Dynamic Output Feedback Control of the Liouville Equation for Discrete-Time SISO Linear Systems

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Abstract—**In this work, we address the so-called Liouville control problem for SISO discrete-time linear systems in the class of Gaussian distributions. In particular, we propose a systematic procedure for the characterization of a dynamic output feedback policy that will transfer the output of the system, which is a known Gaussian random variable, to a goal Gaussian distribution after a finite number of stages. In the proposed approach, the Liouville control problem is reduced to two decoupled (finite-dimensional) quadratic programs, one of which is subject to a single affine constraint, which is a convex program, whereas the other one is subject to a single quadratic equality constraint. Despite the fact that the second optimization problem is not convex, one can characterize its exact solution via a systematic procedure without resorting to convex relaxation techniques, which may yield suboptimal or even infeasible solutions to the original (non-convex) optimization problem. Finally, we present numerical simulations that illustrate the key ideas of this work.**

I. Introduction

We consider the problem of characterizing a dynamic output feedback control policy that will steer the uncertain output variable of a SISO discrete-time linear system to a goal Gaussian distribution. This problem, which we refer to as the *Liouville control problem*, can be put under the umbrella of the so-called Liouville transport problems restricted to the space of Gaussian distributions and subject to linear dynamic constraints. The motivation for the Liouville control problem comes from a variety of real-world applications related to, for instance, quality control and industrial / manufacturing processes, in which the design specifications of the product (output of the process) are described in probabilistic ways [1]. Characteristic examples include industrial processes for thickness control of paper sheets and plastic films [2], [3]. Another envisioned application is the steering control of a multi-agent system whose macroscopic state as a whole can be described in terms of a probability distribution in lieu of a vector formed by concatenating the states of its constituent agents. In this work, we show that the Liouville control problem can be reduced to two decoupled optimization problems. The first problem is a standard convex quadratic program, whereas the second one is a tractable (deterministic) non-convex optimization problem with a convex quadratic performance index but a non-convex quadratic equality constraint. We subsequently utilize a systematic procedure that furnishes the exact solutions to the two optimization problems, which in turn induce the optimal control policy that solves the Liouville control problem.

Literature Review: The Liouville control problem in the continuous-time framework with perfect state information was first proposed by Brockett [4]. In particular, [4] considers the problem of steering the state of a continuous-time linear system, which is initially drawn from a known (multi-variate) Gaussian distribution, to a goal (multi-variate) Gaussian distribution in a given (finite) terminal time and provides conditions for the solvability of this steering problem. It is not surprising that the solution to the latter problem can be expressed as the combination of a feed-forward signal and a time-varying state feedback policy. The results presented in [4] have been extended to the case of static output feedback and the case of parallel interconnections of linear systems in [5]. All the previous problems can be put under the umbrella of the so-called *ensemble control* [6], [7], or control of families of systems, according to [8]. A similar class of problems deal with the steering of the (uncertain) state of a stochastic linear system, which is subject to white Gaussian noise, to a goal Gaussian distribution in either continuous time [1], [9] or discrete time [10]–[14].

Main contributions: The focus of [4], which is the main inspiration of this work, is mainly on the investigation of the question of controllability / reachability under full state feedback without, however, touching upon the practical problem of control design. In particular, [4] does not present a systematic procedure for the characterization of a control law that will realize the required transfer of the state's distribution to the goal Gaussian distribution. In this work, we address the finite-horizon Liouville control problem in the class of Gaussian distributions for discrete-time SISO (deterministic) linear systems. In our approach, we consider control policies that are sequences of control laws which can be expressed as affine functions of any realization of the history of (past and present) output measurements similarly with the so-called affine disturbance feedback parametrization, which is used in control design problems for discrete-time stochastic linear systems [15]. We show that with this particular control policy parametrization, the Liouville control problem considered herein can be reduced to a system of two decoupled optimization problems.

In particular, the first optimization problem corresponds to a convex quadratic program (QP) subject to affine constraints, whose solution will yield the feed-forward input sequence that will steer the mean of the output's distribution to the prescribed goal quantity. On the other hand, the second problem is a quadratic program subject to an equality quadratic constraint, which is a non-convex program, whose solution will yield the gains of a dynamic output feedback control policy that is intended to realize the transfer of the variance of the system's output to the prescribed goal value. While the first problem admits a closed form solution, the second problem poses some significant challenges due to the presence of the quadratic

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equality constraint (terminal variance constraint), which is, in general, non-convex (in the sense that the feasible set it induces is non-convex). We will henceforth refer to the latter quadratic program with the non-convex quadratic (equality) constraint as the GQCQP (Generalized Quadratically Constrained Quadratic Program). It should be mentioned here that problems similar to the GQCQP have received some attention in the literature of optimization and applied statistics due to their relevance to many applications [16]–[20]. It turns out that the exact solution to the GQCQP, and thus the Liouville control problem, can be characterized by means of a systematic procedure which combines analytical and computational techniques. An alternative approach to address our problem would be to employ convex relaxation techniques [21]–[23]. Such techniques are intended to reduce a given non-convex optimization problem to a corresponding "relaxed" convex program, under the assumption that the solution to the latter will be a good approximation of the solution to the original problem. It is very likely, however, that the solution to the relaxed problem may be suboptimal or even fail to satisfy the constraints of the original non-convex program.

Structure of the paper: The rest of the paper is organized as follows. In Section II, we formulate the Liouville control problem, which we subsequently reduce to a finite-dimensional optimization problem in Section III. A systematic approach for the solution of the latter problem, which is equivalent to a decoupled system of two tractable optimization problems, is presented in Section IV. Numerical simulations are presented in Section V and finally, Section VI concludes the paper with a summary of remarks.

II. Problem Formulation

A. Notation and Some Key Background Results and Properties

We denote by $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ the set of n-dimensional real vectors and $m \times n$ real matrices, respectively. We write $|\boldsymbol{\alpha}|$ to denote the 2-norm of a vector $\boldsymbol{\alpha} \in \mathbb{R}^n$. We denote by $\mathrm{int}(\mathcal{S})$ the interior of a set $\mathcal{S} \subseteq \mathbb{R}^n$. We write $\mathbb{Z}_0^+$ and $\mathbb{Z}^+$ to denote the set of non-negative integers and strictly positive integers, respectively. Given $M, N \in \mathbb{Z}_0^+$ with $M \leq N$, we denote by $\mathbb{Z}_M^N$ the discrete set $\{M, \dots, N\} = [M, N] \cap \mathbb{Z}_0^+$. $\mathbb{E}[\cdot]$ denotes the expectation operator. We write $\mathbf{0}_{m\times p}$ (or simply, $\mathbf{0}$) and $\mathbf{I}_m$ (or simply, $\mathbf{I}$) to denote the $m \times p$ zero matrix and the $m \times m$ identity matrix, respectively. We denote by $\mathcal{R}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A})$ the range space and the null space of the matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, respectively. We denote the i-th row and the j-th column of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ by $\mathrm{row}_i(\mathbf{A}) \in \mathbb{R}^n$ and $\mathrm{col}_j(\mathbf{A}) \in \mathbb{R}^m$, respectively. We also denote by $\mathbf{A}^\dagger$ the Moore-Penrose (pseudo-) inverse of $\mathbf{A}$; it holds that $\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}$ and $\mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger$. The space of $n \times n$ symmetric (real) matrices is denoted by $\mathbb{S}_n$. Finally, we will denote the convex cone of $n \times n$ symmetric positive definite and positive semi-definite matrices by $\mathbb{S}_n^{++}$ and $\mathbb{S}_n^{+}$, respectively. Given a square matrix $\mathbf{A}$, we denote its trace by $\mathrm{trace}(\mathbf{A})$. Given $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times m}$, we have that $\mathrm{trace}(\mathbf{AB}) = \mathrm{trace}(\mathbf{BA})$ (cyclic property). By $\otimes$ we denote the Kronecker product operator. Furthermore, we denote by $\mathrm{vec}(\cdot)$ the vectorization (linear) operator that maps a matrix $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{m\times n}$ to a mn-dimensional column vector denoted by $\mathrm{vec}(\mathbf{A})$, where $\mathrm{vec}(\mathbf{A}) := [\mathrm{col}_1(\mathbf{A})^{\mathrm{T}}, \dots, \mathrm{col}_n(\mathbf{A})^{\mathrm{T}}]^{\mathrm{T}}$. Given three real matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ of compatible dimensions, it holds that $\mathrm{vec}(\mathbf{ABC}) = (\mathbf{C}^{\mathrm{T}} \otimes \mathbf{A})\mathrm{vec}(\mathbf{B})$. In addition, $\mathrm{trace}(\mathbf{A}^{\mathrm{T}}\mathbf{B}) = \mathrm{vec}(\mathbf{A})^{\mathrm{T}}\mathrm{vec}(\mathbf{B})$. The set of $N \times N$ lower triangular matrices will be denoted by $\mathfrak{LT}(\mathbb{R}^{N\times N})$, that is, $\mathbf{B} = [b_{ij}] \in \mathfrak{LT}(\mathbb{R}^{N\times N})$ if and only if $b_{ij} = 0$ for all $j > i$. The half-vectorization (linear) operator $\mathrm{vech}(\cdot)$ maps a matrix $\mathbf{B} = [b_{ij}] \in \mathfrak{LT}(\mathbb{R}^{N\times N})$ to an υ-dimensional column vector, with $\upsilon := N(N+1)/2$, which is denoted by $\mathrm{vech}(\mathbf{B})$ and is defined as follows: $\mathrm{vech}(\mathbf{B}) := [\mathrm{colh}_1(\mathbf{B})^{\mathrm{T}}, \dots, \mathrm{colh}_N(\mathbf{B})^{\mathrm{T}}]^{\mathrm{T}}$, where $\mathrm{colh}_j(\mathbf{B}) := [b_{jj}, \dots, b_{Nj}]^{\mathrm{T}} \in \mathbb{R}^{N+1-j}$. Note that there exists a full-column rank matrix $\mathbf{\Omega} \in \mathbb{R}^{N^2\times\upsilon}$, that is, $\mathrm{rank}(\mathbf{\Omega}) = \upsilon$ (note that $N^2 \geq \upsilon$ for all $N \in \mathbb{Z}^+$), whose columns form a set of υ orthonormal vectors (that is, $\mathbf{\Omega}^{\mathrm{T}}\mathbf{\Omega} = \mathbf{I}_\upsilon$) such that $\mathrm{vec}(\mathbf{B}) = \mathbf{\Omega}\mathrm{vech}(\mathbf{B})$ and $\mathrm{vech}(\mathbf{B}) = \mathbf{\Omega}^{\mathrm{T}}\mathrm{vec}(\mathbf{B})$. Finally, we will denote by $\mathrm{vec}^{-1}(\cdot)$ and $\mathrm{vech}^{-1}(\cdot)$ the inverse (linear) operators of $\mathrm{vec}(\cdot)$ and $\mathrm{vech}(\cdot)$, respectively. Finally, given a function $f : \mathcal{D} \subseteq \mathbb{R}^m \to \mathbb{R}^n$, we denote by $\mathfrak{Im}(f|\mathcal{D})$ the image of $\mathcal{D}$ under f, that is, $\mathfrak{Im}(f|\mathcal{D}) = \{y \in \mathbb{R}^n : \ y = f(x), \ x \in \mathcal{D}\}$.

B. Formulation of the Liouville Control Problem for SISO Discrete-Time Linear Systems

For a given $N \in \mathbb{Z}^+$, let $\{\mathbf{A}(t) \in \mathbb{R}^{n\times n} : \ t \in \mathbb{Z}_0^{N-1}\}$, $\{b(t) \in \mathbb{R}^{n\times 1} : \ t \in \mathbb{Z}_0^{N-1}\}$ and $\{c(t) \in \mathbb{R}^{n\times 1} : \ t \in \mathbb{Z}_0^{N-1}\}$ denote known sequences of matrices and column vectors, respectively. Let us also consider the following discrete-time linear system:

$$x(t+1) = \mathbf{A}(t)x(t) + b(t)u(t), \tag{1a}$$

$$y(t) = c(t)^{\mathrm{T}}x(t), \tag{1b}$$

for $t \in \mathbb{Z}_0^{N-1}$, where $x(0) = x_0$ is a random vector drawn from the Gaussian distribution $\mathcal{N}(m_0, \mathbf{\Sigma}_0)$ with $m_0 \in \mathbb{R}^n$ and $\mathbf{\Sigma}_0 \in \mathbb{S}_n^{++}$ be given. In addition, $X_0^N := \{x(t) : \ t \in \mathbb{Z}_0^N\}$, $U_0^{N-1} := \{u(t) : \ t \in \mathbb{Z}_0^{N-1}\}$, and $Y_0^{N-1} := \{y(t) : \ t \in \mathbb{Z}_0^{N-1}\}$ denote, respectively, the state, the control input, and the output *random* (finite-length) sequences[1].

