))}{\partial x_1}=\frac{\partial h_K(u)}{\partial x_1}f_K(u)+\frac{\partial f_K(u)}{\partial x_1}h_K(u).$$

Hence, by Proposition 5.2, the following pair of conditions guarantee equality of smooth symmetric sets K and L:

(1) $h_K(u)f_K(u)=h_K(u)f_K(u)$ at every $u\in\mathbb{S}^{n-1}$;
(2) $\frac{\partial f_K(u)}{\partial x_1}h_K(u)=\frac{\partial f_L(u)}{\partial x_1}h_L(u)$ at every $u\in\mathbb{S}^{n-1}$.

Remark 5.4. Instead of requiring the condition of Proposition 5.2 it is in fact enough to require that there exists a vector v such that for every $u\in\mathbb{S}^{n-1}$,

$$f_K(u)\langle\nabla h_K(u),v\rangle=f_L(u)\langle\nabla h_L(u),v\rangle.$$

In this case we still conclude that $K=L$.

Remark 5.5. The Log-Minkowski problem (see e.g. Böröczky, Lutwak, Yang, Zhang [8], [9], [10], Lutwak, Yang, Zhang [43], Lutwak, Oliker [39], Stancu [53], Huang, Liu, Xu [21]) asks whether a symmetric convex body K is uniquely defined by its cone volume measure $\frac{1}{n}h_K(u)f_K(u)$, where $u\in\mathbb{S}^{n-1}$.

Suppose that symmetric convex bodies K and L satisfy

$$h_K(u)f_K(u)=h_L(u)f_L(u), \tag{38}$$

for every $u\in\mathbb{S}^{n-1}$. Consider a vector field

$$a(u)=\nabla h_K(u)f_K(u)-\nabla h_L(u)f_L(u).$$

Note that by (7), (38) is equivalent to the fact that $a(u)$ is a tangent field, that is $a(u)\perp u$.

In view of Corollary 5.2, unique determination of a smooth convex body would follow if one could show that in fact $a(u)$ has to be identically zero. Moreover, in view of the previous remark it would suffice to show that there exists a vector $v \in \mathbb{S}^{n-1}$ such that $\langle a(u), v\rangle = 0$ for all $u \in \mathbb{S}^{n-1}$.

6. Extensions of the solution to Shephard's problem

We shall follow the scheme of the proof for the classical Shephard problem (see Koldobsky [26]), which suggests glueing together harmonic-analytic results with the Brunn-Minkowski theory.

6.1. General preparatory lemmas

To prove Theorem 1.4, we first need the following Lemma.

Lemma 6.1. *Let μ be a measure with continuous density g, and let K, L be symmetric convex bodies. Assume additionally that L is a projection body. Assume that for a given $t \in [0,1]$ and for every $\theta \in \mathbb{S}^{n-1}$ we have*

$$p_{\mu,K}(\theta, t) \le p_{\mu,L}(\theta, t).$$

Then

$$\mu_1(tK, L) \le \mu_1(tL, L).$$

Proof. Without loss of generality we may assume that K and L are infinitely smooth strictly convex bodies; the general case then follows via standard approximation argument (see, e.g., Koldobsky [26] Section 8).

Consider a symmetrization of μ. Let $\tilde{\mu}$ be the measure with density

$$\tilde{g}(x) = \frac{g(x) + g(-x)}{2}.$$

Since K and L are symmetric, we have for all $\theta \in \mathbb{S}^{n-1}$ and $t \in [0,1]$:

$$p_{\tilde{\mu},K}(\theta, t) = p_{\mu,K}(\theta, t);$$
$$p_{\tilde{\mu},L}(\theta, t) = p_{\mu,L}(\theta, t),$$

and hence

$$p_{\tilde{\mu},K}(\theta, t) \le p_{\tilde{\mu},L}(\theta, t). \tag{39}$$

Assume for a moment that K and L are strictly convex and infinitely smooth. By (16),

$$\widehat{\sigma_{\mu,tL}}(\theta) = -\frac{\pi}{n} p_{\mu,L}(\theta,t).$$

Hence, by Proposition 3.4,

$$p_{\tilde{\mu},K}(\theta,t) = -\frac{n}{\pi} \widehat{f_{tK}\tilde{g}(\nabla h_{tK})}(\theta).$$

