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Parametric Dependence of Large Disturbance Response for Vector Fields with Event-Selected Discontinuities

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Abstract— **The ability of a nonlinear system to recover from a large disturbance to a desired stable equilibrium point depends on system parameter values, which are often uncertain and time varying. A particular disturbance acting for a finite time can be modeled as an implicit map that takes a parameter value to its corresponding post-disturbance initial condition in state space. The system recovers when the post-disturbance initial condition lies inside the region of attraction of the stable equilibrium point. Critical parameter values are defined to be parameter values whose corresponding post-disturbance initial condition lies on the boundary of the region of attraction. Computing such values is important in numerous applications because they represent the boundary between desirable and undesirable system behavior. Many realistic system models involve controller clipping limits and other forms of switching. Furthermore, these hybrid dynamics are closely linked to the ability of a system to recover from disturbances. The paper develops theory which underpins a novel algorithm for numerically computing critical parameter values for nonlinear systems with clipping limits and switching. For an almost generic class of vector fields with event-selected discontinuities, it is shown that the boundary of the region of attraction is equal to a union of the stable manifolds of the equilibria and periodic orbits it contains, and that this decomposition persists and the boundary varies continuously under small changes in parameter.**

I. Introduction

Engineered systems, such as power systems, are subject to disturbances. In the power system case, for example, a disturbance may occur due to a lightning strike on a transmission line. Disturbances drive systems away from their desired operating point, typically a stable equilibrium point. Whether a system is able to recover from a particular disturbance, back to an equilibrium point, depends upon the parameters of the system, which are often uncertain and time varying.

The disturbance can be thought of as providing an initial condition to a nonlinear system, with that initial condition being a function of parameter value. We call this the post-disturbance initial condition (PDIC). The system recovers precisely when the PDIC lies within the region of attraction (RoA) of the desired stable equilibrium point, and it fails to recover when the PDIC is outside the RoA. The boundary case typically (as the results later show) occurs when the PDIC is in the boundary of the RoA. Parameter values which give rise to a PDIC that lies in the boundary of the RoA are therefore referred to as *critical parameter values*. In practice, the implicit map from parameter value to PDIC is typically not known explicitly. The PDIC, as well as the ensuing system behavior, must be obtained using numerical simulation.

Knowledge of critical parameter values is valuable for many applications, such as power systems, because it provides a measure of the margin for safe operation. However, numerous attempts over the past several decades to develop algorithms for numerically computing critical parameter values have met with limited success. The key challenge is that the boundary of the RoA is typically a high dimensional object, not necessarily a manifold, which is very difficult to locate, and even if it could be known exactly at one instant, it is a function of parameter value so will vary as parameter values change in time.

The stable manifold of a hyperbolic equilibrium point or periodic orbit is the set of initial conditions which converge to that equilibrium point or periodic orbit in forward time. Older work attempted to develop a characterization of the boundary of the RoA for nearly generic, or typical, C^1 vector fields as the union of the stable manifolds of the equilibria and periodic orbits it contains [1], [2]. This structure was exploited to develop Lyapunov function based algorithms to compute critical parameter values [3], [4]. This work met with challenges since it did not adequately consider the dependence of the boundary of the RoA on parameter values, and known Lyapunov functions were typically too conservative to be of value for practical power system models.

More recent work has shown that, for nearly generic parametrized C^1 vector fields, the boundary of the RoA varies continuously, and the boundary decomposition discussed above persists, under small changes in parameter values [5]. Furthermore, under these assumptions there exists a special equilibrium point or periodic orbit on the boundary of the RoA, hereafter referred to as the controlling critical element, such that the time the system spends in a neighborhood of it is continuous in parameter value and approaches infinity as the parameter value approaches its critical value. This has been used to provide theoretical justification for algorithms which numerically compute critical parameter values by varying a parameter so as to maximize the time the system trajectory spends inside a neighborhood of the controlling critical element [6], [7].

However, practical power system models involve switching events, such as clipping limits on control devices, which lead to vector fields that are not C^1. Furthermore, it has

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been observed that such switching is fundamentally coupled with loss of system recoverability in practice [8], [9]. The purpose of this note is to extend the more recent theoretical work mentioned above to the setting of a large class of vector fields, closely related to what have been called vector fields with C^1 event-selected discontinuities [10], which can exhibit various forms of switching. In turn, this provides justification for the extension of numerical algorithms to systems exhibiting switching. The algorithms previously developed can already be applied to these types of hybrid systems in practice [5], [6]. Vector fields with event-selected discontinuities have received attention recently due to their applicability to many physical and engineering system models which possess a finite number of triggering hypersurfaces where the vector field is discontinuous.

A sketch of a proof was made in [9] for a decomposition of the boundary of the RoA for systems with clipping limits and fixed parameter values. In that work, the vector field was locally Lipschitz. Generalizations to systems exhibiting switching, or any discontinuous vector fields, have not been made even in the case of constant parameters. Therefore, the classification of the boundary of the RoA for vector fields with C^1 event-selected discontinuities, presented here, may be of interest for the fixed parameter case, as well as for its behavior under small variations in parameter. The primary application, though, is the theoretical justification of algorithms for numerically computing critical parameter values.

The paper is organized as follows. Section II gives a motivating example. Section III provides some dynamical systems background. Then Section IV discusses the main results. Section V provides a sketch of the key proofs, although some are nearly identical to prior work in [5] and, therefore, omitted for brevity. Finally, Section VI offers some concluding remarks.

II. Example

The following example serves to illustrate a mechanism whereby the RoA boundary can fail to vary continuously for a parameterized vector field, and how this can lead to situations where no critical parameter value exists. Existence of critical parameter values is important because, together with their relevant properties, they provide theoretical justification for the numerical algorithm described in the introduction. Without critical parameter values, predicting the transition from desirable to undesirable system behavior under parameter variation becomes much more difficult in both theory and practice. The theory developed in subsequent sections will prove that this example is not typical; in particular, a large class of hybrid dynamical systems does not exhibit the behavior depicted in this example for small changes in parameter.