Our objective is to find a control policy that will steer the last element of Y_0^{N-1}, which is a normal random variable, to a goal Gaussian distribution while minimizing the expected value of a finite sum of cost-per-stage functions, which are convex quadratic functions of the output $y(t)$ and the input $u(t)$ of the linear system (1a)-(1b) as t runs through $\mathbb{Z}_0^{N-1}$. It is assumed that the set of admissible control policies consists of all control policies π which can be expressed as sequences of control laws $\kappa(\cdot; t)$ that are causal (non-anticipative), measurable functions of the σ-field generated by $Y_0^t := \{y(\tau) : \tau \in \mathbb{Z}_0^t\}$. In other words, $u(t)$ can be computed as a (measurable) function of the realization of the t-truncated (random) output sequence Y_0^t. We will henceforth restrict our attention to admissible control policies $\pi = \{\kappa(\cdot; t) : \ t \in \mathbb{Z}_0^{N-1}\}$ for which the feedback control law $\kappa(\cdot; t)$ at stage t can be expressed as an *affine* combination of the elements of the realization of Y_0^t, that is,

$$\kappa(Y_0^t; t) := \nu(t) + \sum_{\tau=0}^{t} k_y(t,\tau)y(\tau), \quad \text{for all} \ \ t \in \mathbb{Z}_0^{N-1},$$

[1]Although (1a)-(1b) describe a deterministic system, the fact that the initial state is a random vector renders the previous sequences random.

where $k_y(t,\tau) \in \mathbb{R}$ is the gain that determines the effect of the output measurement $y(\tau)$ on the control input $u(t)$, for all $(t,\tau) \in \mathbb{Z}_0^{N-1} \times \mathbb{Z}_0^{N-1}$ with $t \geq \tau$. We will denote the set comprised of these control policies by Π. Next, we give the precise formulation of the Liouville control problem for the system (1a)–(1b).

Problem 1: Assume that $\mathbb{E}[x_0] = m_0 = 0$ and $\mathbb{E}[x_0 x_0^{\mathrm{T}}] = \mathbf{\Sigma}_0$, where $\mathbf{\Sigma}_0 \in \mathbb{S}_n^{++}$, and let also $\mu_{\mathsf{f}} \in \mathbb{R}$, $\sigma_{\mathsf{f}} > 0$ and $N \in \mathbb{Z}^+$ be given. In addition, let $Q_0^{N-1} := \{q(t) : \ t \in \mathbb{Z}_0^{N-1}\}$ and $R_0^{N-1} := \{r(t) : \ t \in \mathbb{Z}_0^{N-1}\}$ be known (finite-length) sequences of non-negative and positive numbers, respectively. Then, find a control policy $\pi := \{\kappa(\cdot; t) : \ t \in \mathbb{Z}_0^{N-1}\} \in \Pi$ that minimizes the performance index:

$$J(\pi) := \mathbb{E}\Big[\sum_{t=0}^{N-1} q(t)y(t)^2 + r(t)u(t)^2\Big], \tag{2}$$

subject to the equality constraints induced by (1a)-(1b), and the following boundary conditions: $y(0) \sim \mathcal{N}(0, \sigma_0)$ and $y(N-1) \sim \mathcal{N}(\mu_{\mathsf{f}}, \sigma_{\mathsf{f}})$, or equivalently,

$$\mathbb{E}\left[y(0)\right] = 0, \qquad \mathbb{E}\left[y(0)^2\right] = \sigma_0^2, \tag{3a}$$

$$\mathbb{E}\left[y(N-1)\right] = \mu_{\mathsf{f}}, \qquad \mathbb{E}\left[(y(N-1) - \mu_{\mathsf{f}})^2\right] = \sigma_{\mathsf{f}}^2, \tag{3b}$$

where $\sigma_0^2 := c^{\mathrm{T}} \mathbf{\Sigma}_0 c$.

Remark 1 The assumption that $\mathbb{E}[x(0)] = 0$, and thus $\mathbb{E}[y(0)] = 0$, is made to simplify the subsequent analysis and streamline the presentation. A similar assumption is typically made in the standard formulation of the basic reachability or controllability problem in the literature of deterministic linear systems [24], according to which the initial or terminal (deterministic) state, respectively, is taken to be the origin. The case when $\mathbb{E}[x(0)] \neq 0$ can be treated similarly, after the necessary modifications have been carried out.

III. Conversion of the Stochastic Optimal Control Problem into an Optimization Problem

A. Preliminary Analysis

In this section, we will show how to convert Problem 1 to an equivalent optimization problem that is computationally tractable. To this aim, we will first express the solution to the recursion equation (1a) and the output equation (1b) in the following compact form:

$$\boldsymbol{x} = \boldsymbol{\mathcal{B}}\boldsymbol{u} + \mathbf{\Gamma} x_0, \qquad \boldsymbol{y} = \boldsymbol{\mathcal{C}}\boldsymbol{x}, \tag{4}$$

where $\boldsymbol{x} \in \mathbb{R}^{(N+1)n}$, $\boldsymbol{u} \in \mathbb{R}^N$ and $\boldsymbol{y} \in \mathbb{R}^N$ correspond, respectively, to the concatenations of the elements of X_0^N, U_0^{N-1} and Y_0^{N-1}. In particular, $\boldsymbol{x} := [x(0)^{\mathrm{T}}, \ldots, x(N)^{\mathrm{T}}]^{\mathrm{T}}$, $\boldsymbol{u} := [u(0), \ldots, u(N-1)]^{\mathrm{T}}$, and $\boldsymbol{y} := [y(0), \ldots, y(N-1)]^{\mathrm{T}}$. In addition, the matrix $\boldsymbol{\mathcal{B}} \in \mathbb{R}^{(N+1)n \times N}$ is defined as follows: $\boldsymbol{\mathcal{B}} := \begin{bmatrix}\mathbf{0}_{N\times 1}, \ \boldsymbol{\mathcal{B}}_2^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$ with

$$\boldsymbol{\mathcal{B}}_2 := \begin{bmatrix} b(0) & 0 & \ldots & 0 \\ \mathbf{\Phi}(2,1)b(0) & b(1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Phi}(N,1)b(0) & \mathbf{\Phi}(N,2)b(1) & \ldots & b(N-1) \end{bmatrix},$$

where

$$\mathbf{\Phi}(t,\tau) := \mathbf{A}(t-1)\ldots\mathbf{A}(\tau), \qquad \mathbf{\Phi}(\tau,\tau) = \mathbf{I},$$

for $(t,\tau) \in \mathbb{Z}_1^N \times \mathbb{Z}_0^N$ with $t \geq \tau$ (note that $\mathbf{\Phi}(t,\tau) = \mathbf{A}(t-1) = \mathbf{A}(\tau)$, when $t = \tau + 1$). Furthermore, we consider the matrices $\boldsymbol{\mathcal{C}} = \begin{bmatrix}\boldsymbol{\mathcal{C}}_1, \ \mathbf{0}_{N\times n}\end{bmatrix} \in \mathbb{R}^{N\times(N+1)n}$ and $\mathbf{\Gamma} \in \mathbb{R}^{(N+1)n\times n}$, which are defined, respectively, as follows:

$$\boldsymbol{\mathcal{C}}_1 := \begin{bmatrix} c(0)^{\mathrm{T}} & \mathbf{0}_{1\times n} & \ldots & \mathbf{0}_{1\times n} \\ \mathbf{0}_{1\times n} & c(1)^{\mathrm{T}} & \ldots & \mathbf{0}_{1\times n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \ldots & c(N-1)^{\mathrm{T}} \end{bmatrix}, \quad \mathbf{\Gamma} := \begin{bmatrix} \mathbf{I} \\ \mathbf{\Phi}(1,0) \\ \vdots \\ \mathbf{\Phi}(N,0) \end{bmatrix}.$$

Under the assumption that $\pi = \{\kappa(\cdot; t) : \ t \in \mathbb{Z}_0^{N-1}\} \in \Pi$, we can express the control input $u(t)$ as follows:

$$u(t) = \kappa(Y_0^t; t) = \nu(t) + \sum_{\tau=0}^{t} k_y(t,\tau) y(\tau), \tag{5}$$

for all $t \in \mathbb{Z}_0^{N-1}$. The previous equation can be written in compact form as follows:

$$\boldsymbol{u} = \boldsymbol{\nu} + \boldsymbol{\mathcal{K}}_{\boldsymbol{y}}\boldsymbol{y}, \tag{6}$$

where $\boldsymbol{\nu} := [\nu(0), \ldots, \nu(N-1)]^{\mathrm{T}} \in \mathbb{R}^N$ and $\boldsymbol{\mathcal{K}}_{\boldsymbol{y}} \in \mathfrak{LT}(\mathbb{R}^{N\times N})$, $\boldsymbol{\mathcal{K}}_{\boldsymbol{y}} = [\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}^{(i,j)}]$ (gain matrix) with

$$\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}^{(i,j)} := \begin{cases} k_y(i-1, j-1), & \text{if } i \geq j, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

In view of (4) and Eq. (6), we have

$$\boldsymbol{x} = \boldsymbol{\mathcal{B}}\boldsymbol{\nu} + \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}\boldsymbol{\mathcal{C}}\boldsymbol{x} + \mathbf{\Gamma} x_0, \tag{8}$$

which implies that

$$\boldsymbol{x} = \boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}(\boldsymbol{\mathcal{K}}_{\boldsymbol{y}})\boldsymbol{\nu} + \boldsymbol{\mathcal{X}}_{\mathbf{0}}(\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}) x_0, \tag{9a}$$

$$\begin{aligned}\boldsymbol{\mathcal{X}}_{\mathbf{0}}(\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}) &:= (\mathbf{I} - \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}\boldsymbol{\mathcal{C}})^{-1}\mathbf{\Gamma} \\ &= \mathbf{\Gamma} + \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}_{\boldsymbol{y}})^{-1}\boldsymbol{\mathcal{C}}\mathbf{\Gamma},\end{aligned} \tag{9b}$$

$$\begin{aligned}\boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}(\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}) &:= (\mathbf{I} - \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}\boldsymbol{\mathcal{C}})^{-1}\boldsymbol{\mathcal{B}} \\ &= \boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}_{\boldsymbol{y}})^{-1}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}.\end{aligned} \tag{9c}$$

Then, in view of (4) and (9a)–(9c) and the push-through identity, $\boldsymbol{y}$ can be written compactly as follows:

$$\boldsymbol{y} = \boldsymbol{\mathcal{Y}}_{\boldsymbol{\nu}}(\boldsymbol{\mathcal{K}}_{\boldsymbol{y}})\boldsymbol{\nu} + \boldsymbol{\mathcal{Y}}_{\mathbf{0}}(\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}) x_0, \tag{10a}$$

$$\boldsymbol{\mathcal{Y}}_{\mathbf{0}}(\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}) := (\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}_{\boldsymbol{y}})^{-1}\boldsymbol{\mathcal{C}}\mathbf{\Gamma}, \tag{10b}$$

$$\boldsymbol{\mathcal{Y}}_{\boldsymbol{\nu}}(\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}) := (\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}_{\boldsymbol{y}})^{-1}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}. \tag{10c}$$

Finally, in view of (6) and (10a)-(10c), we can express $\boldsymbol{u}$ as

$$\boldsymbol{u} = \boldsymbol{\mathcal{U}}_{\boldsymbol{\nu}}(\boldsymbol{\mathcal{K}}_{\boldsymbol{y}})\boldsymbol{\nu} + \boldsymbol{\mathcal{U}}_{\mathbf{0}}(\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}) x_0, \tag{11a}$$

$$\boldsymbol{\mathcal{U}}_{\mathbf{0}}(\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}) := \boldsymbol{\mathcal{K}}_{\boldsymbol{y}}(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}_{\boldsymbol{y}})^{-1}\boldsymbol{\mathcal{C}}\mathbf{\Gamma}, \tag{11b}$$

$$\boldsymbol{\mathcal{U}}_{\boldsymbol{\nu}}(\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}) := \mathbf{I} + \boldsymbol{\mathcal{K}}_{\boldsymbol{y}}(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}_{\boldsymbol{y}})^{-1}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}. \tag{11c}$$

Note that $(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}_{\boldsymbol{y}})^{-1}$ is always well defined given that $(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}) \in \mathfrak{LT}(\mathbb{R}^{N\times N})$ and the diagonal elements of the latter are equal to $\mathbf{I}$. In addition, $(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}_{\boldsymbol{y}})^{-1} \in \mathfrak{LT}(\mathbb{R}^{N\times N})$.

