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Knowledge You Can Act on: Optimal Policies for Assembly Systems with Expediting and Advance Demand Information

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We consider a nonstationary, stochastic, multistage supply system with a general assembly structure, in which customers can place orders in advance of their future demand requirements. This advance demand information is now recognized in both theory and practice as an important strategy for managing the mismatch between supply and demand. In conjunction, we allow expediting of components and partially completed subassemblies in the system to provide the supply chain with the means to manage the stockout risk and significantly enhance cost savings realized through advance demand information. To solve the resulting assembly system, we develop a new method based on identifying local properties of optimal decisions. This new method allows us to solve assembly systems with multiple product flows. We derive the structure of the optimal policy, which represents a double-tiered echelon basestock policy whose basestock levels depend on the state of advance demand information. This form of the optimal policy allows us to: (i) provide actionable policies for firms to manage large-scale assembly systems with expediting and advance demand information; (ii) prove that advance demand information and expediting of stock both reduce the amount of inventory optimally held in the system; and (iii) numerically solve such assembly systems, and quantify the savings realized. In contrast to the conventional wisdom, we discover that advance demand information and expediting of stock are complementary under short demand information horizons. They are substitutes only under longer information horizons.

Keywords: multiechelon inventory; assembly system; advance demand information; expediting; optimal policy.
Subject classifications: inventory/production: multi-item/echelon/stage; inventory/production: uncertainty: stochastic; dynamic programming: applications.
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1. Introduction

When customers place orders in advance of their future demand requirements, the result is a portfolio of customers with different demand lead times and demand requirements. Such a portfolio generates what is known as *advance demand information*. The ability of companies to collect and utilize advance demand information (ADI) has come to be recognized, in both industry and research literature, as an important strategy for managing the mismatch between supply and demand. One example of such a strategy is Dell's "Intelligent Fulfillment," which offers customers four different levels of response time: precision delivery with a specific date, premium delivery that arrives the next day, standard five-day delivery, and value delivery with longer deliver times (Özer and Wei 2004). Another example is found in initiatives undertaken by automobile manufacturers, such as Renault's "Projet Nouvelle Distribution" and BMW's Customer-Oriented Sales Processing (COSP), that seek to utilize advance demand information to shift production toward build-to-order (Miemczyk and Holweg 2004). Other large manufacturers, such as General Motors and Boeing, have also undertaken initiatives to take advantage of advance demand information in the face of uncertain demand and capture the value of sharing that information (Gayon et al. 2009).

Advance demand information allows a company to increase profits by shifting its production from build-to-stock to build-to-order, and thus deal with high variability of demand with lower inventory requirements. At the same time, lower inventory levels leave the supply chain vulnerable to potential high realizations of demand and the resulting stockout costs. One strategy to mitigate exposure to high demand realizations is to allow expediting of inventory through the supply chain. Dell, for example, is quite explicit in its annual 10-K that "…our business model generally gives us flexibility to manage backlog at any point in time by expediting…customer orders" (Dell 2013, p. 11). In addition, because the build-to-order production enabled by ADI "increases process complexity and consequently causes more expensive processes" (Ericsson et al. 2010, p. 11), it becomes important to maximize cost savings made possible by ADI. Existing research on ADI achieves cost reductions by adapting the regularly scheduled, periodic replenishments to the acquired demand information in each period. Although reasonable savings can be realized that way, as shown in Gayon et al. (2009) and other studies, more significant cost reductions can potentially be achieved by giving a firm additional options, such as expediting, to act on its knowledge of advance demand. Consequently, expediting of inventory in the supply chain can also serve to mitigate the cost impact of more expensive processes by enhancing savings generated through advance demand information. The key to realizing such savings, however, is being able to manage the supply chain with both expediting and advance demand information, especially given that expediting usually comes at a higher cost.

The main difficulties associated with making use of advance demand information and shifting production to build-to-order tend to come from a company's assembly processes (Weber 2006). Most companies cited in the literature for implementing ADI have significant assembly operations. The academic literature has not yet proposed strategies for managing assembly operations in the presence of either ADI or expediting of stock. Consequently, in this paper we focus on supply chains with the assembly structure. In particular, we consider the problem of assembling products of a single type from multiple components, each of which can be replenished through both regular orders and expedited orders. Our model allows for a general assembly structure, in which some components may be assemblies, some may be subassemblies, some may be (simple) parts, and some may be activities carried out in parallel. Because companies' cost structures tend to change over the length of the product cycle or planning horizon, in departure from the prevailing trend in the assembly literature, we allow for nonstationary model parameters. Formally, we consider a nonstationary, periodic review, finite-horizon, assemble-to-stock model with ADI and the option to expedite inventories of components and partially completed assemblies at any stage in the system (in addition to using regularly scheduled replenishments).

The challenge of (optimally) managing any kind of assembly system is considerable, due to the severe curse of dimensionality created by a very large state space. As a result, assembly systems tend to be analyzed using simulation methods (see, e.g., Sabuncuoglu et al. 2002). For the assembly system considered in our paper, this challenge is augmented by having two ordering decisions at each node in the system. Because of this dual flow of components, and because unit ordering costs and holding costs can vary over time, standard approaches to solving assembly systems by reducing them to equivalent series systems by means of balancing echelon inventories do not yield fruit. Our first contribution, therefore, is a new analytical approach for solving complex assembly systems based on establishing local properties of optimal decisions. We make use of this approach to significantly reduce the state space of the assembly problem with expediting and ADI, and the associated curse of dimensionality. Our second contribution is to identify the form of the optimal policy, which represents a state-dependent, double-tiered, echelon basestock policy. This form of the optimal policy allows the system to be decomposed into a nested sequence of solvable convex subproblems. Our third contribution is to establish key monotonicity properties, and, also, to prove that both advance demand information and expediting of stock reduce the amount of inventory optimally held throughout the supply chain. This inventory reduction is the main driver of cost savings realized through advance demand information and expediting of stock.

Our analytical results also make it possible to numerically solve assembly systems with expediting and advance demand information, and to quantify savings realized by giving companies an additional option to act on their knowledge of ADI by expediting stock. We find ranges of model parameters (hence assembly system characteristics) under which expediting is especially valuable to supply chains with advance demand information. Furthermore, because both ADI and expediting of stock represent strategies to deal with uncertain demand, and because they both lower total inventory in the system, ADI and expediting tend to be considered as substitute strategies. What we discover, however, is that the demand information horizon plays a key role: with short information horizons, ADI and expediting are *complements*. The substitution effect takes place only with longer information horizons. Thus, economic complementarity and substitutability of advance demand information and expediting of stock are shown not to be absolute characteristics, but rather functions of the supply chain structure. Our paper also provides the most comprehensive numerical study of assembly systems to date, allowing us to quantify the effect of structural factors such as the length of the supply chain, demand correlation, and the demand information horizon.

1.1. Literature Review

Our work is related to three streams of research. The first of those pertains to serial multiechelon inventory systems. The classic paper of Clark and Scarf (1960) shows how a multistage inventory model can be reformulated to achieve a decomposition of this multidimensional problem into a sequence of single-dimensional problems. Federgruen and Zipkin (1984) extend these results to the stationary, infinite horizon case. Federgruen (1993) and Angelus (2011) provide reviews of this literature. Of particular relevance to our work is an important paper by Lawson and Porteus (2000), who introduce expediting to a multiechelon series system, show that such a system achieves the Clark-Scarf decomposition, and prove the structure of the optimal policy is a top-down echelon basestock policy. One of our contributions is to generalize Lawson and Porteus (2000) in two important ways: (i) by considering a more general supply chain structure, in the form of an assembly system; and (ii) by incorporating advance demand information.

The second related stream of literature studies systems with advance demand information. Gallego and Özer (2001) establish the optimality of state-dependent (s, S) and basestock policies for single-stage systems with ADI and with and without fixed costs. Chen (2001) studies how, by offering different prices and delivery schedules, a firm is able to segment customers by different advance demand leadtimes. Hu et al. (2003) consider a manufacturer with ADI who can meet customer demand by in-house production or outsourcing, and show that the optimal policy is a double-threshold policy. Wang and Toktay (2008) consider single-stage inventory models with ADI and flexible delivery schedules. With homogeneous customers, state-dependent (s, S) policies are optimal, whereas heterogeneous customers also necessitate an allocation decision. Gayon et al. (2009) address production with limited capacity and several demand classes that share advance demand information with the supplier. The optimal production policy is a state-dependent basestock policy, whereas the optimal inventory-allocation policy is a multilevel rationing policy. Benjafaar et al. (2011a) consider finite production capacity, stochastic production times, and imperfect advance demand information, and show the optimality of a basestock policy whose levels depend on the number of outstanding orders. Bernstein and DeCroix (2014) study different types of advance demand information, such as commonality, mix and volume information, and determine the relationships among them.

When the multiechelon series system of Clark and Scarf (1960) is extended to allow for advanced demand information, Gallego and Özer (2003) show that, for sufficiently short information horizons, the optimal policy is the echelon basestock policy; for longer information horizons, basestock levels become state dependent. In the case of periodic-review distributions systems, Özer (2003) proposes an effective heuristic to manage such a system under ADI, and shows how ADI can be a substitute for leadtimes and inventory. Under continuous review of such a distribution system with ADI and when each installation replenishes its stock using basestock policies, Marklund (2006) provides exact and approximate cost evaluation techniques under various stock reservation and allocation policies for different retailers. We are not aware of any research literature on assembly systems with either advance demand information or expediting of stock.

The third related stream studies assemble-to-stock systems under stochastic demand. Schmidt and Nahmias (1985) show that a two-component assembly problem with joint ordering and assembly decisions can be decomposed into component ordering and finished good assembly decisions. The optimal policy, however, has a very complex structure, with the optimal order for one component depending on the inventory of the other. Benjafaar et al. (2011b) address a complex assembly system with multiple items, stages, and customer classes, where demand from each class follows a compound Poisson process. The optimal production policy is shown to be an inventory state-dependent basestock policy, whereas the optimal allocation policy is a multilevel inventory state-dependent rationing policy.

Assemble-to-stock systems tend to have very large state space and decision space because of the need to keep track of inventory at a large number of locations (i.e., nodes) in the system, and make decisions pertaining to each one of those in each period. Because of the resulting curse of dimensionality, the literature in this field has mostly been focused on developing approaches to reduce the difficulty of managing such systems. Two such approaches are currently available in the literature. The first approach to solving general assembly systems was introduced in the classic work of Rosling (1989), which considers a stationary, infinite horizon assembly system (with regular flow only). Rosling's unit of analysis is an item, which represents either a subassembly in which multiple preceding items are assembled into another item that continues to flow downstream toward the final assembly. Rosling's formulation of the problem makes use of echelon variables, introduced by Clark and Scarf (1960), to account for the echelon inventory level in each period and for each item, as well as a vector of echelon inventory positions based on a set of orders placed from the preceding item but not yet received. Rosling determines inventory holding costs associated with those echelon inventory levels and echelon inventory positions for each item in each period, and sums the discounted expected value of those costs over the infinite time horizon. He makes use of that infinite sum to show that the optimal policy for the system satisfies the "long-run balance" under which all echelon inventory positions for an item closer to the final assembly are lower than corresponding echelon inventory positions for an item farther from the final assembly. Hence, the optimal policy is such that, for any two items, their echelon inventory positions pertaining to those orders that are the same number of periods away from the final assembly must be identical for any number of such periods. Consequently, all echelon states and decisions in the system become identical for any given number of periods that orders are away from the final assembly. As a result, if the initial state of the system satisfies the same balanced condition, the optimal policy for the assembly system can be reduced to that of an equivalent series system, in which all items equally far from the final assembly are aggregated together. Chen and Zheng (1994) offer another derivation of this result. Rosling's approach has subsequently been used by DeCroix and Zipkin (2005) to address an important extension of Rosling's stationary, infinite-horizon assembly model. They allow for uncertain product (and component) returns from customers and describe the item-recovery pattern and restrictions on the inventory policy under which an equivalent series system is shown to exist. DeCroix (2013) also makes use of Rosling's approach in considering an assembly system subject to random supply disruptions, and shows how such system can be simplified by replacing some subsystems with a series structure. Chen and Muharremoglu (2014) consider an assembly system identical to Rosling's (with only the regular flow of components) but in which the initial state may not be balanced. Using the customer-unit decomposition approach, they establish the optimality of echelon basestock policies whose basestock levels are dynamically changing in time as a function of cumulative customer demands.

To analyze assembly systems with *nonstationary* costs, Angelus and Porteus (2008) develop a different approach to solving assembly systems (with regular flow of orders only). They also make use of an echelon formulation of the problem, but instead of using items as units of analysis, they analyze the system by means of components, which represent items with no predecessors. Furthermore, they work explicitly with stages through which components flow downstream, one stage per period, toward final assembly, rather than capturing stages through a set of echelon inventory positions. The basic unit of analysis in Angelus and Porteus (2008) is thus a component assembly system in which there is only a single (final) assembly at the most downstream stage. They formulate a dynamic program for the objective cost function for the problem and identify a new property of the objective cost function that is preserved under minimization. Under this property, referred to as "balance-inducing," the objective cost function increases if any stage in the system has an excess of any component relative to the component with the smallest echelon inventory at that stage. In other words, given an underlying echelon state and the corresponding objective cost function, if additional inventory is added to any component at any stage (other than the one with the smallest echelon inventory at that stage), the objective cost function increases. Angelus and Porteus make use of this property to show that, if the system starts out in a balanced state, the optimal policy for the system is balanced in every subsequent period, so that there is an equal amount of every component at every stage. This result, in turn, implies that a nonstationary assembly system with (only) the regular flow of product can be reduced to an equivalent series system.

When it comes to nonstationary assembly systems with multiple flows of product, such as having both regular and expedited flow of product as allowed in our paper, neither of the two existing approaches in the literature yields fruit. First, the classic approach of Rosling (1989) is limited by its assumption of stationary costs. Indeed, Rosling (1989, p. 574) states that the "series interpretation generally cannot be expected to carry over when holding costs or production costs are non-stationary." With regard to the approach of Angelus and Porteus (2008) on the other hand, achieving the preservation of the balance-inducing property identified in their paper when expediting is added to the system necessitates imposing overly restrictive assumptions on cost parameters in the model. In particular, it can be shown that the preservation of the balance-inducing property for an assembly system with expediting requires that the unit expediting cost for every component at each stage be so low that it becomes optimal to never place a regular order in the system. Second, both existing approaches make use of the echelon formulation which is imbued with a certain limitation when it comes to systems with multiple flow of product. In particular, under echelon formulation, echelon inventory positions in both Rosling (1989) and Angelus and Porteus (2008) would be obtained by including both the expedited and the regular flow orders. Therefore, with expediting, the balanced echelon states found to be optimal in those papers can be reached by *unbalanced* combinations of regular and expedited orders. As a result, even though inventory states may be balanced in each period, such unbalanced optimal decisions would make it impossible to reduce the system to an equivalent series one. In other words, the echelon formulation of assembly systems aggregates expedited and regular flow of product in a way that renders it impossible to balance both of them at the same time, under the optimal policy. Thus, instead of using echelon variables, we formulate the assembly model by means of installation stocks and replenishment decisions related to those. Furthermore, instead of directly establishing system-wide properties of the problem (long run balance in Rosling 1989, balance-inducing in Angelus and Porteus 2008), we tackle the problem stage by stage and establish local (i.e., stage-specific) properties of the optimal policy. As shown in Section 3, these two aspects of our approach make it possible to disaggregate those two product flows (and, for that matter any number of flows), as well as stages in the system, so that each type of order at each stage can be shown to be balanced under the optimal policy. In that manner, our approach makes it possible to solve more general assembly systems.

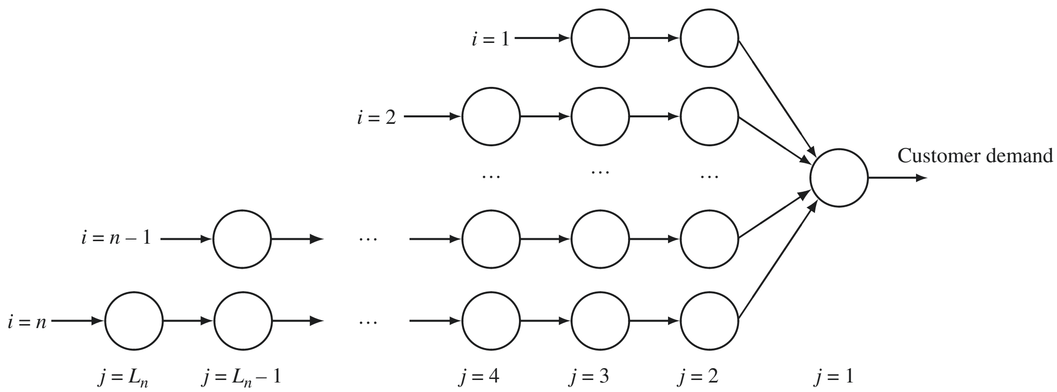
In summary, our paper contributes to the assembly literature by (i) incorporating both advance information and expediting of stock (neither of which was considered either alone or together in the context of assembly systems); (ii) allowing for nonstationary model parameters and finite time horizons; (iii) characterizing the optimal policy that makes it possible to manage the resulting assembly system; and (iv) proposing a new approach to solve complex assembly systems with multiple flows of product.

2. Model Description

2.1. Component Assembly System

We begin our analysis with a simplified assembly model (see Figure 1), where components are assembled only once, at the most downstream stage. We refer to this system as *the component* (*assembly*) *system* because it is the components that flow downstream until they are assembled into the final product. Later we show that the component system plays a key role in solving more complex assembly systems.

Figure 1. Component assembly system.



Following the literature on assemble-to-stock systems (e.g., Rosling 1989, DeCroix and Zipkin 2005), we assume, without loss of generality, that assembling a product requires exactly one unit of each of n components. Each component i, $i \in (1, 2, \ldots n)$, has a standard leadtime of L_i periods between the placing of a regular order for the component and its becoming a part of a final product. Our model allows for a component to stay at a single physical location for more than one period, as work-in-progress, which can be used to represent multiperiod leadtimes between adjacent physical stages (in which case a single physical location would be represented by multiple completion stages). At the same time, if expedited, a unit of any component can move through adjacent stages in the system within a single period, as elaborated below. Without loss of generality, we order component types so that $L_1 \leqslant L_2 \leqslant \cdots \leqslant L_n$. Note that some components can have the same leadtime as others, and that all possible leadtimes need not be represented: In Figure 1, for example, there is no component with a leadtime of one period. Furthermore, we do not combine different components with the same leadtime into a single aggregate component because it is not necessarily optimal to manage them identically. Thus, our starting model is a periodic review, nonstationary, finite-horizon, component assembly system.

2.2. Advance Demand Information

Following Gallego and Özer (2001, 2003), the demand seen during period t is of the form $D_t = (D_{t,t}, \ldots, D_{t,t+N})$, where $D_{t,s}$ is the demand observed in period t for delivery in period s. $D_{t,s}$ is assumed to have a continuous probability distribution for each t and s. Period s is such that $s \in [t, \ldots, t+N]$, where N is the longest available delivery time offered to the customer, referred to as the (*demand*) *information horizon*. This formulation captures demands that are realized now, but need to be fulfilled s periods later. Such advance demand information helps the company better manage the supply-demand mismatch by providing exact information on a portion of demand that will need to be satisfied in the future. Thus, at the beginning of period t, the observed demand to be fulfilled in future period s is $O_{t,s} = \sum_{q=s-N}^{t-1} D_{q,s}$.

At the beginning of each period t, the available demand information is the N dimensional vector $\tilde{O}_t := (O_{t,t}, O_{t,t+1}, \ldots, O_{t,t+N-1})$, and the actual demand to be satisfied in period t is $O_{t,t} = \sum_{q=t-N}^{t-1} D_{q,t}$. By the end of period t, the pending demand (i.e., to be satisfied in period t) is $O_{t+1,t} = O_{t,t} + D_{t,t}$.

We allow components $D_{t,t+i}$ and $D_{t,t+j}$ of the process D_t to be correlated with each other, *and* the process D_t to be correlated across time. In particular, we allow the process D_t to depend on $\tilde{O}_t$, which is known at the beginning of period t and before D_t is realized. This provides the model with the capability to capture the dynamic evolution of demand distributions based on previously realized demands.

2.3. Dynamics

For each component i, at each stage, two decisions are made regarding the flow of inventory: (1) how many units to *expedite* downstream; and (2) how many units to move to the next stage of completion. For convenience, we will refer to the latter as *regular order*. Starting at the upstream stage L_i and proceeding downstream, expedited orders arrive at the next stage downstream immediately (i.e., before any other decision is made). Regular orders reach the next stage downstream at the end of the period (but before customer demand has been realized). The following parameters describe the system:

x_{ijt} = on-hand inventory of component i in stage j if $j > 1$ (net inventory of component i if $j = 1$), at the beginning of period t, prior to making any decisions;

X_{ijt}^E = the number of units of component i expedited into stage j from stage $j+1$ in period t;

X_{ijt}^R = the number of units of component i regular ordered into stage j from stage $j+1$ in period t.

The collection x_t of all on-hand inventories x_{ijt}, will be referred to as *the on-hand* (*inventory*) *state*; the collection of all expedited orders X_{ijt}^E, denoted by X_t^E, will be referred to as *the expedited order schedule*; the collection of all regular orders X^R_{ijt}, denoted by X^R_t will be referred to as *the regular order schedule*. The sequence of events in each period t is as follows: (1) on-hand inventory state x_t is observed; (2) expedited order schedule X^E_t is selected, starting with the most upstream stage and moving down; (3) expedited amounts are received; (4) regular order schedule X^R_t is selected; (5) regular orders are received; (6) vector D_t of customer demands is observed, outstanding demand $O_{t,t+1}$ is satisfied to the extent possible, and unmet demand is backlogged; and, (7) costs are incurred.

Since each expedited order arrives at the destination stage before any other decision is made, then the expedited order into stage $j+1$ arrives before the decision on the expedited order into stage j has to be made; thus, $X^E_{ijt} \leqslant x_{i,j+1,t} + X^E_{i,j+1,t}$, for each i and every $j < L_i$. Since regular orders are placed after all expedited quantities have arrived, then $X^R_{ijt} \leqslant x_{i,j+1,t} + X^E_{i,j+1,t} - X^E_{ijt}$, for each component i and stage $j < L_i$. Note that the delivery of expedited units from any stage to any downstream stage occurs within a single period, and before regular orders are placed. Given x_t, the set of feasible decisions $\mathbb{X}(x_t)$ becomes

$$\mathbb{X}(x_t) = \{X^E_t, X^R_t \geqslant 0 \mid X^E_{ijt} \leqslant x_{i,j+1,t} + X^E_{i,j+1,t}; X^R_{ijt} \leqslant x_{i,j+1,t} + X^E_{i,j+1,t} - X^E_{ijt} \text{ for } 1 \leqslant i \leqslant n, 1 \leqslant j < L_i\}.$$

The state transition equations are given by

$$x_{ij,t+1} = \begin{cases} x_{i1t} + X^E_{i1t} + X^R_{i1t} - O_{t+1,t} & \text{if } j = 1, \\ x_{ijt} + X^E_{ijt} - X^E_{i,j-1,t} + X^R_{ijt} - X^R_{i,j-1,t} & \\ \quad \text{if } j = 2, \ldots, L_i. \end{cases} \tag{1}$$

The constraints of the feasible set $\mathbb{X}(x_t)$ imply that $x_{ijt} \geqslant 0$ for every i and $j \geqslant 2$ in each period t.

2.4. Costs

A backlogging cost p_t is charged for each unit of demand not satisfied by the end of any period; a unit inventory holding cost H_{ijt} is incurred on the amount of component i located at stage j at the end of a period. Holding a unit of assembled product incurs a holding cost H^A_t. Each unit of component i expedited into stage j incurs a unit expediting cost k^E_{ijt}, whereas each unit of component i regular ordered into stage j incurs the unit ordering cost of k^R_{ijt} in period t. All unit costs are positive. Let α be the discount factor.

If the unit cost of a regular order were to exceed the unit cost of expediting for a component at a particular stage, it would always be optimal to expedite every single unit of that component through that stage; such a stage would thus effectively not exist in the system, as the associated single-period leadtime would disappear for all units arriving into that stage (relevant costs could be allocated to the next stage upstream). We assume that all such stages have already been folded into the system, so that the remaining ones are those for which the expedited order cost exceeds the regular order cost. The following assumption thus preserves the structure of the system.

Assumption 1. *$k^E_{ijt} > k^R_{ijt}$ for every component i, stage j, and period t.*

We also assume that unit inventory holding costs are increasing going downstream, because increasing unit inventory holding costs "...reflect higher physical and financial holding cost typically associated with items that have progressed farther through the system" (DeCroix and Zipkin 2005).

Assumption 2. *$H_{ijt} \geqslant H_{i,j+1,t}$, for every component i, stage j and period t.*

To avoid inventory replenishment decisions being made for speculative purposes (that is, solely for the purpose of exploiting the time variability of regular order costs rather than satisfying customer demand), we impose the following restriction on how quickly those unit costs can change in time.

Assumption 3. *$k^R_{ijt} + H_{ijt} - H_{i,j+1,t} > \alpha k^R_{ij,t+1}$ for every component i, stage j and period t.*

Next period's unit regular order cost for any component cannot exceed this period's regular order cost by more than the difference in unit holding costs. (Under Assumption 2, Assumption 3 is satisfied for any stationary costs). If the unit regular order cost of some component were to increase excessively from one time period to another, then it would be optimal to stockpile that component at a downstream stage despite higher inventory holding costs, solely for the purpose of saving on future regular order costs. Assumption 3 rules out such stockpiling through regular orders. Furthermore, because of Assumption 1, Assumption 3 also acts to rule out such stockpiling through expedited orders. Note that Assumption 3 is automatically satisfied in Rosling (1989), both because the only costs assumed in his paper are inventory holding costs, and because all model parameters are stationary. Thus, in addition to generalizing Rosling (1989) to allow for expediting of stock and advance demand information, we also extend his results to include nonstationary model parameters. Hence, these results also bring the assembly literature one step closer to practical implementation.

Because assembly requires one unit of each component, the quantity of assembled products at the end of period t, prior to demand realization, is $\min(x_{11t} + X^E_{11t} + X^R_{11t}, x_{21t} + X^E_{21t} + X^R_{21t}, \ldots, x_{n1t} + X^E_{n1t} + X^R_{n1t})$. Let $h_{ij} := H_{ijt} - H_{i,j+1,t}$, with $H_{i,L_i+1,t} := 0$. One-period costs in period t therefore become

$$\begin{aligned} &\gamma_t\Big(\tilde{O}_t, \min_i(x_{i1t} + X^E_{i1t} + X^R_{i1t})\Big) + \sum_{i=1}^{n}\sum_{j=1}^{L_i}\big[k^E_{ijt}X^E_{ijt} + k^R_{ijt}X^R_{ijt} \\ &\qquad + H_{ijt}(x_{ijt} - X^E_{i,j-1,t} - X^R_{i,j-1,t} + X^E_{ijt} + X^R_{ijt})\big] \\ &= \gamma_t\Big(\tilde{O}_t, \min_i(x_{i1t} + X^E_{i1t} + X^R_{i1t})\Big) \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{L_i}[(k^E_{ijt} + h_{ijt})X^E_{ijt} + (k^R_{ijt} + h_{ijt})X^R_{ijt} + H_{ijt}x_{ijt}], \end{aligned} \tag{2}$$

where $\gamma_t(\tilde{O}_t, x) := \mathbf{E}_{D_{t,t}\mid\tilde{O}_t}[p_t(O_{t+1,t} - x)^+ + H_t^A(x - O_{t+1,t})^+]$, which is convex. The expectation is taken over $D_{t,t}$, given $\tilde{O}_t$ to account for correlation across periods. Let $G_t(\tilde{O}_t, x_t, X_t^E, X_t^R)$ denote the minimum expected present value of the costs over periods t through T, as of the beginning of period t, given that the state x_t is observed and schedules X_t^E and X_t^R are selected. The optimality equations become

$$G_t(\tilde{O}_t, x_t, X_t^E, X_t^R) = \gamma_t\Big(\tilde{O}_t, \min_i(x_{i1t} + X_{i1t}^E + X_{i1t}^R)\Big) + \sum_{i=1}^{n}\sum_{j=1}^{L_i}[(k_{ijt}^E + h_{ijt})X_{ijt}^E + (k_{ijt}^R + h_{ijt})X_{ijt}^R + H_{ijt}x_{ijt}] + \alpha\mathbf{E}[F_{t+1}(\tilde{O}_{t+1}, x_{t+1})], \tag{3}$$

where the expectation is with respect to the entire vector of demands $\{D_{t,t}, \ldots, D_{t,t+N}\}$, given $\tilde{O}_t$. Furthermore,

$$F_t(\tilde{O}_t, x_t) = \min_{X_t^E, X_t^R \in \mathbb{X}(x_t)} G_t(\tilde{O}_t, x_t, X_t^E, X_t^R), \tag{4}$$

where the time horizon is T periods. The terminal value function $F_{T+1}(\tilde{O}_{T+1}, x_{T+1})$ is assumed to be zero.

The optimality equations given for $F_t(\tilde{O}_t, x_t)$ bear a severe curse of dimensionality: the state space has $\sum_{i=1}^n L_i$ inventory dimensions plus N advance demand dimensions. Furthermore, optimal order quantities X_{ijt}^E and X_{ijt}^R for each component i and stage j depend on all the variables in the state space. This formidable dimensionality of the decision space and the state space renders the problem practically impossible to solve numerically, even for systems with very few components and very few nodes in the system.

What makes this problem difficult are two flows of components in the system, the regular flow and the expedited flow, and the nonstationary cost parameters in the model. Because of these factors, neither the classic methods of Rosling (1989) for collapsing the state space (also employed in DeCroix and Zipkin 2005, DeCroix 2013) nor the balance-inducing approach of Angelus and Porteus (2008) are conducive to solving the problem. This is due to the echelon formulation used in both of those papers, under which expedited and regular orders are aggregated to form echelon inventory positions.

3. Optimality Results for Assembly Systems

3.1. Component Assembly System

Our first set of results addresses some fundamental properties of the optimal policy for the component assembly system described by Equations (3) and (4). All proofs are deferred to the appendix.

Lemma 1. *It is never optimal to expedite any component i into stage* 1.

Because both expedited orders and regular orders into stage 1 arrive before the demand is realized in each period, then, given the smaller unit regular order cost, it will always be cheaper to move product by regular order than by expedited order into stage 1. Thus, expedited orders will never be used to replenish inventory at stage 1, and any expediting of components will only take place at most down to stage 2.