Let $J = (-1, 1)$ and for $p \in J$ define f_p as a vector field on $\mathbb{R}^2$ piecewise as follows. Let $r := r(x, y) = \sqrt{x^2 + y^2}$. Let $h_{(a,b)} : \mathbb{R} \to \mathbb{R}$ be a smooth bump function with $h^{-1}(1) = [0, a]$ and $h^{-1}(0) = [b, \infty)$ (see [11, Lemma 2.21] for a specific example). For $|r| \le 1$, let $f_p(x, y) = (-x, -y) - p\frac{h_{(2.5,3)}(r)}{r}(x, y)$, and for $|r| > 1$, let $f_p(x, y) = \frac{h_{(1,2)}(r)}{2r}(-x - y, x - y) - p\frac{h_{(2.5,3)}(r)}{r}(x, y)$. This family of vector fields are each piecewise C^1 with a switching surface on the circle $|r| = 1$. For $p \in J$, let $s_p(x, y) = (r - 1)^2$. Then $s_p^{-1}(0) = \{(x, y) : |r(x, y)| = 1\}$. Thus, for each $p \in J$, $\{f_p, s_p\}$ defines a vector field with event-selected C^1 discontinuities, as will be defined below. Furthermore, $\{f_p, s_p\}_{p \in J}$ is a strong C^1 continuous family of event-selected C^1 vector fields, also defined below.

Figure 1 shows the vector field f_p for $p = 0.3$. It has a stable equilibrium point at the origin whose RoA is the open ball of radius $r = 3$ with boundary the circle of radius $r = 3$. This qualitative picture, in particular the RoA and its boundary, remain the same for any $p > 0$. The red and green curves in the figure show the images of two sets of parameter dependent initial conditions (ICs). The ICs corresponding to $p = 0.3$ are shown as red and green circles, and the ICs move along the red and green lines as p decreases. Note that both the red and green ICs lie inside the RoA for $p = 0.3$ (and any $p > 0$).

Figure 2 shows the vector field f_p for $p = 0$. Note that the RoA of its stable equilibrium point is the open ball of radius $r = 2$, with boundary the circle of radius $r = 2$. Here the red circle, denoting the red IC, lies inside the RoA whereas the green circle, denoting the green IC, lies outside the RoA.

Figure 3 shows the vector field f_p for $p = -0.3$. Note that the RoA of its stable equilibrium point is the open ball of radius $r \approx 1.5$, with boundary the circle (now a periodic orbit) of radius $r \approx 1.5$. Here the red circle, denoting the red IC, lies on the boundary of the RoA, and hence $p = -0.3$ is a critical parameter value for the red ICs, whereas the green circle, denoting the green IC, lies outside the RoA.

In summary, the red ICs intersect the RoA boundary at $p = -0.3$, so that $p = -0.3$ is a critical parameter value for them. However, the green ICs pass from inside the RoA (for $p > 0$) to outside the RoA (for $p \le 0$) without ever lying on the RoA boundary. So, there is no critical parameter value for the green ICs. This is possible because the RoA boundary varies discontinuously at $p = 0$.

The theory developed in this paper will prove that discontinuous variation of the RoA boundary under small parameter variation is not possible for a large class of practical hybrid dynamical systems, that these systems do possess critical parameter values for a particular disturbance, and that a previously designed algorithm can be used to numerically compute those critical values.

III. Background

Characterization of the boundary of the region of attraction in terms of stable manifolds for fixed parameter values [1], [2] was carried out using tools that were developed for a class of C^1 vector fields known as Morse-Smale vector fields [12]. More recently, similar techniques were used to develop arguments for continuity of the boundary under small changes in parameter values for C^1 vector fields [5].

We will assume that all vector fields, C^1 or not, are defined on $M := \mathbb{R}^n$ for some $n > 0$. First we review some terminology of nonlinear systems with a C^1 vector field. Let $J \subset \mathbb{R}$ be an open interval containing p_0 and let $\{f_p\}_{p \in J}$

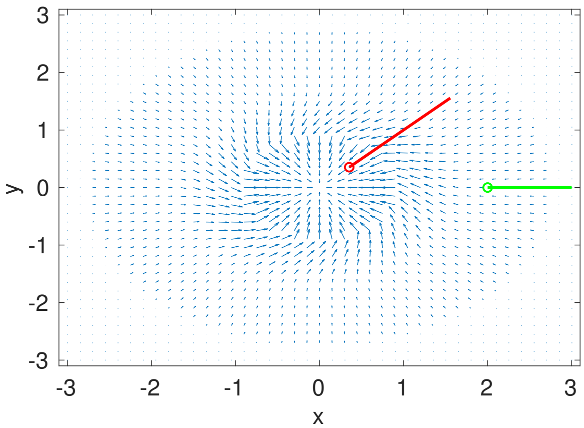

Fig. 1. The vector field f_p for $p = 0.3$. The red and green lines show two sets of parameter dependent initial conditions, with the initial conditions for $p = 0.3$ shown as red and green circles.

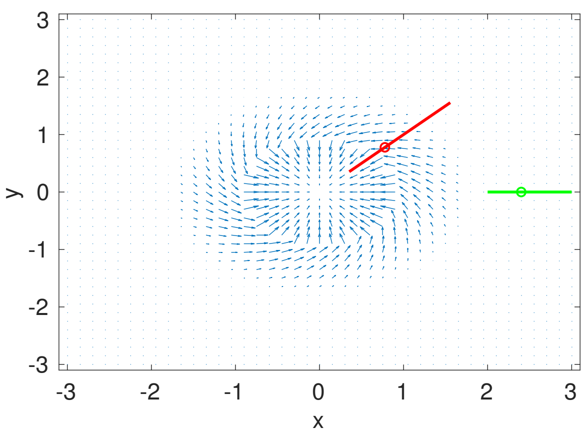

Fig. 2. The vector field f_p for $p = 0$. The red and green lines show two sets of parameter dependent initial conditions, with the initial conditions for $p = 0$ shown as red and green circles.

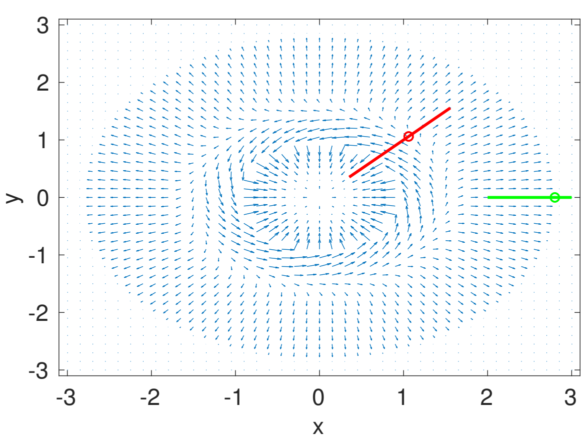

Fig. 3. The vector field f_p for $p = -0.3$. The red and green lines show two sets of parameter dependent initial conditions, with the initial conditions for $p = -0.3$ shown as red and green circles.

be a parameterized family of vector fields on M. Define f a vector field on $M \times J$ by $f(x,p) = (f_p(x), 0)$. Let ϕ denote the flow of f such that $\phi(x,t,p)$ is the flow of the system from initial condition x under the vector field f_p for a time t. Let $\phi_{(t,p)} : M \to M$ by $\phi_{(t,p)}(x) = \phi(x,t,p)$. If $X(p) \subset M$ is an equilibrium point or periodic orbit of f_p for some $p \in J$, we call $X(p)$ a critical element of f_p.