B. Introduction of the New Decision Variables

In this section, we will introduce two new decision variables, which we denote by $\mathbf{\Psi}$ and $\boldsymbol{v}$, that will allow us to reduce Problem 1 to a tractable optimization problem.

To this end, following [25], we first introduce the new decision variable $\mathbf{\Psi} \in \mathfrak{LT}(\mathbb{R}^{N\times N})$, where

$$\mathbf{\Psi} := \mathfrak{f}(\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}), \qquad \mathfrak{f}(\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}) := \boldsymbol{\mathcal{K}}_{\boldsymbol{y}}(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}_{\boldsymbol{y}})^{-1}. \tag{12}$$

In addition, $\mathfrak{f}(\boldsymbol{\mathcal{K}}_{\boldsymbol{y}})$ will belong to $\mathfrak{LT}(\mathbb{R}^{N\times N})$ as the product of two matrices in $\mathfrak{LT}(\mathbb{R}^{N\times N})$. Furthermore,

$$\boldsymbol{\mathcal{K}}_{\boldsymbol{y}} = \mathfrak{g}(\boldsymbol{\Psi}), \quad \mathfrak{g}(\boldsymbol{\Psi}) := (\mathbf{I} + \boldsymbol{\Psi}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}})^{-1}\boldsymbol{\Psi}. \tag{13}$$

Again, the expression of $\mathfrak{g}(\boldsymbol{\Psi})$ given in (13) is well defined for all $\boldsymbol{\Psi} \in \mathfrak{LT}(\mathbb{R}^{N\times N})$. We can then express $\boldsymbol{x}$ in terms of $\boldsymbol{\Psi}$ as follows:

$$\boldsymbol{x} \ = \boldsymbol{X}_{\boldsymbol{\nu}}(\boldsymbol{\Psi})\boldsymbol{\nu} + \boldsymbol{X}_{\mathbf{0}}(\boldsymbol{\Psi})x_0, \tag{14a}$$

$$\boldsymbol{X}_{\mathbf{0}}(\boldsymbol{\Psi}) := \boldsymbol{\mathcal{X}}_{\mathbf{0}}(\mathfrak{g}(\boldsymbol{\Psi})) = (\mathbf{I} + \boldsymbol{\mathcal{B}}\boldsymbol{\Psi}\boldsymbol{\mathcal{C}})\boldsymbol{\Gamma}, \tag{14b}$$

$$\boldsymbol{X}_{\boldsymbol{\nu}}(\boldsymbol{\Psi}) := \boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}(\mathfrak{g}(\boldsymbol{\Psi})) = \boldsymbol{\mathcal{B}}(\mathbf{I} + \boldsymbol{\Psi}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}). \tag{14c}$$

The second decision variable, $\boldsymbol{v} \in \mathbb{R}^N$, is defined as follows:

$$\boldsymbol{v} := (\mathbf{I} + \boldsymbol{\Psi}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}})\boldsymbol{\nu}. \tag{15}$$

In view of (14c) and (15), we have that $\boldsymbol{X}_{\boldsymbol{\nu}}(\boldsymbol{\Psi})\boldsymbol{\nu} = \boldsymbol{\mathcal{B}}\boldsymbol{v}$, and thus, we can express $\boldsymbol{x}$ in terms of $\boldsymbol{\Psi}$ and $\boldsymbol{v}$, in view of (14a)-(14c), as follows:

$$\boldsymbol{x} = \boldsymbol{X}_{\boldsymbol{v}}\boldsymbol{v} + \boldsymbol{X}_{\mathbf{0}}(\boldsymbol{\Psi})x_0, \tag{16}$$

where $\boldsymbol{X}_{\boldsymbol{v}} := \boldsymbol{\mathcal{B}}$ (note that $\boldsymbol{X}_{\boldsymbol{v}}$ is independent of the new decision variables). Similarly, we can first express $\boldsymbol{y}$ in terms of $\boldsymbol{\Psi}$ and $\boldsymbol{v}$, in view of (4) and (16), as follows:

$$\boldsymbol{y} = \boldsymbol{Y}_{\boldsymbol{v}}\boldsymbol{v} + \boldsymbol{Y}_{\mathbf{0}}(\boldsymbol{\Psi})x_0, \tag{17a}$$

$$\boldsymbol{Y}_{\mathbf{0}}(\boldsymbol{\Psi}) := \boldsymbol{\mathcal{C}}\boldsymbol{X}_{\mathbf{0}}(\boldsymbol{\Psi}) = \boldsymbol{\mathcal{C}}(\mathbf{I} + \boldsymbol{\mathcal{B}}\boldsymbol{\Psi}\boldsymbol{\mathcal{C}})\boldsymbol{\Gamma}, \tag{17b}$$

$$\boldsymbol{Y}_{\boldsymbol{v}} := \boldsymbol{\mathcal{C}}\boldsymbol{X}_{\boldsymbol{v}} = \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}. \tag{17c}$$

Finally, in view of (11a)-(11c), we have

$$\boldsymbol{u} \ = \boldsymbol{U}_{\boldsymbol{\nu}}(\boldsymbol{\Psi})\boldsymbol{\nu} + \boldsymbol{U}_{\mathbf{0}}(\boldsymbol{\Psi})x_0, \tag{18a}$$

$$\boldsymbol{U}_{\mathbf{0}}(\boldsymbol{\Psi}) := \boldsymbol{\mathcal{U}}_{\mathbf{0}}(\mathfrak{g}(\boldsymbol{\Psi})) = \boldsymbol{\Psi}\boldsymbol{\mathcal{C}}\boldsymbol{\Gamma}, \tag{18b}$$

$$\boldsymbol{U}_{\boldsymbol{\nu}}(\boldsymbol{\Psi}) := \boldsymbol{\mathcal{U}}_{\boldsymbol{\nu}}(\mathfrak{g}(\boldsymbol{\Psi})) = \mathbf{I} + \boldsymbol{\Psi}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}. \tag{18c}$$

In view of (15) and (18c), we have that $\boldsymbol{U}_{\boldsymbol{\nu}}(\boldsymbol{\Psi})\boldsymbol{\nu} = \boldsymbol{v}$, and thus the expression of $\boldsymbol{u}$ in terms of $\boldsymbol{\Psi}$ and $\boldsymbol{v}$ is given by

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{U}_{\mathbf{0}}(\boldsymbol{\Psi})x_0. \tag{19}$$

Remark 2 It should be highlighted that the expressions of $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{u}$ given in (16), (17a) and (19), respectively, correspond to linear or affine functions of the new decision variables $\boldsymbol{\Psi}$ and $\boldsymbol{v}$.

C. Decomposition of the Performance Index

Next, we will determine the expression of the performance index, $J(\pi)$, which is defined in (2) for $\pi \in \Pi$, in terms of the new decision variables $\boldsymbol{\Psi}$ and $\boldsymbol{v}$. We will denote this expression as $\mathcal{J}(\boldsymbol{v}, \boldsymbol{\Psi})$, where, in view of (17a) and (19),

$$\begin{aligned}\mathcal{J}(\boldsymbol{v}, \boldsymbol{\Psi}) &:= \mathbb{E}\big[\mathrm{trace}\big(\boldsymbol{y}\boldsymbol{y}^{\mathrm{T}}\boldsymbol{\mathcal{Q}} + \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{\mathcal{R}}\big)\big] \\ &= \mathbb{E}\big[\mathrm{trace}\big(\big(\boldsymbol{Y}_{\boldsymbol{v}}\boldsymbol{v} + \boldsymbol{Y}_{\mathbf{0}}(\boldsymbol{\Psi})x_0\big)\big(\boldsymbol{Y}_{\boldsymbol{v}}\boldsymbol{v} + \boldsymbol{Y}_{\mathbf{0}}(\boldsymbol{\Psi})x_0\big)^{\mathrm{T}}\boldsymbol{\mathcal{Q}} \\ &\quad + \big(\boldsymbol{v} + \boldsymbol{U}_{\mathbf{0}}(\boldsymbol{\Psi})x_0\big)\big(\boldsymbol{v} + \boldsymbol{U}_{\mathbf{0}}(\boldsymbol{\Psi})x_0\big)^{\mathrm{T}}\boldsymbol{\mathcal{R}}\big)\big],\end{aligned} \tag{20}$$

where $\boldsymbol{\mathcal{Q}} := \mathrm{diag}(q(0), \ldots, q(N-1))$ and $\boldsymbol{\mathcal{R}} := \mathrm{diag}(r(0), \ldots, r(N-1))$. By using the fact that $\mathbb{E}[x_0] = m_0 = 0$ and $\mathbb{E}[x_0 x_0^{\mathrm{T}}] = \boldsymbol{\Sigma}_0$, it follows readily that

$$\begin{aligned}\mathcal{J}(\boldsymbol{v}, \boldsymbol{\Psi}) = \mathrm{trace}\big(&\big(\boldsymbol{Y}_{\boldsymbol{v}}\boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{Y}_{\boldsymbol{v}}^{\mathrm{T}} + \boldsymbol{Y}_{\mathbf{0}}(\boldsymbol{\Psi})\boldsymbol{\Sigma}_0\boldsymbol{Y}_{\mathbf{0}}(\boldsymbol{\Psi})^{\mathrm{T}}\big)\boldsymbol{\mathcal{Q}} \\ &+ \big(\boldsymbol{v}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{U}_{\mathbf{0}}(\boldsymbol{\Psi})\boldsymbol{\Sigma}_0\boldsymbol{U}_{\mathbf{0}}(\boldsymbol{\Psi})^{\mathrm{T}}\big)\boldsymbol{\mathcal{R}}\big).\end{aligned} \tag{21}$$

An interesting observation here is that $\mathcal{J}(\boldsymbol{v}, \boldsymbol{\Psi})$ can be written as the sum of two terms as follows:

$$\mathcal{J}(\boldsymbol{v}, \boldsymbol{\Psi}) \ = \mathcal{J}_1(\boldsymbol{v}) + \mathcal{J}_2(\boldsymbol{\Psi}), \tag{22a}$$

$$\begin{aligned}\mathcal{J}_1(\boldsymbol{v}) &:= \mathrm{trace}\big(\boldsymbol{Y}_{\boldsymbol{v}}\boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{Y}_{\boldsymbol{v}}^{\mathrm{T}}\boldsymbol{\mathcal{Q}} + \boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{\mathcal{R}}\big) \\ &= \boldsymbol{v}^{\mathrm{T}}\big(\boldsymbol{Y}_{\boldsymbol{v}}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}\boldsymbol{Y}_{\boldsymbol{v}} + \boldsymbol{\mathcal{R}}\big)\boldsymbol{v},\end{aligned} \tag{22b}$$

$$\begin{aligned}\mathcal{J}_2(\boldsymbol{\Psi}) := \mathrm{trace}\big(&\boldsymbol{Y}_{\mathbf{0}}(\boldsymbol{\Psi})\boldsymbol{\Sigma}_0\boldsymbol{Y}_{\mathbf{0}}(\boldsymbol{\Psi})^{\mathrm{T}}\boldsymbol{\mathcal{Q}} \\ &+ \boldsymbol{U}_{\mathbf{0}}(\boldsymbol{\Psi})\boldsymbol{\Sigma}_0\boldsymbol{U}_{\mathbf{0}}(\boldsymbol{\Psi})^{\mathrm{T}}\boldsymbol{\mathcal{R}}\big).\end{aligned} \tag{22c}$$

Proposition 1: The function $(\boldsymbol{v}, \boldsymbol{\Psi}) \mapsto \mathcal{J}(\boldsymbol{v}, \boldsymbol{\Psi})$, where $\mathcal{J}(\boldsymbol{v}, \boldsymbol{\Psi})$ is defined in (21), is jointly convex in $\boldsymbol{v}$ and $\boldsymbol{\Psi}$.