By (39), we get

$$\widehat{f_{tK}\tilde{g}(\nabla h_{tK})}(\theta) \geq \widehat{f_{tL}\tilde{g}(\nabla h_{tL})}(\theta),$$

for every $\theta \in \mathbb{S}^{n-1}$ and for every $t \in [0,1]$. As L is a projection body, we have $\widehat{h_L}(\theta) \leq 0$. Thus

$$\widehat{h_L}(\theta)\widehat{f_{tK}\tilde{g}(\nabla h_{tK})}(\theta) \leq \widehat{h_L}(\theta)\widehat{f_{tL}\tilde{g}(\nabla h_{tL})}(\theta), \tag{40}$$

for every $\theta \in \mathbb{S}^{n-1}$ and for every $t \in [0,1]$. Integrating (40) over the unit sphere, and applying Parseval's identity (13) on both sides of the inequality, we get

$$\int\limits_{\mathbb{S}^{n-1}} h_L(\theta) f_{tK}(\theta)\tilde{g}(\nabla h_{tK}(\theta))d\theta \leq \int\limits_{\mathbb{S}^{n-1}} h_L(\theta) f_{tL}(\theta)\tilde{g}(\nabla h_{tL}(\theta))d\theta. \tag{41}$$

Lemma 3.3 applied along with (41) implies that

$$\tilde{\mu}_1(tK,L) \leq \tilde{\mu}_1(tL,L).$$

Using symmetry of K and L once again, we note that

$$\tilde{\mu}_1(tK,L) = \mu_1(tK,L);$$
$$\tilde{\mu}_1(tL,L) = \mu_1(tL,L),$$

and the lemma follows. □

Via the same scheme as above, invoking Lemma 3.6 along with the fact that $V_{\mu,1}(L,L) = \mu(L)$, we get the following

Lemma 6.2. *Let μ be a measure with continuous density g, and let K, L be symmetric convex bodies. Assume additionally that L is a projection body. Assume that for every $\theta \in \mathbb{S}^{n-1}$ we have*

$$P_{\mu,K}(\theta) \leq P_{\mu,L}(\theta).$$

Then

$$V_{\mu,1}(K,L) \leq \mu(L).$$

6.2. Proof of the Theorem 1.4

Proof. As is shown in Proposition A.2 of the Appendix, if a non-negative function has a positive degree of homogeneity and a positive degree of concavity, then there exists $p \geq 0$ such that g is p-concave and $\frac{1}{p}$-homogenous. Additionally, such function is necessarily supported on a convex cone.

The assumptions of the Theorem allow us to apply Lemma 6.2 and obtain:

$$V_{1,\mu}(K, L) \leq \mu(L). \tag{42}$$

On the other hand, we apply part (26) of Corollary 3.10 and write

$$\mu(L) \geq V_{1,\mu}(K, L) \geq \mu(K)^{1-q}\mu(L)^q,$$

where $q = \frac{p}{np+1}$. Hence $\mu(L) \geq \mu(K)$. □

Remark 6.3. Theorem 1.4 does not hold for all measures. Indeed, consider measure μ with density $1_{B_2^n}$ and convex bodies $L = rB_2^n$, $K = RB_2^n$ such that $r \leq 1 \leq R$ and $R \geq r^{-\frac{1}{n-1}}$. Then $P_{\mu,K}(\theta) \leq P_{\mu,L}(\theta)$ for all $\theta \in \mathbb{S}^{n-1}$ but $\mu(K) \geq \mu(L)$. However, requiring the inequality $p_{\mu,K}(\theta, t) \leq p_{\mu,L}(\theta, t)$ for all $\theta \in \mathbb{S}^{n-1}$ and for all $t \in [0, 1]$ may suffice to conclude that $\mu(K) \leq \mu(L)$ for a wide class of measures with some basic concavity properties.

6.3. A general statement

Finally, we present a measure comparison-type result for a more general class of measures. It may prove useful for considering this problem in greater generality.