An equilibrium point $X(p)$ of f_p is hyperbolic if $d(\phi_{(1,p)})_{X(p)}$ has no imaginary eigenvalues. If $X(p)$ is a periodic orbit and $x \in X(p)$ then there exists a hypersurface $S \supset \{x\}$ and a C^1 map $\tau : S \to S$ such that τ is the first return (Poincare) map. The periodic orbit $X(p)$ is hyperbolic if $d\tau_x$ has no imaginary eigenvalues. For a hyperbolic critical element, let $n^u(X(p_0))$ denote the unstable dimension of $T_{X(p_0)}M$, and let $n^s(X(p_0))$ denote the stable dimension of $T_{X(p_0)}M$. A hyperbolic critical element $X(p)$ possesses local stable and unstable manifolds, denoted $W^s_{\text{loc}}(X(p))$ and $W^u_{\text{loc}}(X(p))$, respectively, such that the flow restricted to $W^s_{\text{loc}}(X(p))$ is a contraction in forwards time, and the flow restricted to $W^u_{\text{loc}}(X(p))$ is a contraction in backwards time. The stable and unstable manifolds, $W^s(X(p))$ and $W^u(X(p))$, respectively, are then constructed by flowing $W^s_{\text{loc}}(X(p))$ and $W^u_{\text{loc}}(X(p))$ backwards and forwards in time, respectively. The RoA of a stable hyperbolic equilibrium point is its stable manifold, and the RoA boundary is the boundary of its stable manifold.

A point $x \in M$ is wandering under f_p if there exists an open neighborhood U of x in M and some $T > 0$ such that $|t| > T$ implies that $\phi(U,t,p) \bigcap U = \emptyset$. A point $x \in M$ is nonwandering if it is not wandering. Let $\Omega(f_p)$ denote the set of all nonwandering points of f_p. These include all critical elements of f_p. A pair of C^1 immersed submanifolds X and Y are transverse if for every $q \in X \cap Y$, $T_qX + T_qY = T_qM \cong M$. A Morse-Smale vector field g is a C^1 vector field such that $\Omega(g)$ is equal to a finite union of hyperbolic critical elements whose stable and unstable manifolds intersect transversely [13].

Let $C^1(M,N)$ denote C^1 maps from M to N, both C^1 manifolds. The strong C^1 topology on $C^1(M,N)$ is defined in [14, Chapter 2]. A parametrized family of vector fields $\{f_p\}_{p\in J}$ is strong C^1 continuous if the map $p \to f_p$ is continuous as a map from J to $C^1(M,TM)$ equipped with the strong C^1 topology. In particular, if $f_p \to g$ as $p \to \hat{p}$ in the strong C^1 topology then there exists $K \subset M$ compact such that for any $\epsilon > 0$, p sufficiently close to $\hat{p}$ implies that f_p and g agree outside of K and that f_p and g along with their first derivatives are ϵ-close on K. A property is generic for vector fields on M if the set of vector fields possessing this property contains a countable intersection of open, dense sets in $C^1(M,TM)$. A family of critical elements $\{X(p)\}_{p\in J}$ of $\{f_p\}_{p\in J}$ is C^1 continuous if there exists a C^1 manifold S and a C^1 function $F : S \times J \to M$ such that $F|_{S\times\{p\}}$ is injective onto its image $X(p)$ for all $p \in J$. We say $\{A_p\}_{p\in J}$ is a Chabauty continuous family of subsets of $\mathbb{R}^n$ if for every $p_0 \in J$ and every $\{p_n\}_{n=1}^{\infty} \subset J$ such that $p_n \to p_0$, for every $x \in A_{p_0}$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in A_{p_n}$ and $x_n \to x$, and every sequence

$\{x_n\}_{n=1}^{\infty}$ with $x_n \in A_{p_n}$ has all of its limit points contained in A_{p_0}.

Next we introduce some additional notation. Let $X(p) \subset M$ and $X_p := X(p) \times \{p\} \subset M \times J$. For $Q \subset J$, write $X(Q) = \bigcup_{p \in Q} X(p) \subset M$ and $X_Q = \bigcup_{p \in Q} X_p \subset M \times J$. For any set A, let ∂A denote its topological boundary, $\overline{A}$ its topological closure, and int A its topological interior. For any set S, let $\sqcup_{p \in J} S := S \times J$. If $D \subset M$, let $D_\epsilon := \{x \in M : d(x, D) < \epsilon\}$ where d is the Euclidean distance.

In order to incorporate the effects of clipping limits and switching, we will employ a class of hybrid dynamical systems which we call vector fields with event-selected C^1 discontinuities. Note though that the equivalent concept in [10] is slightly more general. This class of vector fields, which is defined below, has several desirable properties including the existence of a global flow which is piecewise C^1.

A rough vector field is a vector field that is not assumed to be C^1 or even C^0. This includes vector fields exhibiting limits or switching behavior, but also includes C^1 vector fields. We next define event functions, whose zero level sets represent switching surfaces and are the points of discontinuity of the vector field. The definition ensures that the flow of the vector field will cross switching surfaces transversely. This definition rules out grazing of the switching surfaces and Zeno type behavior.

Definition. Let f be a rough vector field on M, $s \in C^1(M, \mathbb{R}^m)$, and U an open set in M. Then s is an event function for f on U if for each component s_j of s, either $s_j(U)$ is disjoint from zero or, if not, then there exists $c > 0$ such that either $d(s_j)_x(f(x)) \geq c$ or $d(s_j)_x(f(x)) \leq -c$, for all $x \in U$. Such a U is referred to as an event neighborhood of f.

Let $B_m = \{-1, +1\}^m$ be m copies of $\{-1, +1\}$, meant to denote whether each event function is positive or negative, and hence indicate the status of each switching event. For each $j \in \{1, ..., m\}$, let $Z_j = s_j^{-1}(0)$ denote the switching surface corresponding to s_j. Let $Z = \bigcup_{j=1}^{m} Z_j$ be the union of the switching surfaces. Let $\sigma_j : M \to \{-1, +1\}$ that sends x to the sign of $s_j(x)$, or to 1 if $s_j(x) = 0$. Let $\sigma : M \to \{-1, +1\}^m$ be the product of σ_j for $j \in \{1, ..., m\}$. Then $\sigma(x)$ denotes the switching states of the system at x.

The following definition of event-selected C^1 ensures both that the flow encounters switching surfaces transversely and that the flow through points of intersection of multiple switching surfaces is well-defined and C^1.