Next, for each $j = 1, \ldots, L_n$, let $\mathbb{C}(j)$ be the set of components with a leadtime greater than or equal to j, referred to as *the relevant components* at stage j. The following definitions are stated in terms of the on-hand state x_t, but also apply to X_t^E and X_t^R. We say that x_t is *balanced at stage j* in period t if $x_{ijt} = x_{kjt}$ for all $i, k \in \mathbb{C}(j)$. An inventory state balanced at stage j has exactly the same number of units of each relevant component at stage j. Thus, there are no unmatched components at stage j. We say that x_t is *balanced through stage j* if x_t is balanced at each stage l, $l \in \{1, \ldots, j\}$. If x_t is balanced through the very last stage L_n, we simply say that x_t is *balanced*.

Assumption 4. *For each $j = 1, \ldots, L_n$, $x_{ij1} = x_{kj1}$ for all $i, k \in \mathbb{C}(j)$.*

Thus, the system starts out balanced in the very first period. The initial on-hand inventory state in period 1 has, for each stage j, the same number of relevant components scheduled to complete assembly in j periods. This assumption is satisfied, for example, if we start out with no components of any kind.

Lemma 2. *The optimal regular order schedule X_t^R is balanced at stage* 1 *in every period $t = 1, \ldots, T$. The on-hand inventory state x_t is balanced at stage* 1 *in every period $t = 1, \ldots, T + 1$.*

Because of the complexity of the assembly model considered in this paper, and the nonstationary nature of model parameters, it is not feasible to balance the whole system at once as is done in Rosling (1989), DeCroix and Zipkin (2005), or Angelus and Porteus (2008). Instead, it becomes necessary to balance the system stage by stage, and this necessitates application of a methodology different from those found in the existing literature. Lemma 2 represents the first step in this approach.

Lemma 3. *If the on-hand inventory state x_t is balanced through stage j in period t, for any $j \leqslant L_n$, then the optimal expedited order schedule X_t^E is also balanced through stage j in period t.*

A balanced inventory state x_t thus results in a balanced optimal expedited order schedule X_t^E.

Theorem 1. *Optimal order schedules (X_t^E, X_t^R) are balanced in every period $t = 1, \ldots, T$. The on-hand inventory state x_t is balanced in every period $t = 1, \ldots, T + 1$.*

To provide a better understanding of our approach, and motivate its application to other assembly problems, we sketch what is involved in proving our last two results. Briefly, if we can show that the optimal policy (X_t^E, X_t^R) is balanced in each period, so that $X^E_{ijt} = X^E_{i'jt}$ and $X^R_{ijt} = X^R_{i'jt}$ for any components i and i' relevant at any stage j, then, given that x_t is balanced in period 1 by Assumption 4, this would imply that all decisions and states are balanced under the optimal policy for all subsequent periods. This would allow the collapse of dimension i, and thus establish the existence of an equivalent series system.

The proof proceeds by a double induction on t and j, where x_1 being balanced by assumption provides the base case for the induction. If X^E_t is not balanced in some period t, then there exists a component i whose expedited order X^E_{ijt} at stage j is in excess of some other component's expedited order (say, that of component i') at that same stage j. In that case, move the excess of that expedited order to the regular order for component i at stage j in period t. More specifically, if $X^E_{ijt} - X^E_{i'jt} = \delta > 0$, we modify component i's orders to $\bar{X}^E_{ijt} = X^E_{ijt} - \delta$ and $\bar{X}^R_{ijt} = X^R_{ijt} + \delta$. Since $k^E_{ijt} > k^R_{ijt}$ by Assumption 1, this modification leads to a lower total cost. If, on the other hand, X^R_t is not balanced in some period t, so that there exist components i and i' such that $X^R_{ijt} - X^R_{i'jt} = \delta > 0$ at some stage j, we move the excess of that regular order in period t to component i's regular order in period $t+1$. Thus, in this case, we modify the regular orders for component i to $\bar{X}^R_{ijt} = X^R_{ijt} - \delta$ and $\bar{X}^R_{ij,t+1} = X^R_{ij,t+1} + \delta$. Assumption 3 that $k^R_{ijt} + h_{ijt} > \alpha k^R_{ij,t+1}$ then implies that this modification leads to lower total cost. This shows that the optimal policy at stage j in period t should be balanced, and therefore x_{t+1} is balanced, which completes the induction step. (An important part of the proof is showing that modified orders remain feasible.) In this manner, for each component flow, we move progressively upstream one stage at a time, and then one period at a time, until optimal decisions are shown to be balanced at each stage and in every time period.

As a consequence of Theorem 1, it is no longer necessary to manage each component separately; instead, those components that are at the same stage can be managed together as a kit, where the kit for stage j has one each of every component in $\mathbb{C}(j)$. We can therefore represent the on-hand inventory for each relevant component at every stage j by a single variable y_{jt}; thus, $x_{ijt} = y_{jt}$ for every $i \in \mathbb{C}(j)$. Let $y_t := (y_{1t}, \ldots, y_{L_n,t})$. Because the optimal order schedule is balanced, the optimal decisions X^E_{ijt} (and X^R_{ijt}) are the same for every $i \in \mathbb{C}(j)$, and they can each, therefore, be represented by a single variable Y^E_{jt} (and Y^R_{jt}) at each stage j. Let $Y^E_t := (Y^E_{1t}, \ldots, Y^E_{L_n,t})$ and $Y^R_t := (Y^R_{1t}, \ldots, Y^R_{L_n,t})$. The state space of inventory dimensions thus collapses from $\sum_{i=1}^n L_i$ dimensions to only $L =: L_n$.

It is worth noting that the key feature of our proof of the collapse of the state space involves balancing components one flow at a time, first the expedited flow and then the regular flow. What enables each flow to be disaggregated in this way is our formulation of the problem based on installation-stocks. That is, instead of considering echelon inventory levels (of items, as in Rosling 1989, or components as in Angelus and Porteus 2008), we work directly with on-hand inventory levels and the replenishment decisions pertaining to those. Since the landmark paper of Clark and Scarf (1960), research in multiechelon inventory theory has primarily focused on managing echelon inventory levels, rather than installation stocks. The disadvantage of the multiechelon formulation in the context of an assembly system with multiple flows of product is that those flows cannot be disaggregated, since each echelon inventory decision variable will inevitably represent a sum of both regular and expedited orders at all stages downstream of a particular stage. Our approach, by means of which we can disaggregate flows and stages in the system, thus demonstrates that there is still merit in working with installation stocks even in a multistage setting. To the best of our knowledge, this is the first time that an assembly system has been solved by working with the installation stocks, rather than echelon inventory levels. We believe this approach can help solve other complex assembly systems that may not be amenable to previous solution approaches.

3.2. Generalization to Systems with Subassemblies

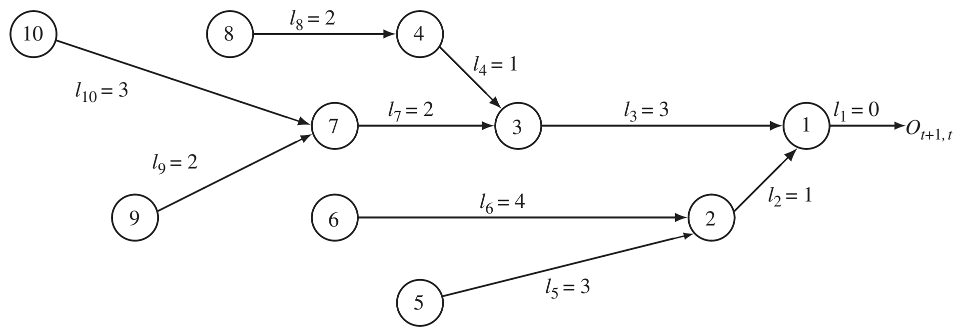
Our second set of results concerns assembly systems where subassemblies are allowed at any stage prior to the final assembly. One example is shown in Figure 2. We refer to each node $k = 1, 2, \ldots, N$ in the system as "(sub)assembly," regardless of whether the node involves processing a single or multiple components.

For each k, let l_k be the *incremental leadtime* required to complete assembly k, where Assembly 1 is the finished product. Let $s(k)$ be the unique *immediate successor* assembly to assembly k. Let $L_1 := l_1 = 0$, and, for each $k > 1$, let $L_k := l_k + L_{s(k)}$ be the *leadtime* for assembly k. Let $P(k)$ be the set of *immediate predecessor* assemblies of assembly k. The set of *components* of such a system is the set of assemblies that have no predecessor assemblies. Let $\mathscr{A}(k)$ denote the set of components required in the composition of assembly k. In Figure 2, for example, $s(6) = 2$ and $L_9 = 8$ $(= l_9 + l_7 + l_3 + L_1)$, while $P(1) = \{2, 3\}$, $P(2) = \{5, 6\}$, $P(3) = \{4, 7\}$, $P(4) = \{8\}$, $P(7) = \{9, 10\}$, $P(5) = P(6) = P(8) = P(9) = P(10) = \varnothing$, and $\mathscr{A}(1) = \{5, 6, 8, 9, 10\}$, $\mathscr{A}(2) = \{5, 6\}$, $\mathscr{A}(3) = \{8, 9, 10\}$, $\mathscr{A}(4) = \{8\}$, $\mathscr{A}(7) = \{9, 10\}$, and $\mathscr{A}(k) = \{k\}$, for $k = 5, 6, 8, 9, 10$. We refer to such a system as *the general assembly system*.

Assume, for convenience, that the only extra costs incurred in this system, in addition to the costs already introduced, are the assembly costs: Let c^A_{kt} be the discounted present value of the costs related to assembling/purchasing/transforming assembly k, evaluated at the beginning of period t. This cost may be the purchasing cost for components (e.g., 5, 6, 8, 9, and 10), as they have no predecessor assemblies.

We now introduce the constraint-relaxation approach from Atkins (1990) and the cost allocation ideas of Atkins (1994) and Chen and Zheng (1994). The Atkins papers develop an approach to relaxing some of the constraints in the problem to provide lower bounds on the cost for certain

Figure 2. A system with subassemblies.



multistage inventory models with deterministic demand. Such a relaxed version of the problem will have lower costs because of fewer constraints in the system, and thus provide a reasonable lower bound on the cost for the original problem. With the right choice of constraints to relax, a problem can be broken down into simpler subproblems. By choosing an optimal cost allocation, it is possible to derive tight lower bounds on the cost function for the original problem. Chen and Zheng (1994) extend this method to systems with stochastic demands, and prove the original results of Clark and Scarf (1960) and Rosling (1989) in a more direct manner. They relax the constraints, and find the optimal policy for the simpler, relaxed problem thus obtained. This optimal policy for the relaxed system is then shown to be feasible in the original system. Since this policy is optimal for the less constrained system, and found to be feasible in the original, more constrained system, then the policy must also be optimal for the original system.

To apply the constraint-relaxation and cost-allocation methods to our problem, we first relax the constraint(s) that there must be exactly the right number of each component ready when any (sub)assembly takes place (other than the final assembly at stage 1). We refer to those constraints as *the component-matching constraints*. Relaxing those constraints provides a system in which the performance is at least as good as in the original one. Furthermore, this relaxed system has exactly the same constraints as the component assembly system analyzed in Section 3.1, so that a general system with subassemblies (with relaxed component-matching constraints) can be treated as an equivalent component assembly system. Thus, for example, with relaxed component-matching constraints, the system with subassemblies shown in Figure 2 becomes structurally equivalent to a five-component assembly system (components 5, 6, 8, 9, and 10 in the original system) with component leadtimes of 4, 5, 6, 7, and 8 periods.

Next, we design a cost allocation scheme so that, if the solution is balanced, then the total expected cost is the same in the equivalent component system and the original system with subassemblies. If there exists an allocation of assembly costs to its components that satisfies the required conditions, then, by Theorem 1, there exists an optimal policy for the equivalent component system that is balanced. Since it is balanced, this policy is feasible in the original constrained system, because every balanced policy satisfies component-matching constraints. Because this policy is optimal for the equivalent component system, and feasible for the original constrained system, then it must be optimal for it.

Let β_{ikt} denote the portion of the assembly cost for assembly k that is allocated to component i, for each $i \in \mathcal{A}(k)$ in period t (e.g., allocate the assembly costs of assembly 1 equally to each required component, so that $\beta_{51t} = \beta_{61t} = \beta_{81t} = \beta_{91t} = \beta_{10,1t} = 1/5$). In general, we require full allocations, namely, that $\sum_{i \in \mathcal{A}(k)} \beta_{ikt} = 1$ for each k and t. For example, we require that $\beta_{84t} = 1$ because assembly 4 requires only component 8. Thus, each component i is allocated $\beta_{ikt} c_{kt}^A$, for each k for which $i \in \mathcal{A}(k)$. If assembly k is initiated at the beginning of period t, it will require one unit of component i for each $i \in \mathcal{A}(k)$, and the amount $\beta_{ikt} c_{kt}^A$ will be allocated to each such component i for stage L_k in period t.

The allocated assembly costs for the system in Figure 2 are summarized in Table 1. Component 6 is shown as bearing the full cost of assembly 6—it is the only component needed for that particular assembly. The cost of assembly 7, by contrast, is shared among components 9 and 10. The following theorem describes how to allocate subassembly costs to ensure the existence of a balanced optimal policy.

Theorem 2. *Let $\mathbb{L}(j) := \{k \mid L_k = j\}$ be the set of assemblies with leadtime L_k equal to j. Let $\theta_{ijt} := \sum_{k \in \mathbb{L}(j)} \beta_{ikt} c_{kt}^A$ for each i and j. If for each component i at stage j in each period t there exists an allocation of assembly costs such that $\theta_{ijt} \geqslant \alpha \theta_{ij,t+1}$, then the optimal policy is balanced in every period.*

Table 1. Allocation of assembly costs to components and stages.

i	L_i	$k = 1$	2	3	4	5	6	7	8	9	10
5	4	$\beta_{51} c_1^A$	$\beta_{52} c_2^A$			c_5^A					
6	5	$\beta_{61} c_1^A$	$\beta_{62} c_2^A$				c_6^A				
8	6	$\beta_{81} c_1^A$		$\beta_{83} c_3^A$	c_4^A				c_8^A		
9	7	$\beta_{91} c_1^A$		$\beta_{93} c_3^A$				$\beta_{97} c_7^A$		c_9^A	
10	8	$\beta_{10,1} c_1^A$		$\beta_{10,3} c_3^A$				$\beta_{10,7} c_7^A$			c^A

If a policy is balanced, then, when an allocation of costs satisfies the condition in Theorem 2, the actual costs incurred in the general system with subassemblies are the same as those captured by the equivalent component model. Since we allow each of the components to be independently managed in a component assembly system, then the optimal solution of the equivalent component system is at least as good as the best one in the original (i.e., general assembly) system. Under Assumptions 1–4, the optimal policy for the equivalent component system is balanced; therefore, that policy is feasible in the original system with subassemblies, because, being balanced, it satisfies the component-matching constraints. Thus, this balanced policy must be optimal for the original system.

The condition required by Theorem 2 states that the allocation of assembly costs across components cannot change too quickly in time. Otherwise, if, for example, in one period assembly costs are allocated evenly across components and in the next period they are all allocated to some component i, it may be optimal to withhold that particular component from assembly for one or more periods, thus unbalancing the system. The requirement that $\theta_{ijt} \geqslant \alpha\theta_{ij,t+1}$ eliminates such pathological changes of cost allocations across time. This condition is only mildly restrictive: it holds, for example, for the case of stationary assembly costs or when the assembly costs are allocated equally to each required component in each period. Thus, under the specified set of permissible cost allocations, the optimal policy for a nonstationary assembly system with subassemblies is balanced, and the system can thus be reduced to an equivalent series system.

4. The Characterization of the Optimal Policy

By Theorems 1 and 2, those components in the assembly system that are at the same stage can be managed together as a kit, where the kit for stage j has one each of every component in $\mathbb{C}(j)$. We can thus represent the on-hand inventory for each relevant component at stage j by a single variable y_{jt}.

Since the optimal expedite order and regular order schedules are balanced, then the optimal expedite order decision for component i at stage j, X^E_{ijt}, is the same for every $i \in \mathbb{C}(j)$, and can therefore be denoted by a single variable, Y^E_{jt}. Similarly, because the optimal regular order decision for component i at stage j, X^R_{ijt}, is also the same for every $i \in \mathbb{C}(j)$, it can be denoted by a single variable, Y^R_{jt}.

Let the new cost parameters k^E_{jt}, k^R_{jt}, H_{jt}, and h_{jt} for each stage j be defined as follows:

$$k^E_{jt} := \sum_{i\in\mathbb{C}(j)} k^E_{ijt}; \quad k^R_{jt} := \sum_{i\in\mathbb{C}(j)} k^R_{ijt};$$
$$H_{jt} := \sum_{i\in\mathbb{C}(j)} H_{ijt}; \quad \text{and} \quad h_{jt} := \sum_{i\in\mathbb{C}(j)} h_{ijt}.$$

We can formulate the new optimality equations for the assembly model with ADI and expediting as follows:

$$\mathcal{F}_t(\tilde{O}_t, y_t) = \min_{Y^E_t,\, Y^R_t \in \mathbb{Y}(y_t)} \mathcal{G}_t(\tilde{O}_t, y_t, Y^E_t, Y^R_t) \tag{5}$$

$$\begin{aligned}\mathcal{G}_t(\tilde{O}_t, y_t, Y^E_t, Y^R_t) &= \gamma_t(\tilde{O}_t, y_{1t} + Y^E_{1t} + Y^R_{1t}) \\ &\quad + \sum_{j=1}^{L} [(k^E_{jt} + h_{jt})Y^E_{jt} + (k^R_{jt} + h_{jt})Y^R_{jt} + H_{jt} y_{jt}] \\ &\quad + \alpha\mathbf{E}[\mathcal{F}_{t+1}(\tilde{O}_{t+1}, y_{t+1})].\end{aligned}$$

The feasible decision set $\mathbb{Y}(y_t)$ for a given on-hand inventory state y_t becomes

$$\mathbb{Y}(y_t) = \{Y^E_t, Y^R_t \mid Y^E_{jt} \leqslant y_{j+1,t} + Y^E_{j+1,t},\, Y^R_{jt} \leqslant y_{j+1,t} + Y^E_{j+1,t} - Y^E_{jt},\, 1 \leqslant j < L\}.$$

The following corollary completes the reduction of an assembly system with advance demand information and expediting to an equivalent series system.

Corollary 1. *Fix t and y_t. If $x_{ijt} = y_{jt}$ for all j and $i \in \mathbb{C}(j)$, then $F_t(\tilde{O}_t, x_t) = \mathcal{F}_t(\tilde{O}_t, y_t)$ for every $\tilde{O}_t$.*

Note that the only subscript in the optimality equations given above, other than the time label, is the stage of the system in sequential order, so that the assembly model with expediting and ADI has now been reduced to an equivalent series system with $L = L_n$ stages.

4.1. Formulation in Terms of Echelons

We now reformulate the series system given in Equation (5) using the following echelon variables:

$z_{jt} := y_{1t} + \cdots + y_{jt}$—Echelon j (on-hand) inventory, at the beginning of period t;

$Z^E_{jt} := z_{jt} + Y^E_{jt}$—Echelon j inventory position after expedited orders have arrived;

$Z^R_{jt} := Z^E_{jt} + Y^R_{jt}$—Echelon j inventory position after both expedited and regular orders have arrived.

Updated echelon inventories are $z_{j,t+1} = Z^R_{jt} - O_{t+1,t}$. Let $z_t := (z_{1t}, \ldots, z_{Lt})$ be *the echelon (inventory) state*, $Z^E_t := (Z^E_{1t}, \ldots, Z^E_{Lt})$ be *the post-expedite echelon schedule*, and $Z^R_t := (Z^R_{1t}, \ldots, Z^R_{Lt})$ be *the post-regular order echelon schedule*. Thus, Z^E_t and Z^R_t are the decision variables of the model for each t.

The single period cost function becomes $\gamma_t(\tilde{O}_t, Z^R_{1t}) + \sum_{j=1}^{L}[(k^E_{jt} - k^R_{jt})Z^E_{jt} + (k^R_{jt} + h_{jt})Z^R_{jt} - k^E_{jt} z_{jt}]$. Our assembly system can thus be reduced to an equivalent series one with these optimality equations:

$$\begin{aligned}&f_t(\tilde{O}_t, z_t) \\ &= -\sum_{j=1}^{L} k^E_j z_{jt} + \min_{Z^E_t,\, Z^R_t \in \mathbb{Z}(z_t)} \Bigg\{ \gamma_t(\tilde{O}_t, Z^R_{1t}) + \sum_{j=1}^{L} (c^E_{jt} Z^E_{jt} + c^R_{jt} Z^R_{jt}) \\ &\quad + \alpha\mathbf{E}[f_{t+1}(\tilde{O}_{t+1}, Z^R_t - O_{t+1,t})] \Bigg\},\end{aligned} \tag{6}$$

where, for notational convenience, $c^E_{jt} := k^E_{jt} - k^R_{jt}$, and $c^R_{jt} := k^R_{jt} + h_{jt}$. The feasible set $\mathbb{Z}(z_t)$ becomes

$$\mathbb{Z}(z_t) = \{Z^E_t, Z^R_t \mid z_{jt} \leqslant Z^E_{jt} \leqslant Z^R_{jt} \leqslant Z^E_{j+1,\,t}, 1 \leqslant j \leqslant L\}. \quad (7)$$

The echelon inventory position after expediting is, at each stage, bounded from below by the echelon inventory at that stage, and, from above, by the echelon inventory position after expediting at the next stage upstream. Those two echelon positions then determine the feasible interval for the echelon position after regular ordering. Our third set of results concerns the solution to the dynamic program given in (6).

4.2. Preservation of Additive Convexity

In what follows, if f is an arbitrary smooth and convex function on $\mathbb{R}$ with a finite unconstrained minimizer S, then we define functions f^+ and f^- as

$$f^+(x) := \begin{cases} f(S) & \text{if } x \leqslant S; \\ f(x) & \text{otherwise;} \end{cases} \quad \text{and}$$
$$f^-(y) := \begin{cases} f(y) - f(S) & \text{if } y \leqslant S; \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

Clearly, f^+ is increasing, and f^- is decreasing for every such function f. Furthermore, if f is increasing, then $f^-(y) = 0$, and $f^+(x) = f(x)$; if f is decreasing, $f^+(x) = 0$, and $f^-(y) = f(y)$. The following lemma presents a new result in convex optimization pertaining to the preservation of additive convexity of multidimensional objective functions minimized over the feasible set with linear, ordered boundaries.

Lemma 4. *Let $\mathbf{u} := (u_1, u_2, \ldots, u_M)$ be a given vector in $\mathbb{R}^M$ such that $u_j \leqslant u_{j+1}$ for every j. Let $\phi_1, \phi_2, \ldots, \phi_M$: $\mathbb{R} \to \mathbb{R}$ be smooth convex functions. Then, for any $m < M$,*

$$\min_{\substack{u_j \leqslant \zeta_j \leqslant \zeta_{j+1} \\ j=1,2,\ldots,M}} \sum_{j=1}^{M} \phi_j(\zeta_j)$$
$$= \sum_{j=1}^{m} \varphi_j(u_j) + \min_{\substack{u_j \leqslant \zeta_j \leqslant \zeta_{j+1} \\ j=m+1,\ldots,M}} \left\{ \varphi_{m+1}(\zeta_{m+1}) + \sum_{j=m+2}^{M} \phi_j(\zeta_j) \right\}, \quad (9)$$

where the minimization is over the vector $\zeta := (\zeta_1, \ldots, \zeta_M)$, and functions φ_j are defined recursively by $\varphi_j := \phi_j + \varphi^-_{j-1}$, with $\varphi_0 := 0$. Furthermore, φ_j is smooth and convex for every j.

The following intermediate results introduce a set of new functions needed for subsequent analysis and establish their smoothness and convexity properties.

Lemma 5. *Assume that $f_{t+1}(\tilde{O}_{t+1}, \cdot)$, as defined in Equation (6), is smooth and additively convex for each $\tilde{O}_{t+1}$, so that there exist smooth convex functions $\{f_{1,\,t+1}, \ldots, f_{L,\,t+1}\}$ such that $f_{t+1}(\tilde{O}_{t+1}, z_{t+1}) = \sum_{j=1}^{L} f_{j,\,t+1}(\tilde{O}_{t+1}, z_{j,\,t+1})$ for each $\tilde{O}_{t+1}$.*

(i) *Let functions $\{g_{1,\,t}, \ldots, g_{N,\,t}\}$ be defined, for each $\tilde{O}_t$, as*

$$g_{jt}(\tilde{O}_t, Z) := \begin{cases} \gamma_t(Z) + c^R_{1t} Z + \alpha \mathbf{E}[f_{1,\,t+1}(\tilde{O}_{t+1}, Z - O_{t+1,\,t})] \\ \qquad \text{if } j = 1; \\ c^R_{jt} Z + \alpha \mathbf{E}[f_{j,\,t+1}(\tilde{O}_{t+1}, Z - O_{t+1,\,t})] \\ \qquad \text{if } j > 1, \end{cases} \quad (10)$$

Then, $g_{jt}(\tilde{O}_t, \cdot)$ is smooth and convex for every $\tilde{O}_t$.

(ii) *Given $g_{1,\,t}, \ldots, g_{N,\,t}$ as above, let $g^+_{jt}(\tilde{O}_t, \cdot)$ and $g^-_{jt}(\tilde{O}_t, \cdot)$ be as defined in Lemma 13. For each $\tilde{O}_t$, let*

$$U_{jt}(\tilde{O}_t, Z) := \begin{cases} c^E_{1t} Z + g^+_{1,\,t}(\tilde{O}_t, Z) \\ \qquad \text{if } j = 1; \\ c^E_{jt} Z + g^+_{jt}(\tilde{O}_t, Z) + g^-_{j-1,\,t}(\tilde{O}_t, Z) \\ \qquad \text{if } 1 < j \leqslant L. \end{cases} \quad (11)$$

Then, $U_{jt}(\tilde{O}_t, \cdot)$ is smooth, convex, and

$$f_t(\tilde{O}_t, z_t) = \min_{\substack{z_{jt} \leqslant Z^E_{jt} \leqslant Z^E_{j+1,\,t} \\ 1 \leqslant j \leqslant L}} \left\{ \sum_{j=1}^{L} U_{jt}(\tilde{O}_t, Z^E_{jt}) \right\}.$$

Theorem 3. *For every z_t and $\tilde{O}_t$ in each period t, let $U_{jt}(\tilde{O}_t, z_{jt})$ be as defined in Lemma 5. Let functions $V_{1t}(\tilde{O}_t, \cdot), \ldots, V_{Lt}(\tilde{O}_t, \cdot)$ be defined recursively as $V_{jt}(\tilde{O}_t, z_{jt}) := U_{jt}(\tilde{O}_t, z_{jt}) + V^-_{j-1,\,t}(\tilde{O}_t, z_{jt})$, with $V_{0t}(\tilde{O}_t, z_{jt}) := 0$. Then, the function $V_{jt}(\tilde{O}_t, \cdot)$ is smooth and convex, and $f_t(\tilde{O}_t, z_t) = \sum_{j=1}^{L} V^+_{jt}(\tilde{O}_t, z_{jt}) - k^E_j z_{jt}$.*

The multivariable objective cost function for a multiechelon inventory system with ADI and expediting can therefore be reduced to a sum of single-variable smooth convex functions. This result acts to significantly reduce the curse of dimensionality inherent in the original problem, to the point where each component function depends only on a single (echelon) inventory dimension (in addition to any advance demand information state variables). Theorem 3 completes our objective of decomposing a complex assembly system with advance information and expediting of stock into a nested sequence of solvable convex subproblems. Each subproblem now contains only $N+1$ dimensions, and becomes numerically tractable because the demand information horizon N is typically not very large in practice, and because the optimization of each component function is only over the single (echelon) variable z_{jt}.

4.3. Structure of the Optimal Policy

In what follows, we use the symbols "$\vee$" and "$\wedge$," in the conventional sense, to represent the maximum and the minimum, respectively, of two or more numbers or variables.

Theorem 4. *For any given $\tilde{O}_t$, let $g_{jt}(\tilde{O}_t, \cdot)$ and $V_{jt}(\tilde{O}_t, \cdot)$ be as Lemma 5 and Theorem 3, respectively. Let*

$$S^R_{jt}(\tilde{O}_t) := \max\Big(\arg\min_Z g_{jt}(\tilde{O}_t, Z)\Big) \quad \text{and}$$
$$S^E_{jt}(\tilde{O}_t) := \max\Big(\arg\min_Z V_{jt}(\tilde{O}_t, Z)\Big)$$

for each j.

(i) *Then, the optimal echelon j position after expediting, $\hat{Z}^E_{jt}(\tilde{O}_t, z_t)$ is given by*

$$\hat{Z}^E_{jt}(\tilde{O}_t, z_t) = \bigwedge_{i=j}^{L} [z_{it} \vee S^E_{it}(\tilde{O}_t)]. \tag{12}$$

(ii) *Given $\hat{Z}^E_t(\tilde{O}_t, z_t)$, the optimal echelon j position after regular ordering, $\hat{Z}^R_{jt}(\tilde{O}_t, z_t)$, is given by*

$$\hat{Z}^R_{jt}(\tilde{O}_t, z_t) := \begin{cases} [\hat{Z}^E_{jt}(\tilde{O}_t) \vee S^R_{jt}(\tilde{O}_t)] \wedge \hat{Z}^E_{j+1,t}(\tilde{O}_t) & \text{if } j < L; \\ \hat{Z}^E_{Lt}(\tilde{O}_t) \vee S^R_{Lt}(\tilde{O}_t) & \text{if } j = L. \end{cases} \tag{13}$$

Thus, the optimal policy for a multiechelon system with ADI and expediting is a double-tiered, echelon basestock policy whose basestock levels depend on the state of ADI.