Proposition 6.4. *Let μ be an absolutely continuous measure on $\mathbb{R}^n$. Suppose that μ is $F(t)$-concave for some invertible C^1 function $F : \mathbb{R}^+ \to \mathbb{R}$. Let K and L be convex symmetric bodies, and let L in addition be a projection body. Assume that for every $\theta \in \mathbb{S}^{n-1}$ and for every $t \in [0, 1]$ we have*

$$p_{\mu,L}(\theta, t) \geq p_{\mu,K}(\theta, t).$$

Then

(i) $\mu(L) \geq \mu(K) + \int_0^1 \frac{F(\mu(tL)) - F(\mu(tK))}{tF'(\mu(tK))} dt;$

(ii) $\mu(L) \geq \mu(K) + \int_0^1 \left[\mu(tL) - \mu(tK) + \frac{F(\mu(tL)) - F(\mu(tK))}{F'(\mu(tK))}\right] dt.$

Proof. By Lemma 6.1, we get that $\mu_1(tK, L) \leq \mu_1(tL, L)$ for every $t \in [0, 1]$, and therefore

$$\mu_1(tK, tL) = t\mu_1(tK, L) \geq t\mu_1(tL, L) = \mu_1(tL, tL). \tag{43}$$

Applying (43) along with Theorem 3.8 we get

$$t\mu_1(tL, L) \geq t\mu_1(tK, K) + \frac{F(\mu(tL)) - F(\mu(tK))}{F'(\mu(tK))}. \tag{44}$$

After dividing both sides by t and integrating we get

$$\int_0^1 \mu_1(tL, L)dt \geq \int_0^1 \mu_1(tK, K)dt + \int_0^1 \frac{F(\mu(tL)) - F(\mu(tK))}{tF'(\mu(tK))}dt, \tag{45}$$

hence (i) follows from (19) and (45).

Next, we integrate by parts:

$$\int_0^1 t\mu_1(tL, L)dt = \mu(L) - \int_0^1 \mu(tL)dt. \tag{46}$$

Thus (44) and (46) imply (ii). $\square$

6.4. Proof of Theorem 1.5

Proof. Without loss of generality we may assume that the boundary of L is infinitely smooth (see the approximation argument in Koldobsky [26], Section 8). Inasmuch as L is not a projection body we have that $\widehat{h_L}$ is positive on an open set $\Omega \subset \mathbb{S}^{n-1}$; recall as well that, per our assumptions, the curvature function f_L is positive everywhere on the sphere, and L is symmetric. Let $v : \mathbb{S}^{n-1} \to \mathbb{R}$ be a non-negative infinitely smooth even function supported on Ω. Let $\tilde{g}(x)$ be the restriction of $g(x)$ on the half space where is had positive homogeneity, and let $\tilde{\mu}$ be the measure with density $\tilde{g}$. Since we assume that g is supported on the whole space, $\tilde{g}$ is fully supported on a half space.

Define a symmetric convex body K via the relation

$$d\sigma_{\mu,K}(u) = d\sigma_{\mu,L}(u) - \epsilon\widehat{v}(u) \tag{47}$$

for every $u \in \mathbb{S}^{n-1}$. Here $\epsilon > 0$ is chosen small enough so that the right hand side of (47) stays non-negative. Theorem 4.2 guarantees that such convex body exists. Applying Fourier transform to $-(n+1)$-homogenous extensions of both sides of (47), we get

$$-\frac{\pi}{nq}P_{\tilde{\mu},K}(\theta) = -\frac{\pi}{nq}P_{\tilde{\mu},L}(\theta) - \epsilon v(\theta),$$

and hence, by symmetry of K and L,

$$-P_{\mu,K}(\theta) = -P_{\mu,L}(\theta) - \frac{nq}{\pi}\epsilon v(\theta). \tag{48}$$

Recall that

$$V_{\mu,1}(K,L) = \int_0^1 \int_{\mathbb{S}^{n-1}} h_L(u) f_{tK}(u) g(\nabla h_{tK}(u)) du dt,$$

and that $P_{\mu,K}(\theta)$ is the Fourier transform of the $-(n+1)$-homogenous extension of

$$-\frac{\pi}{n}\int_0^1 f_{tK}(u) g(\nabla h_{tK}(u)) dt.$$

Note that $\widehat{h_L}(u)v(u)$ is positive for all $u \in \Omega$. Therefore, by Parseval's type formula (13),

$$\begin{aligned} V_{\mu,1}(K,L) = V_{\mu,1}(K,L) &= -(2\pi)^{-n}\frac{\pi}{n}\int_{\mathbb{S}^{n-1}} \widehat{h_L}(u) P_{\mu,K}(u) du \\ &= -(2\pi)^{-n}\frac{\pi}{n}\int_{\mathbb{S}^{n-1}} \widehat{h_L}(u) P_{\mu,L}(u) du - (2\pi)^{-n} q\epsilon \int_{\Omega} \widehat{h_L}(u) v(u) du \\ &< -(2\pi)^{-n}\frac{\pi}{n}\int_{\mathbb{S}^{n-1}} \widehat{h_L}(u) P_{\mu,L}(u) du = \mu(L). \end{aligned}$$