Definition. Let f be a rough vector field on M and $s \in C^1(\mathbb{R}^m, \mathbb{R})$. Then f and s determine an event-selected C^1 vector field if the following conditions are satisfied:

1) For every point $x \in M$, there exists an open set U_x containing x such that s is an event function for f on U_x.
2) For every $j, j' \in \{1, ..., m\}$, Z_j and $Z_{j'}$ are transverse.
3) There exists a set of C^1 vector fields $\{f^b\}_{b \in B_m}$ on M, called selection functions, such that for every $x \in M$, $f(x) = f^{\sigma(x)}(x)$.

An example of an event-selected C^1 vector field was shown in Section II with event function $s_1(x, y) = (\sqrt{x^2 + y^2} - 1)^2$. By [10, Corollary 1], if f and s determine an event-selected C^1 vector field over M then there exists a piecewise C^1 global flow.

Definition. A piecewise C^1 immersed submanifold is a topological manifold $T \subset M$ together with a C^1 immersed submanifold, denoted $\tilde{T}$, in $M \backslash Z$ which is an open and dense topological submanifold of T.

Definition. Two piecewise C^1 immersed submanifolds T and T' are transverse if $\tilde{T}$ and $\tilde{T}'$ are transverse in $M \backslash Z$.

Definition. Let $\{f_p\}_{p \in J}$ be a family of rough vector fields on M and let $\{s_p\}_{p \in J}$ be a family of C^1 functions from M to $\mathbb{R}^m$. We say that $\{f_p, s_p\}_{p \in J}$ is a strong C^1 continuous family of event-selected C^1 vector fields on M if the following conditions are satisfied:

1) There exists $p_0 \in J$ such that f_{p_0} and s_{p_0} determine an event-selected discontinuous C^1 vector field on M.
2) $\{s_p\}_{p \in J}$ is a strong C^1 continuous family of functions in $C^1(M, \mathbb{R}^m)$.
3) There exists a set of families of vector fields on M, $\{f_p^b\}_{b \in B^m, p \in J}$, such that for each $p \in J$ and $x \in M$, $f(x) = f_p^{\sigma(x)}(x)$.
4) For each $b \in B^m$, $\{f_p^b\}_{p \in J}$ is a C^1 continuous family of C^1 vector fields on M.

We will see in Lemma 2 that if $\{f_p, s_p\}_{p \in J}$ is a strong C^1 continuous family of event-selected C^1 vector fields on M then for J sufficiently small, (f_p, s_p) determines an event-selected C^1 vector field for each $p \in J$. Define f, a rough vector field on $M \times J$, by $f(x, p) = (f_p(x), 0)$. Then Lemma 2 will show that for J sufficiently small, f has a piecewise C^1 flow.

For $j \in \{1, ..., m\}$ and $p \in J$, let $Z_j^p = (s_p^j)^{-1}(0)$. For $p \in J$ let $Z^p = \bigcup_{j=1}^{m} Z_j^p$, and for $j \in \{1, ..., m\}$ let $Z_j = \sqcup_{p \in J} Z_j^p$. Let $Z^J = \sqcup_{p \in J} Z^p$ and let $C^J = M \times J \backslash Z^J$.

IV. Results

Let $J \subset \mathbb{R}$ be an open interval and let $p_0 \in J$. Let $\{f_p, s_p\}_{p \in J}$ be a strong C^1 continuous family of event-selected C^1 vector fields on M. Let $X^s(p_0)$ be a stable hyperbolic equilibrium point of f_{p_0} which lies in an open neighborhood on which f_{p_0} is C^1. For J sufficiently small we make the following assumptions:

1) Every equilibrium point of f_{p_0} is disjoint from Z^{p_0}.
2) There exists a neighborhood V of the projection $\pi_M(\partial W^s(X_J^s) \bigcap M_{p_0})$ onto M such that $\Omega(f) \bigcap V$ consists of a finite union of critical elements of f_{p_0}; call them $\{X^i(p_0)\}_{i=1}^{k}$. Shrink V if necessary so that $X_{p_0}^i \subset \partial W^s(X_J^s)$ for every $i \in I := \{1, ..., k\}$.
3) Let $\gamma \subset \partial W^s(X_J^s)$ be an orbit. Then γ converges to X^i for some $i \in I$.
4) With respect to f_{p_0}, every equilibrium point $X^i(p_0)$ is hyperbolic in the sense that it is hyperbolic in its neighborhood on which f_{p_0} is C^1. For any $X^i(p_0)$ a periodic orbit, we will see below that it possesses a point $x \in M \backslash Z^{p_0}$ and a C^1 cross section S with a C^1 first return map with x as a fixed point. Then for any

periodic orbit $X^i(p_0)$, assume that x is a hyperbolic fixed point of its first return map.

5) We will see below that every $X^i(p_0)$ possesses stable and unstable manifolds, call them $W^s(X^i(p_0))$ and $W^u(X^i(p_0))$, respectively, which are piecewise C^1 immersed submanifolds. For every $i, j \in I$, $W^u(X^i(p_0))$ and $W^s(X^j(p_0))$ are transversal in M.

Assumptions 2, 4, and 5 ensure that f_{p_0} is Morse-Smale along $\partial W^s(X_J^s) \bigcap M \times \{p_0\}$. Assumpton 3 ensures that no trajectories in $\partial W^s(X_J^s)$ escape to infinity, which is required since M is not compact, and that no new nonwandering elements enter $\partial W^s(X_J^s)$ for $p \in J$, although we will see that the perturbations of the critical elements $\{X^i(p_0)\}_{i\in I}$ will be contained in the boundary. Assumptions 1, 4, and 5 are generic. For the components on which f_{p_0} is C^1, it is generically true that $\Omega(f_{p_0})$ is equal to the closure of the union of critical elements of f_{p_0}. However, it is not generic that there exists a neighborhood V containing a finite number of critical elements, so Assumption 2 is not generic.

Theorem 1 shows that the boundary of the RoA is equal to the union over $p \in J$ of the family of boundaries of the RoAs, and that the decomposition of the boundary of the RoA into a union of stable manifolds persists for $p \in J$. Corollary 1 then states that the family of boundaries varies continuously with $p \in J$.

Theorem 1. For J sufficiently small, $\partial W^s(X_J^s) = \sqcup_{p\in J}\partial W^s(X^s(p)) = \bigcup_{i\in I} W^s(X_J^i)$.

Corollary 1. $\{\partial W^s(X^s(p))\}_{p\in J}$ is a Chabauty continuous family of subsets of M.

Let $y : J \to M$ denote a set of post-disturbance initial conditions with $y_p = (y(p), p) \subset M \times J$, and let y be C^1. Lemma 1 states that if y_J intersects both the RoA and its complement, then y_J intersects the boundary of the RoA.