Proof: In the light of (22a) and (22b)-(22c), it suffices to show that the functions $\boldsymbol{v} \mapsto \mathcal{J}_1(\boldsymbol{v})$ and $\boldsymbol{\Psi} \mapsto \mathcal{J}_2(\boldsymbol{\Psi})$ are convex (in $\boldsymbol{v}$ and $\boldsymbol{\Psi}$, respectively). First, we observe that $\mathcal{J}_1(\boldsymbol{v})$ is a quadratic form in $\boldsymbol{v}$ whose Hessian $\nabla_{\boldsymbol{v}}^2\mathcal{J}_1(\boldsymbol{v})$, which is equal to $2(\boldsymbol{Y}_{\boldsymbol{v}}\boldsymbol{\mathcal{Q}}\boldsymbol{Y}_{\boldsymbol{v}}^{\mathrm{T}} + \boldsymbol{\mathcal{R}})$, belongs to $\mathbb{S}_N^{++}$. We conclude that $\mathcal{J}_1(\cdot)$ is (strictly) convex in $\boldsymbol{v}$. In addition, $\mathcal{J}_2(\boldsymbol{\Psi})$ can be expressed as the sum of two composite functions, namely the composition of $\mathbf{Z} \mapsto f_1(\mathbf{Z}) := \mathrm{trace}(\mathbf{Z}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\mathcal{Q}})$ with the function $\boldsymbol{\Psi} \mapsto g_1(\boldsymbol{\Psi}) := \boldsymbol{Y}_0(\boldsymbol{\Psi})\boldsymbol{\Sigma}_0^{1/2}$ and the composition of $\mathbf{Z} \mapsto f_2(\mathbf{Z}) := \mathrm{trace}(\mathbf{Z}\mathbf{Z}^{\mathrm{T}}\boldsymbol{\mathcal{R}})$ with the function $\boldsymbol{\Psi} \mapsto g_2(\boldsymbol{\Psi}) := \boldsymbol{U}_0(\boldsymbol{\Psi})\boldsymbol{\Sigma}_0^{1/2}$. Note that $f_1(\cdot)$ and $f_2(\cdot)$ are both convex (in $\mathbf{Z}$) whereas $g_1(\cdot)$ and $g_2(\cdot)$ are, respectively, affine and linear functions of $\boldsymbol{\Psi}$ in view of (17b) and (18b). We conclude that both $\boldsymbol{\Psi} \mapsto f_1(g_1(\boldsymbol{\Psi}))$ and $\boldsymbol{\Psi} \mapsto f_2(g_2(\boldsymbol{\Psi}))$ are convex functions (in $\boldsymbol{\Psi}$) as the compositions of convex functions with, respectively, an affine and a linear function [26]. Consequently, $\mathcal{J}_2(\cdot)$, which is equal to the sum of the functions $f_1(g_1(\cdot))$ and $f_2(g_2(\cdot))$ that are both convex in $\boldsymbol{\Psi}$, will also be a convex function (in $\boldsymbol{\Psi}$). We conclude that $\mathcal{J}(\boldsymbol{v}, \boldsymbol{\Psi})$ is jointly convex in $\boldsymbol{v}$ and $\boldsymbol{\Psi}$ and the proof is now complete. ■

D. Terminal Constraints

In view of (17a), we have that $\mathbb{E}[\boldsymbol{y}] = \boldsymbol{Y}_{\boldsymbol{v}}\boldsymbol{v}$ and $\mathbb{E}[y(N-1)] = \boldsymbol{e}_N^{\mathrm{T}}\mathbb{E}[\boldsymbol{y}]$, where $\boldsymbol{e}_N$ is a unit vector in $\mathbb{R}^N$, whose elements are equal to zero except from the N-th element which is equal to one. Therefore, the terminal condition $\mathbb{E}[y(N-1)] = \mu_{\mathsf{f}}$ can be written as

$$h(\boldsymbol{v}) = 0, \quad h(\boldsymbol{v}) := \mathfrak{C}_N^{\mathrm{T}}\boldsymbol{v} - \mu_{\mathsf{f}}. \tag{23}$$

where $\mathfrak{C}_N := \boldsymbol{Y}_{\boldsymbol{v}}^{\mathrm{T}}\boldsymbol{e}_N$, or equivalently, in view of (17c),

$$\begin{aligned}\mathfrak{C}_N := [&c(N-1)^{\mathrm{T}}\boldsymbol{\Phi}(N-1,1)b(0), \ldots, \\ &c(N-1)^{\mathrm{T}}b(N-2),\ 0]^{\mathrm{T}}.\end{aligned} \tag{24}$$

Note that the constraint function $h(\cdot)$ is affine in $\boldsymbol{v}$.

Proposition 2: Equation (23) admits a solution for any $\mu_{\mathsf{f}} \in \mathbb{R}$, if and only if $\mathfrak{C}_N \neq \mathbf{0}$, where $\mathfrak{C}_N$ is defined in (24). In addition, if $\mathfrak{C}_N = \mathbf{0}$, then (23) admits a solution if and only if $\mu_{\mathsf{f}} = 0$.

Proof: Because $h(\cdot)$ is an affine function (in $\boldsymbol{v}$), we have that $\mathfrak{Im}(h|\mathbb{R}^N) = \mathbb{R}$, or equivalently, the equation $h(\boldsymbol{v}) = 0$ admits a solution for all $\mu_{\mathsf{f}} \in \mathbb{R}$, if and only if $\mathfrak{C}_N \neq \mathbf{0}$. Finally, if $\mathfrak{C}_N = \mathbf{0}$, then $h(\boldsymbol{v}) \equiv -\mu_{\mathsf{f}}$ and consequently, the constraint equation $h(\boldsymbol{v}) = 0$ admits a solution if and only if $\mu_{\mathsf{f}} = 0$. This completes the proof. ■

Remark 3 Note that $\mathfrak{C}_N$ satisfies the following equation:

$$\mathfrak{C}_N^{\mathrm{T}} = c(N-1)^{\mathrm{T}}[\mathbf{C}_{N-1},\ 0], \tag{25}$$

where $\mathbf{C}_{N-1} = [\boldsymbol{\Phi}(N-1,1)b(0), \ldots, b(N-2)]$. Therefore, if the matrix $\mathbf{C}_{N-1}$ is full row rank (provided $N-1 \geq n$) and the vector $c(N-1) \neq \mathbf{0}$, then $\mathfrak{C}_N \neq \mathbf{0}$ and thus, in view of

Proposition 2, (23) always admits a solution. It is interesting to note that if $N-1 \geq n$, then the assumption that $\mathbf{C}_{N-1}$ is full row rank is essentially a controllability assumption for the time-varying system (1a)-(1b).

In addition, we can write $\mathbb{E}\big[\boldsymbol{y} - \mathbb{E}[\boldsymbol{y}]\big] = \boldsymbol{Y_0}(\boldsymbol{\Psi})x_0$, from which it follows that

$$\mathbb{E}\big[(\boldsymbol{y} - \mathbb{E}[\boldsymbol{y}])(\boldsymbol{y} - \mathbb{E}[\boldsymbol{y}])^{\mathrm{T}}\big] = \mathbb{E}\big[\boldsymbol{Y_0}(\boldsymbol{\Psi})x_0x_0^{\mathrm{T}}\boldsymbol{Y_0}(\boldsymbol{\Psi})^{\mathrm{T}}\big] = \boldsymbol{Y_0}(\boldsymbol{\Psi})\boldsymbol{\Sigma}_0\boldsymbol{Y_0}(\boldsymbol{\Psi})^{\mathrm{T}}. \tag{26}$$

Next, we observe that $\mathbb{E}\big[(y(N{-}1){-}\mathbb{E}[y(N{-}1)])^2\big] = \boldsymbol{e}_N^{\mathrm{T}}\mathbb{E}\big[(\boldsymbol{y}{-}\mathbb{E}[\boldsymbol{y}])(\boldsymbol{y}{-}\mathbb{E}[\boldsymbol{y}])^{\mathrm{T}}\big]\boldsymbol{e}_N$. Consequently, the terminal constraint $\mathbb{E}\big[(y(N-1)-\mathbb{E}[y(N-1)])^2\big] = \sigma_{\mathsf{f}}^2$ can be written as follows:

$$\boldsymbol{z}_N(\boldsymbol{\Psi})^{\mathrm{T}}\boldsymbol{z}_N(\boldsymbol{\Psi}) = \sigma_{\mathsf{f}}^2, \qquad \boldsymbol{z}_N(\boldsymbol{\Psi}) := \boldsymbol{\Sigma}_0^{1/2}\boldsymbol{Y_0}(\boldsymbol{\Psi})^{\mathrm{T}}\boldsymbol{e}_N. \tag{27}$$

Note that $\boldsymbol{z}_N(\boldsymbol{\Psi})$ is, in view of Eq. (17b), an affine function of $\boldsymbol{\Psi}$.

IV. Reduction of the Liouville Control Problem to a Decoupled System of Two Tractable Optimization Problems

We are now in position to formulate an optimization problem in terms of the pair of decision variables $(\boldsymbol{v}, \boldsymbol{\Psi})$ that is equivalent to Problem 1 in the sense that the solution $(\boldsymbol{v}^\star, \boldsymbol{\Psi}^\star)$ to the former problem (provided that it admits a solution) will uniquely determine a solution $\pi^\star$ to Problem 1, and vice versa.

Problem 2: Given $\mu_{\mathsf{f}} \in \mathbb{R}$ and $\sigma_{\mathsf{f}} > 0$, find an optimal pair $(\boldsymbol{v}^\star, \boldsymbol{\Psi}^\star) \in \mathbb{R}^N \times \mathfrak{LT}(\mathbb{R}^{N\times N})$ that minimizes the performance index $\mathcal{J}(\boldsymbol{v}, \boldsymbol{\Psi})$, which is defined in (21), subject to the equality constraints given in (23) and (27).

It is important to note that, in the light of (22a)–(22c), $\mathcal{J}(\boldsymbol{v}, \boldsymbol{\Psi})$ can be written as the sum of two cost terms, namely $\mathcal{J}_1(\boldsymbol{v})$ and $\mathcal{J}_2(\boldsymbol{\Psi})$. In addition, the constraint given in (23) involves only the decision variable $\boldsymbol{v}$ whereas the constraint in (27) involves only $\boldsymbol{\Psi}$. Thus, Problem 2 can be decomposed into two decoupled subproblems, one in $\boldsymbol{v}$ and another in $\boldsymbol{\Psi}$.

Problem 3: Find $\boldsymbol{v}^\star \in \mathbb{R}^N$ that minimizes the performance index $\mathcal{J}_1(\boldsymbol{v})$, which is given in (22b), subject to the equality constraint (23) (*Subproblem 3.1*). In addition, find $\boldsymbol{\Psi}^\star \in \mathfrak{LT}(\mathbb{R}^{N\times N})$ that minimizes the performance index $\mathcal{J}_2(\boldsymbol{\Psi})$, which is given in (22c), subject to the equality constraint given in (27) (*Subproblem 3.2*).

Subproblem 3.1 corresponds to a standard controllability / minimum energy linear control problem, which turns out to be a (strictly) convex quadratic program subject to an affine equality constraint Next, we provide the solution to Subproblem 3.1 for completeness of our exposition.

Proposition 3: Suppose that the vector $\mathfrak{C}_N$, which is defined in (24), is non-zero, that is, $\mathfrak{C}_N \neq \mathbf{0}$ for a given $N \in \mathbb{Z}^+$. Then, for any $\mu_{\mathsf{f}} \in \mathbb{R}$, Subproblem 3.1 admits a unique solution $\boldsymbol{v}^\star$, which is given by

$$\boldsymbol{v}^\star = (\mu_{\mathsf{f}}/\mathfrak{C}_N^{\mathrm{T}}\mathbf{R}\mathfrak{C}_N)\mathbf{R}\mathfrak{C}_N, \tag{28}$$

where $\mathbf{R} := \big(\boldsymbol{Y}_{\boldsymbol{v}}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}\boldsymbol{Y}_{\boldsymbol{v}} + \boldsymbol{\mathcal{R}}\big)^{-1}$.

