An Analytical Approach for Loss Minimization and Voltage Profile Improvement in Distribution Systems with Renewable Energy Sources

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***Abstract*—Loss minimization and voltage profile improvement are of prime importance in distribution system operation. This paper presents an analytical approach for coordinating distributed energy resources to reduce the active power loss in distribution networks. The proposed approach is based on the network admittance matrix and has an explicit solution if all loads in the network are constant current loads. Furthermore, it is shown that when loads are modeled with other characteristics such as constant power loads, a numerically efficient solution of loss minimization can also be obtained. The resulting solution reduces network losses while ensuring a nominal voltage at the point of common coupling. Case studies on a 19-node distribution system are presented to validate the proposed approach.**

I. Introduction

Integrating distributed energy resources (DERs) with distribution networks has a significant impact on system performance [1] and hence necessitates proper coordination. Coordination of RESs involves multiple challenges due to the nature of DERs and their geographical distribution. In order to overcome these challenges, grid codes/standards for integrating DERs have been developed. The guidelines for integrating a DER in distribution systems are outlined in IEEE Std. 1547 [2].

In early installations, DERs were not required to actively regulate the voltage at the point of common coupling (PCC) [2]. Accordingly, the DERs were designed to operate in maximum power point tracking (MPPT) mode with no reactive power support, i.e., unity power factor (UPF) mode. However, under revised guidelines, the active and reactive power outputs of DERs must be coordinated to actively regulate the voltage at PCC [3]. From an operators prospective, voltage regulation and loss minimization are important technical issues in distribution system operation. In this paper, we propose an algorithm to coordinate the DERs to reduce the active power loss while ensuring the voltage regulation at PCC.

Various algorithms proposed in the literature for loss minimization can be categorized into optimization-based approaches [4]–[6] and analytical approaches [7], [8]. In the optimization framework, an optimal power flow (OPF) problem is formulated with the objective of minimizing the active power loss subject to various network and operational constraints. On the other hand, analytical approaches attempt to arrive at an optimal solution based on network properties. In [7], an analytical approach to identify the optimal generation schedule based on the $[\mathbf{F_{LG}}]$ matrix is reported. A mathematical proof of optimality is presented in [8].

A. Aim and contributions of this paper

This paper departs from the semidefinite programming (SDP)-based approach to solving loss minization in distribution system, and provides a one step analytical approach to solving the problem. The underlying assumptions in the analytical framework are clearly stated and a simple mathematical proof of optimality is developed. Further, additional modifications required to adapt the analytical framework to DER coordination are outlined. Finally, we prove that utilizing the proposed framework for DER coordination will ensure voltage regulation at the PCC (required by [3]).

II. Proposed approach

The characteristic equations of the network (distribution system) in admittance form can be written as,

$$[\mathbf{I}] = [\mathbf{Y_{bus}}]\,[\mathbf{V}], \tag{1}$$

where $\mathbf{I}$ and $\mathbf{V}$ are respectively the vectors (in $\mathbb{C}^N$) of nodal current injection and voltage. $\mathbf{Y_{bus}}$ is the admittance matrix of the network ($\mathbb{C}^N \times \mathbb{C}^N$). Separating the network equations (given by (1)) corresponding to the generators and loads results in

$$\begin{bmatrix} \mathbf{I_G} \\ \mathbf{I_L} \end{bmatrix} = \begin{bmatrix} \mathbf{Y_{GG}} & \mathbf{Y_{GL}} \\ \mathbf{Y_{LG}} & \mathbf{Y_{LL}} \end{bmatrix} \begin{bmatrix} \mathbf{V_G} \\ \mathbf{V_L} \end{bmatrix}, \tag{2}$$

where G and L represent the set of generators and loads respectively.

To start with, we neglect the shunt admittances of transmission lines (negligibly small for distribution networks) and other reactive power shunts (like fixed capacitor banks, SVC) while forming $\mathbf{Y_{bus}}$. The elements neglected while forming $\mathbf{Y_{bus}}$ are treated as equivalent sources (discussed in Section II-B). Under this scenario

1) The network admittance matrix is a weighted Laplacian matrix [9] and is rank-one deficient.
2) The sub-matrices $[\mathbf{Y_{GG}}]$, $[\mathbf{Y_{LL}}]$ are symmetric and $[\mathbf{Y_{GL}}]^t = [\mathbf{Y_{LG}}]$ (this holds true even when the shunt elements are considered while forming the $[\mathbf{Y_{bus}}]$).