Lemma 1. Suppose Assumptions 1–5 hold and there exist $p_0, p_1 \in J$ with $y_{p_0} \in W^s(X_J^s)$ and $y_{p_1} \notin W^s(X_J^s)$. Without loss of generality suppose $p_0 < p_1$. Then there exists a unique minimum parameter value $p^* \in J$ and $i \in I$ such that $y_{p^*} \in \partial W^s(X_J^s)$ and $y(p^*) \in W^s(X^i(p^*))$.

Let $X^* = X^i$ where i is chosen as in Lemma 1. Then X^* is called the controlling critical element. If $N \subset M$ is any set, let S_N^p be the set in $[0, \infty)$ of times that the flow starting from $y(p)$ is contained in N. Let $\tau_N : J \to \mathbb{R}$ send p to $\lambda(S_N^p)$ where λ is Lebesgue measure. Note that τ_N is not always well-defined since S_N^p may not be Lebesgue measurable.

If X^* is an equilibrium point, let N be a compact, connected, codimension zero C^1 manifold with boundary such that $X^*(J) \subset N \subset M \times J \backslash Z^J$. If X^* is a periodic orbit, let $x(p_0) \in X^* \bigcap M \backslash Z^{p_0}$ be a point which permits a C^1 section S and a C^1 first return map. Shrink S so that it is contained in $M \backslash Z^{p_0}$. We will see below that for J and S sufficiently small, there is a C^1 first return map for S corresponding to f_p for $p \in J$ with a unique hyperbolic fixed point $x(p)$ such that $\{x(p)\}_{p\in J}$ is C^1 continuous. Let N be a flow out of a neighborhood of x_J in S_J with respect to f, and choose N such that it is compact and $N \subset M \times J \backslash Z^J$. Note that N is a codimension zero C^1 manifold with boundary. Theorem 2 states that τ_N is well-defined and continuous over $[p_0, p^*]$ with τ_N diverging to infinity as p approaches p^* from below.

Theorem 2. Assume the conditions of Lemma 1 hold and that N is constructed as above with $X_J^s \bigcap N = \emptyset$ and such that the orbit with initial condition y_p has nonempty, transversal intersection with N for $p \in [p_0, p^*]$. Then $\tau_N : [p_0, p^*] \to [0, \infty]$ is well-defined and continuous. In particular, $\lim_{p_0 \nearrow p^*} \tau_N(p) = \infty$.

Theorem 2 provides a theoretical justification for numerical algorithms which compute critical parameter values by varying a parameter so as to maximize the time the trajectory spends inside a neighborhood of the controlling critical element. This algorithm is described in detail in [6], [7].

The proofs of these results rely on a large number of technical lemmas. Many proofs have been omitted because they are nearly identical to prior proofs in [5]. Otherwise, sketches of the proofs are provided, but complete proofs are omitted for brevity.

V. Proofs

Lemma 2. If $\{f_p, s_p\}_{p\in J}$ is a strong C^1 continuous family of event-selected vector fields on M, then for J sufficiently small, $p \in J$ implies that f_p and s_p determine an event-selected C^1 vector field over M.

Proof Sketch of Lemma 2. Since $\{f_p, s_p\}_{p\in J}$ is a strong C^1 continuous family of event-selected vector fields on M, by definition there exists a set of families of vector fields $\{f_p^b\}_{b\in B^m, p\in J}$ such that for each $p \in J$ and $x \in M$, $f(x) = f_p^{\sigma(x)}(x)$. Furthermore, as $\{f_p^b\}_{p\in J}$ is strong C^1 continuous for each $b \in B^m$, B^m is finite, and $\{s_p\}_{p\in J}$ is strong C^1 continuous, shrinking J if necessary implies that there exists a compact set $S \subset M$ such that for $p \in J$, $f_p|_{M\backslash S} = f_{p_0}|_{M\backslash S}$. Hence, as f_{p_0} and s_{p_0} determine an event-selected C^1 vector field, to show that f_p and s_p determine an event-selected C^1 vector field for each $p \in J$ it suffices to show that their restriction to S is event-selected C^1.

To do so, first it must be shown that the switching surfaces Z_j^p for $j \in \{1, ..., m\}$ are transverse over S. But, since the switching surfaces intersected with S are compact submanifolds, strong C^1 perturbations to s_{p_0} (such as s_p for $p \in J$) result in C^1 perturbations to the switching surfaces, and since transversal intersections of compact submanifolds (possibly with boundary) is preserved under C^1 perturbations to the submanifolds, it can be shown that for J sufficiently small, transverse intersections of the switching surfaces on S can be preserved.

Finally, it must be shown that for every $p \in J$ and $x \in S$, there exists an open set $U_{(x,p)}$ containing x in M such that $U_{(x,p)}$ is an event neighborhood for f_p. To do so, first fix $x \in S$ and let $K \subset \{1, ..., m\}$ denote the switching surfaces of Z^{p_0} which x is contained in. Then the distance of x from every switching surface which x is not contained in must be positive and, choosing $J_x \subset J$ sufficiently small, will remain positive. Therefore, there exists U_x an open neighborhood in M containing x such that for $p \in J_x$ the intersection of

U_x with the switching surfaces consists at most of subsets of switching surfaces which contained x for p_0. As the switching surfaces in K meet transversely at x for p_0, it can be shown that for J_x sufficiently small, the switching surfaces in K still have nonempty, transversal intersection in U_x.

Further argument involving continuity of the derivatives of s and the derivatives of the selection functions shows that the bounds on $d(s_j^p)_x(f(x))$ for $x \in U_x$, which hold for $p = p_0$ since f_{p_0} and s_{p_0} determine an event-selected vector field, will also hold for $p \in J_x$ after shrinking J_x if necessary (and after small changes to the constant c in the definition). Then U_x is an event neighborhood for f_p for any $p \in J_x$. As $\{U_x\}_{x\in S}$ is an open cover of S compact, there exists a finite subcover $\{U_{x_i}\}_{i=1}^l$. Then let $J = \bigcap_{i=1}^l J_{x_i}$. For every $y \in S$ and $p \in J$, $y \in U_{x_i}$ for some i, and U_{x_i} is an event neighborhood for f_p by the construction above. □

Let f be the rough vector field on $M \times J$ defined by $f(x, p) = (f_p(x), 0)$.

Lemma 3. For J sufficiently small, f possesses a piecewise C^1 flow $\phi : M \times \mathbb{R} \times J \to M$.