Proof: By hypothesis and in view of Proposition 2, the affine constraint given in (23) is feasible for all $\boldsymbol{v} \in \mathbb{R}^N$. With the change of variable $\tilde{\boldsymbol{v}} = \mathbf{R}^{-1/2}\boldsymbol{v}$, Subproblem 3.1 reduces to the problem of minimizing $|\tilde{\boldsymbol{v}}|^2$ subject to $\mathfrak{C}_N^{\mathrm{T}}\mathbf{R}^{1/2}\tilde{\boldsymbol{v}} = \mu_{\mathsf{f}}$. The solution to the latter problem corresponds to the minimum norm solution of the (single) linear equation $\mathfrak{C}_N^{\mathrm{T}}\mathbf{R}^{1/2}\tilde{\boldsymbol{v}} = \mu_{\mathsf{f}}$, whose solution is given by $\tilde{\boldsymbol{v}}^\star = (\mu_{\mathsf{f}}/\mathfrak{C}_N^{\mathrm{T}}\mathbf{R}\mathfrak{C}_N)\mathbf{R}^{1/2}\mathfrak{C}_N$, from which we obtain the expression of $\boldsymbol{v}^\star$ given in Eq. (28), where $\boldsymbol{v}^\star = \mathbf{R}^{1/2}\tilde{\boldsymbol{v}}^\star$. ∎

Proposition 4: Subproblem 3.2 is equivalent to the following generalized quadratically constrained quadratic problem (GQCQP):

$$\min_{\mathbf{x}\in\mathbb{R}^{\upsilon}} \mathbf{x}^{\mathrm{T}}\mathbf{H}_0\mathbf{x} + \mathbf{c}_0^{\mathrm{T}}\mathbf{x}, \quad \text{subject to } f_c(\mathbf{x}) = 0, \tag{29a}$$

$$f_c(\mathbf{x}) := \mathbf{x}^{\mathrm{T}}\mathbf{H}_c\mathbf{x} + \mathbf{c}_c^{\mathrm{T}}\mathbf{x} + d_c, \tag{29b}$$

where $\mathbf{x} := \mathrm{vech}(\boldsymbol{\Psi}) = \boldsymbol{\Omega}^{\mathrm{T}}\mathrm{vec}(\boldsymbol{\Psi})$ and $\mathbf{H}_0 \in \mathbb{S}_{\upsilon}^{++}$, $\mathbf{c}_0 \in \mathbb{R}^{\upsilon}$, with $\upsilon := N(N+1)/2$, $\mathbf{H}_0 := \mathbf{M}_0^{\mathrm{T}}\mathbf{M}_0 + \mathbf{N}_0^{\mathrm{T}}\mathbf{N}_0$ and $\mathbf{c}_0 := 2\mathbf{M}_0^{\mathrm{T}}\boldsymbol{\alpha}_0$, where

$$\mathbf{M}_0 := \big(\boldsymbol{\Sigma}_0^{1/2} \otimes \boldsymbol{\mathcal{Q}}^{1/2}\big)\big((\boldsymbol{\mathcal{C}}\boldsymbol{\Gamma})^{\mathrm{T}} \otimes (\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}})\big)\boldsymbol{\Omega}, \tag{30a}$$

$$\mathbf{N}_0 := \big(\boldsymbol{\Sigma}_0^{1/2} \otimes \boldsymbol{\mathcal{R}}^{1/2}\big)\big((\boldsymbol{\mathcal{C}}\boldsymbol{\Gamma})^{\mathrm{T}} \otimes \mathbf{I}\big)\boldsymbol{\Omega}, \tag{30b}$$

$$\boldsymbol{\alpha}_0 := \big(\boldsymbol{\Sigma}_0^{1/2} \otimes \boldsymbol{\mathcal{Q}}^{1/2}\big)\mathrm{vec}(\boldsymbol{\mathcal{C}}\boldsymbol{\Gamma}), \tag{30c}$$

and $\mathbf{H}_c \in \mathbb{S}_{\upsilon}^{+}$, $\mathbf{c}_c \in \mathbb{R}^{\upsilon}$ and $d_c \in \mathbb{R}$, with $\mathbf{H}_c := \mathbf{M}_c^{\mathrm{T}}\mathbf{M}_c$, $\mathbf{c}_c := 2\mathbf{M}_c^{\mathrm{T}}\boldsymbol{\alpha}_c$ and $d_c = |\boldsymbol{\alpha}_c|^2 - \sigma_{\mathsf{f}}^2$, where

$$\mathbf{M}_c := \big(\boldsymbol{\Sigma}_0^{1/2} \otimes \boldsymbol{e}_N^{\mathrm{T}}\big)\big((\boldsymbol{\mathcal{C}}\boldsymbol{\Gamma})^{\mathrm{T}} \otimes (\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}})\big)\boldsymbol{\Omega}, \tag{31a}$$

$$\boldsymbol{\alpha}_c := \big(\boldsymbol{\Sigma}_0^{1/2} \otimes \boldsymbol{e}_N^{\mathrm{T}}\big)\mathrm{vec}(\boldsymbol{\mathcal{C}}\boldsymbol{\Gamma}). \tag{31b}$$

In addition, Problem 1 admits a solution in the class of admissible control policies Π, if and only if the GQCQP is feasible.

Proof: We can write (27) as follows:

$$\begin{aligned} 0 &= \mathrm{trace}\big(\boldsymbol{e}_N^{\mathrm{T}}\boldsymbol{Y_0}(\boldsymbol{\Psi})\boldsymbol{\Sigma}_0\boldsymbol{Y_0}(\boldsymbol{\Psi})^{\mathrm{T}}\boldsymbol{e}_N\big) - \sigma_{\mathsf{f}}^2 \\ &= \mathrm{trace}\big(\boldsymbol{\Sigma}_0^{1/2}\boldsymbol{Y_0}(\boldsymbol{\Psi})^{\mathrm{T}}\boldsymbol{e}_N\boldsymbol{e}_N^{\mathrm{T}}\boldsymbol{Y_0}(\boldsymbol{\Psi})\boldsymbol{\Sigma}_0^{1/2}\big) - \sigma_{\mathsf{f}}^2 \\ &= \mathrm{trace}\big(\mathbf{Z}^{\mathrm{T}}\mathbf{Z}\big) - \sigma_{\mathsf{f}}^2 = \mathrm{vec}(\mathbf{Z})^{\mathrm{T}}\mathrm{vec}(\mathbf{Z}) - \sigma_{\mathsf{f}}^2, \end{aligned}$$

where $\mathbf{Z} := \boldsymbol{e}_N^{\mathrm{T}}\boldsymbol{Y_0}(\boldsymbol{\Psi})\boldsymbol{\Sigma}_0^{1/2}$. It follows that $\mathrm{vec}(\mathbf{Z}) = \big(\boldsymbol{\Sigma}_0^{1/2} \otimes \boldsymbol{e}_N^{\mathrm{T}}\big)\mathrm{vec}(\boldsymbol{Y_0}(\boldsymbol{\Psi}))$, where in view of (17b) we have that

$$\begin{aligned} \mathrm{vec}(\boldsymbol{Y_0}(\boldsymbol{\Psi})) &= \mathrm{vec}(\boldsymbol{\mathcal{C}}\boldsymbol{\Gamma}) + \mathrm{vec}\big((\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}})\boldsymbol{\Psi}(\boldsymbol{\mathcal{C}}\boldsymbol{\Gamma})\big) \\ &= \mathrm{vec}(\boldsymbol{\mathcal{C}}\boldsymbol{\Gamma}) + \big((\boldsymbol{\mathcal{C}}\boldsymbol{\Gamma})^{\mathrm{T}} \otimes (\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}})\big)\boldsymbol{\Omega}\mathrm{vech}(\boldsymbol{\Psi}), \end{aligned}$$

where $\boldsymbol{\Omega} \in \mathbb{R}^{N^2\times\upsilon}$ is defined as in Section II-A. Now, let $\mathbf{x} := \mathrm{vech}(\boldsymbol{\Psi})$. Then, (27) can be written as $f_c(\mathbf{x}) = 0$ with $f_c(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{H}_c\mathbf{x} + \mathbf{c}_c^{\mathrm{T}}\mathbf{x} + d_c$, where $\mathbf{H}_c := \mathbf{M}_c^{\mathrm{T}}\mathbf{M}_c$, $\mathbf{c}_c := 2\mathbf{M}_c^{\mathrm{T}}\boldsymbol{\alpha}_c$, and $d_c = |\boldsymbol{\alpha}_c|^2 - \sigma_{\mathsf{f}}^2$, where $\mathbf{M}_c$ and $\boldsymbol{\alpha}_c$ are given in (31a)–(31b). Thus, we have shown that the constraint given in (27) is equivalent to the equality constraint $f_c(\mathbf{x}) = 0$.

Using similar arguments as before, one can show that the performance index $\mathcal{J}_2(\boldsymbol{\Psi})$ can be written as follows:

$$\mathcal{J}_2(\boldsymbol{\Psi}) = \mathrm{vec}(\mathbf{Y})^{\mathrm{T}}\mathrm{vec}(\mathbf{Y}) + \mathrm{vec}(\mathbf{U})^{\mathrm{T}}\mathrm{vec}(\mathbf{U}),$$

with $\mathrm{vec}(\mathbf{Y}) := \big(\boldsymbol{\Sigma}_0^{1/2} \otimes \boldsymbol{\mathcal{Q}}^{1/2}\big)\mathrm{vec}(\boldsymbol{Y_0}(\boldsymbol{\Psi}))$ and $\mathrm{vec}(\mathbf{U}) := \big(\boldsymbol{\Sigma}_0^{1/2} \otimes \boldsymbol{\mathcal{R}}^{1/2}\big)\mathrm{vec}(\boldsymbol{U_0}(\boldsymbol{\Psi}))$, where in view of (18b)

$$\mathrm{vec}(\boldsymbol{U_0}(\boldsymbol{\Psi})) = \mathrm{vec}\big(\mathbf{I}\boldsymbol{\Psi}(\boldsymbol{\mathcal{C}}\boldsymbol{\Gamma})\big) = \big((\boldsymbol{\mathcal{C}}\boldsymbol{\Gamma})^{\mathrm{T}} \otimes \mathbf{I}\big)\boldsymbol{\Omega}\mathrm{vech}(\boldsymbol{\Psi}).$$

Thus, $\mathcal{J}_2(\boldsymbol{\Psi})$ is equal to the quadratic function $\mathbf{x}^{\mathrm{T}}\mathbf{H}_0\mathbf{x} + \mathbf{c}_0^{\mathrm{T}}\mathbf{x}$ (modulo a constant term) with $\mathbf{x} := \mathrm{vech}(\boldsymbol{\Psi})$, $\mathbf{H}_0 := \mathbf{M}_0^{\mathrm{T}}\mathbf{M}_0 + \mathbf{N}_0^{\mathrm{T}}\mathbf{N}_0$ and $\mathbf{c}_0 := 2\mathbf{M}_0^{\mathrm{T}}\boldsymbol{\alpha}_0$, where $\mathbf{M}_0$, $\mathbf{N}_0$ and $\boldsymbol{\alpha}_0$ are given in (30a)–(30c), respectively. This completes the proof. ∎

Next, we study the feasibility of the GQCQP (and thus, the feasibility of Subproblem 3.2 as well).