An alternate representation of the network can be obtained by pre-multiplying the equations corresponding to the load with $[\mathbf{Z_{LL}}]$ (i.e., $[\mathbf{Y_{LL}}]^{-1}$) and rearranging,

$$[\mathbf{I_G}] = [\mathbf{Y_{GG}}]\,[\mathbf{V_G}] + [\mathbf{Y_{GL}}]\,[\mathbf{V_L}]\,, \tag{3a}$$

$$[\mathbf{V_L}] = [\mathbf{Z_{LL}}]\,[\mathbf{I_L}] \;\; + [\mathbf{F_{LG}}]\,[\mathbf{V_G}]\,, \tag{3b}$$

where

$$[\mathbf{F_{LG}}] = -[\mathbf{Z_{LL}}][\mathbf{Y_{LG}}]. \tag{4}$$

Under no-load operating conditions (i.e., $[\mathbf{I_L}] = [0]$), the voltages at all the load buses are given by (5) (provided the shunt elements are considered),

$$\left[\mathbf{V_L}^0\right] = [\mathbf{F_{LG}}]\,[\mathbf{V_G}]\,. \tag{5}$$

However, under the chosen scenario, the submatrix $[\mathbf{F_{LG}}]$ exhibits *unitary row property*:

Theorem 1. *The row sum of submatrix $[\mathbf{F_{LG}}]$ equals to unity (i.e., $\sum_j \mathbf{F_{ij}} = 1 \;\forall\; i$), if the shunt elements of the network are neglected while forming $\mathbf{Y_{bus}}$.*

Proof: Given that $\mathbf{Y_{bus}}$ is a weighted Laplacian matrix, the following vector is clearly in the null space ($\mathbf{x_n}$)

$$\mathbf{x_n} = [1, \ldots, 1]^t\,. \tag{6}$$

Since $\mathbf{Y_{bus}}\mathbf{x_n} = [0]$, the elements of the partitioned $\mathbf{Y_{bus}}$ can be written as

$$\sum_{j\in G} Y_{ij} + \sum_{j\in L} Y_{ij} \;\; = 0 \;\forall i \in G, \tag{7a}$$

$$\sum_{j\in G} Y_{ij} + \sum_{j\in L} Y_{ij} \;\; = 0 \;\forall i \in L. \tag{7b}$$

Proceeding further, we pre-multiply the $\mathbf{Y_{bus}}$ with a full rank block diagonal matrix. The chosen block diagonal matrix ($\mathbf{T}$) is

$$\mathbf{T} \;=\; \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z_{LL}} \end{bmatrix}. \tag{8}$$

In general, for distribution networks the inverse of the partitioned matrix i.e., $\mathbf{Z_{LL}}$ $(= \mathbf{Y_{LL}}^{-1})$ exists and is of full rank. Consequently, the resulting matrix $\mathbf{Y}^{'} = \mathbf{T}\mathbf{Y_{bus}}$ is rank-one deficient:

$$\mathbf{Y}^{'} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z_{LL}} \end{bmatrix} \begin{bmatrix} \mathbf{Y_{GG}} & \mathbf{Y_{GL}} \\ \mathbf{Y_{LG}} & \mathbf{Y_{LL}} \end{bmatrix} = \begin{bmatrix} \mathbf{Y_{GG}} & \mathbf{Y_{GL}} \\ -\mathbf{F_{LG}} & \mathbf{I} \end{bmatrix} \tag{9}$$

Since $\mathbf{Y}^{'}$ is rank-one deficient, there exists only one unique vector (scalar multiplication) in the null space. Let $\mathbf{z_n}$ be the null vector corresponding to $\mathbf{Y}^{'}$, i.e.,

$$\sum_{j\in G} z_j Y_{ij} + \sum_{j\in L} z_j Y_{ij} \;\; = 0 \;\forall i \in G, \tag{10a}$$

$$-\sum_{j\in G} z_j F_{ij} + \sum_{j\in L} z_j I_{ij} \;\; = 0 \;\forall i \in L. \tag{10b}$$