Proof Sketch of Lemma 3. The main idea of the proof is to modify the proof of existence of a flow for a fixed parameter event-selected C^1 vector field in [10, Section 3] to incorporate parameter variation over $p \in J$. The key challenge is to construct a local flow for f. Once this is accomplished, the local flows can be combined into a global flow in a manner entirely analogous to the fixed parameter case [10, Corollary 1], which itself is analogous to the case of smooth vector fields [11, Theorem 9.12].

By Lemma 2, f_p and s_p determine an event-selected C^1 vector field for every $p \in J$. The event-selected neighborhood condition in the definition of event-selected discontinuous ensures that for each $j \in \{1, ..., m\}$ and for each $p \in J$, if $x(t)$ is an integral curve of f_p then $\sigma_j(x(t))$ can only transition in one direction as time passes forwards - either from $+1$ to -1 or from -1 to $+1$ (depending on whether the bound in the definition is for $+c$ or $-c$). This ensures that switching surfaces are crossed monotonically with respect to time, ie. they can only be crossed in one direction. Using this fact, the local flow is constructed near a point $(x, p) \in M \times J$ as follows.

The monotonicity discussed above makes it possible to identify a unique selection function f_p^b whose flow will dictate the flow of f starting at (x, p) for a small interval in forwards time (similarly in backwards time). The first intersection time of (x, p) under the flow of f_p^b with each switching surface is then found (if it exists). The switching surface with the earliest such intersection time is then identified, say Z_j^p. Then the flow of f_p^b is used to flow (x, p) forwards in time until it intersects Z_j^p. Afterwards, the above process is repeated. It cannot persist forever, even for large flow times, because there are only a finite number of sign transitions permitted. In [10, Section 3], this flow is shown to be piecewise C^1 by showing that the the intersection times of the flow of each selection function with each switching surface are each C^1 functions of initial condition. This follows by the implicit function theorem and since the selection functions have flows which are C^1 with respect to time and initial condition.

The key change required here is to note that the selection functions considered here have flows that are C^1 with respect to time, initial condition, and to parameter. Therefore, the implicit function theorem shows that the intersection times of the flow of each selection function with each switching surface are each C^1 functions of both initial condition and parameter. Ultimately, this is then used to show that the local flow which results from the above construction is piecewise C^1 with respect to initial condition, time, and parameter. These local flows are then combined into a global flow as discussed above. □

Lemma 4. Z^J is closed and C^J is open in $M \times J$.

Proof of Lemma 4. Since $\{s_p\}_{p\in J}$ is C^1 continuous, there exists $s : M \times J \to \mathbb{R}^m$ such that s is C^1. Then $Z^J = s^{-1}(0)$ is closed in $M \times J$ because s is continuous. Hence, $C^J = M \times J \backslash Z^J$ is open in $M \times J$. □

Lemma 5. For any $(x, t, p) \in M \times \mathbb{R} \times J$ such that $x, \phi(x, t, p) \in C^J$, there exist open neighborhoods U^x, T^x, and J^x of x, t, and p, respectively, such that $\phi|_{U^x \times T^x \times J^x}$ is C^1.

Proof Sketch of Lemma 5. By Lemma 4, C^J is open in $M \times J$. By Lemma 3, there exists a piecewise C^1 flow ϕ for f which has discontinuous derivative only on Z^J. As $x, \phi(x, t, p) \in C^J$ open and ϕ is continuous, it can be shown that there exist open neighborhoods U^x, T^x, and J^x of x, t, and p such that $U^x \times T^x \times J^x$ and $\phi(U^x \times T^x \times J^x)$ are contained in C^J. The above implies that $\phi|_{U^x \times T^x \times J^x}$ is C^1. □

Lemma 6. For any $p \in J$, $x \in M$, and $T > 0$ finite, $\phi(x, [0, T], p)$ intersects Z^p in only finitely many isolated points.

Proof Sketch of Lemma 6. For any $t \in [0, T]$, $\phi(x, t, p)$ is contained in an event neighborhood, so the time derivative of $s_p^j \circ \phi(x, t, p)$ is bounded away from zero almost everywhere for every $j \in \{1, ..., m\}$. This can then be used to show that intersection points of $\phi(x, [0, T], p)$ with Z^p, which are points where $s_p^j \circ \phi(x, t, p) = 0$ for some j, must be isolated. As $\phi(x, [0, T], p)$ is compact, this implies that it can only have finitely many. □

Lemma 7. Let $X^i(p_0)$ be a periodic orbit for f_{p_0}. Then there exists $x \in X^i(p_0) \bigcap C^J$ and a C^1 cross section S containing x in M such that the first return map is well-defined and C^1 on S. Furthermore, for J and S sufficiently small, $p \in J$ implies that the first return map is well-defined and C^1 on S, that it varies C^1 with p, and that f_p is transverse to S.

Proof Sketch of Lemma 7. By Lemma 6, since $X^i(p_0)$ is the image of the flow over a finite length of time, its intersection with Z^{p_0} is finite. So, there exists $x \in X^i(p_0) \backslash Z^{p_0}$. Let S be a cross section containing x, transverse to f^{p_0}, and contained in the connected component of $M \backslash Z^{p_0}$ which contains x. By continuity of the flow, shrinking S if necessary the first return map is well-defined on S. Let τ be the period of $X^i(p_0)$. By Lemma 5, for J sufficiently small there exists a neighborhood U of x such that $\phi(\cdot, \tau, \cdot)$ is C^1 over $U \times J$. Shrinking S such that $S \subset U$ then implies that the first return map restricted to S is C^1 for any $p \in J$ and that it varies C^1 with p. Transversality of f_p for $p \in J$ follows since f_{p_0} is transverse to S and since f_p is C^1 close to f_{p_0} for $p \in J$. □

Corollary 2. For J sufficiently small and for any $i \in I$ and $p \in J$, there exists a unique critical element $X^i(p)$ C^0-close to $X^i(p_0)$ such that $X^i(p)$ is hyperbolic in the sense described in Assumption 4.

Proof Sketch of Corollary 2. If $X^i(p_0)$ is an equilibrium point then it lies in C^J open. Since f is C^1 over C^J and $X^i(p_0)$ is hyperbolic with respect to f_{p_0}, this implies that for J sufficiently small, $p \in J$ implies $X^i(p)$ is a hyperbolic equilibrium point C^1 close to $X^i(p_0)$. If $X^i(p_0)$ is a periodic orbit then there exists a point $x \in X^i(p_0) \bigcap M \backslash Z^{p_0}$ such that there exists a cross section S containing x as a hyperbolic fixed point. By Lemma 7 above, the first return map is C^1 for any $p \in J$. Hence, for J sufficiently small S possesses a unique hyperbolic fixed point of the first return map for f_p for any $p \in J$. Under the flow of f_p this gives rise to a unique periodic orbit $X^i(p)$ which is C^0 close to $X^i(p_0)$. □

Lemma 8. For every $i \in I$ and $p \in J$, $W^s(X^i(p))$ and $W^u(X^i(p))$ are invariant, piecewise C^1 immersed submanifolds with $\tilde{W}^s(X^i(p)) = W^s(X^i(p)) \bigcap C^J$ and $\tilde{W}^u(X^i(p)) = W^u(X^i(p)) \bigcap C^J$.