4.4. Implementation of the Optimal Policy

Imagine an organization with two departments charged with managing replenishment: "the expediting" department and "the regular order" department, each with a manager at every stage. The implementation of the optimal policy is top-down, starting with the expediting manager at echelon L who observes his echelon inventory z_{Lt}, and compares it to his basestock level $S^E_{Lt}(\tilde{O}_t)$. If $S^E_{Lt}(\tilde{O}_t) - z_{Lt}$ is positive, he expedites that difference to stage $L-1$. Otherwise, he stays put. Finally, he informs his regular order counterpart, as well as both managers at stage $L-1$, of his decision $\hat{Z}^E_{L,t}$. Having been informed about $\hat{Z}^E_{L-1,t}$, the expediting manager at stage $L-1$ observes his own echelon inventory level $z_{L-1,t}$, selects the point within the interval $[z_{L-1,t}, \hat{Z}^E_{Lt}]$ closest to $S^E_{L-1,t}(\tilde{O}_t)$, and orders the (positive) difference between that point and $z_{L-1,t}$. Next, he communicates his decision $\hat{Z}^E_{L-1,t}$ to the two replenishment managers at stage $L-2$. The implementation of the optimal expediting schedule thus starts at the top, and continues downstream.

Once information on optimal echelon j and $j+1$ positions after expediting reaches the regular order manager at stage j, he holds those decisions as fixed and ignores all other decisions in the system. He chooses a point in the interval $[\hat{Z}^E_{jt}, \hat{Z}^E_{j+1,t}]$ closest to his corresponding basestock level $S^R_{jt}(\tilde{O}_t)$, and places a regular order for the (positive) difference between that point and $\hat{Z}^E_{jt}$.

We now demonstrate this process using a set of actual basestock levels for a five-echelon system shown in Table 2. For simplicity, we assume no advance demand information. Suppose that regular basestock levels were achieved in period $t-1$ so that $\hat{Z}^R_{j,t-1}$ in Table 2 is identical to S^R_j for each j. Let the realized demand at stage 1 in period $t-1$ be 9. The resulting state z_{jt} at the beginning of period t is in column 5. Thus, going in the direction of the optimal policy implementation, which is top-down, $\hat{Z}^E_{5t} = \max(z_{5t}, S^E_5) = 18$; and $\hat{Z}^E_4 = \min[\max(z_{4t}, S^E_4), \hat{Z}^E_5] = 11$; and so on. The optimal number of units $\hat{Y}^E_{jt}$ expedited into each stage is shown in column 7: stage 5 receives 3 expedited units; stage 4 does not expedite units; stage 3 receives 1 expedited unit, and stage 2 receives 4 expedited units.

Table 2. Implementation of the optimal policy.

Echelon (j)	S^E_j	S^R_j	$\hat{Z}^R_{j,t-1}$	z_{jt}	$\hat{Z}^E_{jt}$	$\hat{Y}^E_{jt}$	$\hat{Z}^R_{jt}$	$\hat{Y}^R_{jt}$
1	—	8	8	−1	−1	0	7	8
2	8	12	12	3	7	4	7	0
3	7	15	15	6	7	1	11	4
4	10	20	20	11	11	0	18	7
5	18	24	24	15	18	3	24	6

Because $S^R_j > \hat{Z}^E_{j+1,t}$ for every $j < L$ in Table 2, basestock levels S^R_j cannot be achieved anywhere in the system except at the uppermost stage, where the regular order decision is not constrained. Thus, $\hat{Z}^E_{j+1,t}$ represents the highest level that $\hat{Z}^R_{jt}$, the optimal echelon j inventory position after regular ordering, can feasibly achieve. Thus, column 8 shows that all but the uppermost echelon inventory positions after regular ordering are exactly equal to echelon inventory positions after expediting at the next stage upstream. The number $\hat{Y}^R_{jt}$ of actual units optimally ordered through regular flow into each stage is shown in column 9.

5. Properties of the Optimal Policy

This section establishes the monotonicity properties of the optimal policy with regard to the parameters that characterize advance demand information and expediting of stock. Those properties have both theoretical and practical value. The value to theory is from understanding the behavior of optimal decisions, which facilitates both the implementation of the optimal policy and its calculation. Practical value comes from helping companies understand how key parameters of those two strategies drive cost savings in assembly systems, so that they can make better decisions when it comes to investing in those capabilities.

Lemma 6. *Optimal order schedules $(\hat{Z}^E_t, \hat{Z}^R_t)$ are such that $\hat{Z}^E_{j+1,t} - z_{j+1,t} \leqslant \hat{Z}^R_{jt} - z_{jt}$ for each t and j.*

Since $\hat{Z}^E_{j+1,t} - z_{j+1,t} = Y^E_{j+1,t}$ and $\hat{Z}^R_{jt} - z_{jt} = Y^E_{jt} + Y^R_{jt}$, Lemma 6 has an important implication: a unit expedited into any stage in the system never stays there—it is always moved at least one more stage downstream within the same period, either by expedited or regular flow. In other words, expediting is used only when it is optimal to move a unit downstream more than one stage within a single period. This is because, if a unit were to be moved downstream only a single stage within a period, then it would be cheaper to do so by regular order rather than an expedited one, because regular order is less costly.

LEMMA 7. *For any given $\tilde{O}_t$, $S^R_{jt}(\tilde{O}_t) \geqslant S^E_{jt}(\tilde{O}_t)$ for each j in every period t.*

Lemma 7 establishes explicit ordering between basestock levels for regular orders and basestock levels for expedited orders at each stage. It ensures that regular orders will occur at each stage in the system, for if $S^R_{jt}(\tilde{O}_t)$ were less than $S^E_{jt}(\tilde{O}_t)$ at some stage j, only expedited orders would be placed at that stage.

5.1. Impact of Advance Demand Information

We now address the monotonicity properties of the optimal policy relative to advance demand information. In particular, we explore what happens in period t when an amount ϵ of advance demand that needs to be fulfilled j periods ahead (in period $t+j$) is reassigned to be fulfilled $j+1$ periods ahead (in period $t+j+1$). This analysis helps companies decide whether to focus on collecting advance demand information for shorter horizon orders or longer horizon orders, and understand their relative impact on cost savings.

For that purpose, for each $j = 1, \ldots, N$, we define an N-dimensional unit vector e_j whose jth entry is 1 (the rest of the elements are zeros). Thus, $\tilde{O}_t + \epsilon e_j$ adds ϵ units to the demand to be fulfilled in period $t+j$; in other words, $O_{t,\,t+j}$ becomes $O_{t,\,t+j} + \epsilon$. Our interest is in uncovering the relationship between $\hat{Z}^E_{jt}(\tilde{O}_t, z_t)$ and $\hat{Z}^E_{jt}(\tilde{O}_t - \epsilon e_j + \epsilon e_{j+1}, z_t)$, as well as between $\hat{Z}^R_{jt}(\tilde{O}_t, z_t)$ and $\hat{Z}^R_{jt}(\tilde{O}_t - \epsilon e_j + \epsilon e_{j+1}, z_t)$, for every $j < N$ and $\epsilon > 0$. We use $D_y f(\mathbf{x}, y)$ to refer to the first derivative of f with respect to y.

LEMMA 8. *Let f, g: $\mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be two functions of $(\mathbf{x}, y)$ that are smooth, convex and coercive in y. Assume that $D_y f(\mathbf{x}, y) \leqslant D_y g(\mathbf{x}, y)$ for all $(\mathbf{x}, y)$. Let $s_f(\mathbf{x})$ and $s_g(\mathbf{x})$ be the largest minimizers of f and g, respectively, over y, for any given value of $\mathbf{x}$. Then, the following hold.*

(i) *$s_f(\mathbf{x}) \geqslant s_g(\mathbf{x})$ for every $\mathbf{x}$;*

(ii) *$D_y f^+(\mathbf{x}, y) \leqslant D_y g^+(\mathbf{x}, y)$ and $D_y f^-(\mathbf{x}, y) \leqslant D_y g^-(\mathbf{x}, y)$ for all $(\mathbf{x}, y)$.*

We use the notation $\tilde{O}^2_t \geqslant \tilde{O}^1_t$ for two advance demand information vectors $\tilde{O}^2_t$ and $\tilde{O}^1_t$ such that each element of $\tilde{O}^2_t$ is greater than or equal to the corresponding element of $\tilde{O}^1_t$.

THEOREM 5. *The following hold in every period t.*

(i) *For any $\tilde{O}^2_t \geqslant \tilde{O}^1_t$, $S^E_{jt}(\tilde{O}^2_t) \geqslant S^E_{jt}(\tilde{O}^1_t)$, and $S^R_{jt}(\tilde{O}^2_t) \geqslant S^R_{jt}(\tilde{O}^1_t)$ for every j;*

(ii) *For any $\tilde{O}_t$ and $\delta > 0$, $S^E_{jt}(\tilde{O}_t + \delta e_k) \geqslant S^E_{jt}(\tilde{O}_t + \delta e_{k+1})$ and $S^R_{jt}(\tilde{O}_t + \delta e_k) \geqslant S^R_{jt}(\tilde{O}_t + \delta e_{k+1})$ for every $j = 1, \ldots, N$ and $k = 1, \ldots, N-1$;*

(iii) *For any $\tilde{O}_t$ and $\delta > 0$, $S^E_{jt}(\tilde{O}_t + \delta e_k) - S^E_{jt}(\tilde{O}_t) \leqslant \delta$ and $S^R_{jt}(\tilde{O}_t + \delta e_k) - S^R_{jt}(\tilde{O}_t) \leqslant \delta$ for any j, k.*

Therefore, optimal basestock levels for both regular order and expediting decisions are increasing in observed advance demand. Furthermore, the observed advance demand that is closer to the current period has more impact on optimal basestock levels than the observed demand further in the future. In other words, an additional unit of advance order to be delivered τ periods later increases the optimal basestock levels for the current period more than an additional unit of advance order to be delivered $\tau' > \tau$ periods later. Furthermore, as shown in part (iii) of Theorem 5, an increase of δ units in the observed advance demand increases the basestock levels for either expedited or regular flow order by less than δ.

THEOREM 6. *The following hold for every period echelon state z_t in each period t.*

(i) *For any $\tilde{O}^2_t \geqslant \tilde{O}^1_t$ and every $j = 1, \ldots, N$:*
   (a) *$\hat{Z}^R_{jt}(\tilde{O}^2_t, z_t) \geqslant \hat{Z}^R_{jt}(\tilde{O}^1_t, z_t)$;*
   (b) *$\hat{Z}^E_{jt}(\tilde{O}^2_t, z_t) \geqslant \hat{Z}^E_{jt}(\tilde{O}^1_t, z_t)$; and*
   (c) *$f_t(\tilde{O}^2_t, z_t) \geqslant f_t(\tilde{O}^1_t, z_t)$;*

(ii) *For any $\tilde{O}_t$, $\delta > 0$, every $j = 1, \ldots, N$ and $k \leqslant q$:*
   (a) *$\hat{Z}^R_{jt}(\tilde{O}_t, z_t) \geqslant \hat{Z}^R_{jt}(\tilde{O}_t - \delta e_k + \delta e_q, z_t)$;*
   (b) *$\hat{Z}^E_{jt}(\tilde{O}_t, z_t) \geqslant \hat{Z}^E_{jt}(\tilde{O}_t - \delta e_k + \delta e_q, z_t)$; and*
   (c) *$f_t(\tilde{O}_t, z_t) \geqslant f_t(\tilde{O}_t - \delta e_k + \delta e_q, z_t)$.*

Theorem 6 has two important implications. First, it shows that having advance demand information in the system decreases the amount of inventory held in the supply chain. This result reveals the main mechanism responsible for cost savings generated with ADI: having advance demand information requires less inventory at each stage, and that reduces inventory holding costs. Second, in the sense of allocating advance demand across the information horizon, the objective cost function is decreasing in the length N of that horizon, because a longer demand information horizon allows allocations further into the future, which, by Theorem 6, has the effect of reducing costs. This conclusion provides a measure of justification for companies to invest in information and sales technologies that enable longer advance demand horizons.

5.2. Impact of Expediting

For notational convenience, and without loss of generality, going forward we assume that unit expediting costs are stationary, so that $k^E_{jt} = k^E_j$ at each stage j and for all periods t.

LEMMA 9. *Let $f_{jt}(\tilde{O}_t, z_{jt}) := V^+_{jt}(\tilde{O}_t, z_{jt}) - k^E_j z_{jt}$ for each j. Then, for each $\tilde{O}_t$ in every period t, $g_{jt}(\tilde{O}_t, \cdot)$, $U_{jt}(\tilde{O}_t, \cdot)$, $V_{jt}(\tilde{O}_t, \cdot)$, and $f_{jt}(\tilde{O}_t, \cdot)$ are independent of k^E_m for every $j > m$.*

Thus, components of the objective cost function vary only with downstream unit expediting costs. We now investigate, at each stage j, the behavior of basestock levels and optimal decisions as functions of the unit expediting cost k^E_m at a given stage $m \leqslant j$, while keeping all other unit costs constant. We use the notation $S^E_{jt}(\tilde{O}_t \mid k^E_m)$, $S^R_{jt}(\tilde{O}_t \mid k^E_m)$,

$\hat{Z}^E_{jt}(\tilde{O}, z_t \mid k^E_m)$, $\hat{Z}^R_{jt}(\tilde{O}, z_t \mid k^E_m)$, and $f_{jt}(\tilde{O}_t, z \mid k^E_m)$ to indicate that basestock levels, optimal decisions, and cost functions at every stage j vary with the unit expediting cost at stage m. We denote only the dependence on a single k^E_m because we vary only a single k^E_m at a time.

Lemma 10. *Fix j, and suppose that χ^E_j and k^E_j are two different unit expediting costs at stage j such that $\chi^E_j > k^E_j$. Then, the following hold for every $\tilde{O}_t$ in every period t.*
(i) $k^E_j - \chi^E_j \leqslant D_z f_{jt}(\tilde{O}_t, z \mid \chi^E_j) - D_z f_{jt}(\tilde{O}_t, z \mid k^E_j) \leqslant 0$;
(ii) $k^E_j - \chi^E_j \leqslant D_z g_{jt}(\tilde{O}_t, z \mid \chi^E_j) - D_z g_{jt}(\tilde{O}_t, z \mid k^E_j) \leqslant 0$;
(iii) $0 \leqslant D_z V_{jt}(\tilde{O}_t, z \mid \chi^E_j) - D_z V_{jt}(\tilde{O}_t, z \mid k^E_j) \leqslant \chi^E_j - k^E_j$;
(iv) $S^R_{jt}(\tilde{O}_t \mid \chi^E_j) \geqslant S^R_{jt}(\tilde{O}_t \mid k^E_j)$ *and* $S^E_{jt}(\tilde{O}_t \mid \chi^E_j) \leqslant S^E_{jt}(\tilde{O}_t \mid k^E_j)$.

As we increase the unit expediting cost at a particular stage, basestock levels for regular orders at that stage increase, and those for expedited orders decrease.

Lemma 11. *Fix $j > 1$, and suppose that χ^E_{j-1} and k^E_{j-1} are two different unit expediting costs at stage $j-1$ such that $\chi^E_{j-1} > k^E_{j-1}$. Then, the following hold for every $\tilde{O}_t$ in every period t.*
(i) $D_z f_{jt}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_z f_{jt}(\tilde{O}_t, z \mid k^E_{j-1}) \leqslant 0$;
(ii) $D_z g_{jt}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_z g_{jt}(\tilde{O}_t, z \mid k^E_{j-1}) \leqslant 0$;
(iii) $D_z V_{jt}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_z V_{jt}(\tilde{O}_t, z \mid k^E_{j-1}) \leqslant 0$;
(iv) $S^R_{jt}(\tilde{O}_t \mid \chi^E_{j-1}) \geqslant S^R_{jt}(\tilde{O}_t \mid k^E_{j-1})$ *and* $S^E_{jt}(\tilde{O}_t \mid \chi^E_{j-1}) \geqslant S^E_{jt}(\tilde{O}_t \mid k^E_{j-1})$.

Lemma 12. *Fix j, and suppose that χ^E_m and k^E_m are two different unit expediting costs at a given stage m, $m < j$, such that $\chi^E_m > k^E_m$. Then, the following hold for every $\tilde{O}_t$ in every period t.*
(i) $D_z f_{jt}(\tilde{O}_t, z \mid \chi^E_m) - D_z f_{jt}(\tilde{O}_t, z \mid k^E_m) \leqslant 0$;
(ii) $S^R_{jt}(\tilde{O}_t \mid \chi^E_m) \geqslant S^R_{jt}(\tilde{O}_t \mid k^E_m)$ *and* $S^E_{jt}(\tilde{O}_t \mid \chi^E_m) \geqslant S^E_{jt}(\tilde{O}_t \mid k^E_m)$.

These monotonicity results concerning expediting of stock in the system allow us to establish the following key relationship between expediting and the optimal amount of inventory held in the system.

Theorem 7. *Suppose that χ^E_m and k^E_m are two different unit expediting costs at any given stage m such that $\chi^E_m > k^E_m$. Then, $\hat{Z}^R_{jt}(\tilde{O}_t, z_t \mid \chi^E_m) \geqslant \hat{Z}^R_{jt}(\tilde{O}^1_t, z_t \mid k^E_m)$ for every stage j.*

The amount of inventory optimally held in the system increases as expediting costs increase. Because, for sufficiently high unit expediting costs, the option to expedite is no longer exercised and the system reduces to one with regular flow only, Theorem 7 implies that having the option to expedite stock in the supply chain acts to reduce the amount of inventory held in the system (relative to the corresponding system with regular flow only). With expediting, companies can therefore expect to hold less inventory throughout the supply chain, and this is the main source of savings realized by the option to expedite inventory. Although this conclusion may not be surprising, what is less intuitive is the combination of monotonicity properties needed to make it happen, such as the fact, not found in the research literature, that increasing unit expediting costs at a particular stage increases expedited orders at every upstream stage.

6. Quantifying the Value of ADI and Expediting

In this section, we carry out numerical studies to investigate the benefit of having both ADI and stock expediting in a supply chain, and the nature of their mutual interaction. We quantify cost savings from having (i) only ADI; (ii) only expediting; (iii) both ADI and expediting. In what follows, all cost savings numbers are reported in percentage terms, relative to the classical multiechelon model without either ADI or expediting. We also explore the sensitivity of those cost savings to: (i) the unit backlogging cost; (ii) unit expediting costs; (iii) demand variability and correlation; (iv) allocations of total demand to ADI; (v) total leadtime of the assembly system; and (vi) the length of the demand information horizon.

We use Poisson random variables to model the demand vector $(D_{t,t}, D_{t,t+1}, \ldots, D_{t,t+N})$, following Özer (2003) and Levi and Shi (2013). In many practical situations, demand behaves like a Poisson process, especially when it comes from many small, nearly independent sources such as individual customers (Zipkin 2000, p. 179). In our basic model, demand vectors $\{D_{t,t}, D_{t,t+1}, \ldots, D_{t,t+N}\}$ are not correlated across time.

We begin by analyzing the four-component assembly system displayed in Figure 3, referred to as *the basic assembly model*. This system has one subassembly at node 4, a system leadtime of 3 periods (i.e., $L = 3$), and a single period of advance demand ($N = 1$). Nodes 2, 3, 5, and 6 are components. To facilitate interpretation of numerical outputs, we use stationary model parameters and demands. Table 3 shows unit costs for each component at each stage. Subassembly costs at node 4 are assumed to have already been allocated to components 5 and 6 in the manner prescribed by Theorem 2.

Figure 3. The basic assembly model.

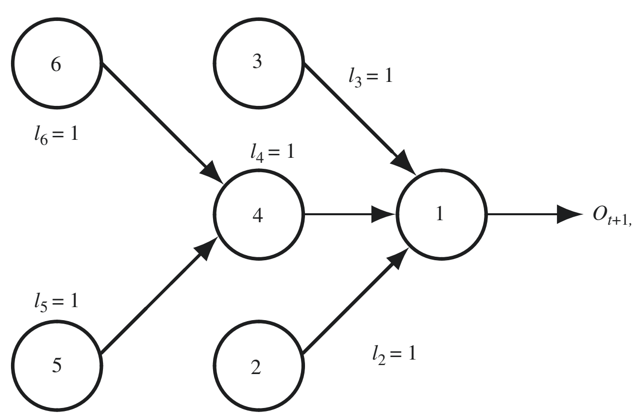


Table 3. Unit costs in the basic assembly system.

Comp.	$H_{i,1}$	$H_{i,2}$	$H_{i,3}$	$k^R_{i,1}$	$k^R_{i,2}$	$k^R_{i,3}$	$k^E_{i,1}$	$k^E_{i,2}$	$k^E_{i,3}$
2	0.3	0.1	0	0.5	0.2	0	0.9	0.4	0
3	0.4	0.2	0	0.7	0.4	0	1.2	0.7	0
5	0.5	0.3	0.1	1.2	1.0	0.8	2.0	1.8	1.6
6	0.6	0.4	0.3	1.6	1.4	1.2	2.7	2.2	1.8

Mean demands used are $\{(5,0),(4,1),(3,2),(2,3),(1,4),(0,5)\}$, where $(\mu_0,\mu_1)=(5,0)$ represents a system with no ADI, and, in the absence of expediting, such a system is the reference point for the percentage savings presented below; with $(\mu_0,\mu_1)=(0,5)$, all of the demand is realized one period in advance. By Theorems 1 and 2, the basic assembly system in Figure 3 can be reduced to an equivalent series system whose unit costs are obtained by aggregating unit assembly costs over relevant components as follows: $k^E_j := \sum_{i\in\mathbb{C}(j)} k^E_{ij}$, $k^R_j := \sum_{i\in\mathbb{C}(j)} k^R_{ij}$, and $h_j := \sum_{i\in\mathbb{C}(j)} h_{ij}$. Thus, for the equivalent series system, echelon inventory holding costs are $h_1=0.8$, $h_2=0.6$, and $h_3=0.4$; ordering costs are $k^R_1=4.0$, $k^R_2=3.0$, and $k^R_3=2.0$; and expediting costs are $k^E_1=6.8$, $k^E_2=5.1$, and $k^E_3=3.4$. Also, $H^A=1.5$.

6.1. Cost Savings as a Function of the Unit Backlogging Cost

Table 4 presents cost savings from using ADI *only*, whereas Table 5 displays cost savings from using *both* ADI and expediting, as a function of the unit backlogging cost. The cost savings shown in both tables are increasing as a greater proportion of total demand is known in advance (i.e., as μ_1 is increasing and $\mu_0=\mu^{\max}-\mu_1$ decreasing). Our results in Table 4 are in line with numerical studies conducted on systems with ADI only in Özer (2003). Although the monotonicity of cost savings as a function of the greater proportion of total demand known in advance is predicted by Theorem 6(ii), the value of quantifying those cost savings lies in their actual values. For example, at the unit backlogging cost of 10, cost savings from expediting almost double when half of the total average demand becomes known one period in advance.

As far as we know, the value of the option to expedite stock in a multistage system has not been quantified before, even though Lawson and Porteus (2000) highlight the need for it. Our results show that this value can be substantial. Even when unit expediting costs are 70% higher than regular order costs, savings from expediting alone range from 5.7%–23.5% of total costs (the top row of Table 5).

To quantify the value of adding expediting to a system with ADI, Table 6 presents cost savings from the model with *both* ADI and expediting relative to the model with ADI *only*. As shown in Table 6, this *marginal value of* (*adding*) *expediting* to a system with ADI is significant and increasing in the backlogging cost and the amount of ADI. Thus, the greater the portion of demand that a company receives in the form of ADI, the more beneficial it is to also develop expediting capability in the supply chain.

When it comes to the cost savings displayed in Tables 4 and 5, one would expect to see a strong substitution effect between ADI and expediting of stock, since they both serve to reduce the mismatch between the supply and demand, and they both reduce total inventory held in the system. What we observe in Tables 4 and 5, however, is something different. For example, when the backlogging cost is 20, the option to expedite stock, without ADI, generates 11.3% in cost savings. Having only ADI, without expediting, results in the cost reduction of 4.1% when mean demands are $(2,3)$. Having both ADI and expediting at those same demand means results in 16.7% cost savings—more than the sum of cost savings from ADI alone and from expediting alone. In other words, what we observe is complementarity.

Table 4. Cost savings—ADI only.

Mean demand	Unit backlogging cost (p) (%)				
(μ_0,μ_1)	10	20	30	40	50
(5, 0)	0.0	0.0	0.0	0.0	0.0
(4, 1)	0.9	1.2	1.4	1.4	1.2
(3, 2)	2.1	2.7	2.6	2.8	2.8
(2, 3)	3.4	4.1	4.4	4.4	4.2
(1, 4)	4.8	6.2	6.2	6.6	6.5
(0, 5)	8.6	10.8	11.5	11.9	11.8

Note. As a function of the backlogging cost.

Table 5. Cost savings—ADI with expediting.

Mean demand	Unit backlogging cost (p) (%)				
(μ_0,μ_1)	10	20	30	40	50
(5, 0)	5.7	11.3	15.8	19.9	23.5
(4, 1)	7.0	12.6	17.5	21.4	25.0
(3, 2)	8.5	14.6	19.3	23.4	26.8
(2, 3)	10.2	16.7	21.3	25.6	29.2
(1, 4)	12.5	19.4	24.4	28.5	31.9
(0, 5)	17.6	26.3	31.9	36.3	39.9

Note. As a function of the backlogging cost.

Table 6. The marginal value of expediting.

Mean demand	Unit backlogging cost (p) (%)				
(μ_0,μ_1)	10	20	30	40	50
(5, 0)	5.7	11.3	15.8	19.9	23.5
(4, 1)	6.1	11.6	16.4	20.3	24.1
(3, 2)	6.5	12.3	17.1	21.2	24.7
(2, 3)	7.1	13.1	17.6	22.1	26.0
(1, 4)	8.0	14.1	19.4	23.5	27.2
(0, 5)	9.9	17.3	23.0	27.8	31.9

Note. As a function of the backlogging cost.

Table 7. The synergy differential.

Mean demand (μ_0, μ_1)	Unit backlogging cost (p) (%)				
	10	20	30	40	50
(5, 0)	0.00	0.00	0.00	0.00	0.00
(4, 1)	0.35	0.20	0.38	0.10	0.35
(3, 2)	0.69	0.67	0.88	0.72	0.54
(2, 3)	1.16	1.35	1.08	1.30	1.45
(1, 4)	1.95	1.93	2.45	2.12	1.91
(0, 5)	3.35	4.22	4.59	4.59	4.65

Note. As a function of the backlogging cost.

To assess the strength of this complementarity effect, we take the difference between percentage cost savings with both ADI and expediting (i.e., the entries in Table 5) on the one hand, and the sum of the cost savings from ADI alone (i.e., the entries in Table 4) and cost savings from expediting alone (i.e., the top row of Table 5) on the other, for each choice of model parameters. We refer to this difference as *the synergy differential*. Equivalently, if we define F_{classic} to be the optimal cost with *neither* ADI *nor* expediting, F_{ADI} to be the optimal cost with ADI *only*, F_{EXP} to be the optimal cost with expediting *only*, and $F_{\text{ADI+EXP}}$ to be the optimal cost with *both* ADI and expediting, then the synergy differential becomes

$$\frac{(F_{\text{classic}} - F_{\text{ADI+EXP}}) - (F_{\text{classic}} - F_{\text{ADI}}) - (F_{\text{classic}} - F_{\text{EXP}})}{F_{\text{classic}}}. \quad (14)$$

Table 7 presents this synergy differential. Positive values indicate complementarity between ADI and expediting (negative values would imply that they are substitutes). Contrary to the conventional wisdom, ADI and expediting are found to be complements with regard to cost savings in the system.

Understanding this complementarity requires a closer examination of the dynamics between ADI and expediting. With ADI, a portion of future demand becomes known in advance, so that it becomes optimal to keep less inventory in the system. Keeping less inventory, however, leaves the supply chain more vulnerable to high realizations of demand and resulting stock-outs. Expediting of stock protects the supply chain against such stock-outs and the associated backlogging costs by making it possible to expedite inventory downstream, and making it available for customer demand sooner than with only regular flow of stock. The vulnerability created by ADI is thus "hedged" by the option to expedite stock, and this hedging acts to create the synergistic effect between ADI and expediting observed in Table 7. Furthermore, the more total demand is realized through advance demand, the less likely it is for the system to stock out, so that, as μ_1 increases, the less costly it becomes for expediting to provide a hedge to ADI, and the complementarity effect grows stronger. One may therefore expect that a longer demand information horizon would have an even more negatively correlated impact on the complementarity of ADI and expediting.

6.2. Cost Savings as a Function of Unit Expediting Costs

We now investigate cost savings for the basic assembly model in Figure 3 as a function of unit expediting costs. The unit backlogging cost is 30. Let $\rho_j := k_j^E / k_j^R$ be the ratio of the unit expediting cost to the unit regular order cost for each stage j. Instead of varying each of the three k_j^E parameters individually, we now vary them at the same time by choosing identical ratios for each j. We set $\rho = \rho_1 = \rho_2 = \rho_3$, and we refer to ρ as the *expedite-to-regular cost ratio*, or E-to-R Cost Ratio. We evaluate cost savings as ρ takes on the values $\{1.3, 1.5, 1.7, 1.9, 2.1\}$. (In Tables 4–7, $\rho = 1.7$.) Table 6 remains unchanged because changing ρ does not impact cost savings from ADI alone. Thus, what is of interest are cost savings with both ADI and expediting, as well as the synergy differential, as a function of ρ. The results are in Tables 8 and 9.

Cost savings from having both ADI and expediting are decreasing in unit expediting costs. Although this monotonicity result follows analytically from the single-period cost function, the benefit of having numerical results in Table 8 is in knowing the exact rate at which the optimal cost is increasing in ρ. The synergy differential remains positive for all model parameters explored in Table 9. For each value of unit expediting costs, the synergy differential is also monotonically increasing in the amount of ADI.

Table 8. Cost savings—ADI with expediting.

Mean demand (μ_0, μ_1)	E-to-R cost ratio (ρ) (%)				
	1.3	1.5	1.7	1.9	2.1
(5, 0)	20.5	17.4	15.8	14.7	13.9
(4, 1)	22.2	19.1	17.5	16.4	15.5
(3, 2)	24.2	21.1	19.3	18.0	17.1
(2, 3)	26.1	23.0	21.3	20.3	19.5
(1, 4)	29.2	26.2	24.4	23.2	22.3
(0, 5)	36.6	33.6	31.9	30.8	29.9

Note. As a function of expediting costs.

Table 9. The synergy differential.

Mean demand (μ_0, μ_1)	E-to-R cost ratio (ρ) (%)				
	1.3	1.5	1.7	1.9	2.1
(5, 0)	0.00	0.00	0.00	0.00	0.00
(4, 1)	0.28	0.41	0.38	0.29	0.26
(3, 2)	1.01	1.05	0.88	0.69	0.57
(2, 3)	1.17	1.20	1.08	1.18	1.26
(1, 4)	2.51	2.60	2.45	2.31	2.23
(0, 5)	4.55	4.69	4.59	4.50	4.48

Note. As a function of expediting costs.