Using the above along with Corollary 3.10 we get that

$$\mu(L) > V_{\mu,1}(K,L) \geq \mu(K)^{1-q}\mu(L)^q,$$

and hence $\mu(L) > \mu(K)$. On the other hand, (48) implies that $P_{\mu,L}(\theta) \leq P_{\mu,K}(\theta)$ for every $\theta \in \mathbb{S}^{n-1}$. $\square$

7. Stability and separation for Shephard's problem extension

7.1. Separation result for Theorem 1.4

Theorem 7.1. *Fix $n \geq 1$, $p \in [0,\infty)$ and consider a measure μ on $\mathbb{R}^n$ whose density $g : \mathbb{R}^n \to \mathbb{R}^+$ is p-concave and $\frac{1}{p}$-homogenous function. Set $q = \frac{p}{np+1}$.*

Let K and L be symmetric convex bodies, and let L additionally be a projection body. Fix $\epsilon > 0$. Assume that for every $\theta \in \mathbb{S}^{n-1}$ we have

$$P_{\mu,K}(\theta) \leq P_{\mu,L}(\theta) - \epsilon.$$

Then

$$\mu(K)^{1-q} \leq \mu(L)^{1-q} - C(\mu)\epsilon,$$

where $C(\mu)$ is a constant which only depends on the measure μ.

We formulate the following notable corollary of Theorem 7.1.

Corollary 7.2. *Fix $n \geq 1$, $p \in [0, \infty)$ and consider a measure μ on $\mathbb{R}^n$ whose density $g : \mathbb{R}^n \to \mathbb{R}^+$ is p-concave and $\frac{1}{p}$-homogenous function. Set $q = \frac{p}{np+1}$.*

Let L be a strictly convex symmetric projection body. Then

$$\mu(L)^{1-q} \geq C(\mu) \min_{\theta \in \mathbb{S}^{n-1}} P_{\mu,L}(\theta),$$

where $C(\mu)$ is a constant which only depends on the measure μ.

Corollary 7.2 is an analogue of a hyperplane inequality for Lebesgue measure of projections (see Gadrner [15], or Koldobsky [27]).

Proof of Theorem 7.1. Let $\tilde{\mu}$ be, as before, the symmetrization of μ, i.e. the measure with the density $g(x) = \frac{g(x)+g(-x)}{2}$.

Assume without loss of generality that K and L are infinitely smooth. The assumptions $\widehat{h_L} \leq 0$ and

$$P_{\mu,K}(\theta) \leq P_{\mu,L}(\theta) - \epsilon,$$

lead to the following chain of inequalities:

$$\begin{aligned} V_{\tilde{\mu},1}(K, L) &= -(2\pi)^{-n}\frac{\pi}{n} \int\limits_{\mathbb{S}^{n-1}} \widehat{h_L}(u) P_{\tilde{\mu},K}(u)du \\ &\leq -(2\pi)^{-n}\frac{\pi}{n} \int\limits_{\mathbb{S}^{n-1}} \widehat{h_L}(u) P_{\tilde{\mu},L}(u)du + \epsilon(2\pi)^{-n}\frac{\pi}{n} \int\limits_{\mathbb{S}^{n-1}} \widehat{h_L}(u)du \\ &= \tilde{\mu}(L) + \epsilon(2\pi)^{-n}\frac{\pi}{n} \int\limits_{\mathbb{S}^{n-1}} \widehat{h_L}(u)du. \end{aligned}$$

By Corollary 3.10, we have

$$\tilde{\mu}(L) + \epsilon(2\pi)^{-n}\frac{\pi}{n} \int\limits_{\mathbb{S}^{n-1}} \widehat{h_L}(u)du \geq \tilde{\mu}(K)^{1-q}\tilde{\mu}(L)^q. \tag{49}$$