Equation (7a) indicates that an obvious choice for the vector that takes the linear combination of elements in submatrices $\mathbf{Y_{GG}}$ and $\mathbf{Y_{GL}}$ to zero is $\mathbf{x_N}$. Stated equivalently, $\mathbf{x_N}$ also lies in the null space of $Y^{'}$ i.e., $\mathbf{z_N} = \mathbf{x_N}$. As a result,

$$\begin{aligned} -\sum_{j\in G} F_{ij} + \sum_{j\in L} I_{ij} \;&=\; 0 \;\forall i \in L \\ \Longrightarrow \sum_{j\in G} F_{ij} \;&=\; 1 \;\forall i. \end{aligned} \tag{11}$$

■

In this work, the proof of the *unitary row property* is established using the rank-one deficiency of the $[\mathbf{Y_{bus}}]$ matrix. An alternate proof is reported in [8], [10].

A. Analytic criterion for loss minimization

Pre-multiplying (3b) with $[\mathbf{I_L}]^*$, and the conjugate of (3b) with $[\mathbf{V_G}]$ results in

$$[\mathbf{I_L}^*]^t\,[\mathbf{V_L}] = [\mathbf{I_L}^*]^t\,[\mathbf{Z_{LL}}]\,[\mathbf{I_L}] + [\mathbf{I_L}^*]^t\,[\mathbf{F_{LG}}]\,[\mathbf{V_G}] \tag{12a}$$

$$[\mathbf{I_G}^*]^t\,[\mathbf{V_G}] = \Big([\mathbf{V_G}^*]^t\,[\mathbf{Y}^*_{\mathbf{GG}}] + [\mathbf{V_L}^*]^t\,[\mathbf{Y}^*_{\mathbf{LG}}]\Big)\,[\mathbf{V_g}]\,. \tag{12b}$$

The total network losses ($S_{\text{loss}} = P_{\text{loss}} + jQ_{\text{loss}}$) can be obtained by adding (12a) and (12b), i.e.,

$$S_{\text{loss}} = [\mathbf{I_L}^*]^t\,[\mathbf{Z_{LL}}]\,[\mathbf{I_L}] + [\mathbf{I_L}^*]^t\,[\mathbf{F_{LG}}]\,[\mathbf{V_G}] + [\mathbf{I_G}^*]^t\,[\mathbf{V_G}]\,. \tag{13}$$

In (13), the first term is dependent on the load current. The load and generator currents are by convention negative and positive (respectively), and hence the terms $[\mathbf{I_L}^*]^t\,[\mathbf{F_{LG}}]\,[\mathbf{V_G}]$ and $[\mathbf{I_G}^*]^t\,[\mathbf{V_G}]$ have negative and positive real parts respectively. Assuming the load current at any given operating point to be fixed, the minimum active power loss in the system is attained when the real part of the sum of last two terms in (13) attains a minimum value.

Theorem 2. *For any operating point, the real part of the sum of last two terms in (13) cannot attain a negative value, i.e., $[\mathbf{I_L}^*]^t\,[\mathbf{F_{LG}}]\,[\mathbf{V_G}] + [\mathbf{I_G}^*]^t\,[\mathbf{V_G}] \geq 0$.*

Proof: To start with, we presume that the voltage at terminals of generators is equal to the nominal value. Then, $[\mathbf{F_{LG}}]\,[\mathbf{V_G}]$ is almost identical to row sum of $[\mathbf{F_{LG}}]$. As a result,

$$[\mathbf{I_L}^*]^t\,[\mathbf{F_{LG}}]\,[\mathbf{V_G}] = \sum_{j\in L} I_j^* \quad \text{(using (11))}. \tag{14}$$

Under the same presumption,

$$[\mathbf{I_G}^*]^t\,[\mathbf{V_G}] = \sum_{j\in G} I_j^*. \tag{15}$$

Since the real parts of generator currents are in general greater than or equal to the real parts of load currents, the real part of the sum $[\mathbf{I_L}^*]^t\,[\mathbf{F_{LG}}]\,[\mathbf{V_G}] + [\mathbf{I_G}^*]^t\,[\mathbf{V_G}]$ cannot have a negative value.