6.3. Impact of Variability and Correlation

Next, we investigate the impact of demand variability and demand correlation (across time periods) on realized savings and the synergy differential. Regarding demand variability, the coefficient variation for a Poisson Process with mean μ is given by $1/\sqrt{\mu}$. Consequently, to examine the impact of variability in the context of our basic assembly system, we vary the coefficient of variation of the Poisson distribution for both the current period's demand and the one period of advance demand. In particular, we start with $(\mu_0, \mu_1) = (2, 2)$ and raise both distribution means in increments of one until we reach $(\mu_0, \mu_1) = (7, 7)$. Thus, we vary the coefficient of variation from 0.707 to 0.378. Relevant results are in Tables 10 and 11.

As observed in Table 10, the impact of variability on cost savings with both ADI and expediting depends on the unit backlogging cost: for small unit backlogging costs, higher variability leads to higher percentage cost savings, whereas with larger unit backlogging costs the opposite happens. With regard to the synergy differential (Table 11), demand variability does not seem to have a monotonic impact.

To evaluate the impact of demand correlation across time periods on realized savings and the synergy differential, we model the joint distribution of $D_{t,t}$, the demand observed in period t for delivery in period t, and $D_{t-1,t}$, the demand observed in period $t-1$ for delivery in period t, with bivariate Poisson distribution. Bivariate Poisson Distribution has been used, for example, to study multi-item inventory systems (Song 1998), and assemble-to-order systems with multiple product types (Lu et al. 2003)

Table 10. Cost savings—ADI with expediting.

Mean demand (μ_0, μ_1)	Unit backlogging cost (p) (%)				
	10	20	30	40	50
(7, 7)	8.9	16.8	23.0	28.2	32.6
(6, 6)	8.9	16.7	22.7	27.9	32.2
(5, 5)	9.0	16.5	22.4	27.3	31.7
(4, 4)	9.0	16.3	21.9	26.5	30.6
(3, 3)	9.2	15.8	21.0	25.4	29.1
(2, 2)	9.5	15.1	19.1	23.1	26.4

Note. Impact of variability.

Table 11. The synergy differential.

Mean demand (μ_0, μ_1)	Unit backlogging cost (p) (%)				
	10	20	30	40	50
(7, 7)	0.48	0.61	0.63	0.67	0.61
(6, 6)	0.53	0.71	0.71	0.71	0.67
(5, 5)	0.60	0.77	0.72	0.78	0.78
(4, 4)	0.76	0.83	0.84	0.79	0.90
(3, 3)	0.79	0.90	1.11	0.94	0.87
(2, 2)	0.86	1.26	0.69	1.14	0.95

Note. Impact of variability.

The joint probability mass function $\phi(y_1, y_0)$ of the bivariate Poisson distribution of random variables Y_0 and Y_1 is defined by three (positive) parameters, λ_1, λ_2, and λ_3 as follows:

$$P\phi(y_0, y_1 \mid \lambda_1, \lambda_2, \lambda_3) = e^{-(\lambda_1+\lambda_2+\lambda_3)} \frac{\lambda_1^{y_1}}{y_0!} \frac{\lambda_2^{y_1}}{y_1!} \sum_{i=0}^{\min(y_0, y_1)} \binom{y_0}{i} \binom{y_1}{i} i! \left(\frac{\lambda_3}{\lambda_1 \lambda_2} \right)^i.$$

It can be shown that Y_0 and Y_1 are then marginally distributed as Poisson with means $\mu_0 = \lambda_1 + \lambda_3$ and $\mu_1 = \lambda_2 + \lambda_3$, respectively (see, e.g., Kocherlakota and Kocherlakota 1992). The covariance of Y_0 and Y_1 is λ_3, so that the correlation coefficient becomes $\lambda_3/(\sqrt{\lambda_1+\lambda_3}\sqrt{\lambda_2+\lambda_3})$. Using joint and marginal probability distributions, we derive the conditional distribution of Y_0 given Y_1 to be the following.

$$\phi_{Y_0}(y_0 \mid y_1) = \sum_{i=0}^{\min(y_0, y_1)} \binom{y_1}{i} r^i (1-r)^{y_1 - i} \frac{e^{-\lambda_1} \lambda_1^{y_0 - i}}{(y_0 - i)!}, \quad (15)$$

where $r := \lambda_3/(\lambda_3 + \lambda_2)$. Letting $Y_0 := D_{t,t}$, $Y_1 := D_{t-1,t}$, and $\lambda_1 = \lambda_2 = 2$, we are now in the position to capture the correlation between advance demand information realized in the previous period for delivery this period and the distribution of demand realized this period for delivery this period. We evaluate cost savings and the synergy differential as λ_3 varies from 0 to 5 in increments of 1. (This corresponds to varying the covariance from 0 to 5, and the correlation coefficient from 0.05 to 0.2.) Thus, $\mu_0 = \mu_1$ varies from 2 to 7. Results are in Tables 12 and 13, where for convenience, we label the first column with $(\lambda_1+\lambda_3, \lambda_2+\lambda_3)$.

Table 12 displays cost savings from having both ADI and expediting as a function of the covariance λ_3 and the unit backlogging cost. Percentage cost savings are uniformly increasing in λ_3 across all values of the unit backlogging cost. Furthermore, by comparing those value with corresponding values without correlation in Table 10, which displays cost saving for the system with identical mean demands but no correlation, having positive correlation across time can be seen to uniformly increase relative cost savings generated by expediting and ADI in the system. One interpretation for this effect is that, with positive correlation, advance demand information provides "a signal"

Table 12. Cost savings—ADI with expediting.

Mean demand $(\lambda_1+\lambda_3, \lambda_2+\lambda_3)$	Unit backlogging cost (p) (%)				
	10	20	30	40	50
(7, 7)	10.0	18.4	24.7	30.0	34.5
(6, 6)	9.9	18.1	24.3	29.5	33.9
(5, 5)	9.8	17.7	23.7	28.7	33.0
(4, 4)	9.5	17.0	22.7	27.6	31.6
(3, 3)	9.5	16.0	21.3	25.7	29.5
(2, 2)	9.5	15.1	19.1	23.1	26.4

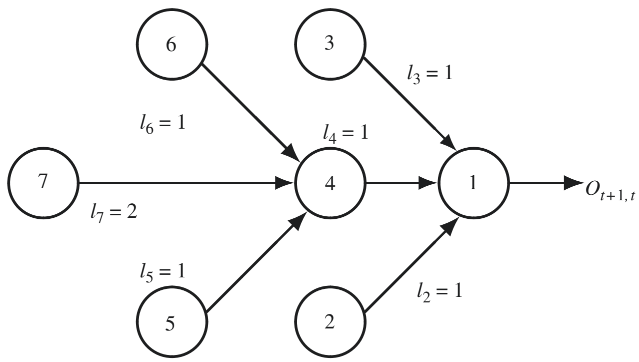
Note. Impact of correlation.

Table 13. The synergy differential.

Mean demand $(\lambda_1+\lambda_3, \lambda_2+\lambda_3)$	Unit backlogging cost (p) (%) 10	20	30	40	50
(7, 7)	1.81	2.26	2.37	2.43	2.35
(6, 6)	1.81	2.30	2.44	2.43	2.41
(5, 5)	1.82	2.32	2.41	2.41	2.40
(4, 4)	1.77	2.16	2.29	2.39	2.32
(3, 3)	1.32	1.92	2.06	2.01	2.10
(2, 2)	0.86	1.26	0.69	1.14	0.95

Note. Impact of correlation.

Figure 4. A four-period assembly model.



about next period's demand realization and thus serves to reduce the uncertainty of future demand (and related costs). Furthermore, as seen by comparing Table 13 with Table 11, incorporating positive demand correlation also acts to significantly increase the synergy differential, up to fourfold for small values of the unit backlogging cost. Therefore, companies that find positive covariance between advance demand information and current demand realization are especially well positioned to reap the benefits of having both expediting and ADI capabilities in their supply chains.

6.4. Higher Echelon Systems

Next, we quantify cost savings from ADI and expediting in higher echelon assembly systems. We consider the four- and five-period assembly systems shown in Figures 4 and 5, respectively. In the spirit of changing only one parameter at a time, we maintain one period of advance demand. To make fair cost comparisons across assembly systems with different total leadtimes, the sum of each category's costs (i.e., holding costs, regular order costs, and expediting costs) across the total leadtime for the system is kept constant. The holding cost for the assembled product is also the same: $H^A = 1.5$. The assembly system in Figure 4 has a total leadtime of four periods ($L = 4$), and, like our basic assembly system, it has one subassembly at node 4. This subassembly now also requires a component 7, which has a two-period leadtime. For this system, nodes 2, 3, 5, 6, 7 are components. All stage-dependent unit costs are shown in Table 14.

Figure 5. A five-period assembly model.

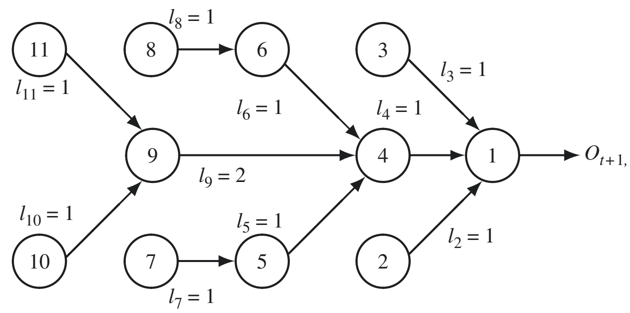


Table 15. Cost savings—ADI only.

Mean demand (μ_0, μ_1)	Unit backlogging cost (p) (%) 10	20	30	40	50
(5, 0)	0.0	0.0	0.0	0.0	0.0
(4, 1)	0.8	0.9	1.1	0.9	0.9
(3, 2)	1.7	2.0	2.1	2.1	2.0
(2, 3)	2.8	3.2	3.4	3.1	3.1
(1, 4)	4.1	4.8	4.9	4.8	4.7
(0, 5)	7.1	8.5	8.8	8.6	8.3

Note. Four-period assembly model.

Tables 15 and 16 present cost savings for the four-period assembly system with ADI only, and with both ADI and expediting. Although cost savings with ADI alone are still increasing in the percentage of demand allocated to ADI, they are less than those for the basic assembly system. The longer the supply chain, the more inventory there is in the system, so that reductions in inventory from ADI are about the same in total terms, but lesser relative to the basic three-period assembly system.

With both ADI and expediting, cost savings increase with the total leadtime (Table 16). The longer the supply chain, the more options there are for expediting—inventory can be expedited into stage 2 from an increasing number of

Table 14. Unit costs for the four-period assembly model.

Comp.	$H_{i,1}$	$H_{i,2}$	$H_{i,3}$	$H_{i,4}$	$k^R_{i,1}$	$k^R_{i,2}$	$k^R_{i,3}$	$k^R_{i,4}$	$k^E_{i,1}$	$k^E_{i,2}$	$k^E_{i,3}$	$k^E_{i,4}$
2	0.2	0.1	0	0	0.3	0.1	0	0	0.6	0.3	0	0
3	0.2	0.1	0	0	0.5	0.2	0	0	1.0	0.5	0	0
5	0.3	0.2	0.1	0	0.6	0.4	0.2	0	1.2	0.7	0.3	0
6	0.3	0.2	0.1	0	1.0	0.6	0.3	0	1.5	0.8	0.5	0.0
7	0.5	0.4	0.3	0.2	1.6	1.2	1.0	1.0	2.5	2.0	1.8	1.7

Table 16. Cost savings—ADI with expediting.

Mean demand	Unit backlogging cost (p) (%)				
(μ_0, μ_1)	10	20	30	40	50
(5, 0)	11.2	20.5	27.6	33.5	38.4
(4, 1)	12.3	21.7	29.0	34.7	39.7
(3, 2)	13.5	23.1	30.6	36.2	41.0
(2, 3)	15.1	25.1	32.2	38.0	42.8
(1, 4)	17.4	27.5	34.8	40.3	44.8
(0, 5)	22.2	33.4	41.0	46.6	51.1

Note. Four-period assembly model.

Table 18. Cost savings—ADI only.

Mean demand	Unit backlogging cost (p) (%)				
(μ_0, μ_1)	10	20	30	40	50
(5, 0)	0.0	0.0	0.0	0.0	0.0
(4, 1)	0.7	0.7	0.9	0.7	0.7
(3, 2)	1.4	1.6	1.7	1.6	1.5
(2, 3)	2.4	2.7	2.7	2.4	2.3
(1, 4)	3.6	3.9	3.8	3.7	3.5
(0, 5)	6.2	7.0	6.9	6.6	6.2

Note. Five-period assembly model.

origination stages, and this additional optionality directly results in correspondingly larger cost savings for both the system with expediting alone and with the system with both ADI and expediting. As a consequence, the synergy differential remains (roughly) unchanged.

Figure 5 displays a five-period assembly model with two subassemblies, at nodes 4 and 9, while nodes 2, 3, 7, 8, 10, and 11 are components. Cost parameters for this assembly model are in Table 17. Cost savings for the five-period assembly model with only ADI (Table 18) continue to decrease with the length of the supply chain: each entry in Table 18 is smaller than the corresponding entry in Table 15.

The benefit of having only ADI is decreasing in the supply chain length, whereas cost savings with both ADI and expediting still increase in the length of the supply chain, reaching more than 60% (Table 19). Thus, there exists a structural difference between systems with ADI only and those with both ADI and expediting—the savings from having ADI only are decreasing in the length of the supply chain, whereas those from having both ADI and expediting are increasing. As a result, in practice, with ADI, longer supply chains have more to benefit from adding the option to expedite inventory than shorter ones do.

The synergy differential for the five-period assembly model is shown in Table 21. As observed, the complementarity effect persists unabated even as we increase the length of the supply chain.

6.5. Longer Demand Information Horizons

We now explore the impact of longer demand information horizons on cost savings. We use the four-period assembly system whose structure is shown in Figure 5 and costs in

Table 19. Cost savings—ADI with expediting.

Mean demand	Unit backlogging cost (p) (%)				
(μ_0, μ_1)	10	20	30	40	50
(5, 0)	16.1	28.7	37.6	44.6	50.0
(4, 1)	17.1	29.8	38.8	45.6	51.0
(3, 2)	18.3	31.0	40.2	46.8	52.0
(2, 3)	19.7	32.7	41.5	48.3	53.5
(1, 4)	21.8	34.8	43.7	50.1	55.1
(0, 5)	26.3	40.0	48.9	55.2	60.1

Note. Five-period assembly model.

Table 20. The synergy differential.

Mean demand	Unit backlogging cost (p) (%)				
(μ_0, μ_1)	10	20	30	40	50
(5, 0)	0.00	0.00	0.00	0.00	0.00
(4, 1)	0.30	0.35	0.36	0.28	0.41
(3, 2)	0.64	0.55	0.91	0.61	0.60
(2, 3)	1.12	1.38	1.26	1.43	1.35
(1, 4)	2.07	2.23	2.38	1.98	1.75
(0, 5)	3.91	4.46	4.59	4.44	4.42

Note. Four-period assembly model.

Table 14, and present results for the case of two periods of advance demand ($N = 2$).

Tables 22 and 23 display cost savings with ADI only and with both ADI and expediting for the four-period assembly system with two periods of advance demand information. By Theorem 6, since costs in the system are decreasing with N, the demand information horizon, those cost savings can be expected to exceed those in Tables 15 and 16 (when the system has only one period of advance

Table 17. Unit costs for the five-period assembly model.

Comp.	$H_{i,1}$	$H_{i,2}$	$H_{i,3}$	$H_{i,4}$	$H_{i,5}$	$k^R_{i,1}$	$k^R_{i,2}$	$k^R_{i,3}$	$k^R_{i,4}$	$k^R_{i,5}$	$k^E_{i,1}$	$k^E_{i,2}$	$k^E_{i,3}$	$k^E_{i,4}$	$k^E_{i,4}$
2	0.6	0.3	0	0	0	0.2	0.1	0	0	0	0.4	0.1	0	0	0
3	0.7	0.4	0	0	0	0.3	0.1	0	0	0	0.5	0.3	0	0	0
7	0.8	0.5	0.3	0.1	0	0.4	0.2	0.1	0.1	0	0.6	0.4	0.2	0.1	0
8	1.0	0.6	0.4	0.1	0	0.5	0.3	0.2	0.1	0	0.7	0.5	0.3	0.2	0
10	1.3	0.7	0.5	0.2	0.1	0.5	0.4	0.3	0.2	0.1	0.8	0.6	0.5	0.3	0.1
11	1.6	0.8	0.6	0.3	0.2	0.6	0.5	0.4	0.3	0.1	1.0	0.7	0.6	0.5	0.2

Table 21. The synergy differential.

Mean demand (μ_0, μ_1)	Unit backlogging cost (p) (%) 10	20	30	40	50
(5, 0)	0.00	0.00	0.00	0.00	0.00
(4, 1)	0.33	0.36	0.31	0.28	0.39
(3, 2)	0.74	0.65	0.87	0.56	0.63
(2, 3)	1.18	1.36	1.26	1.28	1.24
(1, 4)	2.12	2.23	2.24	1.81	1.65
(0, 5)	3.92	4.36	4.30	4.04	3.91

Note. Five-period assembly model.

Table 22. Cost savings—ADI only.

Mean demand (μ_0, μ_1, μ_2)	Unit backlogging cost (p) (%) 10	20	30	40	50
(5, 0, 0)	0.0	0.0	0.0	0.0	0.0
(2, 2, 1)	5.5	7.1	8.2	8.8	9.4
(2, 1, 2)	8.0	10.8	12.9	14.1	15.5
(1, 1, 3)	11.9	16.1	18.9	21.1	23.0
(0, 1, 4)	17.5	23.5	27.2	29.9	32.0
(0, 0, 5)	20.6	27.4	31.8	34.9	37.5

Note. Four-period assembly model, with $N = 2$.

Table 23. Cost savings—ADI with expediting.

Mean demand (μ_0, μ_1, μ_2)	Unit backlogging cost (p) (%) 10	20	30	40	50
(5, 0, 0)	11.2	20.5	27.6	33.5	38.4
(2, 2, 1)	16.1	25.9	33.0	38.7	43.5
(2, 1, 2)	17.1	26.8	33.7	39.4	44.1
(1, 1, 3)	20.3	30.0	37.0	42.3	46.7
(0, 1, 4)	26.1	36.7	43.9	49.2	53.5
(0, 0, 5)	26.9	37.5	44.5	49.8	54.1

Note. Four-period assembly model, with $N = 2$.

demand information), and that is exactly what we observe in Tables 22 and 23. The size of the increase in the savings from ADI alone is particularly noteworthy, as the demand information horizon is increased from one to two periods.

Table 24 displays the synergy differential for this assembly model with two periods of advance demand information, respectively. When it comes to the interaction between ADI and expediting of stock, the length of the demand information horizon plays a key role in determining if the two are complements or substitutes. With a single period of demand information horizon, ADI and expediting are complements; as the information horizon increases, so does the substitution effect between them. Economic complementarity and substitutability of advance demand information and expediting of stock are thus shown not to be absolute characteristics, but rather functions of the supply chain structure.

Table 24. The synergy differential.

Mean demand (μ_0, μ_1, μ_2)	Unit backlogging cost (p) (%) 10	20	30	40	50
(5, 0, 0)	0.0	0.0	0.0	0.0	0.0
(2, 2, 1)	−0.5	−1.7	−2.8	−3.6	−4.3
(2, 1, 2)	−2.1	−4.6	−6.7	−8.3	−9.8
(1, 1, 3)	−2.7	−6.6	−9.5	−12.4	−14.7
(0, 1, 4)	−2.8	−7.2	−10.9	−14.2	−16.8
(0, 0, 5)	−4.9	−10.4	−14.8	−18.7	−21.7

Note. Four-period assembly model, with $N = 2$.

This observation concerning economic complementarity between advance demand information and expediting of stock being a function of the demand information horizon is further confirmed by our study of the identical four-period assembly model with *three* periods of advance demand information (not shown). In that study, savings from ADI alone and with both ADI and expediting continue to increase, whereas the synergy differential becomes more negative. In particular, in a system with three periods of advance information, most of the savings from having both ADI and expediting are captured by having ADI alone, as the system comes close to running in the make-to-order mode.

7. Capacity Constraints on Expediting

We now demonstrate how the approach developed in this paper for solving assembly systems with ADI and expediting can be applied to analyze additional structural features of those systems. In particular, we now allow limits on expedited orders, in the form of capacity constraints on the amount of each component expedited into each stage, throughout the assembly process. Referring to our component assembly system in Figure 1, let K_{ij} be the capacity constraint on orders for component i expedited into stage j, so that the new set of feasible decisions $\tilde{\mathbb{X}}(x_t)$ becomes

$$\begin{aligned}\tilde{\mathbb{X}}(x_t) = \{X_t^E, X_t^R \geqslant 0 \mid X_{ijt}^E \leqslant \min(x_{i,\,j+1,\,t} + X_{i,\,j+1,\,t}^E, K_{ij});\\ X_{ijt}^R \leqslant x_{i,\,j+1,\,t} + X_{i,\,j+1,\,t}^E - X_{ijt}^E\\ \text{for } 1 \leqslant i \leqslant n,\ 1 \leqslant j < L_i\}.\end{aligned}$$

We will refer to the component assembly system under those capacity constraints on expedited orders as the *capacitated component* (*assembly*) *system*.

Theorem 8. *For the capacitated component assembly system, the following hold.*

(i) *Optimal order schedules* (X_t^E, X_t^R) *are balanced in each period* $t = 1, \ldots, T$;

(ii) *The on-hand inventory state* x_t *is balanced in each period* $t = 1, \ldots, T+1$;

(iii) *Let* $K_j^* := \min_{i \in \mathbb{C}(j)} K_{ij}$ *for* $j = 1, \ldots, L_n$. *For any* j *and every* $i \in \mathbb{C}(j)$, $X_{ijt}^E \leqslant K_j^*$ *in each period* t.

Thus, in a capacitated component assembly system, it is no longer necessary to manage each component separately; instead, those components that are at the same stage can again be managed together as a kit. We can therefore represent the on-hand inventory for each relevant component at every stage j by a single variable y_{jt} (i.e., $x_{ijt} = y_{jt}$ for every $i \in \mathbb{C}(j)$) and optimal decisions X^E_{ijt} and X^R_{ijt} for all $i \in \mathbb{C}(j)$ by single variables Y^E_{jt} and Y^R_{jt} for each j, respectively. Furthermore, at each stage j, we can also replace capacity constraints K_{ij} for all $i \in \mathbb{C}(j)$ with a single capacity constraint K^*_j. In that manner, a capacitated component system can be reduced to an equivalent series system with ADI and expediting and capacity constraint K^*_j on all orders expedited into every stage j. Next, we extend this result to a general assembly system, such as the one shown in Figure 2, in which subassemblies are allowed throughout the system, and in which every expedited order (for any component i into any stage j) may be subject to the capacity constraint K_{ij}. We will refer to such a system as the *general capacitated* (*assembly*) *system.*

Theorem 9. *If, in a general capacitated system there exists an allocation of assembly costs such that $\theta_{ijt} \geqslant \alpha\theta_{ij,t+1}$ for every component i, stage j, and period t, then the optimal policy is balanced in every period. Given j, all capacity constraints K_{ij} for $i \in \mathbb{C}(j)$ can be replaced with the single constraint K^*_j.*

A general capacitated assembly system with ADI and expediting can thus be reduced to an equivalent series system with ADI and capacity constraints K^*_j on orders expedited into each stage j. The optimality equations for the resulting series system are

$$\tilde{f}_t(\tilde{O}_t, z_t) = -\sum_{j=1}^{L} k^E_j z_{jt} + \min_{Z^E_t, Z^R_t \in \tilde{\mathbb{Z}}(z_t)} \left\{ \gamma_t(Z^R_{1t}) + \sum_{j=1}^{L} (c^E_{jt} Z^E_{jt} + c^R_{jt} Z^R_{jt}) + \alpha \mathbf{E}[\tilde{f}_{t+1}(\tilde{O}_{t+1}, Z^R_t - O_{t+1,t})] \right\}, \quad (16)$$

where

$$\tilde{\mathbb{Z}}(z_t) = \{Z^E_t, Z^R_t \mid z_{jt} \leqslant Z^E_{jt} \leqslant \min(z_{jt} + K^*_j, Z^E_{j+1,t}), \; Z^E_{jt} \leqslant Z^R_{jt} \leqslant Z^E_{j+1,t}; 1 \leqslant j \leqslant L\}. \quad (17)$$

Note that, in the above expression, the upper boundary of the feasible region for each Z^E_{jt}, the echelon j inventory position after expediting, depends on both z_{jt} and $Z^E_{j+1,t}$, which, in effect, implies that this boundary depends on all echelon inventory levels at stage j and all stages upstream of it. As a result, the system cannot be expected to achieve the decomposition of the objective cost function into a sum of single-variable convex functions. We remark that even for a series system with only capacitated regular orders the general form of the optimal inventory policy is not known. The solution of the dynamic program given in Equations (16) and (17) therefore remains outside the scope of this work. Although the series-equivalent results of Theorems 8 and 9 therefore have limited practical value, especially for longer assembly systems, their main contributions is to highlight the importance of making progress on multiechelon systems with capacity-constrained expediting, which have so far not been addressed in the literature.

There exists a special case of the general problem given in (16) and (17) amenable to analysis: when only the last stage in the series system has a capacity constraint (K) on expedited orders, we are able to solve the problem and establish the form of the optimal policy, as shown in the following theorem.

Theorem 10. *For every z_t and $\tilde{O}_t$ in each period t, let $g_{jt}(\tilde{O}_t, z_{jt})$ and $U_{jt}(\tilde{O}_t, z_{jt})$ be as defined in Lemma 5 (with $f_{j,t+1}(\tilde{O}_{t+1}, \cdot)$ replaced by $\tilde{f}_{j,t+1}(\tilde{O}_{t+1}, \cdot)$). Let functions $V_{1t}(\tilde{O}_t, \cdot), \ldots, V_{Lt}(\tilde{O}_t, \cdot)$ be defined recursively as $V_{jt}(\tilde{O}_t, z_{jt}) := U_{jt}(\tilde{O}_t, z_{jt}) + V^-_{j-1,t}(\tilde{O}_t, z_{jt})$, with $V_{0t}(\tilde{O}_t, z_{jt}) := 0$. Then the following hold.*

(i) The function $V_{jt}(\tilde{O}_t, \cdot)$ is smooth and convex, and $f_t(\tilde{O}_t, z_t) = \sum_{j=1}^{L} V^+_{jt}(\tilde{O}_t, z_{jt}) + V^-_{Lt}(\tilde{O}_t, z_{Lt} + K)$;

(ii) Let $S^E_{jt}(\tilde{O}_t) := \max(\operatorname{argmin}_Z V_{jt}(\tilde{O}_t, Z))$ for every $j = 1, \ldots, L$. Then, $\hat{Z}^E_{jt}(\tilde{O}_t, z_t)$ is given by

$$\hat{Z}^E_{jt}(\tilde{O}_t, z_t) = \bigwedge_{i=j}^{L} [z_{it} \vee S^E_{it}(\tilde{O}_t)] \wedge (z_{Lt} + K). \quad (18)$$

(iii) Let $S^R_{jt}(\tilde{O}_t) := \max(\operatorname{argmin}_Z g_{jt}(\tilde{O}_t, Z))$ for every $j = 1, \ldots, L$. Then, given the optimal echelon positions after expediting, $\hat{Z}^E_{jt}(\tilde{O}_t, z_t)$, $\hat{Z}^R_{jt}(\tilde{O}_t, z_t)$ is given by

$$\hat{Z}^R_{jt}(\tilde{O}_t, z_t) := \begin{cases} [\hat{Z}^E_{jt}(\tilde{O}_t) \vee S^R_{jt}(\tilde{O}_t)] \wedge \hat{Z}^E_{j+1t}(\tilde{O}_t) & \text{if } j < L; \\ \hat{Z}^E_{Lt}(\tilde{O}_t) \vee S^R_{Lt}(\tilde{O}_t) & \text{if } j = L. \end{cases} \quad (19)$$

8. Concluding Remarks

Companies with assembly operations are increasingly using ADI systems in an effort to shift production to make-to-order, and thus reduce variability of demand. Allowing expediting of stock in such systems provides an opportunity to significantly enhance the resulting cost savings. Assembly systems with ADI and expediting, however, are very difficult to solve because of the curse of dimensionality of a large state space. We approach this problem by introducing a new way to analyze assembly systems, which is based on disaggregating product flows and identifying local properties of optimal decisions satisfied at each stage in the system. This new approach enables us to characterize the structure of the optimal policy that makes it possible to optimally manage such systems in an analytically and numerically tractable manner.

The key feature of the optimal policy is that it is no longer necessary to manage each component (or subassembly) separately; instead, those components that are at the same stage can be managed together as a "kit," where the kit for a particular stage has one each of every relevant component. The concept of a "kit-in-time" in which a supplier provides all parts or components (of a subassembly system) just-in-time to the manufacturer is not new to industry. What is novel, from the perspective of practical implementation, about the balanced policy found to be optimal for the assembly system considered in this paper, is the idea that balancing each kit has to extend across multiple suppliers of subassemblies. Thus, for example, in Figure 2, it is not enough that the supplier of subassembly 7 provide a kit with balanced amounts of components 9 and 10; in addition, those components also have to be balanced across the supplier of component 8 and the supplier of component 6. In other words, it is not just one supplier that has to provide a balanced kit for subassembly at a particular stage, but rather his component orders also have to be matched among all other suppliers relevant at all stages that precede and include that subassembly. What enables that implementation to be carried out in practice are cutting-edge enterprise resource planning (ERP) systems that are wide enough to offer transparency across the entire supply chain, and deep enough to provide information about component-level decisions.

In our study, advance demand information and expediting of stock are found to be complements with regard to the realized savings when demand information horizon is short, and substitutes under longer information horizons. Therefore, companies looking to shift their production/assembly operations from make-to-stock to make-to-order by gathering advance demand information could find it profitable to also implement expediting, much like Dell has done, especially when they are able to collect advance demand for only the near future, and when advance demand information is positively correlated with future demands.