Let $S = \mathbb{S}^{n-1} \cap supp(g)$. By Theorem 4.2 there exists a symmetric convex body Q (depending on the measure μ) with

$$d\sigma_{\tilde{\mu},Q} = \frac{1}{q}\widehat{1_S},$$

and therefore satisfying

$$P_{\tilde{\mu},Q}(\theta) = 1_S(\theta).$$

We then estimate

$$(2\pi)^{-n}\frac{\pi}{n}\int_{\mathbb{S}^{n-1}} \widehat{h_L}(u)du \leq (2\pi)^{-n}\frac{\pi}{n}\int_{S} \widehat{h_L}(u)du = (2\pi)^{-n}\frac{\pi}{n}\int_{\mathbb{S}^{n-1}} \widehat{h_L}(u)P_{\tilde{\mu},Q}(\theta)du =$$
$$-V_{\tilde{\mu},1}(Q,L) \leq -\tilde{\mu}(Q)^{1-q}\tilde{\mu}(L)^q. \tag{50}$$

Letting $C(\mu) = \tilde{\mu}(Q)^{1-q}$, by (49) and (50), we get

$$\tilde{\mu}(L) - \epsilon C(\mu)\tilde{\mu}(L)^q \geq \tilde{\mu}(K)^{1-q}\tilde{\mu}(L)^q,$$

which implies the statement of the Theorem for $\tilde{\mu}$ in place of μ, and hence the Theorem follows for μ as well. □

7.2. Stability for Theorem 1.4

Finally, we prove the stability result.

Theorem 7.3. *Fix $n \geq 1$, $p \in [0, \infty)$ and consider a measure μ on $\mathbb{R}^n$ whose density $g : \mathbb{R}^n \to \mathbb{R}^+$ is p-concave and $\frac{1}{p}$-homogenous function. Set $q = \frac{p}{np+1}$.*

Let K and L be symmetric convex bodies, and let L additionally be a projection body. Let $\epsilon > 0$. Assume that for every $\theta \in \mathbb{S}^{n-1}$ we have

$$P_{\mu,K}(\theta) \leq P_{\mu,L}(\theta) + \epsilon.$$

Then $\mu(K)^{1-q} \leq \mu(L)^{1-q} + C(\mu, L)\epsilon$, where $C(\mu, L)$ is a constant which depends on the measure μ and the body L.

Proof. Suppose that

$$P_{\mu,K}(\theta) \leq P_{\mu,L}(\theta) + \epsilon.$$

Assume without loss of generality that K and L are infinitely smooth. Then, similarly to the proof of Theorem 7.1, we have

$$\mu(L) - \epsilon(2\pi)^{-n}\frac{\pi}{n}\int_{\mathbb{S}^{n-1}} \widehat{h_L}(u)du \geq \mu(K)^{1-q}\mu(L)^q.$$

For the unit ball B_2^n we have

$$(2\pi)^{-n}\frac{\pi}{n}\int\limits_{\mathbb{S}^{n-1}}\widehat{h_L}(u)du=-\nu_{n-1}^{-1}V_1(B_2^n,L).$$

Let $R(L)$ be the smallest positive number such that $L\subset R(L)B_2^n$. Note that

$$V_1(B_2^n,L)=\lim_{\epsilon\to 0}\frac{|B_2^n+\epsilon L|-|B_2^n|}{n\epsilon}\le \nu_n\frac{(1+\epsilon R(L))^n-1}{n\epsilon}=\nu_n R(L).$$

Letting $C(L,\mu)=\frac{\nu_n}{\nu_{n-1}}R(L)\mu(L)^{-q}$, we get the statement of the Theorem. □

Appendix A

Lemma A.1. *Given convex bodies K and L containing the origin, and a measure μ with continuous density g on $\mathbb{R}^n$, we have*

$$\mu_1(K,L)=\int\limits_{\mathbb{S}^{n-1}}h_L(u)d\sigma_{\mu,K}(u).$$

Here h_L is the support function of L and $\sigma_{\mu,K}$ is the surface area measure of K.