On the other hand, if the generator voltages are not close to nominal values, a proof of non-negativity of

$[\mathbf{I_L}^*]^t\,[\mathbf{F_{LG}}]\,[\mathbf{V_G}] + [\mathbf{I_G}^*]^t\,[\mathbf{V_G}]$ can be obtained by contradiction. Consider an arbitrary operating point such that

$$\text{Re}\left\{-\,[\mathbf{I_L}^*]^t\,[\mathbf{F_{LG}}]\,[\mathbf{V_G}]\right\} > \text{Re}\left\{[\mathbf{I_G}^*]^t\,[\mathbf{V_G}]\right\}. \quad (16)$$

It is to be noted that, $-\,[\mathbf{I_L}^*]^t\,[\mathbf{F_{LG}}]\,[\mathbf{V_G}]$ represents the sum of the product of no-load voltage and load current (using (5)). Accordingly, (16) translates to

$$\text{Re}\left\{\sum \mid V_L^0 I_L \mid\right\} > \text{Re}\left\{\sum \mid V_G I_g \mid\right\}. \quad (17)$$

Equation (17) can only hold true if

1) The voltages at load buses under no-load are higher than the generator voltage.
2) The sum of load currents is greater than the sum of the generator currents.

The above arguments generally do not hold true for distribution networks since they are dominated by resistance and inductive reactance. As a result, (16) cannot hold true. ∎

As a consequence of Theorem 2, minimum active power loss in the system is attained when the real part of the sum of the last two terms in (13) equals to zero. Hence, the criterion for loss minimization is

$$\begin{aligned}[\mathbf{I_G^*}]^{\mathbf{t}}\,[\mathbf{V_G}] &= -[\mathbf{I_L^*}]^{\mathbf{t}}\,[\mathbf{F_{LG}}]\,[\mathbf{V_G}]\\ [\mathbf{I_G}] &= -[\mathbf{F_{LG}^*}]^{\mathbf{t}}\,[\mathbf{I_L}].\end{aligned} \quad (18)$$

In terms of generator power, the optimality criterion is

$$\begin{aligned}[\mathbf{V_g}][\mathbf{I_G^*}] &= -[\mathbf{V_g}][\mathbf{F_{lg}}]^t\,[I_l^*]\\ [\mathbf{S_g}] &= -[\mathbf{V_g}][\mathbf{F_{lg}}]^t[\mathbf{I_l^*}].\end{aligned} \quad (19)$$

where $[\mathbf{V_g}]$ is a diagonal matrix with its elements set to the voltages at generator terminals.

Theorem 3. *Scheduling the generators according to (18) results in a nominal voltage profile ($1\angle 0$) at the PCC of the distributed generators.*

Proof: Using (2) and (3b), an expression for the currents injected at the generator nodes in terms of load currents can be obtained as

$$[\mathbf{I_G}] = [\mathbf{Y_{GG}}]\,[\mathbf{V_G}]+[\mathbf{Y_{GL}}]\,[\mathbf{F_{LG}}]\,[\mathbf{V_G}]+[\mathbf{Y_{GL}}]\,[\mathbf{Z_{LL}}]\,[\mathbf{I_L}]. \quad (20)$$

Since the admittance matrix is symmetric,

$$[\mathbf{Y_{GL}}]\,[\mathbf{Z_{LL}}] = [\mathbf{Y_{LG}}]^t\,[\mathbf{Z_{LL}}] = -\,[\mathbf{F_{LG}}]^t. \quad (21)$$

Using (20) and (21), the relationship describing the voltage at the terminals of generators under optimal scheduling (i.e., $[\mathbf{I_G}] = -[\mathbf{F_{LG}^*}]^{\mathbf{t}}\,[\mathbf{I_L}]$) is

$$\begin{aligned}[\mathbf{Y_{GG}}]\,[\mathbf{V_G}] + [\mathbf{Y_{GL}}]\,[\mathbf{F_{LG}}]\,[\mathbf{V_G}] &= \mathbf{0},\\ \text{i.e., } [\mathbf{Y_{GG}}]\,[\mathbf{V_G}] + [\mathbf{Y_{GL}}]\,[\mathbf{V_L}^0] &= \mathbf{0}.\end{aligned} \quad (22)$$