In practice, advanced demand information (ADI) typically reduces to having information about the timing and quantity of future customer orders. This information can be obtained by satisfying customers who are willing to pay higher prices for shorter leadtimes, by offering price discounts to those customers willing to accept longer leadtimes, and by employing information technologies, such as electronic data interchange and Internet-based software. Advance demand information can also be assessed through clickstream data, as shown empirically in Huang and Van Mieghem (2014). Collaborative planning and forecasting and replenishment (CPFR) enables supply-chain partners to receive better information on demand, and can thus also serve as effective means of collecting advance demand information (Hu et al. 2003). Another strategy to ascertain future demand is through recent advances in supply chain integration, which are making advance demand information commonly available in a wide range of industries (see, e.g., Gallego and Özer 2001, Wang and Toktay 2008, Huang and Van Mieghem 2014).

Expediting of stock, by comparison, is already a well established service provided by a number of logistics, freight-forwarding, shipping, and 3PL companies. The necessary task for the supply chain function of an organization, when it comes to maximizing the value of that logistics service, is to integrate expediting of stock with the company's ADI capabilities and related technologies.

In conclusion, with advance demand information, companies obtain valuable knowledge about customers needs; with the option to expedite stock, they can better act on this information.

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Appendix.

Proof of Lemma 1. Suppose that it were optimal to expedite some component i into stage 1 in some period t. Let X_{it}^E and X_{it}^R be the optimal order schedules for component i in period t. Thus, $X_{i1t}^E > 0$.

Consider another policy for component i in period t given by a set of order schedules $(\bar{X}_{it}^E, \bar{X}_{it}^R)$ such that $\bar{X}_{i1t}^E = 0$ and $\bar{X}_{i1t}^R = X_{i1t}^E + X_{i1t}^R$. For every $j > 1$, let $\bar{X}_{ijt}^E = X_{ijt}^E$, and $\bar{X}_{ijt}^R = X_{ijt}^R$. (Optimal schedules for component i remain unchanged in all future periods.) Because the resulting state in period $t+1$ is identical under both sets of schedules, the difference in the cost between two sets of order schedules becomes

$$\begin{aligned} G_t(\tilde{O}_t, x_t, X_t^E, X_t^R) &- G_t(\tilde{O}_t, x_t, \bar{X}_t^E, \bar{X}_t^R) \\ &= (k_{i1t}^E + h_{i1t})X_{i1t}^E + (k_{i1t}^R + h_{i1t})X_{i1t}^R - (k_{i1t}^R + h_{i1t})(X_{i1t}^R + X_{i1t}^E) \\ &= (k_{i1t}^E - k_{i1t}^R)X_{i1t}^E > 0, \end{aligned}$$

since $k_{i1t}^E > k_{i1t}^R$ by Assumption 1. Thus, it can never be optimal to expedite any component i into stage 1.

Proof of Lemma 2. By Assumption 4, x_t is balanced in period 1. Assume inductively that x_t is balanced at stage 1 for some $t < T+1$. Thus, $x_{i1t} = x_{k1t}$ for all $i, k \in \mathbb{C}(1)$. Let X_t^E and X_t^R be optimal schedules in period t, and $\pi_t := \{(X_t^E, X_t^R), (X_{t+1}^E, X_{t+1}^R), \ldots, (X_T^E, X_T^R)\}$ be an optimal policy for periods t through T. By Lemma 1, $X_{i1t}^E = 0$ for every component i for each t. Let $q = q(1, X_t^R)$ be a component in $\mathbb{C}(1)$ such that $X_{q1t}^R = \min_{i \in \mathbb{C}(1)} X_{i1t}^R$. Thus, component q is the component with the smallest regular order into stage 1 in period t. If $X_{i1t}^R = X_{q1t}^R$ for every i, then regular orders into stage 1 for all components are balanced, and by state transition equation given in (1), it follows that x_{t+1} is balanced at stage 1.

Assume there exists a component $i \in \mathbb{C}(1)$ such that $X_{i1t}^R > X_{q1t}^R$. Consider a policy $\bar{\pi}_t$ completely identical to π_t except for the following: $\bar{X}_{i1t}^R = X_{i1t}^R - \delta$ and $\bar{X}_{i1,t+1}^R = X_{i1,t+1}^R + \delta$ for some $0 < \delta \leqslant X_{i1t}^R - X_{q1t}^R$. Let x_{t+1} and $\bar{x}_{t+1}$ be on-hand states generated by starting with x_t, and applying schedules X_t^R and $\bar{X}_t^R$, respectively in period t, for some given $O_{t+1,t}$. Similarly, let x_{t+2}

and $\bar{x}_{t+2}$ be on-hand states generated by starting with x_{t+1} and $\bar{x}_{t+1}$, respectively, and applying schedules X^R_{t+1} and $\bar{X}^R_{t+1}$, in period $t+1$. We have

$$\begin{aligned}\bar{x}_{i1,\,t+1} &= x_{i1,\,t} + \bar{X}^R_{i1t} - O_{t+1,\,t} = x_{i1,\,t} + X^R_{i1t} - \delta - O_{t+1,\,t}\\ &= x_{i1,\,t+1} - \delta. \qquad (20)\end{aligned}$$

$$\begin{aligned}\bar{x}_{i2,\,t+1} &= x_{i2,\,t} + \bar{X}^E_{i2t} + \bar{X}^R_{i2t} - \bar{X}^R_{i1t}\\ &= x_{i2,\,t} + X^E_{i2t} + X^R_{i2t} - (X^R_{i1t} - \delta) = x_{i2,\,t+1} + \delta. \qquad (21)\end{aligned}$$

Next, we show that $\bar{X}^R_{i1t}$ and $\bar{X}^R_{i1,\,t+1}$ are feasible in periods t and $t+1$.

$$\begin{aligned}\bar{X}^R_{i1t} &< X^R_{i1t} && [\text{definition of } \bar{X}^R_{i1t}]\\ &\leqslant x_{i2t} + X^E_{i2t} && [(X^E_t, X^R_t) \text{ is feasible and optimal}]\\ &= x_{i2t} + \bar{X}^E_{i2t}. && [\text{definition of } \bar{\pi}_t]\end{aligned}$$

Thus, $\bar{X}^R_{i1t}$ is feasible in period t. Furthermore,

$$\begin{aligned}\bar{X}^R_{i1,\,t+1} &= X^R_{i1,\,t+1} + \delta && [\text{definition of } \bar{X}^R_{i1,\,t+1}]\\ &\leqslant x_{i2,\,t+1} + X^E_{i2,\,t+1} + \delta && [(X^E_t, X^R_t) \text{ is feasible and optimal}]\\ &= x_{i2,\,t+1} + \delta + \bar{X}^E_{i2,\,t+1} && [\text{definition of } \bar{\pi}_t]\\ &= \bar{x}_{i2,\,t+1} + \bar{X}^E_{i2,\,t+1} && [\text{by (21)}]\end{aligned}$$

Therefore, $\bar{X}^R_{i1,\,t+1}$ is feasible in period $t+1$. Finally, we get

$$\begin{aligned}\bar{x}_{i1,\,t+2} &= \bar{x}_{i1,\,t+1} + \bar{X}^R_{i1,\,t+1} - O_{t+2,\,t+1} && [\text{by (1)}]\\ &= \bar{x}_{i1,\,t+1} + X^R_{i1,\,t+1} + \delta - O_{t+2,\,t+1} && [\text{definition of } \bar{X}^R_{i1,\,t+1}]\\ &= x_{i1,\,t+1} - \delta + X^R_{i1,\,t+1} + \delta - O_{t+2,\,t+1} && [\text{by (20)}]\\ &= x_{i1,\,t+2}. && [\text{by (1)}]\\ \bar{x}_{i2,\,t+2} &= \bar{x}_{i2,\,t+1} + \bar{X}^E_{i2,\,t+1} + \bar{X}^R_{i2,\,t+1} - \bar{X}^R_{i1,\,t+1} && [\text{by (1)}]\\ &= \bar{x}_{i2,\,t+1} + X^E_{i2,\,t+1} + X^R_{i2,\,t+1} - (X^R_{i1,\,t+1} + \delta) && [\text{definition of } \bar{\pi}_t]\\ &= x_{i2,\,t+1} + \delta + X^E_{i2,\,t+1} + X^R_{i2t} - X^R_{i1t} - \delta && [\text{by (21)}]\\ &= x_{i2,\,t+2}. && [\text{by (1)}]\end{aligned}$$

In period $t+2$, the two policies result in the same on-hand states. The two policies are identical starting in period $t+2$, and they start out in identical states in period $t+2$; thus, they produce identical states for the rest of the time horizon. We now evaluate the difference in cost between the two policies.

Since x_t is balanced at stage 1 by assumption, then, because $\bar{X}^R_{i1t} \geqslant X^R_{q1t}$, we have $\gamma_t(\tilde{O}_t, \min_i, (x_{i1t} + X^R_{i1t})) = \gamma_t(\tilde{O}_t, \min_i(x_{i1t} + \bar{X}^R_{i1t})) = \gamma_t(\tilde{O}_t, (x_{q1t} + X^R_{q1t}))$, by definition of q. Then, by means of (3),

$$\begin{aligned}&G_t(\tilde{O}_t, x_t, \pi) - G_t(\tilde{O}_t, x_t, \bar{\pi})\\ &\quad = \delta(k^R_{i1t} + h_{i1t}) + \alpha \mathbf{E}_{D_t} \mid \tilde{O}_t[G_{t+1}(\tilde{O}_{t+1}, x_{t+1}, \pi_{t+1})]\\ &\qquad - \alpha \mathbf{E}_{D_t} \mid \tilde{O}_t[G_{t+1}(\tilde{O}_{t+1}, \bar{x}_{t+1}, \bar{\pi}_{t+1})].\end{aligned}$$

Next, by (20) and definition of $\bar{X}^R_{i1,\,t+1}$, we get $\bar{x}_{i1,\,t+1} + \bar{X}^R_{i1,\,t+1} = x_{i1,\,t+1} - \delta + X^R_{i1,\,t+1} + \delta = x_{i1,\,t+1} + X^R_{i1,\,t+1}$. Thus, for any given $\tilde{O}_{t+1}$, $\gamma_{t+1}(\tilde{O}_{t+1}, \min_i (x_{i1,\,t+1} + X^R_{i1,\,t+1})) = \gamma_{t+1}(\tilde{O}_{t+1}, \min_i (x_{i1,\,t+1} + \bar{X}^R_{i1,\,t+1}))$. As a result, by using the definition of $\bar{X}^R_{i1,\,t+1}$ and expressions (20) and (21), we get

$$\begin{aligned}&G_{t+1}(\tilde{O}_{t+1}, x_{t+1}), \pi_{t+1}) - G_{t+1}(\tilde{O}_{t+1}, \bar{x}_{t+1}, \bar{\pi}_{t+1})\\ &\quad = \delta(k^R_{i1,\,t+1} + h_{i1,\,t+1}) + \delta(H_{i1,\,t+1} - H_{i2,\,t+1})\\ &\qquad + \alpha \mathbf{E}_{D_{t+1} \mid \tilde{O}_{t+1}}[G_{t+2}(\tilde{O}_{t+2}, x_{t+2}, \pi_{t+2})]\\ &\qquad - \alpha \mathbf{E}_{D_{t+1} \mid \tilde{O}_{t+1}}[G_{t+2}(\tilde{O}_{t+2}, \bar{x}_{t+2}, \bar{\pi}_{t+2})]. \qquad (22)\end{aligned}$$

As already shown, $x_{t+2} = \bar{x}_{t+2}$. Therefore, since $\pi_{t+2} = \bar{\pi}_{t+2}$, then the bottom line of (22) becomes identically zero, and the RHS of (22) reduces to $-\delta k^R_{i1,\,t+1}$. Substituting this result, we get

$$G_t(\tilde{O}_t, x_t, \pi) - G_t(\tilde{O}_t, x_t, \bar{\pi}) = \delta(k^R_{i1t} + h_{i1t} - \alpha k^R_{i1,\,t+1}) > 0,$$

by Assumption 3. Thus, it cannot be optimal in period t for any component i to place a regular order at stage 1 in excess of the regular order placed by any other component at stage 1. In other words, if x_t is balanced at stage 1 in any period t, then it is optimal for X^R_t to also be balanced at stage 1 in period t.

Since x_t is balanced at stage 1 by inductive assumption, and because it is optimal for X^R_t to be balanced at stage 1, then by (1) and Lemma 1, x_{t+1} is balanced at stage 1. This completes the proof.

Proof of Lemma 3. We begin with $j = 2$. Thus, assuming that x_t is balanced through stage 2, we show that X^E_t is also balanced *through* stage 2. By Lemma 1, it is optimal not to expedite any component i into stage 1. Thus, $X^E_{i1t} = 0$ for all i and t, and X^E_t is balanced at stage 1. We only need to show that X^E_t it is balanced *at* stage 2. Let $q = q(2, X^E_t)$ be a component in $\mathbb{C}(2)$ such that $X^E_{q2t} = \min_{i \in \mathbb{C}(2)} X^E_{i2t}$. Thus, component q is the component with the smallest expedited order into stage 2 in period t. If $X^E_{i2t} = X^E_{q1t}$ for every i, then X^E_t is balanced at stage 2.

Assume that there exists a component $i \in \mathbb{C}(2)$ such that $X^E_{i2t} > X^E_{q2t}$. Consider another set of order schedules $\bar{X}^E_t$ and $\bar{X}^R_t$ identical to X^E_t and X^R_t at all stages and for all components except for the following: $\bar{X}^E_{i2t} = X^E_{i2t} - \delta$, and $\bar{X}^R_{i2t} = X^R_{i2t} + \delta$, for some $0 < \delta \leqslant X^E_{i2t} - X^E_{q2t}$. First, we show that $\bar{X}^E_t$ and $\bar{X}^R_t$ are feasible for x_t. Since $\bar{X}^E_{i2t} < X^E_{i2t} \leqslant x_{i3t} + X^E_{i3t} = x_{i3t} + \bar{X}^E_{i3t}$, by definition of $\bar{X}^E_t$, then $\bar{X}^E_{i2t}$ is feasible. And,

$$\begin{aligned}\bar{X}^R_{i1t} &= X^R_{i1t} && [\text{definition of } \bar{X}^R_t]\\ &= X^R_{q1t} && [X^R_t \text{ is balanced at stage 1}]\\ &\leqslant x_{q2t} + X^E_{q2t} - X^E_{q1t} && [X^R_t \text{ is feasible for } x_t]\\ &= x_{q2t} + X^E_{q2t} && [\text{by Lemma 1}]\\ &= x_{i2t} + X^E_{q2t} && [x_t \text{ is balanced at stage 2}]\\ &\leqslant x_{i2t} + X^E_{i2t} - \delta && [\text{definition of } \delta]\\ &= x_{i2t} + \bar{X}^E_{i2t}. && [\text{definition of } \bar{X}^E_{i2t}]\\ \bar{X}^R_{i2t} &= X^R_{i2t} + \delta && [\text{definition of } \bar{X}^R_t]\\ &\leqslant x_{i3t} + X^E_{i3t} - X^E_{i2t} + \delta && [X^R_t \text{ is feasible for } x_t]\\ &= x_{i3t} + \bar{X}^E_{i3t} - \bar{X}^E_{i2t} && [\text{definition of } \bar{X}^E_t]\end{aligned}$$

Since $\bar{X}^E_t$ and $\bar{X}^R_t$ satisfy the required lower and upper bounds, they are feasible for x_t.

Next, let x_{t+1} and $\bar{x}_{t+1}$ be on-hand states generated by starting with x_t, and applying schedules (X^E_t, X^R_t) and $(\bar{X}^E_t, \bar{X}^R_t)$, respectively in period t, for some given value of $O_{t+1,\,t}$. Since $\bar{X}^R_{i1t} = X^R_{i1t}$ for every $i \in \mathbb{C}(1)$, it follows that $x_{i1,\,t+1} = \bar{x}_{i1,\,t+1}$ for every

$i \in \mathbb{C}(1)$. Since the decisions at all stages above stage 2 are identical for (X_t^E, X_t^R) and $(\bar{X}_t^E, \bar{X}_t^R)$ then $\bar{x}_{ij,t+1} = x_{ij,t+1}$ for every $j \geqslant 4$, and every $i \in \mathbb{C}(j)$. Furthermore,

$$\begin{aligned}
\bar{x}_{i2,t+1} &= x_{i2t} + \bar{X}_{i2t}^E - \bar{X}_{i1t}^E + \bar{X}_{i2t}^R - \bar{X}_{i1,t}^R \quad \text{[by (1)]} \\
&= x_{i2t} + \bar{X}_{i2t}^E + \bar{X}_{i2t}^R - \bar{X}_{i1,t}^R \quad \text{[by Lemma 1]} \\
&= x_{i2t} + (X_{i2t}^E - \delta) + (X_{i2t}^R + \delta) - X_{i1t}^R \\
&\qquad\qquad \text{[definition of } \bar{X}_t^E \text{ and } \bar{X}_t^R] \\
&= x_{i2,t+1}. \quad \text{[by (1)]} \\
\bar{x}_{i3,t+1} &= x_{i3t} + \bar{X}_{i3t}^E - \bar{X}_{i2t}^E + \bar{X}_{i3t}^R - \bar{X}_{i2,t}^R \quad \text{[by (1)]} \\
&= x_{i3t} + X_{i3t}^E - (X_{i2t}^E - \delta) + X_{i3t}^R - (X_{i2,t}^R + \delta) \\
&\qquad\qquad \text{[definition of } \bar{X}_t^E \text{ and } \bar{X}_t^R] \\
&= x_{i3,t+1}. \quad \text{[by (1)]}
\end{aligned}$$

Therefore, $\bar{x}_{t+1} = x_{t+1}$. As a result the cost difference between the two sets of order schedules becomes

$$G_t(\tilde{O}_t, x_t, X_t^E, X_t^R) - G_t(\tilde{O}_t, x_t, \bar{X}_t^E, \bar{X}_t^R) = \delta(k_{i2t}^E - k_{i2t}^R) > 0,$$

by Assumption 1. Thus, it cannot be optimal in period t for any component i to place an expedited order at stage 2 in excess of the expedited order placed by any other component at stage 2. Consequently, the optimal expedited order schedule for X_t^E must be balanced through stage 2.

Assume inductively that the lemma holds for some $j-1$, so that, if x_t is balanced through stage $j-1$, the optimal expedited order schedule X_t^E is also balanced through $j-1$. Let x_t be balanced through stage j. Since that implies that x_t is balanced through stage $j-1$, it follows from the inductive assumption that X_t^E is also balanced through stage $j-1$. Thus we only need to show that X_t^E is balanced *at* stage j.

Let $s = s(j, X_t^E)$ be a component in $\mathbb{C}(j)$ such that $X_{sjt}^E = \min_{i\in\mathbb{C}(j)} X_{ijt}^E$. Thus, component s is the component with the smallest expedited order into stage j in period t. If $X_{ijt}^E = X_{sjt}^E$ for every i, then X_t^E is balanced at stage j. Assume there exists a component $i \in \mathbb{C}(j)$ such that $X_{ijt}^E > X_{sjt}^E$.

Let schedules $\bar{X}_t^E$ and $\bar{X}_t^R$ identical to X_t^E and X_t^R except for the following: $\bar{X}_{ijt}^E = X_{ijt}^E - \delta$, and $\bar{X}_{ijt}^R = X_{ijt}^R + \delta$, for some $0 < \delta \leqslant X_{ijt}^E - X_{sjt}^E$. To show that $\bar{X}_t^E$ and $\bar{X}_t^R$ are feasible for x_t, we follow steps identical to those given earlier in the proof, when showing feasibility for stage 2-modified policies. To avoid repetition, we omit the details except to show that $\bar{X}_{i,j-1,t}^R$ and $\bar{X}_{i,j-1,t}^E$ are feasible. First we have

$$\begin{aligned}
\bar{X}_{i,j-1,t}^R &= X_{i,j-1,t}^R \quad \text{[definition of } \bar{X}_t^R] \\
&= X_{s,j-1,t}^R \quad [X_t^R \text{ is balanced at stage } j-1] \\
&\leqslant x_{sjt} + X_{sjt}^E - X_{s,j-1,t}^E \quad [X_t^R \text{ is feasible for } x_t] \\
&= x_{ijt} + X_{sjt}^E - X_{s,j-1,t}^E \quad [x_t \text{is balanced at stage } j] \\
&\leqslant x_{ijt} + X_{ijt}^E - \delta - X_{s,j-1,t}^E \quad \text{[definition of } \delta] \\
&= x_{ijt} + X_{ijt}^E - \delta - X_{i,j-1,t}^E \quad [X_t^E \text{ is balanced at stage } j-1] \\
&= x_{ijt} + \bar{X}_{ijt}^E - \bar{X}_{i,j-1,t}^E. \quad \text{[definition of } \bar{X}_t^E]
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\bar{X}_{i,j-1,t}^E &= X_{i,j-1,t}^E \quad \text{[definition of } \bar{X}_t^E] \\
&= X_{s,j-1,t}^R \quad [X_t^E \text{ is balanced at stage } j-1] \\
&\leqslant x_{sjt} + X_{sjt}^E \quad [X_t^E \text{ is feasible for } x_t] \\
&= x_{ijt} + X_{sjt}^E \quad [x_t \text{ is balanced at stage j}] \\
&\leqslant x_{ijt} + X_{ijt}^E - \delta \quad \text{[definition of } \delta] \\
&= x_{ijt} + \bar{X}_{ijt}^E. \quad \text{[definition of } \bar{X}_t^E]
\end{aligned}$$

Since $\bar{X}_t^E$ and $\bar{X}_t^R$ satisfy the required lower and upper bounds, they are feasible for x_t. Let x_{t+1} and $\bar{x}_{t+1}$ be on-hand states generated by starting with x_t, and applying schedules (X_t^E, X_t^R) and $(\bar{X}_t^E, \bar{X}_t^R)$, respectively. We follow the same steps used earlier in the proof (with stage 2-modified policies) to show that $\bar{x}_{t+1} = x_{t+1}$. (Note that this implies that $\bar{X}_{t+1}^E = X_{t+1}^E$ is feasible at every stage j in period $t+1$).

The cost difference becomes $G_t(\tilde{O}_t, x_t, X_t^E, X_t^R) - G_t(\tilde{O}_t, x_t, \bar{X}_t^E, \bar{X}_t^R) = \delta(k_{ijt}^E - k_{ijt}^R)$, which is positive by Assumption 1. Thus, it cannot be optimal in period t for any component i to place an expedited order at stage j in excess of the expedited order placed by any other component at stage j. Consequently, the optimal expedited order schedule X_t^E must be balanced at stage j. Therefore, if x_t is balanced through any stage j, then so is X_t^E, which completes the proof.

Proof of Theorem 1. We use induction within induction to prove that the optimal schedule X_t^E and X_t^R and the echelon state x_t are balanced through stage j, for every $j \leqslant L_n$ and in every period t. Since x_t is balanced at stage 1 for all t by Lemma 2, we start with stage 2. By Assumption 4, x_t is balanced in period 1. Assume inductively that x_t is balanced in every period t, $t \leqslant t'$.

Because x_t is balanced for all $t \leqslant t'$ then, by Lemma 3, the optimal expedited order schedule X_t^E is also balanced for all $t \leqslant t'$. Since, by Lemma 2, the optimal regular order schedule X_t^R is balanced at stage 1 for all periods, we only need to show that X_t^R is balanced *at* stage 2 for all $t \leqslant t'$. For that purpose, pick any period $t \leqslant t'$, and let $p = p(2, X_t^R)$ be a component in $\mathbb{C}(2)$ such that $X_{p2t}^R = \min_{i\in\mathbb{C}(2)} X_{i2t}^R$. Thus, component p is the component with the smallest regular order into stage 2 in period t. Suppose X_t^R is not balanced at stage 2. Thus, there exists a component i such that $X_{i2t}^R > X_{p2t}^R$.

Let $\pi_t := \{(X_t^E, X_t^R), (X_{t+1}^E, X_{t+1}^R), \ldots, (X_T^E, X_T^R)\}$ be an optimal policy for periods t through T. Consider another policy $\bar{\pi}_t$ completely identical to π_t except for the following: $\bar{X}_{i2t}^R = X_{i2t}^R - \delta$ and $\bar{X}_{i2,t+1}^R = X_{i2,t+1}^R + \delta$ for some $0 < \delta \leqslant X_{i2t}^R - X_{p2t}^R$. Let x_{t+1} and $\bar{x}_{t+1}$ be on-hand states generated by starting with x_t, and applying schedules X_t^R and $\bar{X}_t^R$, respectively. Similarly, let x_{t+2} and $\bar{x}_{t+2}$ be on-hand states generated by starting with x_{t+1} and $\bar{x}_{t+1}$, respectively, and applying schedules X_{t+1}^R and $\bar{X}_{t+1}^R$. We have

$$\begin{aligned}
\bar{x}_{i2,t+1} &= x_{i2t} + \bar{X}_{i2t}^E + \bar{X}_{i2t}^R - \bar{X}_{i1t}^R \\
&= x_{i2t} + X_{i2t}^E + X_{i2t}^R - \delta - X_{i1t}^R = x_{i2,t+1} - \delta. \quad (23)
\end{aligned}$$

$$\begin{aligned}
\bar{x}_{i3,t+1} &= x_{i3t} + \bar{X}_{i3t}^E - \bar{X}_{i2t}^E + \bar{X}_{i3t}^R - \bar{X}_{i2t}^R \\
&= x_{i3t} + X_{i3t}^E - X_{i2t}^E + X_{i3t}^R - (X_{i2t}^R - \delta) = x_{i3,t+1} + \delta. \quad (24)
\end{aligned}$$

Next, we show that $\bar{X}_{i2t}^R$ and $\bar{X}_{i2,t+1}^R$ are feasible in periods t and $t+1$.

$$\begin{aligned}
\bar{X}_{i2t}^R &< X_{i2t}^R \quad \text{[definition of } \bar{X}_{i1t}^R] \\
&\leqslant x_{i3t} + X_{i3t}^E - X_{i2t}^E \quad [X_t^R \text{ is feasible}] \\
&= x_{i3t} + \bar{X}_{i3t}^E - \bar{X}_{i2t}^E. \quad \text{[definition of } \bar{\pi}_t] \\
\bar{X}_{i2,t+1}^R &= X_{i2,t+1}^R + \delta \quad \text{[definition of } \bar{X}_{i2,t+1}^R]
\end{aligned}$$

$$\leqslant x_{i3,t+1}+X^E_{i3,t+1}-X^E_{i2,t+1}+\delta \quad [(X^E_t,X^R_t) \text{ is feasible and optimal}]$$
$$= x_{i3,t+1}+\delta+\bar{X}^E_{i3,t+1}-\bar{X}^E_{i2,t+1} \quad [\text{definition of } \bar{\pi}_t]$$
$$= \bar{x}_{i3,t+1}+\bar{X}^E_{i3,t+1}-\bar{X}^E_{i2,t+1}. \quad [\text{by (24)}]$$

Therefore, $\bar{X}^R_{i2t}$ is feasible in period t, and $\bar{X}^R_{i2,t+1}$ is feasible in period $t+1$. Finally, we get

$$\bar{x}_{i2,t+2} = \bar{x}_{i2,t+1}+\bar{X}^E_{i2,t+1}+\bar{X}^R_{i2,t+1}-\bar{X}^R_{i1,t+1} \quad [\text{by (1) and Lemma 1}]$$
$$= \bar{x}_{i2,t+1}+X^E_{i2,t+1}+X^R_{i2,t+1}+\delta-X^R_{i1,t+1} \quad [\text{definition of } (\bar{X}^E_t,\bar{X}^R_t)]$$
$$= x_{i2,t+1}-\delta+X^E_{i2,t+1}+X^R_{i2,t+1}+\delta-X^R_{i1,t+1}=x_{i2,t+2} \quad [\text{by (23)}]$$
$$\bar{x}_{i3,t+2} = \bar{x}_{i3,t+1}+\bar{X}^E_{i3,t+1}-\bar{X}^E_{i2,t+1}+\bar{X}^R_{i3,t+1}-\bar{X}^R_{i2,t+1} \quad [\text{by (1)}]$$
$$= \bar{x}_{i3,t+1}+X^E_{i3,t+1}-X^E_{i2,t+1}+X^R_{i3,t+1}-(X^R_{i2,t+1}+\delta) \quad [\text{definition of } \bar{\pi}_t]$$
$$= x_{i3,t+1}+\delta+X^E_{i3,t+1}-X^E_{i2,t+1}+X^R_{i3,t+1}-X^R_{i2,t+1}-\delta$$
$$= x_{i3,t+2}. \quad [\text{by (23)}]$$

In period $t+2$, the two policies result in the same on-hand states. The two policies are identical starting in period $t+2$, and they start out in identical states in period $t+2$; thus, they produce identical states for the rest of the time horizon. We now evaluate the difference in cost between the two policies.

Since: (i) x_t is balanced through stage 2 by assumption; (ii) X^R_t is balanced at stage 1 by Lemma 2; and (iii) $\bar{X}^R_t$ is identical to X^R_t at stage 1, we get $\gamma_t(\tilde{O}_t, \min_i(x_{i1t}+X^R_{i1t})) = \gamma_t(\tilde{O}_t, \min_i(x_{i1t}+\bar{X}^R_{i1t}))$. Thus,

$$G_t(\tilde{O}_t,x_t,\pi)-G_t(\tilde{O}_t,x_t,\bar{\pi})$$
$$=\delta(k^R_{i2t}+h_{i2t})+\alpha\mathbf{E}_{D_t\mid\tilde{O}_t}[G_{t+1}(\tilde{O}_{t+1},x_{t+1},\pi_{t+1})]$$
$$-\alpha\mathbf{E}_{D_t\mid\tilde{O}_t}[G_{t+1}(\tilde{O}_{t+1},\bar{x}_{t+1},\bar{\pi}_{t+1})].$$

Next, by definition of $\bar{X}^R_{t+1}$, we get $\bar{x}_{i1,t+1}+\bar{X}^R_{i1,t+1}=x_{i1,t+1}+X^R_{i1,t+1}$. Thus, for any given $\tilde{O}_{t+1}$, $\gamma_{t+1}(\tilde{O}_{t+1},\min_i(x_{i1,t+1}+X^R_{i1,t+1}))=\gamma_{t+1}(\tilde{O}_{t+1},\min_i(x_{i1,t+1}+\bar{X}^R_{i1,t+1}))$. As a result, we get

$$G_{t+1}(\tilde{O}_{t+1},x_{t+1},\pi_{t+1})-G_{t+1}(\tilde{O}_{t+1},\bar{x}_{t+1},\bar{\pi}_{t+1})$$
$$=-\delta(k^R_{i2,t+1}+h_{i2,t+1})+\delta(H_{i2,t+1}-H_{i3,t+1})$$
$$+\alpha\mathbf{E}_{D_{t+1}\mid\tilde{O}_{t+1}}[G_{t+2}(\tilde{O}_{t+2},x_{t+2},\pi_{t+2})]$$
$$-\alpha\mathbf{E}_{D_{t+1}\mid\tilde{O}_{t+1}}[G_{t+2}(\tilde{O}_{t+2},\bar{x}_{t+2},\bar{\pi}_{t+2})]. \tag{25}$$

Since $x_{t+2}=\bar{x}_{t+2}$ and $\pi_{t+2}=\bar{\pi}_{t+2}$, then (25) reduces to $-\delta k^R_{i2,t+1}$. Substituting into the above expression for $G_t(\tilde{O}_t,x_t,\pi)-G_t(\tilde{O}_t,x_t,\bar{\pi})$, we get $G_t(\tilde{O}_t,x_t,\pi)-G_t(\tilde{O}_t,x_t,\bar{\pi})=\delta(k^R_{i2t}+h_{i2t}-\alpha k^R_{i2,t+1})>0$, by Assumption 3. Thus, it cannot be optimal in any period t for any component i to place a regular order at stage 2 in excess of the regular order placed by any other component at stage 2. Thus, if x_t is balanced through stage 2, then it is optimal for X^R_t to also be balanced through stage 2 in every period t.