Proof. Consider a convex compact set K. Recall that a unit normal n_y is well defined, continuous and differentiable H_{n-1}-almost everywhere for $y\in\partial K$; we shall denote the set where it happens by $\widetilde{\partial K}$. Let $X:\widetilde{\partial K}\times(0,\infty)\to\mathbb{R}^n\setminus K$ be the map $X(y,t)=y+tn_y$. Let $D(y,t)$ be the Jacobian of this map. Then

$$\frac{1}{\epsilon}\left(\mu(K+\epsilon L)-\mu(K)\right)=\frac{1}{\epsilon}\int\limits_{\widetilde{\partial K}}\int\limits_0^{\epsilon h_L(n_y)}D(y,t)g(y+tn_y)dtdH_{n-1}(y).$$

First, we show that $X(y,t)$ is an expanding map. Let $y_1,y_2\in\partial K$ and $t_1,t_2\in[0,\infty)$. Then

$$\begin{aligned}|X(y_1,t_1)-X(y_2,t_2)|^2&=|y_1+t_1n_1-y_2-t_2n_2|^2\\&=|y_1-y_2|^2+|t_1n_1-t_2n_2|^2+t_1\langle y_1-y_2,n_1\rangle+t_2\langle y_2-y_1,n_2\rangle.\end{aligned}\tag{51}$$

By convexity,

$$\langle y_1,n_1\rangle\ge\langle y_2,n_1\rangle,$$
$$\langle y_2,n_2\rangle\ge\langle y_1,n_2\rangle.$$

Hence (51) is greater than or equal to

$$|y_1 - y_2|^2 + |t_1 n_1 - t_2 n_2|^2 \geq |y_1 - y_2|^2 + |t_1 - t_2|^2.$$

This implies that $X(y,t)$ is expanding, and hence $D(y,t) \geq 1$. Therefore,

$$\mu_1(K,L) \geq \liminf_{\epsilon\to 0} \frac{1}{\epsilon} \int\limits_{\widetilde{\partial K}} \int\limits_0^{\epsilon h_L(n_y)} g(y + t n_y) dt dH_{n-1}(y)$$
$$= \int\limits_{\widetilde{\partial K}} h_L(n_y) g(y) dH_{n-1}(y). \tag{52}$$

Using the fact that $H_{n-1}(\partial K \setminus \widetilde{\partial K}) = 0$, and applying the Gauss map to pass the integration on the sphere, we get

$$\mu_1(K,L) \geq \int\limits_{\mathbb{S}^{n-1}} h_L(u) d\sigma_{\mu,K}(u).$$

Next, for an arbitrary $\delta > 0$, consider a set

$$(\partial K)_\delta = \{y \in \partial K : \exists a \in \mathbb{R}^n \text{ s.t. } y \in B(a,\delta) \subset K\},$$

where $B(a,\delta)$ stands for a ball of radius δ centered at a. It was shown by Hug [22] (see Besau, Werner [5] for more details), that the Gauss map is Lipschitz for $y \in (\partial K)_\delta$.

For a (small) $\epsilon > 0$, assume that $0 \leq t_1, t_2 \leq \epsilon$, and $y_1, y_2 \in (\partial K)_\delta$. Then (51) is smaller than or equal to

$$|y_1 - y_2|^2 + |t_1 - t_2|^2 + \epsilon^2 |n_1 - n_2|^2 + \epsilon \langle y_1 - y_2, n_1 - n_2 \rangle.$$

Denote by $L(\delta)$ the Lipschitz constant of the Gauss map on $(\partial K)_\delta$. Then

$$\frac{|y_1 - y_2|^2 + |t_1 - t_2|^2 + \epsilon^2 |n_1 - n_2|^2 + \epsilon \langle y_1 - y_2, n_1 - n_2 \rangle}{|y_1 - y_2|^2 + |t_1 - t_2|^2} \leq 1 + L(\delta)\epsilon + L(\delta)^2 \epsilon^2.$$

Therefore,

$$D(y,t) \leq (1 + L(\delta)\epsilon + L(\delta)^2\epsilon^2)^{n-1} \leq 1 + C(K,n,\delta)\epsilon.$$