According to (7a) and (10a) (rank-one deficiency of admittance matrix), the only unique combination that can realize (22) is

$$\begin{aligned}[\mathbf{V_G}] &= [\lambda,\ldots,\lambda]^t,\\ [\mathbf{V_L^0}] &= [\lambda,\ldots\ldots,\lambda]^t.\end{aligned} \quad (23)$$

It must be noted that when $[\mathbf{V_G}] = [\lambda,\ldots,\lambda]^t$,

$$\begin{aligned}[\mathbf{V_L^0}] &= [\mathbf{F_{LG}}]\,[\mathbf{V_G}]\\ &= \lambda \text{ (row sum of } \mathbf{F_{LG}})\\ &= [\lambda,\ldots\ldots,\lambda]^t \text{ (by Theorem 1).}\end{aligned} \quad (24)$$

Equation (24) indicates the consistency of the criterion given by (23).

In general, the voltage at the substation (one of the generators) is always regulated. If this voltage is chosen as the base voltage (i.e., $\lambda = 1\angle 0$ p.u.), then by (23), the voltage at other generators is also maintained at this value. ∎

B. Application to practical networks

The optimality criterion for loss minimization obtained in section II-A assumes the network to have no shunt elements. In addition, the loads were modeled as constant current sources. In practical distribution systems, these assumptions do not hold true. In this section, the modifications needed to employ the analytical solution for practical distribution networks are outlined.

1) For a network comprised of loads with different load characteristics, an optimal solution can be obtained by using the analytical criterion in an iterative manner. The steps involved are the following:
   - Approximate the load at a particular node as a constant current source i.e.
   $$I_l^k = \left(\frac{S_l}{V_l^k}\right)^*. \quad (25)$$
   - Compute the optimal generation schedule using (19).
   - Obtain a power flow solution with the computed generation schedule, and update the generator/load voltages.

   The above iterative procedure is employed till a convergence criterion is satisfied. In this work, the convergence criterion is chosen as iterate when the change in the optimal generator schedule (ΔS_g) obtained between successive iterations and the tolerance is chosen as 10^{-3}(in p.u.). To obtain the power flow solution, the Forward-Backward approach is adopted.
2) The shunt elements in the network (susceptance of transmission line or fixed shunt elements such as capacitor banks) are treated as equivalent current sources.
3) If the optimal generator schedule violates the limits of the generator, it is treated as an equivalent load bus with the load set to the negative of the maximum value. In this scenario, the voltages at the PCC may not be at nominal.
4) For DERs operating in the MPPT mode, the reactive power scheduling can be carried out by considering only the imaginary part of the optimal generation schedule. Our observation is that the resulting network loss obtained by using such an approximate solution is in close agreement with the optimal solution (illustrated in Section III).