Thus, since x_t is balanced through stage 2 by inductive assumption, then it is optimal for X^R_t to be balanced through stage 2. Furthermore, X^E_t is then also balanced through stage 2 by Lemma 3, and therefore, by (1), x_{t+1} is balanced through stage 2. it follows that x_t is balanced through stage 2 for all $t\leqslant t'+1$, and therefore our result holds for $j=2$: (X^E_t,X^R_t) and x_t are balanced through stage 2 in every period t.

Assume inductively that our result holds for some stage $j-1<L_n$, so that X^R_t, X^E_t are balanced through stage $j-1$ in every period t. Since x_t is balanced by inductive assumption for all $t<t'$ then, by Lemma 3, X^E_t is balanced through *all stages* for $t<t'$ (and through stage $j-1$ for all periods, by inductive assumption). Since, by inductive assumption, X^R_t is balanced through stage $j-1$ for all periods, it suffices to show that X^R_t is balanced *at* stage j all $t<t'$. Our proof proceeds by establishing that, under the optimal policy (X^E_t,X^R_t) in each period t and at each stage j:

$$X^R_{qjt}+X^E_{qj,t+1}=X^R_{sjt}+X^E_{sj,t+1}, \tag{26}$$

for any two components q and s in $\mathbb{C}(j)$. In proving that we proceed as follows.

Fix j, consider any period $t\leqslant t'$ and let $u=u(j,X^R_t)$ be a component in $\mathbb{C}(j)$ such that $X^R_{ujt}=\min_{i\in\mathbb{C}(j)}[X^R_{ijt}+X^E_{ij,t+1}]$. Component u is, therefore, a component with the smallest sum of the regular order in period t and expedited order in period $t+1$ into stage j. Suppose there exists a component i such that

$$X^R_{ijt}+X^E_{ij,t+1}>X^R_{ujt}+X^E_{uj,t+1}. \tag{27}$$

Now, consider any δ such that $0<\delta\leqslant X^R_{ijt}+X^E_{ij,t+1}-X^R_{ujt}+X^E_{uj,t+1}$. In what follows, let $\pi_t:=\{(X^E_t,X^R_t),(X^E_{t+1},X^R_{t+1}),\ldots,(X^E_T,X^R_T)\}$ be an optimal policy for periods t through T. We now distinguish three, mutually exclusive, cases.

Case 1: $X^R_{ijt}-X^R_{ujt}\geqslant\delta$ and $X^E_{ij,t+1}\leqslant X^E_{uj,t+1}$.

Consider another policy $\bar{\pi}_t$ identical to π_t except for the following: $\bar{X}^R_{ijt}=X^R_{ijt}-\delta$ and $\bar{X}^R_{ij,t+1}=X^R_{ij,t+1}+\delta$. Let x_{t+1} and $\bar{x}_{t+1}$ be the on-hand states generated by starting with x_t, and applying order schedules (X^E_t,X^R_t) and $(\bar{X}^E_t,\bar{X}^R_t)$, respectively, in period t. Let x_{t+2} and $\bar{x}_{t+2}$ be generated by starting with x_{t+1} and $\bar{x}_{t+1}$, respectively, and applying order schedules (X^E_{t+1},X^R_{t+1}) and $(\bar{X}^E_{t+1},\bar{X}^R_{t+1})$ in period $t+1$.

We first prove that $\bar{\pi}_t$ is feasible. By using the same steps as earlier in this proof, it follows that $\bar{X}^R_{ijt}$ and $\bar{X}^R_{ij,t+1}$ are feasible in periods t and $t+1$ (we omit the details). We also get that: (i) $\bar{x}_{ij,t+1}=x_{ij,t+1}-\delta$; (ii) $\bar{x}_{i3,t+1}=x_{i3,t+1}+\delta$; and (iii) $\bar{x}_{t+2}=x_{t+2}$. Thus, in period $t+2$, the two policies result in the same on-hand states. The two policies start in identical states in period $t+2$, and make identical decision in period $t+2$ (and thereafter); thus, they produce identical states for the rest of the time horizon.

Next we show that $\bar{X}^E_{i,j-1,t+1}$ is feasible for $\bar{x}_{ij,t+1}=x_{ij,t+1}-\delta$, which will imply that that $\bar{X}^E_{t+1}=X^E_{t+1}$ is feasible at every stage j for $\bar{x}_{t+1}$. By Lemma 3 and the inductive assumption, we get the following:

$$\bar{X}^E_{i,j-1,t+1} = X^E_{i,j-1,t+1} \quad [\text{definition of } \bar{\pi}_t]$$
$$= X^E_{u,j-1,t+1} \quad [X^E_{t'} \text{ is balanced through stage } j-1 \text{ for all } t']$$
$$\leqslant x_{uj,t+1}+X^E_{uj,t+1} \quad [X^E_{t+1} \text{ is feasible}]$$
$$= x_{uj,t}+X^E_{ujt}-X^E_{u,j-1,t}-X^R_{u,j-1,t}+X^R_{ujt}+X^E_{uj,t+1} \quad [\text{by (1)}]$$
$$= x_{ijt}+X^E_{ujt}-X^E_{u,j-1,t}-X^R_{u,j-1,t}+X^R_{ujt}+X^E_{uj,t+1} \quad [x_t \text{ is balanced}]$$

$$= x_{ijt} + X^E_{ijt} - X^E_{i,j-1,t} - X^R_{u,j-1,t} + X^R_{ujt} + X^E_{ij,t+1} \quad [X^E_t \text{ is balanced in period } t]$$
$$= x_{ijt} + X^E_{ijt} - X^E_{i,j-1,t} - X^R_{i,j-1,t} + X^R_{ujt} + X^E_{uj,t+1} \quad [X^R_t \text{ is balanced through stage } j-1]$$
$$\leqslant x_{ijt} + X^E_{ijt} - X^E_{i,j-1,t} - X^R_{i,j-1,t} + X^R_{ijt} + X^E_{ij,t+1} - \delta \quad [\text{definition of } \delta]$$
$$= x_{ij,t+1} - \delta + X^E_{ij,t+1} \quad [\text{by (1)}]$$
$$= \bar{x}_{ij,t+1} + X^E_{ij,t+1} \quad [\bar{x}_{ij,t+1} = x_{ij,t+1} - \delta]$$
$$= \bar{x}_{ij,t+1} + \bar{X}^E_{ij,t+1}. \quad [\text{definition of } \bar{\pi}_t]$$

Thus, $\bar{X}^E_{i,j-1,t+1}$ is feasible for $\bar{x}_{ij,t+1}$. Using identical steps, it can be shown that $\bar{X}^R_{i,j-1,t+1}$ is also feasible for $\bar{x}_{ij,t+1}$. Since $\bar{X}^E_t$, $\bar{X}^E_{t+1}$, $\bar{X}^R_t$ and $\bar{X}^R_{t+1}$ satisfy the required lower and upper bounds, they are feasible for x_t, and thus so is $\bar{\pi}_t$. We can now evaluate the cost difference between π_t and $\bar{\pi}_t$.

$$\begin{aligned} &G_t(\tilde{O}_t, x_t, \pi_t) - G_t(\tilde{O}_t, x_t, \bar{\pi}_t) \\ &= \delta(k^R_{ijt} + h_{ijt}) + \alpha \mathbf{E}_{D_t \mid \tilde{O}_t}[G_{t+1}(\tilde{O}_{t+1}, x_{t+1}, \pi_{t+1})] \\ &\quad - \alpha \mathbf{E}_{D_t \mid \tilde{O}_t}[G_{t+1}(\tilde{O}_{t+1}, \bar{x}_{t+1}, \bar{\pi}_{t+1})] \\ &= \delta(k^R_{ijt} + h_{ijt}) \\ &\quad + \alpha \mathbf{E}_{D_t \mid \tilde{O}_t}\{-\delta(k^R_{ij,t+1} + h_{ij,t+1}) + \delta(H_{ij,t+1} - H_{i,j+1,t+1}) \\ &\qquad + \alpha \mathbf{E}_{D_{t+1} \mid \tilde{O}_{t+1}}[G_{t+2}(\tilde{O}_{t+2}, x_{t+2}, \pi_{t+2}) \\ &\qquad - G_{t+2}(\tilde{O}_{t+2}, \bar{x}_{t+2}, \bar{\pi}_{t+2})]\}. \end{aligned}$$

Since $x_{t+2} = \bar{x}_{t+2}$ and $\pi_{t+2} = \bar{\pi}_{t+2}$, then $G_{t+2}(\tilde{O}_{t+2}, x_{t+2}, \pi_{t+2}) = G_{t+2}(\tilde{O}_{t+2}, \bar{x}_{t+2}, \bar{\pi}_{t+2})$, and the above reduces to $G_t(\tilde{O}_t, x_t, \pi) - G_t(\tilde{O}_t, x_t, \bar{\pi}) = \delta(k^R_{ijt} + h_{ijt} - \alpha k^R_{ij,t+1})$, which is strictly positive (Assumption 3). Note, parenthetically, that the result $X^R_{ijt} = X^R_{ujt}$, if $X^E_{ij,t+1} \leqslant X^E_{uj,t+1}$, implies that $X^E_{ij,t+1} = X^E_{uj,t+1}$, so that (26) holds for every $i \in \mathbb{C}(j)$ by definition of u.

Consequently, if $X^E_{ij,t+1} \leqslant X^E_{uj,t+1}$, it cannot be optimal in period t for any component i to place a regular order at stage j in excess of the regular order placed by any other component at stage j, and so $X^R_{ijt} = X^R_{ujt}$ for every component $i \in \mathbb{C}(j)$. Thus, if x_t is balanced through stage j, then, in this Case 1, it is optimal for X^R_t to also be balanced through stage j. Therefore, because X^E_t is balanced in period t by Lemma 3, it follows from state transitions in (1) that x_{t+1} is also balanced through stage j.

Case 2: $X^R_{ijt} \leqslant X^R_{ujt}$ and $X^E_{ij,t+1} - X^E_{uj,t+1} \geqslant \delta$.

Consider another policy $\bar{\pi}_{t+1}$ identical to π_{t+1} with $\bar{X}^E_{t+1}$ and $\bar{X}^R_{t+1}$ identical to X^E_{t+1} and X^R_{t+1} at all stages and for all components except for the following: $\bar{X}^E_{ij,t+1} = X^E_{ij,t+1} - \delta$, and $\bar{X}^R_{ij,t+1} = X^R_{ij,t+1} + \delta$. Using steps identical to those in the proof of Lemma 3, it can be shown that $\bar{X}^E_{t+1}$ and $\bar{X}^R_{t+1}$ are feasible for x_{t+1}. Let x_{t+2} and $\bar{x}_{t+2}$ be the on-hand states generated by starting with x_{t+1}, and applying order schedules (X^E_{t+1}, X^R_{t+1}) and $(\bar{X}^E_{t+1}, \bar{X}^R_{t+1})$, respectively, in period $t+1$.

Following again the steps from the proof of Lemma 3, the cost difference between the two policies becomes $G_{t+1}(\tilde{O}_{t+1}, x_{t+1}, \pi_{t+1}) - G_{t+1}(\tilde{O}_{t+1}, x_{t+1}, \bar{\pi}_{t+1}) = \delta(k^E_{ij,t+1} - k^R_{ij,t+1})$, which is positive by Assumption 1. Thus, it cannot be optimal in period $t+1$ for any component i to place an expedited order at stage j in excess of the expedited order placed by any other component at stage j. Hence, $X^E_{ij,t+1} = X^E_{uj,t+1}$ for each j and every $i \in \mathbb{C}(j)$. Since $X^R_{ijt} \leqslant X^R_{ujt}$ by assumption, and $X^E_{ij,t+1} X^E_{uj,t+1}$ as just obtained, it follows from the definition of component u that $X^R_{ijt} = X^R_{ujt}$ for every $i \in \mathbb{C}(j)$. Consequently, the optimal expedited order schedule X^R_t must be balanced through stage j in period t. Thus, if x_t is balanced through stage j, then, in this Case 2, it is optimal for X^R_t to also be balanced through stage j. Because X^E_t is balanced in period t by Lemma 3, it follows from (1) that x_{t+1} is also balanced through stage j.

Case 3: $0 < X^R_{ijt} - X^R_{ujt} < \delta$ and $0 < X^E_{ij,t+1} - X^E_{uj,t+1} < \delta$.

In this case, it follows from (27) that there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $\delta_1 + \delta_2 = \delta$ and

$$X^R_{ijt} - X^R_{ujt} \geqslant \delta_1 \quad \text{and} \quad X^E_{ij,t+1} - X^E_{uj,t+1} \geqslant \delta_2. \tag{28}$$

We now define $\bar{\pi}_t$ to be identical to π_t except for: $\bar{X}^R_{ijt} = X^R_{ijt} - \delta_1$, $\bar{X}^E_{ij,t+1} = X^E_{ij,t+1} - \delta_2$, and $\bar{X}^R_{ij,t+1} = X^R_{ij,t+1} + \delta_1 + \delta_2$. We follow the usual definitions of x_{t+1} and $\bar{x}_{t+1}$, x_{t+2} and $\bar{x}_{t+2}$. We first get

$$\begin{aligned} \bar{x}_{ij,t+1} &= x_{ijt} + \bar{X}^E_{ijt} + \bar{X}^R_{ijt} - \bar{X}^E_{i,j-1,t} - \bar{X}^R_{i,j-1,t} \\ &= x_{ijt} + X^E_{ijt} + X^R_{ijt} - \delta_1 - X^E_{i,j-1,t} - X^R_{i,j-1,t} \\ &= x_{ij,t+1} - \delta_1. \end{aligned} \tag{29}$$

$$\begin{aligned} \bar{x}_{i,j+1,t+1} &= x_{i,j+1,t} + \bar{X}^E_{i,j+1,t} - \bar{X}^E_{ijt} + \bar{X}^R_{i,j+1,t} - \bar{X}^R_{ijt} \\ &= x_{i,j+1,t} + X^E_{i,j+1,t} - X^E_{ijt} + X^R_{i,j+1,t} - (X^R_{ijt} - \delta_1) \\ &= x_{i,j+1,t+1} + \delta_1. \end{aligned} \tag{30}$$

We first establish that $\bar{\pi}_t$ is feasible for $\bar{x}_t$ and $\bar{x}_{t+1}$. Using familiar steps from earlier in the proof, we find that $\bar{X}^R_{ijt} \leqslant x_{i,j+1,t} + \bar{X}^E_{i,j+1,t} - \bar{X}^E_{ijt}$ and thus $\bar{X}^R_{ijt}$ is feasible for x_t. Next, we have

$$\bar{X}^R_{ij,t+1} = X^R_{ij,t+1} + (\delta_1 + \delta_2) \quad [\text{definition of } \bar{\pi}_t]$$
$$\leqslant x_{i,j+1,t+1} + X^E_{i,j+1,t+1} - X^E_{ij,t+1} + (\delta_1 + \delta_2) \quad [X^R_t \text{ is feasible}]$$
$$= x_{i,j+1,t+1} + \bar{X}^E_{i,j+1,t+1} - (\bar{X}^E_{ij,t+1} + \delta_2) + (\delta_1 + \delta_2) \quad [\text{definition of } \bar{\pi}_t]$$
$$= \bar{x}_{i,j+1,t+1} + \bar{X}^E_{i,j+1,t+1} - \bar{X}^E_{ij,t+1}. \quad [\text{by (30)}]$$

Thus, $\bar{X}^R_{ij,t+1}$ is feasible for $\bar{x}_{t+1}$. Finally, we have

$$\bar{X}^E_{i,j-1,t+1} = X^E_{i,j-1,t+1} \quad [\text{definition of } \bar{\pi}_t]$$
$$= X^E_{u,j-1,t+1} \quad [X^E_{t'} \text{ is balanced for all periods } t']$$
$$\leqslant x_{uj,t+1} + X^E_{uj,t+1} \quad [X^E_t \text{ is feasible}]$$
$$= x_{ujt} + X^E_{ujt} - X^E_{u,j-1,t} + X^R_{ujt} - X^R_{u,j-1,t} + X^E_{uj,t+1} \quad [\text{by (1)}]$$
$$= x_{ijt} + X^E_{ijt} - X^E_{i,j-1,t} - X^R_{u,j-1,t} + X^R_{ujt} + X^E_{uj,t+1} \quad [x_t \text{ and } X^E_t \text{ are balanced}]$$
$$= x_{ijt} + X^E_{ijt} - X^E_{i,j-1,t} - X^R_{i,j-1,t} + X^R_{ujt} + X^E_{uj,t+1} \quad [X^R_t \text{ is balanced through stage } j]$$
$$\leqslant x_{ijt} + X^E_{ijt} - X^E_{i,j-1,t} - X^R_{i,j-1,t} + X^R_{ijt} + X^E_{ij,t+1} - \delta \quad [\text{definition of } \delta]$$

$$= x_{ij,\,t+1} + X^E_{ij,\,t+1} - \delta \quad \text{[by (1)]}$$
$$= x_{ij,\,t+1} + \bar{X}^E_{ij,\,t+1} + \delta_2 - \delta \quad \text{[definition of } \bar{\pi}_t\text{]}$$
$$= x_{ij,\,t+1} + \delta_1 + \bar{X}^E_{ij,\,t+1} \quad [\delta = \delta_1 + \delta_2]$$
$$= \bar{x}_{ij,\,t+1} + \bar{X}^E_{ij,\,t+1}. \quad \text{[by (30)]}$$

Consequently, $\bar{X}^E_{i,\,j-1,\,t+1}$ is feasible for $\bar{x}_{t+1}$. Identical steps demonstrate that $\bar{X}^R_{i,\,j-1,\,t+1}$ is also feasible for $\bar{x}_{t+1}$. Thus, $\bar{\pi}_t$ is feasible. Next, we evaluate the resulting states in period $t+2$.

$$\bar{x}_{ij,\,t+2} = \bar{x}_{ij,\,t+1} + \bar{X}^E_{ij,\,t+1} + \bar{X}^R_{ij,\,t+1} - \bar{X}^E_{i,\,j-1,\,t+1} - \bar{X}^R_{i,\,j-1,\,t+1} \quad \text{[by (1)]}$$
$$= \bar{x}_{ij,\,t+1} + X^E_{ij,\,t+1} - \delta_2 + X^R_{ij,\,t+1} + \delta - X^E_{i,\,j-1,\,t+1} - X^R_{i,\,j-1,\,t+1} \quad \text{[definition of } \bar{\pi}_t\text{]}$$
$$= x_{ij,\,t+1} - \delta_1 + X^E_{i2,\,t+1} + X^R_{i2,\,t+1} - X^R_{i1,\,t+1} + (\delta - \delta_2) \quad \text{[by (29)]}$$
$$= x_{ij,\,t+2}; \quad [\delta = \delta_1 + \delta_2]$$
$$\bar{x}_{i,\,j+1,\,t+2} = \bar{x}_{i,\,j+1,\,t+1} + \bar{X}^E_{i,\,j+1,\,t+1} + \bar{X}^R_{i,\,j+1,\,t+1} - \bar{X}^E_{ij,\,t+1} - \bar{X}^R_{ij,\,t+1} \quad \text{[by (1)]}$$
$$= \bar{x}_{i,\,j+1,\,t+1} + X^E_{i,\,j+1,\,t+1} + X^R_{i,\,j+1,\,t+1} - (X^E_{ij,\,t+1} - \delta_2) - (X^R_{ij,\,t+1} + \delta) \quad \text{[definition of } \bar{\pi}_t\text{]}$$
$$= x_{i,\,j+1,\,t+1} + \delta_1 + X^E_{i,\,j+1,\,t+1} + X^R_{i,\,j+1,\,t+1} - X^E_{ij,\,t+1} - X^R_{ij,\,t+1} - (\delta - \delta_2) \quad \text{[by (30)]}$$
$$= x_{i,\,j+1,\,t+2}. \quad [\delta = \delta_1 + \delta_2]$$

Consequently, the resulting states from the policies π_t and $\bar{\pi}_t$ are in period $t+2$ are identical. As a result we get the following difference in cost between the two policies:

$$\begin{aligned}
&G_t(\tilde{O}_t, x_t, \pi_t) - G_t(\tilde{O}_t, x_t, \bar{\pi}_t)\\
&= \delta_1(k^R_{ijt} + h_{ijt}) + \alpha \mathbf{E}_{D_t \mid \tilde{O}_t}[G_{t+1}(\tilde{O}_{t+1}, x_{t+1}, \pi_{t+1})]\\
&\quad - \alpha \mathbf{E}_{D_t \mid \tilde{O}_t}[G_{t+1}(\tilde{O}_{t+1}, \bar{x}_{t+1}, \bar{\pi}_{t+1})]\\
&= \delta_1(k^R_{ijt} + h_{ijt}) + \alpha \mathbf{E}_{D_t \mid \tilde{O}_t}\{\delta_1(H_{ij,\,t+1} - H_{i,\,j+1,\,t+1})\\
&\quad - \delta(k^R_{ij,\,t+1} + h_{ij,\,t+1}) + \delta_2(k^E_{ij,\,t+1} + h_{ij,\,t+1})\\
&\quad + \alpha \mathbf{E}_{D_{t+1} \mid \tilde{O}_{t+1}}[G_{t+2}(\tilde{O}_{t+2}, x_{t+2}, \pi_{t+2})\\
&\quad - G_{t+2}(\tilde{O}_{t+2}, \bar{x}_{t+2}, \bar{\pi}_{t+2})]\}.
\end{aligned}$$

Since $x_{t+2} = \bar{x}_{t+2}$ and $\pi_{t+2} = \bar{\pi}_{t+2}$, then $G_{t+2}(\tilde{O}_{t+2}, x_{t+2}, \pi_{t+2}) = G_{t+2}(\tilde{O}_{t+2}, \bar{x}_{t+2}, \bar{\pi}_{t+2})$, and the above expression reduces to $G_t(\tilde{O}_t, x_t, \pi) - G_t(\tilde{O}_t, x_t, \bar{\pi}) = \delta_1(k^R_{ijt} + h_{ijt} - \alpha k^R_{ij,\,t+1})$, which is strictly positive by Assumption 3. Because $\bar{\pi}_t$ is feasible and results in a lower cost then $\bar{\pi}_t$, it follows that the optimal policy must be such that $X^R_{ijt} = X^R_{ujt}$ and $X^E_{ij,\,t+1} = X^E_{uj,\,t+1}$ at each stage j and for every $i \in \mathbb{C}(j)$. Because both X^R_t and X^E_t are balanced in period t, it follows from (1) that x_{t+1} is also balanced through stage j.

Therefore, in all three cases that can arise if there exists a component i at stage j such that $X^R_{ijt} + X^E_{ij,\,t+1} > X^R_{ujt} + X^E_{uj,\,t+1}$, the optimal policy is such that $X^R_{ijt} = X^R_{ujt}$ and $X^E_{ij,\,t+1} = X^E_{uj,\,t+1}$ at each stage j and for every $i \in \mathbb{C}(j)$. The only thing remaining to show is that the same is true if $X^R_{ijt} + X^E_{ij,\,t+1} = X^R_{ujt} + X^E_{uj,\,t+1}$, for all $i \in \mathbb{C}(j)$. In that case, we again define $\bar{\pi}_t$ as identical to π_t except for the following: $\bar{X}^R_{ijt} = X^R_{ijt} - \delta$ and $\bar{X}^R_{ij,\,t+1} = X^R_{ij,\,t+1} + \delta$. The proof then proceeds by means of exactly the same steps as proof for Case 1 above. Consequently, optimal order schedules (X^E_t, X^R_t) are balanced in every period $t = 1, \ldots, T$. It follows that the on-hand inventory state x_t is balanced in every period $t = 1, \ldots, T+1$.

Proof of Theorem 2. The total cost allocated to component i at stage j in period t is $\theta_{ijt}(X^R_{ijt} + X^E_{ijt})$. Let $\hat{k}^E_{ijt} := k^E_{ijt} + \theta_{ij}$, and $\hat{k}^R_{ijt} := k^R_{ijt} + \theta_{ij}$. Then, Assumption 1 is satisfied for $\hat{k}^E_{ijt}$ and $\hat{k}^R_{ijt}$ directly. Furthermore, since $\theta_{ijt} \geqslant \alpha \theta_{ij,\,t+1}$ for all t by the assumption of the theorem, then Assumption 3 also holds for $\hat{k}^E_{ijt}$ and $\hat{k}^R_{ijt}$ (given that h_{ijt} remains unchanged). Consequently, by Theorem 1, the optimal policy is balanced in each period t and for every $\tilde{O}_t$, and the resulting state x_{t+1} in period $t+1$ is balanced.

Lemma 13 (Karush 1958). *If f is an arbitrary smooth and convex function on $\mathbb{R}$, then, given $x \leqslant y$, $\min_{x \leqslant \theta \leqslant y} f(\theta)$ can be expressed as $f^+(x) + f^-(y)$, where f^+ is smooth and convex increasing, and f^- is smooth and convex decreasing. In particular, if f has a finite unconstrained minimizer S, then*

$$f^+(x) := \begin{cases} f(S) & \text{if } x \leqslant S;\\ f(x) & \text{otherwise;}\end{cases} \quad \text{and}$$
$$f^-(y) := \begin{cases} f(y) - f(S) & \text{if } y \leqslant S;\\ 0 & \text{otherwise.}\end{cases}$$

If f is increasing, $f^-(y) = 0$, and $f^+(x) = f(x)$; if f is decreasing, $f^+(x) = 0$, and $f^-(y) = f(y)$.

Proof of Lemma 4. Start with $m = 1$. By Lemma 13, we get

$$\min_{\substack{u_j \leqslant \zeta_j \leqslant \zeta_{j+1}\\ j=1,2,\ldots,M}} \sum_{j=1}^{M} \phi_j(\zeta_j) = \phi_1^+(u_1) + \min_{\substack{u_j \leqslant \zeta_j \leqslant \zeta_{j+1}\\ j=2,3\ldots,M}} \left[\phi_1^-(\zeta_2) + \phi_2(\zeta_2) + \sum_{j=3}^{M} \phi_j(\zeta_j)\right].$$

Since $\varphi_1 = \phi_1$, then $\varphi_1^+ = \phi_1^+$ and $\varphi_1^- = \phi_1^-$. Thus, $\varphi_2 = \phi_1^- + \phi_2$, and (9) holds for $m = 1$. Assume inductively that (9) holds for some m. Using the definition of φ_{m+1}, the RHS of (9) becomes

$$\begin{aligned}
&\sum_{j=1}^{m} \varphi_j^+(u_j) + \min_{\substack{u_j \leqslant \zeta_j \leqslant \zeta_{j+1}\\ j=m+1,\ldots,M}} \left[\varphi_{m+1}(\zeta_{m+1}) + \sum_{j=m+2}^{M} \phi_j(\zeta_j)\right]\\
&= \sum_{j=1}^{m+1} \varphi_j^+(u_j) + \min_{\substack{u_j \leqslant \zeta_j \leqslant \zeta_{j+1}\\ j=m+2,\ldots,M}} \left[\varphi_{m+1}^-(\zeta_{m+2}) + \phi_{m+2}(\zeta_{m+2}) + \sum_{j=m+3}^{M} \phi_j(\zeta_j)\right]\\
&= \sum_{j=1}^{m+1} \varphi_j^+(u_j) + \min_{\substack{u_j \zeta_j \leqslant \zeta_{j+1}\\ j=m+2,\ldots,M}} \left[\varphi_{m+2}(\zeta_{m+2}) + \sum_{j=m+3}^{M} \phi_j(\zeta_j)\right],
\end{aligned} \tag{31}$$

where the second equality above follows from Lemma 13. Thus, by (31), it follows that (9) also holds for $m+1$. This concludes the proof, since smoothness and convexity carry over directly from Lemma 13.

Proof of Lemma 5. This lemma follows directly from the assumption that $f_{t+1}(\tilde{O}_{t+1}, \cdot)$ is smooth and additively convex, and from definition of g_{jt} and U_{jt}.