Hence, in view of (52), the limit in ϵ exists, and

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \int\limits_{(\partial K)_\delta} \int\limits_0^{\epsilon h_L(n_y)} D(y,t) g(y + t n_y) dt dH_{n-1}(y) = \int\limits_{(\partial K)_\delta} h_L(n_y) g(y) dH_{n-1}(y),$$

and by dominated convergence theorem and lower-semi continuity,

$$
\begin{aligned}
\mu_1(K,L) &= \liminf_{\epsilon\to 0} \frac{1}{\epsilon} \int\limits_{\partial K} \int\limits_{0}^{\epsilon h_L(n_y)} D(y,t) g(y+tn_y) dt dH_{n-1}(y) \\
&= \lim_{\epsilon\to 0} \lim_{\delta\to 0} \frac{1}{\epsilon} \int\limits_{(\partial K)_\delta} \int\limits_{0}^{\epsilon h_L(n_y)} D(y,t) g(y+tn_y) dt dH_{n-1}(y) \\
&= \lim_{\delta\to 0} \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int\limits_{(\partial K)_\delta} \int\limits_{0}^{\epsilon h_L(n_y)} D(y,t) g(y+tn_y) dt dH_{n-1}(y) \\
&= \lim_{\delta\to 0} \int\limits_{(\partial K)_\delta} h_L(n_y) g(y) dH_{n-1}(y) \\
&= \int\limits_{\widetilde{\partial K}} h_L(n_y) g(y) dH_{n-1}(y) = \int\limits_{\partial K} h_L(n_y) g(y) dH_{n-1}(y) \\
&= \int\limits_{\mathbb{S}^{n-1}} h_L(u) d\sigma_{\mu,K}(u).
\end{aligned}
$$

The last equation is obtained via the application of the Gauss map. □

Proposition A.2. *For $p \geq 0$ and $r \geq 0$, let $g : \mathbb{R}^n \to \mathbb{R}^+$ be p-concave and r-homogenous. Then g is also $\frac{1}{r}$-concave.*

Proof. The proof splits in two cases. Firstly, if $\frac{1}{r} \leq p$, then the statement follows automatically by the standard inequality for q-averages

$$M_q(\lambda, a, b) \leq M_{q'}(\lambda, a, b),$$

whenever $q \leq q'$ (see the definition (5) and Gardner [15] for more details).

Secondly, let $0 \leq r \leq \frac{1}{p}$. Observe, that in the presence of r-homogeneity it is sufficient to show that for every $x, y \in \mathbb{R}^n$ one has

$$g(x+y) \geq \left(g(x)^{\frac{1}{r}} + g(y)^{\frac{1}{r}}\right)^r. \tag{53}$$

By p-concavity, we have for every $\lambda \in [0,1]$:

$$
\begin{aligned}
g(x+y) &= g\left(\lambda \frac{x}{\lambda} + (1-\lambda)\frac{y}{1-\lambda}\right) \geq \left(\lambda g\left(\frac{x}{\lambda}\right)^p + (1-\lambda) g\left(\frac{y}{1-\lambda}\right)^p\right)^{\frac{1}{p}} \\
&= \left(\lambda^{1-pr} g(x)^p + (1-\lambda)^{1-pr} g(y)^p\right)^{\frac{1}{p}}.
\end{aligned} \tag{54}
$$

Observe that for

$$\lambda_0 = \frac{g(x)^{\frac{1}{r}}}{g(x)^{\frac{1}{r}} + g(y)^{\frac{1}{r}}},$$

the expression in (54) is exactly equal to the right hand side of (53), which concludes the proof. $\square$

We remark that λ_0 in the proof above is found as the maximizer for the function from (54).

Proposition A.3. *Let K and L be convex bodies within Hausdorff distance ϵ from each other, $\epsilon > 0$. Let μ be a measure on $\mathbb{R}^n$ with continuous density $g(x)$. Then for every Lipschitz function $a(u)$,*

$$\left| \int_{\mathbb{S}^{n-1}} a(u) d\sigma_{\mu,K}(u) - \int_{\mathbb{S}^{n-1}} a(u) d\sigma_{\mu,L}(u) \right| \leq C(\epsilon),$$

where the constant $C(\epsilon) > 0$ depends on $a(u)$, $g(x)$, K and L, and tends to zero when $\epsilon \to 0$.