Proof Sketch of Lemma 8. Fix $i \in I$ and $p \in J$. By Lemma 2, the flow of f_p is piecewise C^1, hence continuous. For any $x \in W^s(X^i(p))$, it can be shown that the flow yields a C^0 homeomorphism between an open subset of $W^s_{\text{loc}}(X^i(p))$ and a neighborhood of x in $W^s(X^i(p))$, showing that $W^s(X^i(p))$ is a topological submanifold. By Lemma 5, for any $x \in W^s(X^i(p)) \bigcap C^J$, it can be shown that the flow yields a C^1 homeomorphism between an open subset of $W^s_{\text{loc}}(X^i(p))$ and a neighborhood of x in $W^s(X^i(p))$, showing that $\tilde{W}^s(X^i(p))$ is a C^1 immersed submanifold. Density of $\tilde{W}^s(X^i(p))$ in $W^s(X^i(p))$ follows from density of C^J in M. An analogous argument works for $W^u(X^i(p))$. □

Lemma 9. $W^s(X^s_J)$ is open and invariant in $M \times J$.

Proof Sketch of Lemma 9. Invariance follows trivially. Note that $X^s_J \subset C^J$, which is open by Lemma 4. By Lemma 2, the flow of f is C^1 in a neighborhood of $X^s_J \subset C^J$ and continuous everywhere, so the proof of [5, Lemma 2] works to show that $W^s(X^s_J)$ is open in $M \times J$. □

Lemma 10. For any $i \in I$ such that $X^i(p_0)$ is an equilibrium point (periodic orbit with $x \in X^i(p_0)$ possessing a cross section S with a C^1 first return map by Lemma 7) and for $\epsilon > 0$ sufficiently small, there exists a compact set $D \subset W^u_{\text{loc}}(X^i_{J'}) \backslash X^i_{J'}$ ($D \subset W^u_{\text{loc}}(X^i_{J'}) \bigcap S \backslash X^i_{J'}$) and an open neighborhood N of D in $M \times J$ such that $N \subset D_\epsilon$, $D_\epsilon \bigcap X^i_J = \emptyset$, and $\bigcup_{t \le 0} \phi_t(N) \bigcup W^s(X^i_J)$ contains an open neighborhood of $X^i_{p_0}(x)$ in $M \times J$.

Proof Sketch of Lemma 10. Fix $i \in I$. Shrinking J if necessary, if $X^i(p_0)$ is an equilibrium point (periodic orbit) then we may assume $X^i_J \subset C^J$ ($S \times J \subset C^J$ by Lemma 4). The subsequent proof is nearly identical to the proof of [5, Lemma 4] since the flow of $f|_{C^J}$ is C^1, except if $X^i(p_0)$ is a periodic orbit then the first return map should be substituted for the time-one flow $\phi(\cdot, 1, p)$ in that proof, and the resulting neighborhood $N \subset S \times J$ should be flowed out from S to get an open neighborhood in $M \times J$. □

Lemma 11. For any $i \in I$, $(W^u(X^i_{p_0}) - X^i_{p_0}) \bigcap \overline{W}^s(X^s_J) \neq \emptyset$.

Proof Sketch of Lemma 11. The proof is identical to the proof of [5, Lemma 5] except if $X^i_{p_0}$ is a periodic orbit then $\bigcup_{t \le 0} \phi_t(N) \bigcup W^s(X^i_J)$ contains a neighborhood of x in $M \times J$, where $x \in X^i_{p_0}$ possesses a cross section S with a C^1 first return map as in Lemma 7. □

Lemma 12. If $W^s(X^i(p_0)) \bigcap W^u(X^j(p_0)) \neq \emptyset$ then $n^u(X^j(p_0)) \ge n^u(X^i(p_0))$ for any $i, j \in I$. If $X^j(p_0)$ is an equilibrium point, $n^u(X^j(p_0)) > n^u(X^i(p_0))$.

Proof Sketch of Lemma 12. By Lemma 6, intersections of an orbit with Z^J are isolated. Hence, by invariance, intersection of $W^s(X^i(p_0))$ with $W^u(X^j(p_0))$ implies intersection of $\tilde{W}^s(X^i(p_0))$ with $\tilde{W}^u(X^j(p_0))$. As the latter are transverse C^1 immersed submanifolds, the proof proceeds as in [5, Lemma 6]. □

Lemma 13.
If $(W^s(X^i(p_0)) \backslash X^i(p_0)) \bigcap (W^u(X^j(p_0)) \backslash X^j(p_0)) \neq \emptyset$ and $(W^s(X^j(p_0)) \backslash X^j(p_0)) \bigcap (W^u(X^k(p_0)) \backslash X^k(p_0)) \neq \emptyset$ then $(\tilde{W}^s(X^i(p_0)) \backslash X^i(p_0)) \bigcap (\tilde{W}^u(X^k(p_0)) \backslash X^k(p_0)) \neq \emptyset$.

Proof Sketch of Lemma 13. By invariance, there exist $y \in W^s(X^i(p_0)) \bigcap W^u_{\text{loc}}(X^j(p_0))$ and $z \in W^u(X^k(p_0)) \bigcap W^s_{\text{loc}}(X^j(p_0))$. These can be chosen so that $y, z \in C^J$. Hence, by Lemma 8, $y \in \tilde{W}^s(X^i(p_0)) \bigcap \tilde{W}^u_{\text{loc}}(X^j(p_0))$ and $z \in \tilde{W}^u(X^k(p_0)) \bigcap \tilde{W}^s_{\text{loc}}(X^j(p_0))$. As these are transverse C^1 immersed submanifolds, and since C^J is open by Lemma 4, working locally near $X^j(p_0)$ (or a point on its orbit), the proof proceeds identically to the proof of [5, Lemma 8]. □

Corollary 3. Suppose that $W^u(X^i(p_0)) \bigcap W^s(X^s(p_0)) \neq \emptyset$ and $W^s(X^i(p_0)) \bigcap W^u(X^j(p_0)) \neq \emptyset$. Then $\tilde{W}^u(X^j(p_0)) \bigcap \tilde{W}^s(X^s(p_0)) \neq \emptyset$.

Proof Sketch of Corollary 3. Follows immediately from Lemma 13. □

Lemma 14. For any $X^i(p_0)$, $W^s(X^i(p_0)) \bigcap W^u(X^i(p_0)) = X^i(p_0)$.