Proof of Theorem 3. The theorem clearly holds for period $T+1$. Assume that $f_{t+1}(\tilde{O}_{t+1},\cdot)$, as defined in Equation (6), is smooth and additively convex for each $\tilde{O}_{t+1}$, so that there exist smooth convex functions $\{f_{1,t+1},\ldots,f_{L,t+1}\}$ such that $f_{t+1}(\tilde{O}_{t+1},z_{t+1})=\sum_{j=1}^{L} f_{j,t+1}(\tilde{O}_{t+1},z_{j,t+1})$ for each $\tilde{O}_{t+1}$. Using the definitions of g_{jt} and U_{jt} given in Lemma 5, we get

$$\begin{aligned}
&f_t(\tilde{O}_t,z_t)\\
&=-\sum_{j=1}^{L}k_j^E z_{jt}+\min_{\substack{z_{jt}\le Z_{jt}^E\le Z_{j+1,t}^E\\1\le j\le L}}\left\{\sum_{j=1}^{L}c_{jt}^E Z_{jt}^E+\sum_{j=1}^{L}\min_{\substack{Z_{jt}^E\le Z_{jt}^R\le Z_{j+1,t}^E\\1\le j\le L}}[g_{jt}(\tilde{O}_t,Z_{jt}^R)]\right\}\\
&\qquad\qquad\text{[definition of } g_{jt}(\tilde{O}_t,\cdot)]\\
&=-\sum_{j=1}^{L}k_j^E z_{jt}+\min_{\substack{z_{jt}\le Z_{jt}^E\le Z_{j+1,t}^E\\1\le j\le L}}\left\{\sum_{j=1}^{L}c_{jt}^E Z_{jt}^E+\sum_{j=1}^{L-1}[g_{jt}^+(\tilde{O}_t,Z_{jt}^E)+g_{jt}^-(\tilde{O}_t,Z_{j+1,t}^E)]\right.\\
&\qquad\left.+g_{Lt}^+(\tilde{O}_t,Z_{Lt}^E)\right\}\quad\text{[by Lemma 13]}\\
&=-\sum_{j=1}^{L}k_j^E z_{jt}+\min_{\substack{z_{jt}\le Z_{jt}^E\le Z_{j+1,t}^E\\1\le j\le L}}\left\{\sum_{j=1}^{L}U_{jt}(\tilde{O}_t,Z_{jt}^E)\right\}.\quad\text{[definition of } U_{jt}(\tilde{O}_t,\cdot)]
\end{aligned}$$

Let $V_{1t}(\tilde{O}_t,\cdot),\ldots,V_{Lt}(\tilde{O}_t,\cdot)$ be defined recursively as $V_{jt}(\tilde{O}_t,z_{jt}):=U_{jt}(\tilde{O}_t,z_{jt})+V_{j-1,t}^-(\tilde{O}_t,z_{jt})$, with $V_{0t}(\tilde{O}_t,z_{jt}):=0$. Then, $V_{1t}(\tilde{O}_t,\cdot),\ldots,V_{Lt}(\tilde{O}_t,\cdot)$ are smooth and convex, and, by Lemma 4, for any $n<L$,

$$\begin{aligned}
&\min_{\substack{z_{jt}\le Z_{jt}^E\le Z_{j+1,t}^E\\j=1,2,\ldots,L}}\sum_{j=1}^{L}U_{jt}(\tilde{O}_t,Z_{jt}^E)\\
&\quad=\sum_{j=1}^{n}V_{jt}^+(\tilde{O}_t,z_{jt})\\
&\qquad+\min_{\substack{z_{jt}\le Z_{jt}^E\le Z_{j+1,t}^E\\j=n+1,\ldots,L}}\left\{V_{n+1,t}(\tilde{O}_t,Z_{n+1,t}^E)+\sum_{j=n+2}^{L}U_{jt}(\tilde{O}_t,Z_{jt}^E)\right\},
\end{aligned}$$

where the minimization is over $Z_t^E=(Z_{1t}^E,\ldots,Z_{Lt}^E)$. Letting $n=L-1$, the above expression yields

$$\begin{aligned}
f_t(\tilde{O}_t,z_t)&=-\sum_{j=1}^{L}k_j^E z_{jt}+\sum_{j=1}^{L-1}V_{jt}^+(\tilde{O}_t,z_{jt})\\
&\quad+\min_{z_{Lt}\le Z_{Lt}^E}[V_{L-1,t}^-(\tilde{O}_t,Z_{Lt}^E)+U_{Lt}(\tilde{O}_t,Z_{Lt}^E)]\\
&=-\sum_{j=1}^{L}k_j^E z_{jt}+\sum_{j=1}^{L-1}V_{jt}^+(\tilde{O}_t,z_{jt})\\
&\quad+\min_{z_{Lt}\le Z_{Lt}^E}[V_{Lt}(\tilde{O}_t,Z_{Lt}^E)]\quad\text{[definition of } V_{Lt}]\\
&=\sum_{j=1}^{L}[V_{jt}^+(\tilde{O}_t,z_{jt})-k_j^E z_{jt}].\quad\text{[Lemma 13]}
\end{aligned}$$

Proof of Theorem 4. Let $\tilde{O}_t$ in period t be given. As shown in the proof of Theorem 3, for any $n<L$,

$$\begin{aligned}
f_t(\tilde{O}_t,z_t)&=-\sum_{j=1}^{L}k_j^E z_{jt}+\sum_{j=1}^{n}V_{jt}^+(\tilde{O}_t,z_{jt})\\
&\quad+\min_{\substack{z_{jt}\le Z_{jt}^E\le Z_{j+1,t}^E\\j=n+1,\ldots,L}}\left[V_{n+1,t}(\tilde{O}_t,Z_{n+1,t}^E)+\sum_{j=n+2}^{L}U_{jt}(\tilde{O}_t,Z_{jt}^E)\right].
\end{aligned}\tag{32}$$

Let $S_{jt}^E(\tilde{O}_t):=\arg\min_Z V_{jt}(\tilde{O}_t,Z)$ be the unconstrained minimizer of $V_{jt}(\tilde{O}_t,\cdot)$ over Z for $j=1,\ldots,L$. Thus, by (32), given any $Z_{j+1,t}^E$, $\hat{Z}_{jt}^E(\tilde{O}_t,z_t)=z_{jt}\vee(S_{jt}^E(\tilde{O}_t)\wedge Z_{j+1,t}^E)$. For $n=L-1$, (32) yields $f_t(\tilde{O}_t,z_t)=-\sum_{j=1}^{L}k_j^E z_{jt}+\sum_{j=1}^{L-1}V_{jt}^+(\tilde{O}_t,z_{jt})+\min_{z_{Lt}\le Z_{Lt}^E}V_{Lt}(\tilde{O}_t,Z_{Lt}^E)$. Therefore, $\hat{Z}_{Lt}^E(\tilde{O}_t,z_t)=[z_{Lt}\vee S_{Lt}^E(\tilde{O}_t)]$, and thus part (i) holds for $j=L$. Assume inductively that (i) holds for $j+1,\ldots,L$, so that

$$\hat{Z}_{j+1,t}^E(\tilde{O}_t,z_t)=\bigwedge_{i=j+1}^{L}[z_{it}\vee S_{it}^E(\tilde{O}_t)].\tag{33}$$

It follows that

$$\begin{aligned}
\hat{Z}_{jt}^E(\tilde{O}_t,z_t)&=z_{jt}\vee[S_{jt}^E(\tilde{O}_t)\wedge\hat{Z}_{j+1,t}^E(\tilde{O}_t,z_t)]\\
&\qquad[\hat{Z}_{jt}^E(\tilde{O}_t,z_t)=z_{jt}\vee(S_{jt}^E(\tilde{O}_t)\wedge Z_{j+1,t}^E)]\\
&=[z_{jt}\vee S_{jt}^E(\tilde{O}_t)]\wedge[z_{jt}\vee\hat{Z}_{j+1,t}^E(\tilde{O}_t,z_t)]\\
&\qquad\text{[distributive property of ``}\vee\text{'']}\\
&=[z_{jt}\vee S_{jt}^E(\tilde{O}_t)]\wedge\hat{Z}_{j+1,t}^E(\tilde{O}_t,z_t)\\
&\qquad[\text{since } \hat{Z}_{j+1,t}^E(\tilde{O}_t,z_t)\ge z_{j+1,t}\ge z_{jt}]
\end{aligned}$$

By making use of the inductive hypothesis given in in (33), we then get

$$\begin{aligned}
\hat{Z}_{jt}^E(\tilde{O}_t,z_t)&=[z_{jt}\vee S_{jt}^E(\tilde{O}_t)]\wedge\bigwedge_{i=j+1}^{L}[z_{it}\vee S_{it}^E(\tilde{O}_t)]\\
&=\bigwedge_{i=j}^{L}[z_{it}\vee S_{it}^E(\tilde{O}_t)].
\end{aligned}$$

To prove part (ii), let $S_{jt}^R(\tilde{O}_t)$ be an unconstrained minimizer of $g_{j,t+1}(\tilde{O}_t,\cdot)$ for each j. The result then follows directly from the convexity of $g_{j,t+1}(\tilde{O}_t,\cdot)$, shown in Lemma 4, and the specification of $\mathbb{Z}$ in (7).

Proof of Lemma 6. Let $Y_{jt}^E=\hat{Z}_{jt}^E-z_{jt}$ and $Y_{jt}^R=\hat{Z}_{jt}^R-\hat{Z}_{jt}^E$ for each j. We need to establish that optimal schedules Y_t^E and Y_t^R in period t are such that $Y_{j+1,t}^E\le Y_{jt}^E+Y_{jt}^R$ for each j. Suppose that for some $j<L$, $Y_{j+1,t}^E>Y_{jt}^E+Y_{jt}^R$. Consider schedules $(\bar{Y}_t^E,\bar{Y}_{it}^R)$ identical to (Y_t^E,Y_t^R) except for: $\bar{Y}_{j+1,t}^E=Y_{j+1,t}^E-\delta$ and $\bar{Y}_{j+1,t}^R=Y_{j+1,t}^R+\delta$, for any $0<\delta\le Y_{j+1,t}^E-Y_{jt}^E-Y_{jt}^R$. $\bar{Y}_{j+1,t}^E$ is feasible since $\bar{Y}_{j+1,t}^E<Y_{j+1,t}^E$, and

$$\begin{aligned}
\bar{Y}_{j+1,t}^R&=Y_{j+1,t}^R+\delta\quad[\text{definition of } \bar{Y}_{j+1,t}^R]\\
&\le y_{j+2,t}+Y_{j+2,t}^E-Y_{j+1,t}^E+\delta\quad[Y_{j+1,t}^R\text{ is feasible}]\\
&=y_{j+1,t}+Y_{j+2,t}^E-\bar{Y}_{j+1,t}^E\quad[\bar{Y}_{j+1,t}^E=Y_{j+1,t}^E-\delta\text{ is feasible}]\\
&=y_{j+1,t}+\bar{Y}_{j+2,t}^E-\bar{Y}_{j+1,t}^E.\quad[\text{definition of } \bar{Y}_t^E]\\
\bar{Y}_{jt}^E&=Y_{jt}^E\quad[\text{definition of } \bar{Y}_{jt}^E]\\
&\le Y_{jt}^E+Y_{jt}^R\quad[Y_{jt}^R\ge 0]\\
&\le Y_{j+1,t}^E-\delta\quad[\text{definition of } \delta]\\
&=\bar{Y}_{j+1,t}^E\le y_{j+1,t}+\bar{Y}_{j+1,t}^E.\quad[\text{definition of } \bar{Y}_{j+1,t}^E]
\end{aligned}$$

Thus, $\bar{Y}_{j+1,t}^R$ and $\bar{Y}_{jt}^E$ are feasible. Finally,

$$\begin{aligned}
\bar{Y}_{jt}^R&=Y_{jt}^R\quad[\text{definition of } \bar{Y}_{jt}^R]\\
&\le Y_{j+1,t}^E-Y_{jt}^E-\delta\quad[\text{definition of } \delta]\\
&=\bar{Y}_{j+1,t}^E-\bar{Y}_{jt}^E\quad[\text{definitions of } \bar{Y}_{jt}^E\text{ and } \bar{Y}_{j+1,t}^E]\\
&\le y_{j+1,t}+\bar{Y}_{j+1,t}^E-\bar{Y}_{jt}^E
\end{aligned}$$

Thus, $(\bar{Y}_t^E, \bar{Y}_t^R)$ is feasible for y_t. Furthermore, the resulting state in period $t+1$ is identical under both sets of schedules at each stage j. The difference in the cost between two sets of order schedules becomes

$$\mathcal{G}_t(\tilde{O}_t, y_t, Y_t^E, Y_t^R) - \mathcal{G}_t(\tilde{O}_t, y_t, \bar{Y}_t^E, \bar{Y}_t^R)$$
$$= (k_{j+1,t}^E + h_{j+1,t})(Y_{j+1,t}^E - \bar{Y}_{j+1,t}^E) + (k_{j+1,t}^R + h_{j+1,t})(Y_{j+1,t}^R - \bar{Y}_{j+1,t}^R)$$
$$= (k_{i1t}^E - k_{i1t}^R)\delta > 0,$$

by Assumption 1. Thus, it can never be optimal to have $Y_{jt}^E > Y_{j-1,t}^E + Y_{jt}^R$. This concludes the proof.

Proof of Lemma 7. By Theorem 4, the basestock level for the regular order decision at echelon j is given by $S_{jt}^R(\tilde{O}_t) := \max \arg\min_Z g_{jt}(\tilde{O}_t, Z)$ for each $\tilde{O}_t$. Let $j = L$. Suppose that there exists some advance demand information state $\tilde{O}_t$ such that $S_{Lt}^R(\tilde{O}_t) < S_{Lt}^E(\tilde{O}_t)$. Consider another basestock level $\bar{S}_{Lt}^R(\tilde{O}_t)$ defined as $\bar{S}_{Lt}^R(\tilde{O}_t) = S_{Lt}^R(\tilde{O}_t) + \delta$ for any δ, $0 < \delta \leqslant S_{Lt}^E(\tilde{O}_t) - S_{Lt}^R(\tilde{O}_t)$. Thus, $S_{Lt}^R(\tilde{O}_t) < \bar{S}_{Lt}^R(\tilde{O}_t) \leqslant S_{Lt}^E(\tilde{O}_t)$.

Let $Z_{Lt}^R(\tilde{O}_t)$ be the optimal echelon L regular order decision obtained by $S_{Lt}^R(\tilde{O}_t)$ and $\bar{Z}_{Lt}^R(\tilde{O}_t)$ be the optimal echelon L regular order decision obtained by means of the basestock level $\bar{S}_{Lt}^R(\tilde{O}_t)$. Then, we have

$$\begin{aligned} Z_{Lt}^R(\tilde{O}_t) - \bar{Z}_{Lt}^R(\tilde{O}_t) &= Z_{Lt}^E(\tilde{O}_t) \vee S_{Lt}^R(\tilde{O}_t) - \bar{Z}_{Lt}^E(\tilde{O}_t) \vee \bar{S}_{Lt}^R(\tilde{O}_t) \quad \text{[by Theorem 4(ii)]} \\ &= z_{Lt} \vee S_{Lt}^E(\tilde{O}_t) \vee S_{Lt}^R(\tilde{O}_t) \\ &\quad - z_{Lt} \vee S_{Lt}^E(\tilde{O}_t) \vee \bar{S}_{Lt}^R(\tilde{O}_t) \quad \text{[by Theorem 4(i)]} \\ &= z_{Lt} \vee S_{Lt}^E(\tilde{O}_t) - z_{Lt} \vee S_{Lt}^E(\tilde{O}_t) = 0. \quad [S_{Lt}^R(\tilde{O}_t), \bar{S}_{Lt}^R(\tilde{O}_t) \leqslant S_{Lt}^E(\tilde{O}_t)] \end{aligned}$$

Therefore, the two regular order decisions at echelon L are identical (as are all other regular and expedited decisions made in the system). As a result, all the costs associated with those two different basestock levels are identical. Consequently, because $g_{Lt}(\tilde{O}_t, Z)$ is smooth and convex in Z for every $\tilde{O}_t$, it follows that $\bar{S}_{Lt}^R(\tilde{O}_t) = \arg\min_Z g_{jt}(\tilde{O}_t, Z)$. But, since $S_{jt}^R(\tilde{O}_t) := \max \arg\min_Z g_{jt}(\tilde{O}_t, Z)$ this implies that $S_{Lt}^R(\tilde{O}_t) \geqslant \bar{S}_{Lt}^R(\tilde{O}_t)$ which is a contradiction. It follows that $S_{Lt}^R(\tilde{O}_t) \geqslant S_{Lt}^E(\tilde{O}_t)$ must be true.

Now consider $j < L$. Assume there exists an $\tilde{O}_t$ such that $S_{jt}^R(\tilde{O}_t) < S_{jt}^E(\tilde{O}_t)$. Consider another basestock level $\bar{S}_{jt}^R(\tilde{O}_t)$ defined as $\bar{S}_{jt}^R(\tilde{O}_t) = S_{jt}^R(\tilde{O}_t) + \delta$ for any δ, $0 < \delta \leqslant S_{jt}^E(\tilde{O}_t) - S_{jt}^R(\tilde{O}_t)$. Thus, $S_{jt}^R(\tilde{O}_t) < \bar{S}_{jt}^R(\tilde{O}_t) \leqslant S_{jt}^E(\tilde{O}_t)$. Let $Z_{jt}^R(\tilde{O}_t)$ be the optimal echelon j regular order decision obtained by $S_{jt}^R(\tilde{O}_t)$ and $\bar{Z}_{jt}^R(\tilde{O}_t)$ be the optimal echelon j regular order decision obtained by $\bar{S}_{jt}^R(\tilde{O}_t)$. We have that

$$\begin{aligned} Z_{jt}^R(\tilde{O}_t, z_t) &= [\hat{Z}_{jt}^E(\tilde{O}_t) \vee S_{jt}^R(\tilde{O}_t)] \wedge \hat{Z}_{j+1,t}^E(\tilde{O}_t) \quad \text{[by Theorem 4(ii)]} \\ &= \{[(z_{jt} \vee S_{jt}^E(\tilde{O}_t)) \wedge \hat{Z}_{j+1t}^E(\tilde{O}_t)] \vee S_{jt}^R(\tilde{O}_t)\} \wedge \hat{Z}_{j+1,t}^E(\tilde{O}_t) \quad \text{[by Theorem 4(i)]} \\ &= \{[z_{jt} \vee S_{jt}^E(\tilde{O}_t) \vee S_{jt}^R(\tilde{O}_t)] \wedge [S_{jt}^R(\tilde{O}_t) \vee \hat{Z}_{j+1,t}^E(\tilde{O}_t)]\} \\ &\quad \wedge \hat{Z}_{j+1,t}^E(\tilde{O}_t) \\ &= [z_{jt} \vee S_{jt}^E(\tilde{O}_t)] \wedge \hat{Z}_{j+1,t}^E(\tilde{O}_t) \quad [S_{jt}^R(\tilde{O}_t) \leqslant S_{jt}^E(\tilde{O}_t)]. \end{aligned}$$

By identical steps, $\bar{Z}_{jt}^R(\tilde{O}_t) = [z_{jt} \vee S_{jt}^E(\tilde{O}_t)] \wedge \hat{Z}_{j+1,t}^E(\tilde{O}_t)$. Thus, $Z_{jt}^R(\tilde{O}_t) = \bar{Z}_{jt}^R(\tilde{O}_t)$. Since $g_{jt}(\tilde{O}_t, Z)$ is convex and smooth in Z, it follows that $\bar{S}_{jt}^R(\tilde{O}_t) = \arg\min_Z g_{jt}(\tilde{O}_t, Z)$. But, since $S_{jt}^R(\tilde{O}_t) := \max \arg\min_Z g_{jt}(\tilde{O}_t, Z)$ this implies $S_{jt}^R(\tilde{O}_t) \geqslant \bar{S}_{jt}^R(\tilde{O}_t)$ which is a contradiction. It follows that $S_{jt}^R(\tilde{O}_t) \geqslant S_{jt}^E(\tilde{O}_t)$ must be true.

Proof of Lemma 8. To prove (i), let $D_y f(\mathbf{x}, y) \leqslant D_y g(\mathbf{x}, y)$ for all $(\mathbf{x}, y)$. Let $s_f(\mathbf{x})$ and $s_g(\mathbf{x})$ be the largest minimizers of f and g, respectively, over y, for any $\mathbf{x}$. Suppose there exists an $\mathbf{x}$ such that $s_f(\mathbf{x}) < s_g(\mathbf{x})$. This implies $D_y g(\mathbf{x}, s_g(\mathbf{x})) = 0$. Because $f(\mathbf{x}, y)$ is convex in y, then, since $s_f(\mathbf{x})$ is the largest minimizer of $f(\mathbf{x}, y)$ and $s_f(\mathbf{x}) < s_g(\mathbf{x})$, we must have $D_y f(\mathbf{x}, s_g(\mathbf{x})) > 0$. Since $D_y g(\mathbf{x}, y) = 0$ for $y = s_g(\mathbf{x})$ by definition of $s_g(\mathbf{x})$ ($g(\mathbf{x}, \cdot)$ is convex and smooth), we get that $D_y f(\mathbf{x}, s_g(\mathbf{x})) > D_y g(\mathbf{x}, s_g(\mathbf{x}))$. That contradicts the initial assumption that $D_y f(\mathbf{x}, y) \leqslant D_y g(\mathbf{x}, y)$ for all $(\mathbf{x}, y)$.

To prove (ii), we use part (i). Since $s_f(\mathbf{x}) \geqslant s_g(\mathbf{x})$ for every $\mathbf{x}$, then by Lemma 4,

$$D_y f^+(\mathbf{x}, y) - D_y g^+(\mathbf{x}, y) = \begin{cases} 0 & \text{if } y \leqslant s_g(\mathbf{x}); \\ -D_y g(\mathbf{x}, y) & \text{if } s_g(\mathbf{x}) < y \leqslant s_f(\mathbf{x}); \\ D_y f(\mathbf{x}, y) - D_y g(\mathbf{x}, y) & \text{if } s_f(\mathbf{x}) \leqslant y. \end{cases}$$

Since $-D_y g(\mathbf{x}, y) \leqslant 0$ for $y > s_g(\mathbf{x})$, and $D_y f(\mathbf{x}, y) \leqslant D_y g(\mathbf{x}, y)$ for all $(\mathbf{x}, y)$, by assumption, we conclude that $D_y f^+(\mathbf{x}, y) \leqslant D_y g^+(\mathbf{x}, y)$ for every $(\mathbf{x}, y)$. Similarly, $D_y f^-(\mathbf{x}, y) \leqslant D_y g^-(\mathbf{x}, y)$ for every $(\mathbf{x}, y)$.

Proof of Theorem 5. We show that for every $\tilde{O}_t^2 \geqslant \tilde{O}_t^1$, $D_Z g_{jt}(\tilde{O}_t^2, Z) \leqslant D_Z g_{jt}(\tilde{O}_t^1, Z)$ and $D_Z V_{jt}(\tilde{O}_t^2, Z) \leqslant D_Z V_{jt}(\tilde{O}_t^1, Z)$. The result follows from definitions of $S_{jt}^R(\tilde{O}_t)$ and $S_{jt}^E(\tilde{O}_t)$, by Lemma 8(i).

Let $t = T$, and $\tilde{O}_T^2 \geqslant \tilde{O}_T^1$. Since $g_{jt}(\tilde{O}_t, \cdot)$ is smooth, and the salvage value function is zero, we get $D_Z g_{jT}(\tilde{O}_T^2, Z) = D_Z g_{jT}(\tilde{O}_T^1, Z)$ for every j; thus, $S_{jT}^R(\tilde{O}_T^2) = S_{jT}^R(\tilde{O}_T^1)$. Assume inductively that $D_Z g_{jt}(\tilde{O}_t^2, Z) \leqslant D_Z g_{jt}(\tilde{O}_t^1, Z)$ for $\tilde{O}_t^2 \geqslant \tilde{O}_t^1$ for some t. Then, by Lemma 8(i), $S_{jt}^R(\tilde{O}_t^2) \geqslant S_{jt}^R(\tilde{O}_t^1)$. We get

$$D_Z U_{jt}(\tilde{O}_t^2, Z) - D_Z U_{jt}(\tilde{O}_t^1, Z) = \begin{cases} D_Z g_{1,t}^+(\tilde{O}_t^2, Z) - D_Z g_{1,t}^+(\tilde{O}_t^1, Z) & \text{if } j = 1; \\ D_Z g_{jt}^+(\tilde{O}_t^2, Z) - D_Z g_{jt}^+(\tilde{O}_t^1, Z) + D_Z g_{j-1,t}^-(\tilde{O}_t^2, Z) \\ \quad - D_Z g_{j-1,t}^-(\tilde{O}_t^1, Z) & 1 < j \leqslant L. \end{cases}$$

Consequently, by Lemma 8(ii), $D_Z U_{jt}(\tilde{O}_t^2, Z) \leqslant D_Z U_{jt}(\tilde{O}_t^1, Z)$ for every j and Z, and any $\tilde{O}_t^2 \geqslant \tilde{O}_t^1$. Since, by definition, $V_{1t}(\tilde{O}_t, Z) := U_{1t}(\tilde{O}_t, Z)$, then $D_Z V_{1t}(\tilde{O}_t^2, Z) \leqslant D_Z V_{1t}(\tilde{O}_t^1, Z)$. Assume inductively that $D_Z V_{jt}(\tilde{O}_t^2, Z) \leqslant D_Z V_{jt}(\tilde{O}_t^1, Z)$ for $\tilde{O}_t^2 \geqslant \tilde{O}_t^1$ and some $j < N$. Thus, by Lemma 8(b),

$$D_Z V_{jt}^-(\tilde{O}_t^2, Z) \leqslant D_Z V_{jt}^-(\tilde{O}_t^1, Z). \tag{34}$$

By definition, we have $V_{j+1,t}(\tilde{O}_t, z_{j+1,t}) := U_{j+1,t}(\tilde{O}_t, z_{j+1,t}) + V_{jt}^-(\tilde{O}_t, z_{j+1,t})$. Thus, for any $\tilde{O}_t^2 \geqslant \tilde{O}_t^1$,

$$\begin{aligned} &D_Z V_{j+1,t}^-(\tilde{O}_t^2, Z) - D_Z V_{j+1,t}^-(\tilde{O}_t^1, Z) \\ &\quad = D_Z U_{j+1,t}^-(\tilde{O}_t^2, Z) - D_Z U_{j+1,t}^-(\tilde{O}_t^1, Z) + D_Z V_{jt}^-(\tilde{O}_t^2, Z) \\ &\qquad - D_Z V_{jt}^-(\tilde{O}_t^1, Z) \\ &\quad \leqslant D_Z V_{jt}^-(\tilde{O}_t^2, Z) - D_Z V_{jt}^-(\tilde{O}_t^1, Z) \\ &\qquad\qquad [\text{since } D_Z U_{jt}(\tilde{O}_t^2, Z) \leqslant D_Z U_{jt}(\tilde{O}_t^1, Z)] \\ &\quad \leqslant 0 \quad [\text{by (34)}]. \end{aligned}$$

Thus, $D_Z V_{jt}(\tilde{O}_t^2, Z) \leqslant D_Z V_t(\tilde{O}_t^1, Z)$. By Lemma 8(ii) and Theorem 4(i), this yields $S_{jt}^E(\tilde{O}_t^2) \geqslant S_{jt}^E(\tilde{O}_t^1)$.

It remains to show that the relation $D_Z g_{j,t-1}(\tilde{O}_{t-1}^2, Z) \leqslant D_Z g_{j,t-1}(\tilde{O}_{t-1}^1, Z)$ holds for $\tilde{O}_{t-1}^2 \geqslant \tilde{O}_{t-1}^1$ in period $t-1$. Note that, by Lemma 8(ii), $D_Z V_{jt}(\tilde{O}_t^2, Z) \leqslant D_Z V_t(\tilde{O}_t^1, Z)$ shown above implies $D_Z V_{jt}^+(\tilde{O}_t^2, Z) \leqslant D_Z V_{jt}^+(\tilde{O}_t^1, Z)$ and $D_Z V_{jt}^-(\tilde{O}_t^2, Z) \leqslant D_Z V_{jt}^-(\tilde{O}_t^1, Z)$. By Theorem 3, $f_{jt}(\tilde{O}_t, z_t) = V_{jt}^+(\tilde{O}_t, z_{jt}) - k_j^E z_{jt}$, then, for every j,

$$D_Z f_{jt}(\tilde{O}_t^2, Z) \leqslant D_Z f_{jt}(\tilde{O}_t^1, Z). \tag{35}$$

Next, we make use of (10) to construct $D_Z g_{j,t-1}(\tilde{O}_{t-1}^2, Z) \leqslant D_Z g_{j,t-1}(\tilde{O}_{t-1}^1, Z)$ for $\tilde{O}_{t-1}^2 \geqslant \tilde{O}_{t-1}^1$. Thus,

$$\begin{aligned} D_Z g_{j,t-1}(\tilde{O}_{t-1}^2, Z) &- D_Z g_{j,t-1}(\tilde{O}_{t-1}^1, Z) \\ &= \alpha \mathbf{E}[D_Z f_{jt}(\tilde{O}_t^2, Z - O_{t,t-1}^2) - D_Z f_{jt}(\tilde{O}_t^1, Z - O_{t,t-1}^1)] \\ &\leqslant \alpha \mathbf{E}[D_Z f_{jt}(\tilde{O}_t^2, Z - O_{t,t-1}^1) - D_Z f_{jt}(\tilde{O}_t^1, Z - O_{t,t-1}^1)], \end{aligned} \tag{36}$$

by convexity of $f_{jt}(\tilde{O}_t, Z)$ since $\tilde{O}_{t-1}^2 \geqslant \tilde{O}_{t-1}^1$ implies $Z - O_{t,t-1}^2 \leqslant Z - O_{t,t-1}^1$. Because $\tilde{O}_{t-1}^2 \geqslant \tilde{O}_{t-1}^1$ implies that $\tilde{O}_t^2 \geqslant \tilde{O}_t^1$, and because (35) holds for every $\tilde{O}_t^2 \geqslant \tilde{O}_t^1$, we get $D_Z g_{j,t-1}(\tilde{O}_{t-1}^2, Z) \leqslant D_Z g_{j,t-1}(\tilde{O}_{t-1}^1, Z)$.

To prove (ii), start with $k=1$. Let $\Gamma_{jt}(Z) = \gamma_t(Z)$ if $j=1$ and $\Gamma_{jt}(Z) = 0$ if $j>1$. Use (10) to get

$$\begin{aligned} D_Z g_{jt}(\tilde{O}_t, Z-\delta) &= D_Z \Gamma_{jt}(Z-\delta) + c_{jt}^R + \alpha \mathbf{E}[D_Z f_{j,t+1}(\tilde{O}_{t+1}, Z - \delta - O_{t+1,t})] \\ &\leqslant D_Z \Gamma_{jt}(Z) + c_{jt}^R + \alpha \mathbf{E}[D_Z f_{j,t+1}(\tilde{O}_{t+1}, Z - \delta - O_{t+1,t})] \\ &\qquad [\Gamma_{jt}(\cdot) \text{ is convex}] \\ &= D_Z g_{jt}(\tilde{O}_t + \delta e_1, Z). \end{aligned}$$

Thus, $D_Z g_{jt}(\tilde{O}_t, Z-\delta) \leqslant D_Z g_{jt}(\tilde{O}_t + \delta e_1, Z)$ for every Z, $j = 1, \ldots, N$ and $\delta > 0$. Since the largest minimizer of $g_{jt}(\tilde{O}_t, Z-\delta)$ is exactly $S_{jt}^R(\tilde{O}_t) + \delta$, then, by Lemma 8(i), $S_{jt}^R(\tilde{O}_t + \delta e_1) - S_{jt}^R(\tilde{O}_t) \leqslant \delta$, and part (c) holds for S_{jt}^R when $k=1$. Since $D_Z g_{jt}(\tilde{O}_t, Z-\delta) \leqslant D_Z g_{jt}(\tilde{O}_t + \delta e_1, Z)$, then, by Lemma 8(ii), $D_Z g_{jt}^+(\tilde{O}_t, Z-\delta) \leqslant D_Z g_{jt}^+(\tilde{O}_t + \delta e_1, Z)$ and $D_Z g_{jt}^-(\tilde{O}_t, Z-\delta) \leqslant D_Z g_{jt}^-(\tilde{O}_t + \delta e_1, Z)$.