Proof. We write

$$\begin{aligned} &\left| \int_{\mathbb{S}^{n-1}} a(u) d\sigma_{\mu,K}(u) - \int_{\mathbb{S}^{n-1}} a(u) d\sigma_{\mu,L}(u) \right| \\ &= \left| \int_{\mathbb{S}^{n-1}} a(u) g(\nu_K^{-1}(u)) d\sigma_K(u) - \int_{\mathbb{S}^{n-1}} a(u) g(\nu_L^{-1}(u)) d\sigma_L(u) \right| \\ &\leq \int_{\mathbb{S}^{n-1}} |a(u)| \left| g(\nu_K^{-1}(u)) - g(\nu_L^{-1}(u)) \right| d\sigma_K(u) \qquad (55) \\ &\quad + \left| \int_{\mathbb{S}^{n-1}} a(u) g(\nu_L^{-1}(u)) d\sigma_K(u) - \int_{\mathbb{S}^{n-1}} a(u) g(\nu_L^{-1}(u)) d\sigma_L(u) \right|. \qquad (56) \end{aligned}$$

Since K and L are convex bodies, and hence are bounded, $g(x)$ is uniformly continuous on both ∂K and ∂L. Hence, as the Hausdorff distance between K and L is bounded by ϵ,

$$|g(\nu_K^{-1}(u)) - g(\nu_L^{-1}(u))| \leq C' |\nu_K^{-1}(u) - \nu_L^{-1}(u)|,$$

and thus, by the weak convergence of the inverse Gauss maps of convex bodies converging in Hausdorff distance (see, e.g. Schneider [51]),

$$\int_{\mathbb{S}^{n-1}} |a(u)| \left|g(\nu_K^{-1}(u)) - g(\nu_L^{-1}(u))\right| d\sigma_K(u) \leq C'(\epsilon),$$

where $C'(\epsilon) \to 0$ as $\epsilon \to 0$. As $a(u)$ is a continuous function on $\mathbb{S}^{n-1}$, it attains its maximum. Hence there exists a constant $C''(\epsilon)$, depending on $a(u)$, $g(x)$, K and L such that (55) is bounded from above by $C''(\epsilon)$, and $C''(\epsilon)$ tends to zero as $\epsilon \to 0$.

Next, (56) is bounded from above by

$$\tilde{C}\left|\int_{\mathbb{S}^{n-1}} a(u)d\sigma_K(u) - \int_{\mathbb{S}^{n-1}} a(u)d\sigma_L(u)\right|,$$

which in turn is bounded by $\tilde{C}'(\epsilon) \to_{\epsilon\to 0} 0$, since classical (Lebesgue) surface area measures of convex bodies, which converge in Hausdorff distance, do converge weakly (see, e.g. Schneider [51]). The proposition follows. $\square$

Proposition A.4. *If a body K is C^2-smooth and strictly convex then its surface area measure with respect to a measure μ with continuous density g, has representation*

$$d\sigma_{\mu,K}(u) = f_K(u)g(\nabla h_K(u))du.$$

Proof. Under the assumptions of the proposition, the Gauss map ν_K of K is a bijection, and $\nu_K^{-1}(u) = \nabla h_K(u)$ for every $u \in \mathbb{S}^{n-1}$. Therefore, for every $\Omega \subset \mathbb{S}^{n-1}$,

$$\begin{aligned}\sigma_{\mu,K}(\Omega) &= \int_{\nu_K^{-1}(\Omega)} g(x)d\sigma_K(x) \\ &= \int_{\Omega} g(\nu_K^{-1}(u))f_K(u)du = \int_{\Omega} f_K(u)g(\nabla h_K(u))du. \quad \square\end{aligned}$$

Proposition A.5. *The surface area measure of a convex polytope P with respect to a measure μ has representation*

$$d\sigma_{\mu,P}(u) = \sum_{i=1}^{N} \delta_{u_i}\mu_{n-1}(F_i),$$

where u_i, $i = 1, ..., N$ are the normals to the faces of the polytope, F_i are the corresponding faces, and $\mu_{n-1}(F_i)$ stands for $\int_{F_i} d\mu(x)$.

Proof. For a polytope P with faces F_i and corresponding normals u_i, Gauss map ν_K is defined everywhere in the interior of the faces, and for $x \in int(F_i)$, $\nu_K(x) = u_i$. Hence, for a Borel set $\Omega \subset \mathbb{S}^{n-1}$,

$$\sigma_{\mu,K}(\Omega) = \int\limits_{\nu_K^{-1}(\Omega)} g(x) d\sigma_K(x) = \sum_{i:\, u_i \in \Omega} \int\limits_{F_i} d\mu_{n-1}(x). \quad \square$$

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