Proposition 5: Suppose that $\mathbf{H}_c \in \mathbb{S}_v^+\backslash\{\mathbf{0}\}$. Then, we consider the following two (exhaustive and mutually exclusive) cases: (i) $\mathbf{H}_c \in \mathbb{S}_v^{++}$, that is, $\mathrm{rank}(\mathbf{H}_c) = v$, in which case GQCQP is feasible, if and only if

$$d_c \leq (1/4)\mathbf{c}_c^{\mathrm{T}}\mathbf{H}_c^{-1}\mathbf{c}_c. \tag{32}$$

(ii) $\mathbf{H}_c \in \mathbb{S}_v^+\backslash(\{\mathbf{0}\} \cup \mathbb{S}_v^{++})$, that is, $\mathrm{rank}(\mathbf{H}_c) \in \mathbb{Z}_1^{v-1}$, in which case, we consider the following two sub-cases: (ii.a) $\mathbf{c}_c \in \mathcal{R}^{\perp}(\mathbf{H}_c) = \mathcal{N}(\mathbf{H}_c)$, in which case GQCQP has a non-trivial feasible set; and (ii.b) $\mathbf{c}_c \in \mathcal{R}(\mathbf{H}_c)$, in which case GQCQP is feasible, if and only if

$$d_c \leq (1/4)\mathbf{c}_c^{\mathrm{T}}\mathbf{H}_c^{\dagger}\mathbf{c}_c. \tag{33}$$

Proof: The GQCQP problem is feasible, if and only if $0 \in \mathfrak{Im}(f_c|\mathbb{R}^v)$. Because the function $\mathbf{x} \mapsto f_c(\mathbf{x})$, where $f_c(\mathbf{x})$ is defined in (29b), is a convex quadratic (and thus continuous) function, $\mathfrak{Im}(f_c|\mathbb{R}^v)$ will be an interval in $\mathbb{R}$ that will always contain positive numbers. Thus, in order to show that $0 \in \mathfrak{Im}(f_c|\mathbb{R}^v)$, it suffices to show that either $\mathfrak{Im}(f_c|\mathbb{R}^v)$ is not bounded from below or it is bounded from below and also $\min(\mathfrak{Im}(f_c|\mathbb{R}^v)) = \inf(\mathfrak{Im}(f_c|\mathbb{R}^v)) \leq 0$. To this aim, we will consider the problem of minimizing $f_c(\cdot)$, which is an unconstrained convex QP.

In case (i), $f_c(\cdot)$ admits a unique global minimizer, $\mathbf{x}^\star = -(1/2)\mathbf{H}_c^{-1}\mathbf{c}_c$, with corresponding minimum value $f_c^\star := \min_{\mathbf{x}\in\mathbb{R}^v} f_c(\mathbf{x}) = f_c(\mathbf{x}^\star) = d_c - (1/4)\mathbf{c}_c^{\mathrm{T}}\mathbf{H}_c^{-1}\mathbf{c}_c$. Therefore, $0 \in \mathfrak{Im}(f_c|\mathbb{R}^v)$, if and only if $f_c^\star \leq 0$, which yields (32). In case (ii), $f_c(\cdot)$ admits a global minimizer, if and only if the equation $\mathbf{0} = \frac{\partial}{\partial \mathbf{x}} f_c(\mathbf{x}) = 2\mathbf{H}_c\mathbf{x} + \mathbf{c}_c$ admits a solution, which is in turn equivalent to $\mathbf{c}_c \in \mathcal{R}(\mathbf{H}_c) = \mathcal{N}^{\perp}(\mathbf{H}_c)$. If $\mathbf{c}_c \in \mathcal{R}^{\perp}(\mathbf{H}_c) = \mathcal{N}(\mathbf{H}_c)$ (subcase (ii.a)), then the function $\mathbf{x} \mapsto f_c(\mathbf{x})$ is not lower bounded (note that $f_c(\cdot)$ is lower bounded if and only if $\mathbf{c}_c^{\mathrm{T}}\mathbf{x} = 0$, for all $\mathbf{x} \in \mathcal{N}(\mathbf{H}_c)$, or equivalently, $\mathbf{c}_c \in \mathcal{N}^{\perp}(\mathbf{H}_c)$), which implies that GQCQP has a non-trivial feasible set. On the other hand, if $\mathbf{c}_c \in \mathcal{R}(\mathbf{H}_c) = \mathcal{N}^{\perp}(\mathbf{H}_c)$ (subcase (ii.b)), then the set of minimizers of $f_c(\cdot)$ corresponds to the affine subspace $\mathcal{S}$ that consists of all the solutions to the equation $\mathbf{H}_c\mathbf{x} = -(1/2)\mathbf{c}_c$, which is defined as $\mathcal{S} := \{\mathbf{x} \in \mathbb{R}^v : \ \mathbf{x} = -(1/2)\mathbf{H}_c^{\dagger}\mathbf{c}_c + \mathbf{U}_c^{\mathrm{T}}[\mathbf{0}, \ \mathbf{z}^{\mathrm{T}}]^{\mathrm{T}}, \ \mathbf{z} \in \mathbb{R}^{v-r}\}$, where $\mathbf{H}_c = \mathbf{U}_c^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{U}_c$ corresponds to a spectral decomposition of $\mathbf{H}_c$ (the diagonal elements of $\boldsymbol{\Sigma}$ are the eigenvalues of $\mathbf{H}_c$ in decreasing order and $\mathbf{U}_c$ is an orthogonal matrix). In addition, $f_c^\star := \min_{\mathbf{x}\in\mathbb{R}^v} f_c(\mathbf{x}) = d_c - (1/4)\mathbf{c}_c^{\mathrm{T}}\mathbf{H}_c^{\dagger}\mathbf{c}_c$. Again, $0 \in \mathfrak{Im}(f_c|\mathbb{R}^v)$ if and only if $f_c^\star \leq 0$, which yields (33). ■

A. Computation of the Solution to the GQCQP

In order to address the GQCQP, which is a non-convex problem, one can employ convex relaxation techniques [22], [23], which aim at associating the original (non-convex) problem with a convex, but not necessarily equivalent, program. In this work, we will employ instead a direct solution approach to the GQCQP which will allow us to characterize an optimal control policy that will steer the system's output to the goal Gaussian distribution exactly. The proposed approach leverages some key results from the solution of optimization problems with quadratic but not necessarily convex performance indices subject to a single quadratic equality constraint, which is a non-convex constraint, in the sense that the feasible set it induces is not convex, after tailoring the latter results to the specific structure of the GQCQP. (For more details on generalized quadratic programs subject to a single quadratic equality constraint, in which neither the performance index nor the constraint function are necessarily convex quadratic functions, one may refer to [16]–[18], [20].) To this aim, we will simultaneously diagonalize the matrices $\mathbf{H}_0 \in \mathbb{S}_v^{++}$ and $\mathbf{H}_c \in \mathbb{S}_v^+$. In particular, $\mathbf{H}_0 = \mathbf{U}\mathbf{D}_0\mathbf{U}^{\mathrm{T}}$ and $\mathbf{H}_c = \mathbf{U}\mathbf{D}_c\mathbf{U}^{\mathrm{T}}$, where $\mathbf{U}$ is an $v \times v$ invertible matrix and $\mathbf{D}_0$, $\mathbf{D}_c$ are diagonal matrices in $\mathbb{S}_v^{++}$ and $\mathbb{S}_v^+$, respectively. Now, let $\mathbf{S} = \mathbf{D}_0^{-1/2}\mathbf{U}^{-1}$. Then, we have that $\mathbf{S}\mathbf{H}_0\mathbf{S}^{\mathrm{T}} = \mathbf{I}$ and $\mathbf{S}\mathbf{H}_c\mathbf{S}^{\mathrm{T}} = \mathbf{D}$, where $\mathbf{D} := \mathbf{D}_0^{-1/2}\mathbf{D}_c\mathbf{D}_0^{-1/2}$. Note that $\mathbf{D}$ is a diagonal matrix in $\mathbb{S}_v^+$. By using the transformation $\mathbf{z} := (\mathbf{S}^{\mathrm{T}})^{-1}\mathbf{x} + (1/2)\mathbf{S}\mathbf{c}_0$, the GQCQP can be formulated equivalently (modulo a constant term in the performance index) as follows:

$$\min_{\mathbf{z}\in\mathbb{R}^v} \mathbf{z}^{\mathrm{T}}\mathbf{z}, \quad \text{subject to } \varphi(\mathbf{z}) = 0, \tag{34}$$

where $\varphi(\mathbf{z}) := \mathbf{z}^{\mathrm{T}}\mathbf{D}\mathbf{z} + \boldsymbol{\sigma}^{\mathrm{T}}\mathbf{z} + \vartheta$, $\boldsymbol{\sigma} := \mathbf{S}\mathbf{c}_c - \mathbf{D}\mathbf{S}\mathbf{c}_0$ and $\vartheta := (1/4)\mathbf{c}_0^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\mathbf{D}\mathbf{S}\mathbf{c}_0 - (1/2)\mathbf{c}_c^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\mathbf{S}\mathbf{c}_0 + d_c$. Note that the problem given in (34) corresponds to the geometric optimization problem of determining the closest point of the conic characterized by the quadratic equation $\varphi(\mathbf{z}) = 0$ from the origin $\mathbf{z} = \mathbf{0}$.

Now, let $r := \mathrm{rank}(\mathbf{D})$, with $r \in \mathbb{Z}_1^v$. Then, without loss of generality, we can assume that

$$\mathbf{D} = \begin{bmatrix} \mathbf{0}_{v-r,v-r} & \mathbf{0}_{v-r,r} \\ \mathbf{0}_{r,v-r} & \boldsymbol{\Delta} \end{bmatrix}, \qquad \boldsymbol{\Delta} := \mathrm{diag}(\delta_1, \ldots, \delta_r),$$

where $\delta_1 \geq \cdots \geq \delta_r$ are the non-zero eigenvalues of $\mathbf{D}$. The vector $\boldsymbol{\sigma}$ can be decomposed as follows: $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\mathcal{R}} + \boldsymbol{\sigma}_{\mathcal{N}}$, where $\boldsymbol{\sigma}_{\mathcal{R}} = [\mathbf{0}^{\mathrm{T}}, \ 2\boldsymbol{\delta}^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{R}(\mathbf{D})$ with $\boldsymbol{\delta} \in \mathbb{R}^r$ and $\boldsymbol{\sigma}_{\mathcal{N}} = [\boldsymbol{\gamma}^{\mathrm{T}}, \ \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{N}(\mathbf{D})$ with $\boldsymbol{\gamma} \in \mathbb{R}^{v-r}$. Now, let $\boldsymbol{p} := [\boldsymbol{\xi}^{\mathrm{T}}, \ \boldsymbol{\zeta}^{\mathrm{T}}]^{\mathrm{T}} = \mathbf{z} + [\mathbf{0}^{\mathrm{T}}, \ (\boldsymbol{\Delta}^{-1}\boldsymbol{\delta})^{\mathrm{T}}]^{\mathrm{T}}$ with $\boldsymbol{\xi} \in \mathbb{R}^{v-r}$ and $\boldsymbol{\zeta} \in \mathbb{R}^r$. Then, instead of addressing the constrained minimization problem given in (34), we can equivalently search for the minimizer $\boldsymbol{p}^\star := [(\boldsymbol{\xi}^\star)^{\mathrm{T}}, \ (\boldsymbol{\zeta}^\star)^{\mathrm{T}}]^{\mathrm{T}}$ of the following problem:

$$\min_{\boldsymbol{p}\in\mathbb{R}^v} |\boldsymbol{\xi}|^2 + |\boldsymbol{\zeta} - \boldsymbol{\Delta}^{-1}\boldsymbol{\delta}|^2, \text{ subject to } \phi(\boldsymbol{\xi}, \boldsymbol{\zeta}) = 0, \tag{35}$$

where $\phi(\boldsymbol{\xi}, \boldsymbol{\zeta}) := \boldsymbol{\zeta}^{\mathrm{T}}\boldsymbol{\Delta}\boldsymbol{\zeta} + \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\xi} + \theta$ and $\theta := \vartheta - \boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{\Delta}^{-1}\boldsymbol{\delta}$.

Next we characterize the minimizer $\boldsymbol{p}^\star$ or, more generally, the set of minimizers, $\boldsymbol{\mathcal{P}}^\star$, of the problem given in (35). To this aim, we consider the following three mutually exclusive and exhaustive cases [17], [18].