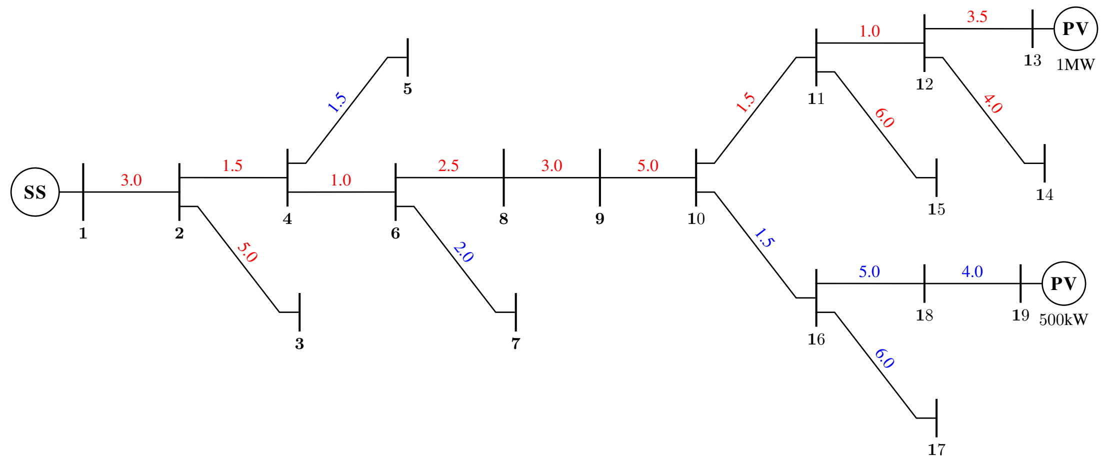

Fig. 1. Single line diagram of the 19-node system

III. Case Studies

This section illustrates the proposed approach on a 19-node distribution network with 2 DERs. The results obtained using the proposed approach are compared with those obtained using exhaustive search. In the exhaustive search approach, the optimal solution is identified by evaluating the network loss over all possible operating points of the DERs. The single line diagram of the system along with the length of the feeders is shown in Fig. 1. The peak load on the system is 2800 MVA with a power factor of 0.9. The loads are presumed to have a constant power characteristic. Two distributed generators having peak capacities of 1 MW and 0.5 MW are presumed to be integrated at nodes 13 and 19 respectively.

The resulting network losses obtained by scheduling the DERs using the proposed and exhaustive search approaches for two scenarios are given in Table I. In addition, the network loss obtained by operating the DERs in a unity power factor (UPF) mode (according to [2]) is also indicated. In the UPF mode of operation, it is presumed that the DERs are operated in MPPT mode.

TABLE I
Comparison of Network Loss for 19-Node Network

	Proposed approach	Exhaustive search	UPF Mode
Case A	90.55 kW (4.30%)	90.13 kW (4.28%)	156.63 kW (7.21%)
Case B	94.35 kW (4.47%)	94.17 kW (4.46%)	158.33 kW (7.28%)

For all test cases, the load factor is presumed to be 0.8. In the first case (Case A), it is presumed that:

a) The maximum possible output of the distributed generators at nodes 13 and 19 are 800 kW and 450 kW respectively.
b) The distributed generators have the flexibility to operate in non-MPPT mode.

The losses obtained using the proposed approach are in close agreement with those obtained using the exhaustive search approach (indicated in Table I). It is to be noted that the resulting loss is significantly lower when compared to a scenario where DERs are operated in UPF mode. The power settings of DERs obtained using the proposed approach along with the resulting voltage at the PCC are given in Table II. The optimal settings obtained using the proposed approach ensure that the voltage at the PCC is close to nominal value (1.0 p.u.).

TABLE II
Settings of DERs for Case-A in 19-Node Network

	Node 19	Node 13
Active power setting	770.98 kW	324.39 kW
Reactive power setting	350.22 kVAR	183.54 kVAR
Voltage at PCC (in p.u.)	$1.000\angle 0.000$	$0.999\angle -0.57°$

In the second case (Case B), it is presumed that:

a) The maximum possible outputs of distributed generators at nodes 13 and 19 are 850 kW and 450 kW respectively.
b) The distributed generators operate in MPPT mode.

Since the DERs are presumed to be operating in MPPT mode, only the reactive power settings are obtained using the proposed approach. Even in this scenario, the resulting network loss is in close agreement with that obtained using exhaustive search. It is interesting to note that the resulting loss is higher than that obtained when the DERs are operated in non-MPPT mode. The optimal settings obtained using the proposed approach along with the resulting voltage at the PCC are given in Table III.

TABLE III
Settings of DERs for Case-B in 19-Node Network

	Node 19	Node 13
Active power setting	850.00 kW	450.00 kW
Reactive power setting	354.80 kVAR	188.45 kVAR
Voltage at PCC (in p.u.)	$1.030\angle 0.013$	$1.041\angle 0.006°$

The final version of the manuscript will include detailed case studies on the 56-node Southern California Edison system [11] along with the convergence characteristics of the proposed approach.

IV. Concluding remarks

In this paper, an analytical framework is reported for loss minimization and voltage profile improvement in distribution systems. The proposed approach utilizes the network connectivity information and reduces network losses while maintaining a nominal voltage at the point of common coupling. Case studies on the 19-node distribution network illustrate the effectiveness of the proposed approach. An interesting observation is that, under certain scenarios, operating the distributed generators in non-MPPT mode results in a better network performance.

Acknowledgement

This work is supported in part by NSF Contracts ECCS-1646449, CNS-1302182, ECCS-1546682, NSF Science & Technology Center Grant CCF-0939370, and the Power Systems Engineering Research Center (PSERC).

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