Proof Sketch of Lemma 14. The proof is identical to the proof of [5, Lemma 7]. □

Lemma 15. There do not exist any transverse heteroclinic cycles of critical elements contained in $\partial W^s(X^s)$. Hence, every heteroclinic sequence of critical elements contained in $\partial W^s(X^s)$ has finite length.

Proof Sketch of Lemma 15. The proof is identical to the proof of [5, Lemma 9]. □

Lemma 16. For any $i \in I$, $W^u(X^i(p_0)) \bigcap W^s(X^s(p_0)) \neq \emptyset$.

Proof Sketch of Lemma 16. The proof is identical to the proof of [5, Lemma 10]. □

Lemma 17. If $W^u(X^i(p)) \bigcap W^s(X^s(p)) \neq \emptyset$ for any $p \in J$ then $W^s(X^i(p)) \subset \partial W^s(X^s(p))$.

Proof Sketch of Lemma 17. By invariance, if $X^i(p)$ is an equilibrium point (periodic orbit) it suffices to show that $W^s_{\text{loc}}(X^i(p))$ ($W^s_{\text{loc}}(X^i(p)) \bigcap S$) is contained in $\partial W^s(X^s(p))$. So, let $y \in W^s_{\text{loc}}(X^i(p))$ ($y \in$

$W^s_{\text{loc}}(X^i(p)) \bigcap S$), let $\epsilon > 0$, and let D be a closed C^1 disk containing y such that D is contained in the ϵ ball centered at y and is transverse to $W^s_{\text{loc}}(X^i(p))$ ($W^s_{\text{loc}}(X^i(p)) \bigcap S$). By the Inclination Lemma, a C^1 embedded submanifold of D converges C^1 to $W^u_{\text{loc}}(X^i(p))$ ($W^u_{\text{loc}}(X^i(p)) \bigcap S$). By invariance, $W^u_{\text{loc}}(X^i(p))$ ($W^u_{\text{loc}}(X^i(p)) \bigcap S$) intersects $W^s(X^s(p))$. Since $W^s(X^s(p))$ is open, $D \cap W^s(X^s(p)) \neq \emptyset$ so the distance $d(y, \overline{W}^s(X^s(p)) \leq \epsilon$. As this holds for all $\epsilon > 0$, the distance must equal zero, which implies $y \in \partial W^s(X^s(p))$. □

Proof Sketch of Theorem 1. That $\partial W^s(X^s_J) \supset \sqcup_{p \in J} \partial W^s(X^s(p))$ follows trivially. By Assumption 3, $\partial W^s(X^s_J) \subset \bigcup_{i \in I} W^s(X^i_J)$. Fix $i \in I$. By Lemma 16, $W^u(X^i(p_0)) \bigcap W^s(X^s(p_0)) \neq \emptyset$. If $X^i(p_0)$ is an equilibrium point let $B(p) = W^s_{\text{loc}}(X^i(p))$. If $X^i(p_0)$ is a periodic orbit let $B(p) = W^s_{\text{loc}}(x(p))$ where $x(p)$ is the fixed point of the first return map for f_p as in Lemma 7. By invariance, $B(p_0) \bigcap W^s(X^s(p_0)) \neq \emptyset$. As $B(p)$ is C^1 continuous and $W^s(X^s_J)$ is open, J sufficiently small implies that $B(p) \bigcap W^s(X^s(p)) \neq \emptyset$. By Lemma 17, this implies that $W^s(X^i(p)) \subset \partial W^s(X^s(p))$. Hence, $\bigcup_{i \in I} W^s(X^i_J) \subset \sqcup_{p \in J} \partial W^s(X^s(p))$. Combining the inclusions above yields the result. □

Proof Sketch of Corollary 1. The proof is identical to the proof of [5, Corollary 1] except for $X^i(p_0)$ a periodic orbit, $W^s_{\text{loc}}(X^i(p_0)) \bigcap S$ is substituted for $W^s_{\text{loc}}(X^i(p_0))$ and the first return map is used in place of the flow. □

Proof Sketch of Lemma 1. The proof is identical to the proof of [5, Lemma 1] since it only requires continuity of the flow. □

Lemma 18. Let $q \in [p_0, p^*)$. Let $T \subset [0, \infty)$ be compact with $\phi(y(q), T, q) \bigcap \partial N = \emptyset$ where N is as in the statement of Theorem 2. Then there exists an open interval $P \subset [p_0, p^*)$ containing q such that $p \in P$ implies $\phi(y(p), T, p) \bigcap \partial N = \emptyset$.

Proof Sketch of Lemma 18. The proof is identical to the proof of [5, Lemma 11] since it only requires continuity of the flow, which follows from Lemma 2. □

Lemma 19. For $p \in [p_0, p^*)$, the number of intersections of the orbit with initial condition $y(p)$ with ∂N is finite, even, and constant. Furthermore, there exist C^1 functions $\{k_l\}_{l=1}^L$ such that $k_l : [p_0, p^*) \to [0, \infty)$ and $k_l(p)$ is the time corresponding to the lth intersection of the orbit with initial condition $y(p)$ with ∂N.

Proof Sketch of Lemma 19. Since $N \subset C^J$, C^J is open by Lemma 4, and the flow of f restricted to C^J is C^1 by Lemma 2, the flow of f is C^1 over a neighborhood of N and C^0 elsewhere. Then the proof is identical to the proof of [5, Lemma 12] since this requires only that the flow is C^1 over a neighborhood of N and C^0 elsewhere. □

Proof Sketch of Theorem 2. By the proof of Lemma 19, the flow of f is C^1 on a neighborhood of N and C^0 elsewhere, so the proof of the Theorem is identical to the proof of [5, Theorem 2]. □

VI. Conclusion

The paper establishes that for almost generic vector fields with C^1 event-selected discontinuities, the boundary of the region of attraction (RoA) of a hyperbolic stable equilibrium point is equal to the union of the stable manifolds of the equilibria and periodic orbits contained in the boundary. Furthermore, this decomposition persists, and the boundary varies Chabauty continuously, under small changes in parameter.

This theory underpins novel algorithms for numerically computing critical parameter values in nonlinear systems with clipping limits and switching, such as power systems. It establishes the existence of a controlling critical element in the boundary of the RoA. A critical parameter value can be found such that the corresponding trajectory lies in the stable manifold of this critical element. For a neighborhood of this critical element, the amount of time the trajectory spends inside this neighborhood is a continuous function of parameter value and diverges to infinity as the parameter value approaches its critical value. This result forms the basis for an algorithm which numerically computes critical parameter values by varying parameters to maximize the time the trajectory spends inside a neighborhood of the critical element.

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