Case 1: This is a degenerate case that occurs when $\boldsymbol{\gamma} = 0$ and $\theta = 0$. In this case, $\phi(\boldsymbol{\xi}, \boldsymbol{\zeta}) = \boldsymbol{\zeta}^{\mathrm{T}}\boldsymbol{\Delta}\boldsymbol{\zeta}$, and thus $\boldsymbol{\zeta} = \mathbf{0}$ is the only feasible point of the problem given in (35). Thus, $\boldsymbol{\mathcal{P}}^\star := \{\boldsymbol{p} \in \mathbb{R}^v : \ \boldsymbol{p} = [\boldsymbol{\xi}^{\mathrm{T}}, \ \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}}, \ \boldsymbol{\xi} \in \mathbb{R}^{v-r}\}$.

Case 2: This case occurs when $\theta \neq 0$, $\boldsymbol{\delta} = \mathbf{0}$, and $\boldsymbol{\gamma} = \mathbf{0}$. In this case, $\phi(\boldsymbol{\xi}, \boldsymbol{\zeta}) = \boldsymbol{\zeta}^{\mathrm{T}}\boldsymbol{\Delta}\boldsymbol{\zeta} + \theta$. We conclude that the problem is feasible only if $\theta < 0$. In addition, $\phi(\cdot)$ is independent of $\boldsymbol{\xi}$ (there are no constraints on $\boldsymbol{\xi}$), which in turn implies that $\boldsymbol{\xi}^\star = \mathbf{0}$. By using the transformation $\boldsymbol{\eta} := (1/\sqrt{|\theta|})\boldsymbol{\Delta}^{1/2}\boldsymbol{\zeta}$ for $\theta < 0$, the problem given in (35) reduces to

$$\min_{\boldsymbol{\eta}\in\mathbb{R}^r} \boldsymbol{\eta}^{\mathrm{T}}(|\theta|\boldsymbol{\Delta}^{-1})\boldsymbol{\eta}, \quad \text{subject to} \quad |\boldsymbol{\eta}| = 1. \tag{36}$$

It follows from the min-max theorem for the Rayleigh quotient that the minimum value of the problem given in (36) is equal to $\lambda_{\min}(|\theta|\mathbf{\Delta}^{-1}) = |\theta|/\lambda_{\max}(\mathbf{\Delta}) = |\theta|/\delta_1$, where δ_1 is the largest eigenvalue of $\mathbf{\Delta}$. The minimum value is attained at $\boldsymbol{\eta}^\star = [(\boldsymbol{\eta}_1^\star)^{\mathrm{T}},\ \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}}$, where $\boldsymbol{\eta}_1^\star$ is an ν_1-dimensional (real) unit vector and ν_1 denotes the algebraic multiplicity of δ_1 (as an eigenvalue of $\mathbf{\Delta}$). Thus, $\boldsymbol{\zeta}^\star = \sqrt{|\theta|}\mathbf{\Delta}^{-1/2}\boldsymbol{\eta}^\star$, from which it follows that $\boldsymbol{\zeta}^\star = [(\boldsymbol{\zeta}_1^\star)^{\mathrm{T}},\ \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}}$ with $\boldsymbol{\zeta}_1^\star \in \mathbb{R}^{\nu_1}$. In addition, because $|\boldsymbol{\eta}_1^\star| = 1$, we have $|\boldsymbol{\zeta}_1^\star|^2 = |\theta|/\delta_1$, which implies that $\boldsymbol{\zeta}_1^\star = \pm\sqrt{|\theta|/\delta_1}$ when $\nu_1 = 1$, or $\boldsymbol{\zeta}_1^\star$ belongs to the $(\nu_1 - 1)$-dimensional sphere $\mathbf{S}^{\nu_1 - 1} := \{\boldsymbol{\varrho} \in \mathbb{R}^{\nu_1} : |\boldsymbol{\varrho}| = \sqrt{|\theta|/\delta_1}\}$, when $\nu_1 > 1$; for both cases, we write $\boldsymbol{\zeta}_1^\star \in \boldsymbol{\mathcal{Z}}_1^\star \subseteq \mathbb{R}^{\nu_1}$. Thus, $\boldsymbol{\mathcal{P}}^\star := \{\boldsymbol{p} \in \mathbb{R}^{\upsilon} :\ \boldsymbol{p} = [\mathbf{0}^{\mathrm{T}},\ \boldsymbol{\zeta}_1^{\mathrm{T}},\ \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}},\ \boldsymbol{\zeta}_1 \in \boldsymbol{\mathcal{Z}}_1^\star\}$.

Case 3: The remaining case, which is the most general one, determines the *normal form* [17] of the problem given in (35). Before we proceed, we define the Lagrangian $\mathcal{L} : \mathbb{R}^{\upsilon - r} \times \mathbb{R}^r \times \mathbb{R}_{\geq 0}$, with $\mathcal{L}(\boldsymbol{\xi}, \boldsymbol{\zeta}, \lambda) := |\boldsymbol{\xi}|^2 + |\boldsymbol{\zeta} - \mathbf{\Delta}^{-1}\boldsymbol{\delta}|^2 - \lambda(\boldsymbol{\zeta}^{\mathrm{T}}\mathbf{\Delta}\boldsymbol{\zeta} + \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\xi} + \theta)$. We observe that $\mathcal{L}(\boldsymbol{\xi}, \boldsymbol{\zeta}, \lambda) = \mathcal{L}_1(\boldsymbol{\xi}, \lambda) + \mathcal{L}_2(\boldsymbol{\zeta}, \lambda)$, where

$$\mathcal{L}_1(\boldsymbol{\xi}, \lambda) := \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi} - \lambda(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\xi} + \theta),$$
$$\mathcal{L}_2(\boldsymbol{\zeta}, \lambda) := \boldsymbol{\zeta}^{\mathrm{T}}(\mathbf{I} - \lambda\mathbf{\Delta})\boldsymbol{\zeta} - 2\boldsymbol{\delta}^{\mathrm{T}}\mathbf{\Delta}^{-1}\boldsymbol{\zeta} + \boldsymbol{\delta}^{\mathrm{T}}\mathbf{\Delta}^{-2}\boldsymbol{\delta}.$$

Based on Lagrangian duality theory [21], the maximum of the dual function $g(\cdot) : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ with

$$g(\lambda) := \inf_{\boldsymbol{\xi} \in \mathbb{R}^{\upsilon - r}} \mathcal{L}_1(\boldsymbol{\xi}, \lambda) + \inf_{\boldsymbol{\zeta} \in \mathbb{R}^r} \mathcal{L}_2(\boldsymbol{\zeta}, \lambda), \tag{37}$$

which is associated to the primal optimization problem formulated in (35), furnishes a lower bound on the optimal value of the latter problem. We observe that $g(\cdot)$ attains finite values for these $\lambda \in \mathbb{R}$ for which the functions $\boldsymbol{\xi} \mapsto \mathcal{L}_1(\boldsymbol{\xi}, \lambda)$ and $\boldsymbol{\zeta} \mapsto \mathcal{L}_2(\boldsymbol{\zeta}, \lambda)$ are lower bounded. The function $\mathcal{L}_1(\cdot, \lambda)$, which is quadratic (in $\boldsymbol{\xi}$) for any given $\lambda \in \mathbb{R}$, will be lower bounded, if and only if the equation $\mathbf{0} = \nabla_{\boldsymbol{\xi}}\mathcal{L}_1(\boldsymbol{\xi}, \lambda) = 2\boldsymbol{\xi} - \lambda\boldsymbol{\gamma}$ admits a solution $\boldsymbol{\xi}_\star(\lambda) \in \mathbb{R}^{\upsilon - r}$ and in addition, its Hessian matrix $\nabla^2_{\boldsymbol{\xi}}\mathcal{L}_1(\boldsymbol{\xi}, \lambda) = 2\mathbf{I}$ belongs to $\mathbb{S}^+_{\upsilon - r}$, which are both trivially satisfied for all $\lambda \in \mathbb{R}$. Similarly, the function $\mathcal{L}_2(\cdot, \lambda)$, which is quadratic (in $\boldsymbol{\zeta}$) for any given $\lambda \in \mathbb{R}$, is lower bounded, if and only if the equation $\mathbf{0} = \nabla_{\boldsymbol{\zeta}}\mathcal{L}_2(\boldsymbol{\zeta}, \lambda) = 2(\mathbf{I} - \lambda\mathbf{\Delta})\boldsymbol{\zeta} - 2\mathbf{\Delta}^{-1}\boldsymbol{\delta}$ admits a solution $\boldsymbol{\zeta}_\star(\lambda) \in \mathbb{R}^r$ and its Hessian $\nabla^2_{\boldsymbol{\zeta}}\mathcal{L}_2(\boldsymbol{\zeta}, \lambda) = 2(\mathbf{I} - \lambda\mathbf{\Delta})$ belongs to $\mathbb{S}^+_r$. Thus, $\mathcal{L}_2(\cdot, \lambda)$ is lower bounded for all $\lambda \in \Lambda := (-\infty, 1/\delta_1]$ for which $\mathbf{\Delta}^{-1}\boldsymbol{\delta} \in \mathcal{R}(\mathbf{I} - \lambda\mathbf{\Delta})$. Therefore, when both $\mathcal{L}_1(\boldsymbol{\xi}, \lambda)$ and $\mathcal{L}_2(\boldsymbol{\zeta}, \lambda)$ are lower bounded, in which case $\mathcal{L}(\boldsymbol{\xi}, \boldsymbol{\zeta})$ is also lower bounded, we have that

$$\boldsymbol{\xi}_\star(\lambda) = (1/2)\lambda\boldsymbol{\gamma}, \quad (\mathbf{I} - \lambda\mathbf{\Delta})\boldsymbol{\zeta}_\star(\lambda) = \mathbf{\Delta}^{-1}\boldsymbol{\delta}. \tag{38}$$

From the strong duality of the GQCQP, which can be proven using similar arguments with those used in the approach presented in, for instance, [21, pp. 653–658], we conclude that for the characterization of $\boldsymbol{p}^\star := [(\boldsymbol{\xi}^\star)^{\mathrm{T}},\ (\boldsymbol{\zeta}^\star)^{\mathrm{T}}]^{\mathrm{T}}$, it suffices to find a $\lambda^\star \in \Lambda$ such that $\boldsymbol{\xi}_\star(\lambda^\star)$ and $\boldsymbol{\zeta}_\star(\lambda^\star)$ satisfy the constraint equation $\phi(\boldsymbol{\xi}_\star(\lambda^\star), \boldsymbol{\zeta}_\star(\lambda^\star)) = 0$ (alternatively, one can find the global minimizer $\lambda^\star$ of $-g(\lambda)$). After finding such $\lambda^\star \in \Lambda$, we can simply set $\boldsymbol{\xi}^\star := \boldsymbol{\xi}_\star(\lambda^\star)$ and $\boldsymbol{\zeta}^\star := \boldsymbol{\zeta}_\star(\lambda^\star)$. To proceed with the characterization of the solution to the problem in (35) in normal form, we will consider the following two subcases:

Case 3.1: $\lambda^\star \in \mathrm{int}(\Lambda) = (-\infty, 1/\delta_1)$, which implies that $(\mathbf{I} - \lambda^\star\mathbf{\Delta}) \in \mathbb{S}^{++}_r$. Consequently, Eq. (38) becomes

$$\boldsymbol{\xi}^\star = (1/2)\lambda^\star\boldsymbol{\gamma}, \quad \boldsymbol{\zeta}^\star = (\mathbf{I} - \lambda^\star\mathbf{\Delta})^{-1}\mathbf{\Delta}^{-1}\boldsymbol{\delta}. \tag{39}$$

Next, we plug the expressions of $\boldsymbol{\xi}^\star$ and $\boldsymbol{\zeta}^\star$ in terms of $\lambda^\star$, which are given in (39), into the constraint equation $\phi(\boldsymbol{\xi}^\star, \boldsymbol{\zeta}^\star) = 0$, where $\phi(\boldsymbol{\xi}^\star, \boldsymbol{\zeta}^\star) := (\boldsymbol{\zeta}^\star)^{\mathrm{T}}\mathbf{\Delta}\boldsymbol{\zeta}^\star + \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\xi}^\star + \theta$, to obtain the following equation: $\Sigma(\lambda^\star) = 0$, where $\Sigma(\cdot) : \mathrm{int}(\Lambda) \to \mathbb{R}$ with

$$\Sigma(\lambda) := \boldsymbol{\delta}^{\mathrm{T}}\mathbf{\Delta}^{-1/2}(\mathbf{I} - \lambda\mathbf{\Delta})^{-2}\mathbf{\Delta}^{-1/2}\boldsymbol{\delta} + (|\boldsymbol{\gamma}|^2/2)\lambda + \theta. \tag{40}$$

The equation $\Sigma(\lambda) = 0$ admits a unique solution in the interior of Λ, if and only if $0 \in \mathfrak{Im}(\Sigma|\mathrm{int}(\Lambda))$ (for more details on the solution of this scalar equation, the reader may refer to [18]). By continuity of $\Sigma(\cdot)$, it suffices to show, in view of the intermediate zero theorem from real analysis, that there are (finite) $\lambda_1, \lambda_2 \in \mathrm{int}(\Lambda)$ such that $\Sigma(\lambda_1) < 0$ and $\Sigma(\lambda_2) > 0$. It is easy to show that $\liminf_{\lambda \to -\infty} \Sigma(\lambda) = -\infty$, which implies the existence of a (finite) λ_1 such that $\Sigma(\lambda_1) < 0$. Now, let $\overline{\Sigma} := \sup\{\Sigma(\lambda) :\ \lambda \in \mathrm{int}(\Lambda)\}$ where $\overline{\Sigma} \in \mathbb{R} \cup \{+\infty\}$ with the convention that $\overline{\Sigma} = +\infty$ when the set $\{\Sigma(\lambda) :\ \lambda \in \mathrm{int}(\Lambda)\}$ is not upper bounded, and let us also simultaneously decompose $\mathbf{\Delta} \in \mathbb{S}^{++}_r$ and $\boldsymbol{\delta} \in \mathbb{R}^r$ as follows:

$$\mathbf{\Delta} = \mathrm{bdiag}(\delta_1\mathbf{I}_{\nu_1}, \tilde{\mathbf{\Delta}}), \quad \boldsymbol{\delta} = [\boldsymbol{\delta}_1^{\mathrm{T}},\ \tilde{\boldsymbol{\delta}}^{\mathrm{T}}]^{\mathrm{T}},$$

where $\tilde{\mathbf{\Delta}} := \mathrm{diag}(\delta_{\nu_1+1}, \ldots, \delta_r) \in \mathbb{S}^{++}_{r-\nu_1}$, $\boldsymbol{\delta}_1 \in \mathbb{R}^{\nu_1}$ and $\tilde{\boldsymbol{\delta}} \in \mathbb{R}^{r-\nu_1}$. On the one hand, when $\boldsymbol{\delta}_1 \neq \mathbf{0}$, then $\overline{\Sigma} = \infty$, which implies the existence of a (finite) λ_2 such that $\Sigma(\lambda_2) > 0$. Thus, in this case, $0 \in \mathfrak{Im}(\Sigma|\mathrm{int}(\Lambda))$ and we conclude that the equation $\Sigma(\lambda) = 0$, where $\Sigma(\lambda)$ is given in (40), admits a solution in the interior of Λ. On the other hand, when $\boldsymbol{\delta}_1 = \mathbf{0}$, then $\overline{\Sigma} = \limsup_{\lambda \to 1/\delta_1} \Sigma(\lambda)$ and in particular

$$\overline{\Sigma} = |(\mathbf{I} - (1/\delta_1)\tilde{\mathbf{\Delta}})^{-1}\tilde{\mathbf{\Delta}}^{-1/2}\tilde{\boldsymbol{\delta}}|^2 + |\boldsymbol{\gamma}|^2/(2\delta_1) + \theta.$$

If $0 < \overline{\Sigma} < +\infty$, then $\Sigma(\lambda)$ admits a unique solution in the interior of Λ. If, however, $\overline{\Sigma} \leq 0$, then $\Sigma(\lambda) < 0$ for all λ in the interior of Λ (even in the special subcase in which $\overline{\Sigma} = \limsup_{\lambda \to 1/\delta_1} \Sigma(\lambda) = 0$, we have that $0 \notin \mathfrak{Im}(\Sigma|\mathrm{int}(\Lambda))$) and thus the GQCQP is not feasible.

Case 3.2: $\lambda^\star = 1/\delta_1$ (note that $1/\delta_1$ belongs to the boundary of Λ). Then, $\mathbf{\Delta}^{-1}\boldsymbol{\delta} \in \mathcal{R}(\mathbf{I} - (1/\delta_1)\mathbf{\Delta})$ only if $\boldsymbol{\delta}_1 = \mathbf{0}$. It follows that $\boldsymbol{\xi}^\star = (1/(2\delta_1))\boldsymbol{\gamma}$ and $\boldsymbol{\zeta}^\star \in \boldsymbol{\mathcal{Z}}^*$, where

$$\boldsymbol{\mathcal{Z}}^\star := \{[\boldsymbol{\zeta}_1^{\mathrm{T}},\ \tilde{\boldsymbol{\zeta}}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^r : \tilde{\boldsymbol{\zeta}} = (\mathbf{I} - (1/\delta_1)\tilde{\mathbf{\Delta}})^{-1}\tilde{\mathbf{\Delta}}^{-1}\tilde{\boldsymbol{\delta}},$$
$$|\boldsymbol{\zeta}_1|^2 = -(1/\delta_1)(\tilde{\boldsymbol{\zeta}}^{\mathrm{T}}\tilde{\mathbf{\Delta}}\tilde{\boldsymbol{\zeta}} + |\boldsymbol{\gamma}|^2/(2\delta_1) + \theta)\},$$

in view of (39) and $\phi(\boldsymbol{\xi}^\star, \boldsymbol{\zeta}^\star) = 0$. Thus, $\boldsymbol{\mathcal{P}}^\star = \{\boldsymbol{p} \in \mathbb{R}^{\upsilon} :\ \boldsymbol{p} = [(1/(2\delta_1))\boldsymbol{\gamma}^{\mathrm{T}},\ \boldsymbol{\zeta}^{\mathrm{T}}]^{\mathrm{T}},\ \boldsymbol{\zeta} \in \boldsymbol{\mathcal{Z}}^\star\}$.

B. The Proposed Algorithm for the Liouville Control Problem

Next, we present the main steps of the proposed algorithm for the characterization of the solution to the Liouville control problem (Problem 1).

Step 1: If $\mu_{\mathrm{f}} \neq 0$, find the solution, $\boldsymbol{v}^\star$, to Subproblem 3.1 by making use of Eq. (28), provided that the latter problem admits a solution. If no solution exists, return failure and stop. If $\mu_{\mathrm{f}} = 0$, set $\boldsymbol{v}^\star = \mathbf{0}$ and go to Step 2.

Step 2: Find a solution $\boldsymbol{p}^\star := [(\boldsymbol{\xi}^\star)^{\mathrm{T}},\ (\boldsymbol{\zeta}^\star)^{\mathrm{T}}]^{\mathrm{T}}$ to the problem given in (35), if the latter admits a solution, and then compute the corresponding solution, $\mathbf{x}^\star$, to GQCQP by using the equations: $\mathbf{x}^\star = \mathbf{S}^{\mathrm{T}}\mathbf{z}^\star - (1/2)\mathbf{S}^{\mathrm{T}}\mathbf{S}\mathbf{c}_0$ and $\mathbf{z}^\star = [(\boldsymbol{\xi}^\star)^{\mathrm{T}},\ (\boldsymbol{\zeta}^\star - \mathbf{\Delta}^{-1}\boldsymbol{\delta})^{\mathrm{T}}]^{\mathrm{T}}$. Otherwise, return failure and stop.

Step 3: Compute the corresponding optimal gain matrix $\mathbf{\Psi}^\star \in \mathfrak{LT}(\mathbb{R}^{N \times N})$ from the equation: $\mathbf{\Psi}^\star = \mathrm{vech}^{-1}(\mathbf{x}^\star)$.

Step 4: Proceed to the computation of the corresponding

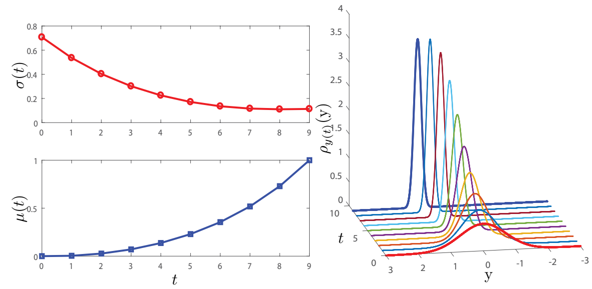

Fig. 1. Evolution of the density function, $\rho_{y(t)}(\mathrm{y})$, the mean, $\mu(t)$, and the standard deviation, $\sigma(t)$, of the output variable $y(t)$ when the system is driven by the output feedback control policy $\pi^\star$ versus the number of stages t.

optimal gain matrix $\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}^\star \in \mathfrak{LT}(\mathbb{R}^{N\times N})$, where $\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}^\star := \mathfrak{g}(\boldsymbol{\Psi}^\star)$.

Step 5: Finally, characterize the optimal policy $\pi^\star \in \Pi$ that solves Problem 1 by extracting (i) the feed-forward inputs $\nu^\star(t)$ from the vector $\boldsymbol{v}^\star$, which was computed in Step 1, and (ii) the optimal control gains $k_y^\star(t,\tau)$ from the corresponding entries of the optimal gain matrix $\boldsymbol{\mathcal{K}}_{\boldsymbol{y}}^\star \in \mathfrak{LT}(\mathbb{R}^{N\times N})$, which was computed in Step 4.

V. Numerical Simulations

To illustrate the main ideas of the proposed solution approach for the Liouville control problem, we present numerical simulations for a damped linear oscillator described by the following equation: $\dot{x}(\tau) = \mathbf{A}_{\mathsf{c}} x(\tau) + b_{\mathsf{c}} u(\tau)$, with output $y(\tau) = c_{\mathsf{c}}^{\mathrm{T}} x(\tau)$ with $\mathbf{A}_{\mathsf{c}} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}$, $b_{\mathsf{c}} = [0,\ 1]^{\mathrm{T}}$ and $c_{\mathsf{c}} = [0,\ 1]^{\mathrm{T}}$, where $\zeta = 0.15$ and $\omega_n = 1$. A corresponding discrete-time model with sampling period $\Delta\tau > 0$ is given by $x(t+1) = \mathbf{A}x(t) + bu(t)$ and $y(t) = c^{\mathrm{T}} x(t)$, where $\mathbf{A} = \exp(\Delta\tau \mathbf{A}_{\mathsf{c}})$, $b = (\int_0^{\Delta\tau} \exp(s\mathbf{A}_{\mathsf{c}})\mathrm{d}s) b_{\mathsf{c}}$, $c = c_{\mathsf{c}}$. For the simulations, we have used the following data: $N = 10$, $\Delta\tau = 0.2$, $q(t) \equiv 1$, $r(t) \equiv 1$, $\boldsymbol{\Sigma}_0 = 0.5\mathbf{I}_2$ (and thus, $\sigma_0^2 = 0.5$), $\mu_{\mathsf{f}} = 1$ and $\sigma_{\mathsf{f}}^2 = 0.01$ (or $\sigma_{\mathsf{f}} = 0.1$). The evolution of the Gaussian density function, $\rho_{y(t)}(\mathrm{y}) := (1/\sqrt{2\pi}\sigma(t)) \exp(-(\mathrm{y} - \mu(t))^2/(2\sigma(t)^2))$, the mean, $\mu(t) := \mathbb{E}\big[y(t)\big]$, and the standard deviation, $\sigma(t) := \sqrt{\mathbb{E}\big[(y(t)-\mu(t))^2\big]}$, of the output variable $y(t)$ versus the number of stages, t, when the system is driven by the optimal policy $\pi^\star$, are illustrated in Fig. 1.

VI. Conclusion

We have proposed a systematic approach for the computation of a dynamic output feedback control policy that solves the Liouville control problem for the class of Gaussian distributions for discrete-time SISO systems with incomplete state information. In the proposed approach, the control problem is reduced into a system of two decoupled optimization problems, namely a convex quadratic program subject to an affine constraint and a non-convex but tractable (generalized) quadratic program subject to a (non-convex) quadratic equality constraint. In our future work, we plan to address similar problems using control policies that depend on state estimates constructed by the available output measurements (separation-based control policies).

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