Thus, by definition of $U_{jt}(\tilde{O}_t, Z)$, we get $D_Z U_{jt}(\tilde{O}_t, Z-\delta) \leqslant D_Z U_{jt}(\tilde{O}_t + \delta e_1, Z)$ for $j = 1, \ldots, N$. By Lemma 8(ii), $D_Z U_{jt}^-(\tilde{O}_t, Z-\delta) \leqslant D_Z U_{jt}^-(\tilde{O}_t + \delta e_1, Z)$ for $j = 1, \ldots, N$. Thus, $D_Z V_{1t}(\tilde{O}_t, Z-\delta) \leqslant D_Z V_{1t}(\tilde{O}_t + \delta e_1, Z)$. As shown above, $U_{jt}^-(\tilde{O}_t, Z-\delta)$ exhibits the same property, hence, by straightforward induction, and definition of $V_{jt}(\tilde{O}_t, Z)$, we get $D_Z V_{jt}(\tilde{O}_t, Z-\delta) \leqslant D_Z V_{jt}(\tilde{O}_t + \delta e_1, Z)$ for $j = 1, \ldots, N$. Thus, $D_Z V_{jt}^+(\tilde{O}_t, Z-\delta) \leqslant D_Z V_{jt}^+(\tilde{O}_t + \delta e_1, Z)$ and $D_Z V_{jt}^-(\tilde{O}_t, Z-\delta) \leqslant D_Z V_{jt}^-(\tilde{O}_t + \delta e_1, Z)$.

Since the largest minimizer of $V_{jt}(\tilde{O}_t, Z-\delta)$ is exactly $S_{jt}^E(\tilde{O}_t) + \delta$, then, by Lemma 8(i), $S_{jt}^E(\tilde{O}_t + \delta e_1) - S_{jt}^E(\tilde{O}_t) \leqslant \delta$, and part (c) holds for S_{jt}^E when $k=1$. Since $f_{jt}(\tilde{O}_t, Z-\delta) = V_{jt}^+(\tilde{O}_t, Z-\delta) - k_j^E$, then

$$D_Z f_{jt}(\tilde{O}_t, Z-\delta) \leqslant D_Z f_{jt}(\tilde{O}_t + \delta e_1, Z), \tag{37}$$

$$\begin{aligned} &D_Z g_{jt}^+(\tilde{O}_t + \delta e_1, Z) \\ &= D_Z \Gamma_{jt}(Z) + C_{jt}^R + \alpha \mathbf{E}[D_Z f_{j,t+1}(\tilde{O}_{t+1}, Z - O_{t+1,t} - \delta)] \\ &\leqslant D_Z \Gamma_{jt}(Z) + C_{jt}^R + \alpha \mathbf{E}[D_Z f_{j,t+1}(\tilde{O}_{t+1} + \delta e_1, Z - O_{t+1,t})] \quad [\text{by (37)}] \\ &= D_Z g_{jt}^+(\tilde{O}_t + \delta e_2, Z). \end{aligned} \tag{38}$$

Thus, $D_Z g_{jt}(\tilde{O}_t + \delta e_k, Z) \leqslant D_Z g_{jt}(\tilde{O}_t + \delta e_{k+1}, Z)$, is true for $k=1$ and $j = 1, \ldots, N$.

Assume inductively that $D_Z g_{jt}(\tilde{O}_t + \delta e_k, Z) \leqslant D_Z g_{jt}(\tilde{O}_t + \delta e_{k+1}, Z)$ is true for some $k < N-1$. Then, by applying the same steps already used, it follows that: (a) $D_Z U_{jt}(\tilde{O}_t + \delta e_k, Z) \leqslant D_Z U_{jt}(\tilde{O}_t + \delta e_{k+1}, Z)$; (b) $D_Z V_{jt}(\tilde{O}_t + \delta e_k, Z) \leqslant D_Z V_{jt}(\tilde{O}_t + \delta e_{k+1}, Z)$; and (c) $D_Z f_{jt}(\tilde{O}_t + \delta e_k, Z) \leqslant D_Z f_{jt}(\tilde{O}_t + \delta e_{k+1}, Z)$. Thus,

$$\begin{aligned} &D_Z g_{jt}(\tilde{O}_t + \delta e_{k+1}, Z) \\ &\quad = D_Z \Gamma_{jt}(Z) + C_{jt}^R + \alpha \mathbf{E}[D_Z f_{j,t+1}(\tilde{O}_{t+1} + \delta e_k, Z - O_{t+1,t})] \\ &\quad \leqslant D_Z \Gamma_{jt}(Z) + C_{jt}^R + \alpha \mathbf{E}[D_Z f_{j,t+1}(\tilde{O}_{t+1} + \delta e_{k+1}, Z - O_{t+1,t})] \\ &\qquad\qquad [\text{by statement } (c) \text{ above}] \\ &\quad = D_Z g_{jt}(\tilde{O}_t + \delta e_{k+2}, Z). \end{aligned}$$

Thus, $D_Z g_{jt}(\tilde{O}_t + \delta e_k, Z) \leqslant D_Z g_{jt}(\tilde{O}_t + \delta e_{k+1}, Z)$ holds for every k; by Lemma 8, $S_{jt}^R(\tilde{O}_t + \delta e_k) \geqslant S_{jt}^R(\tilde{O}_t + \delta e_{k+1})$ for $j = 1, \ldots, N$, and $k = 1, \ldots, N-1$. By identical steps, $D_Z g_{jt}(\tilde{O}_t + \delta e_k, Z) \leqslant D_Z g_{jt}(\tilde{O}_t + \delta e_{k+1}, Z)$ also implies $D_Z U_{jt}(\tilde{O}_t + \delta e_k, Z) \leqslant D_Z U_{jt}(\tilde{O}_t + \delta e_{k+1}, Z)$ and $D_Z V_{jt}(\tilde{O}_t + \delta e_k, Z) \leqslant D_Z V_{jt}(\tilde{O}_t + \delta e_{k+1}, Z)$ for $j = 1, \ldots, N$ and $k = 1, \ldots, N-1$. Thus, by Lemma 8, $S_{jt}^E(\tilde{O}_t + \delta e_k) \geqslant S_{jt}^E(\tilde{O}_t + \delta e_{k+1})$.

To prove part (iii), we make use of the inequalities already established in the proof of part (b):

$$S_{jt}^R(\tilde{O}_t + \delta e_1) - S_{jt}^R(\tilde{O}_t) \leqslant \delta; \tag{39}$$

$$S_{jt}^E(\tilde{O}_t + \delta e_1) - S_{jt}^E(\tilde{O}_t) \leqslant \delta. \tag{40}$$

Then, part (iii) of Theorem 5 follows directly from part (ii), since $S_{jt}^R(\tilde{O}_t + \delta e_j) - S_{jt}^R(\tilde{O}_t) \leqslant S_{jt}^R(\tilde{O}_t + \delta e_{j-1}) - S_{jt}^R(\tilde{O}_t) \leqslant \cdots \leqslant S_{jt}^R(\tilde{O}_t + \delta e_1) - S_{jt}^R(\tilde{O}_t) \leqslant \delta$, where the last inequality is just (39). Similarly, using (40), $S_{jt}^E(\tilde{O}_t + \delta e_j) - S_{jt}^E(\tilde{O}_t) \leqslant S_{jt}^E(\tilde{O}_t + \delta e_{j-1}) - S_{jt}^E(\tilde{O}_t) \leqslant \cdots \leqslant S_{jt}^E(\tilde{O}_t + \delta e_1) - S_{jt}^E(\tilde{O}_t) \leqslant \delta$.

Proof of Theorem 6. Part (i) follows from Theorem 4. Part (ii) follows from Theorem 5, since $S_{jt}^E(\tilde{O}_t) = S_{jt}^E(\tilde{O}_t - \delta e_k + \delta e_k) \geqslant S_{jt}^E(\tilde{O}_t - \delta e_k + \delta e_j)$, as well as $S_{jt}^R(\tilde{O}_t) = S_{jt}^R(\tilde{O}_t - \delta e_k + \delta e_k) \geqslant S_{jt}^R(\tilde{O}_t - \delta e_k + \delta e_j)$.

Proof of Lemma 9. Being zero, $f_{j,T+1}(\tilde{O}_{T+1}, z_{j,T+1})$ satisfies the Lemma for every j. Assume that $f_{j,t+1}(\tilde{O}_{t+1}, \cdot)$ satisfies the Lemma for all j and some $t+1$. Then, $g_{jt}(\tilde{O}_t, \cdot)$, defined in (10) is also independent of k_{qt}^E for every $q > j$, and so are $g_{jt}^+(\tilde{O}_t, \cdot)$ and $g_{jt}^-(\tilde{O}_t, \cdot)$. Therefore, $U_{jt}(\tilde{O}_t, \cdot)$, defined in (11), is also independent of k_{qt}^E for every $q > j$. Furthermore, since $V_{jt}(\tilde{O}_t, z_{jt}) := U_{jt}(\tilde{O}_t, z_{jt}) + V_{j-1,t}^-(\tilde{O}_t, z_{jt})$, with $V_{0t}(\tilde{O}_t, z_{jt}) := 0$, this implies that $V_{jt}(\tilde{O}_t, \cdot)$ is independent of k_{qt}^E for every $q > j$. Since $f_{jt}(\tilde{O}_t, z_t) = V_{jt}^+(\tilde{O}_t, z_{jt}) - k_j^E z_{jt}$ by Theorem 3, then $f_{jt}(\tilde{O}_t, \cdot)$ is independent of k_{qt}^E for every $q > j$.

Proof of Lemma 10. Part (i) holds for period $T+1$ because $f_{T+1} = 0$. Suppose part (i) holds for some $t+1$ so that $k_j^E - \chi_j^E \leqslant D_z f_{j,t+1}(\tilde{O}_t, z \mid \chi_j^E) - D_z f_{j,t+1}(\tilde{O}_t, z \mid k_j^E) \leqslant 0$. By definition of $g_{jt}(\tilde{O}_t, Z)$, we get

$$k_j^E - \chi_j^E \leqslant D_z g_{jt}(\tilde{O}_t, z \mid \chi_j^E) - D_z g_{jt}(\tilde{O}_t, z \mid k_j^E) \leqslant 0. \tag{41}$$

By Lemma 8, this implies that $S^R_{jt}(\tilde{O}_t \mid \chi^E_j) \geqslant S^R_{jt}(\tilde{O}_t \mid k^E_j)$. Next,

$$D_z U_{jt}(\tilde{O}_t, Z \mid \chi^E_j) - D_z U_{jt}(\tilde{O}_t, Z \mid k^E_j)$$
$$= \begin{cases} \chi^E_1 - k^E_1 + D_z g^+_{1,t}(\tilde{O}_t, Z \mid \chi^E_j) - D_z g^+_{1,t}(\tilde{O}_t, Z \mid k^E_j) & \text{if } j = 1; \\ \chi^E_j - k^E_j + D_z g^+_{jt}(\tilde{O}_t, Z \mid \chi^E_j) + D_z g^-_{j-1,t}(\tilde{O}_t, Z \mid \chi^E_j) & \\ \quad - D_z g^+_{jt}(\tilde{O}_t, Z \mid k^E_j) - D_z g^-_{j-1,t}(\tilde{O}_t, Z \mid k^E_j) & \text{if } 1 < j \leqslant L. \end{cases}$$

By Lemma 9, $g^-_{j-1,t}(\tilde{O}_t, Z)$ does not depend on any unit expediting costs above stage $j-1$, and thus, $g^-_{j-1,t}(\tilde{O}_t, Z \mid \chi^E_j) = g^-_{j-1,t}(\tilde{O}_t, Z \mid k^E_j)$, so that the above expression reduces to

$$D_z U_{jt}(\tilde{O}_t, Z \mid \chi^E_j) - D_z U_{jt}(\tilde{O}_t, Z \mid k^E_j)$$
$$= \chi^E_j - k^E_j + D_z g^+_{jt}(\tilde{O}_t, Z \mid \chi^E_j) - D_z g^+_{jt}(\tilde{O}_t, Z \mid k^E_j), \quad (42)$$

for every j. Next, using the definition of g^+ from (8), and $S^R_{jt}(\tilde{O}_t \mid \chi^E_j) \geqslant S^R_{jt}(\tilde{O}_t \mid k^E_j)$ established above, we evaluate the difference $D_z g^+_{jt}(\tilde{O}_t, Z \mid \chi^E_j) - D_z g^+_{jt}(\tilde{O}_t, Z \mid k^E_j)$. We get the following.

$$D_z g^+_{jt}(\tilde{O}_t, Z \mid \chi^E_j) - D_z g^+_{jt}(\tilde{O}_t, Z \mid k^E_j)$$
$$= \begin{cases} 0 & \text{if } Z \leqslant S^R_{jt}(\tilde{O}_t \mid k^E_j); \\ -D_z g_{jt}(\tilde{O}_t, Z \mid k^E_j) & \text{if } S^R_{jt}(\tilde{O}_t \mid k^E_j) < Z \leqslant S^R_{jt}(\tilde{O}_t \mid \chi^E_j); \\ D_z g_{jt}(\tilde{O}_t, Z \mid \chi^E_j) - D_z g_{jt}(\tilde{O}_t, Z \mid k^E_j) & \text{if } Z > S^R_{jt}(\tilde{O}_t \mid \chi^E_j); \end{cases}$$

For $S^R_{jt}(\tilde{O}_t \mid k^E_j) < Z \leqslant S^R_{jt}(\tilde{O}_t \mid \chi^E_j)$, by convexity of $g_{jt}(\tilde{O}_t, \cdot)$, and definition of $S^R_{jt}(\tilde{O}_t \mid k^E_j)$, we have that $D_z g_{jt}(\tilde{O}_t, Z \mid k^E_j) \geqslant 0$, and thus $-D_z g_{jt}(\tilde{O}_t, Z \mid k^E_j) \leqslant 0$. Furthermore, for any such Z, by definition of $S^R_{jt}(\tilde{O}_t \mid \chi^E_j)$, $D_z g_{jt}(\tilde{O}_t, Z \mid \chi^E_j) \geqslant 0$. Since, by (41), $k^E_j - \chi^E_j \leqslant D_z g_{jt}(\tilde{O}_t, z \mid \chi^E_j) - D_z g_{jt}(\tilde{O}_t, z \mid k^E_j)$, this implies that, for $S^R_{jt}(\tilde{O}_t \mid k^E_j) < Z \leqslant S^R_{jt}(\tilde{O}_t \mid \chi^E_j)$, we have $-D_z g_{jt}(\tilde{O}_t, Z \mid k^E_j) \geqslant k^E_j - \chi^E_j$. Thus, by (41),

$$k^E_j - \chi^E_j \leqslant D_z g^+_{jt}(\tilde{O}_t, z \mid \chi^E_j) - D_z g^+_{jt}(\tilde{O}_t, z \mid k^E_j) \leqslant 0. \quad (43)$$

Substituting (43) into (42), we get

$$0 \leqslant D_z U_{jt}(\tilde{O}_t, Z \mid \chi^E_j) - D_z U_{jt}(\tilde{O}_t, Z \mid k^E_j) \leqslant \chi^E_j - k^E_j. \quad (44)$$

By Lemma 8 and Theorem 4, the LHS of the inequality in (44) implies that $S^E_{jt}(\tilde{O}_t \mid \chi^E_j) \leqslant S^E_{jt}(\tilde{O}_t \mid k^E_j)$.

Next, we use the definition of $V_{jt}(\tilde{O}_t, \cdot)$ given in Theorem 3 to get the following.

$$D_z V_{jt}(\tilde{O}_t, Z \mid \chi^E_j) - D_z V_{jt}(\tilde{O}_t, Z \mid k^E_j)$$
$$= D_z U_{jt}(\tilde{O}_t, Z \mid \chi^E_j) - D_z U_{jt}(\tilde{O}_t, Z \mid k^E_j)$$
$$- [D_z V^-_{j-1,t}(\tilde{O}_t, Z \mid \chi^E_j) - D_z V^-_{j-1,t}(\tilde{O}_t, Z \mid k^E_j)]$$
$$= D_z U_{jt}(\tilde{O}_t, Z \mid \chi^E_j) - D_z U_{jt}(\tilde{O}_t, Z \mid k^E_j), \quad (45)$$

by Lemma 9, since $V_{j-1,t}(\tilde{O}_t, \cdot)$ does not vary with k^E_j. By means of (44), (45) then implies that

$$0 \leqslant D_z V_{jt}(\tilde{O}_t, Z \mid \chi^E_j) - D_z V_{jt}(\tilde{O}_t, Z \mid k^E_j) \leqslant \chi^E_j - k^E_j. \quad (46)$$

Next, we use $S^E_{jt}(\tilde{O}_t \mid \chi^E_j) \leqslant S^E_{jt}(\tilde{O}_t \mid k^E_j)$ established above to evaluate $D_z V^+_{jt}(\tilde{O}_t, Z \mid \chi^E_j) - D_z V^+_{jt}(\tilde{O}_t, Z \mid k^E_j)$.

$$D_z V^+_{jt}(\tilde{O}_t, Z \mid \chi^E_j) - D_z V^+_{jt}(\tilde{O}_t, Z \mid k^E_j)$$
$$= \begin{cases} 0 & \text{if } Z \leqslant S^E_{jt}(\tilde{O}_t \mid \chi^E_j); \\ D_z V_{jt}(\tilde{O}_t, Z \mid \chi^E_j) & \text{if } S^E_{jt}(\tilde{O}_t \mid \chi^E_j) < Z \leqslant S^E_{jt}(\tilde{O}_t \mid k^E_j); \\ D_z V_{jt}(\tilde{O}_t, Z \mid \chi^E_j) - D_z V_{jt}(\tilde{O}_t, Z \mid k^E_j) & \\ & \text{if } Z > S^E_{jt}(\tilde{O}_t \mid k^E_j); \end{cases}$$

If $S^E_{jt}(\tilde{O}_t \mid \chi^E_j) < Z \leqslant S^E_{jt}(\tilde{O}_t \mid k^E_j)$, then by convexity of $V_{jt}(\tilde{O}_t, \cdot)$, and definition of $S^E_{jt}(\tilde{O}_t \mid \chi^E_j)$, we have that $D_z V_{jt}(\tilde{O}_t, Z \mid \chi^E_j) \geqslant 0$. Furthermore, for any such Z, by definition of $S^E_{jt}(\tilde{O}_t \mid k^E_j)$, $D_z V_{jt}(\tilde{O}_t, Z \mid k^E_j) \leqslant 0$. Since, by (46), $0 \leqslant D_z V_{jt}(\tilde{O}_t, Z \mid \chi^E_j) - D_z V_{jt}(\tilde{O}_t, Z \mid k^E_j) \leqslant \chi^E_j - k^E_j$, this implies that, for $S^E_{jt}(\tilde{O}_t \mid \chi^E_j) < Z \leqslant S^E_{jt}(\tilde{O}_t \mid k^E_j)$, we must have $D_z V_{jt}(\tilde{O}_t, Z \mid \chi^E_j) \leqslant \chi^E_j - k^E_j$. Consequently, applying (46), we see that the above expression for $D_z V^+(\tilde{O}_t, Z \mid \chi^E_j) - D_z V^+_{jt}(\tilde{O}_t, Z \mid k^E_j)$ implies that

$$0 \leqslant D_z V^+_{jt}(\tilde{O}_t, Z \mid \chi^E_j) - D_z V^+_{jt}(\tilde{O}_t, Z \mid k^E_j) \leqslant \chi^E_j - k^E_j. \quad (47)$$

Finally, because $f_{jt}(\tilde{O}_t, z_{jt}) := V^+_{jt}(\tilde{O}_t, z_{jt}) - k^E_j z_{jt}$ by Theorem 3, then we get

$$D_z f_{jt}(\tilde{O}_t, z \mid \chi^E_j) - D_z f_{jt}(\tilde{O}_t, z \mid k^E_j)$$
$$= D_z V^+_{jt}(\tilde{O}_t, Z \mid \chi^E_j) - D_z V^+_{jt}(\tilde{O}_t, Z \mid k^E_j) - (\chi^E_j - k^E_j),$$

which combined with (47) completes the proof of part (i), and thus parts (ii)–(iv) as well.

Proof of Lemma 11. Being zero, $f_{j,T+1}(\tilde{O}_{T+1}, z_{j,T+1})$ satisfies the Lemma for every j. Assume $D_z f_{j,t+1}(\tilde{O}_{t+1}, z \mid \chi^E_{j-1}) \leqslant D_z f_{j,t+1}(\tilde{O}_{t+1}, z \mid k^E_{j-1})$ for all j in some period $t+1$. Then, by (10) we get $D_z g_{jt}(\tilde{O}_t, z \mid \chi^E_{j-1}) \leqslant D_z g_{jt}(\tilde{O}_t, z \mid k^E_{j-1})$ for all j. By Lemma (8), this implies $S^R_{jt}(\tilde{O}_t \mid \chi^E_{j-1}) \geqslant S^R_{jt}(\tilde{O}_t \mid k^E_{j-1})$ for all j. Following steps identical to those used in the proof of Lemma 10 to obtain (43), we now get

$$D_z g^+_{jt}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_z g^+_{jt}(\tilde{O}_t, z \mid k^E_{j-1}) \leqslant 0. \quad (48)$$

Since $V_{jt}(\tilde{O}_t, z_{jt}) := U_{jt}(\tilde{O}_t, z_{jt}) + V^-_{j-1,t}(\tilde{O}_t, z_{jt})$, we get

$$D_z V_{jt}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_z V_{jt}(\tilde{O}_t, z \mid k^E_{j-1})$$
$$= D_z U_{jt}(\tilde{O}_t, z \mid \chi^E_{j-1}) + D_z V^-_{j-1,t}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_z U_{jt}(\tilde{O}_t, z \mid k^E_{j-1})$$
$$- D_z V^-_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1}). \quad (49)$$

For any k^E_{j-1}, it is $D_z V_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1}) = D_z U_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1}) + D_z V^-_{j-2,t}(\tilde{O}_t, z \mid k^E_{j-1}) = D_z U_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1})$, since, by Lemma 9, $V^-_{j-2,t}(\tilde{O}_t, z)$ does not vary with k^E_{j-1}. Thus, $D_z V^-_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1}) = D_z U^-_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1})$ for every k^E_{j-1}, and, consequently, by (49), $D_z V_{jt}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_z V_{jt}(\tilde{O}_t, z \mid k^E_{j-1})$ reduces to

$$D_z U_{jt}(\tilde{O}_t, z \mid \chi^E_{j-1}) + D_z U^-_{j-1,t}(\tilde{O}_t, z \mid \chi^E_{j-1})$$
$$- D_z U_{jt}(\tilde{O}_t, z \mid k^E_{j-1}) - D_z U^-_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1}). \quad (50)$$

Let $S_{j-1,t}(\tilde{O}_t \mid \chi^E_{j-1}) := \inf(\arg\min_Z U_{jt}(\tilde{O}_t, Z \mid \chi^E_{j-1}))$ and $S_{j-1,t}(\tilde{O}_t \mid k^E_{j-1}) := \inf(\arg\min_Z U_{jt}(\tilde{O}_t, Z \mid k^E_{j-1}))$. Because, by Lemma 10(iii), $D_z U_{j-1,t}(\tilde{O}_t, z \mid \chi^E_{j-1}) \geqslant D_z U_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1})$, by Lemma 8 we must have $S_{j-1,t}(\tilde{O}_t \mid \chi^E_{j-1}) \leqslant S_{j-1,t}(\tilde{O}_t \mid k^E_{j-1})$. Consequently, by (8), we get

$$D_z U^-_{j-1,t}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_z U^-_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1})$$
$$= \begin{cases} D_z U_{j-1,t}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_z U_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1}) \\ \quad \text{if } Z \leqslant S_{j-1,t}(\tilde{O}_t \mid \chi^E_{j-1}); \\ -D_z U_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1}) \\ \quad \text{if } S_{j-1,t}(\tilde{O}_t \mid \chi^E_{j-1}) < Z \leqslant S_{j-1,t}(\tilde{O}_t \mid k^E_{j-1}); \\ 0 \quad \text{if } Z > S_{j-1,t}(\tilde{O}_t \mid k^E_{j-1}). \end{cases}$$

If $Z > S_{j-1,t}(\tilde{O}_t \mid \chi^E_{j-1})$, then $D_z U_{j-1,t}(\tilde{O}_t, z \mid \chi^E_{j-1}) \geqslant 0$. Consequently, if we have $S_{j-1,t}(\tilde{O}_t \mid \chi^E_{j-1}) < Z \leqslant S_{j-1,t}(\tilde{O}_t \mid k^E_{j-1}) -$

$D_zU_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1})$, then also inequality $-D_zU_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1}) \leqslant D_zU_{j-1,t}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_zU_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1})$ holds. Since $D_zU_{j-1,t}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_zU_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1}) \geqslant 0$ by Lemma 10(iii), the above expression implies

$$\begin{aligned} &D_zU^-_{j-1,t}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_zU^-_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1}) \\ &\quad \leqslant D_zU_{j-1,t}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_zU_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1}). \end{aligned} \tag{51}$$

Substituting (50) and (51) into (49), we get

$$\begin{aligned} &D_zV_{jt}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_zV_{jt}(\tilde{O}_t, z \mid k^E_{j-1}) \\ &\quad \leqslant D_zU_{jt}(\tilde{O}_t, z \mid \chi^E_{j-1}) + D_zU^-_{j-1,t}(\tilde{O}_t, z \mid \chi^E_{j-1}) \\ &\quad\quad - D_zU^-_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1}) \\ &\quad = [D_zg^+_{jt}(\tilde{O}_t, z \mid \chi^E_{j-1}) + D_zg^-_{j-1,t}(\tilde{O}_t, z \mid \chi^E_{j-1})] \\ &\quad\quad + [D_zg^+_{j-1,t}(\tilde{O}_t, z \mid \chi^E_{j-1}) + D_zg^-_{j-2,t}(\tilde{O}_t, z \mid \chi^E_{j-1})] \\ &\quad\quad - [D_zg^+_{jt}(\tilde{O}_t, z \mid k^E_{j-1}) + D_zg^-_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1})] \\ &\quad\quad - [D_zg^+_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1}) + D_zg^-_{j-2,t}(\tilde{O}_t, z \mid k^E_{j-1})] \end{aligned} \tag{52}$$

By Lemma 9, $D_zg^-_{j-2,t}(\tilde{O}_t, z \mid \chi^E_{j-1}) = 0$ and $D_zg^-_{j-2,t}(\tilde{O}_t, z \mid k^E_{j-1}) = 0$. Furthermore,

$g^-_{j-1,t}(\tilde{O}_t, z \mid \chi^E_{j-1}) + g^+_{j-1,t}(\tilde{O}_t, z \mid \chi^E_{j-1}) = g_{j-1,t}(\tilde{O}_t, z \mid \chi^E_{j-1})$, and

$g^-_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1}) + g^+_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1}) = g_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1})$.

Therefore, (52) reduces to

$$\begin{aligned} &D_zV_{jt}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_zV_{jt}(\tilde{O}_t, z \mid k^E_{j-1}) \\ &\quad \leqslant D_zg^+_{jt}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_zg^+_{jt}(\tilde{O}_t, z \mid k^E_{j-1}) \\ &\quad\quad + D_zg_{j-1,t}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_zg_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1}). \end{aligned} \tag{53}$$

By (48), $D_zg^+_{jt}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_zg^+_{jt}(\tilde{O}_t, z \mid k^E_{j-1}) \leqslant 0$. By Lemma 10(ii), $D_zg_{j-1,t}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_zg_{j-1,t}(\tilde{O}_t, z \mid k^E_{j-1}) \leqslant 0$. Consequently, $D_zV_{jt}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_zV_{jt}(\tilde{O}_t, z \mid k^E_{j-1}) \leqslant 0$. By Lemma (8), this inequality implies that $S^E_{jt}(\tilde{O}_t \mid \chi^E_{j-1}) \geqslant S^E_{jt}(\tilde{O}_t \mid k^E_{j-1})$ for all j, and, following familiar steps, that

$$D_zV^+_{jt}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_zV^+_{jt}(\tilde{O}_t, z \mid k^E_{j-1}) \leqslant 0, \tag{54}$$

By definition of f_{jt} (Theorem 3 and Lemma 10), expression (54) implies that $D_zf_{jt}(\tilde{O}_t, z \mid \chi^E_{j-1}) - D_zf_{jt}(\tilde{O}_t, z \mid k^E_{j-1}) \leqslant 0$, which therefore completes the proof.

Proof of Lemma 12. By Lemma 11, this result holds for $m = j-1$. So, going forward we consider only $m < j-1$. Being zero, $f_{j,T+1}(\tilde{O}_{T+1}, z_{j,T+1})$ satisfies the lemma for every j. Assume inductively that $D_zf_{j,t+1}(\tilde{O}_{t+1}, z \mid \chi^E_m) \leqslant D_zf_{j,t+1}(\tilde{O}_{t+1}, z \mid k^E_m)$ for all j and $m < j-1$ in some period $t+1$. Then, by (10),

$$D_zg_{jt}(\tilde{O}_t, z \mid \chi^E_m) \leqslant D_zg_{jt}(\tilde{O}_t, z \mid k^E_m), \tag{55}$$

for all j and $m < j-1$ in t. By Lemma (8), it follows that $S^R_{jt}(\tilde{O}_t \mid \chi^E_m) \geqslant S^R_{jt}(\tilde{O}_t \mid k^E_m)$ for all j and $m < j-1$. Following the same steps as in the previous proofs, we then get that, for all j and $m < j-1$,

$$D_zg^+_{jt}(\tilde{O}_t, z \mid \chi^E_m) \leqslant D_zg^+_{jt}(\tilde{O}_t, z \mid k^E_m) \tag{56}$$

$$D_zg^-_{jt}(\tilde{O}_t, z \mid \chi^E_m) \leqslant D_zg^-_{jt}(\tilde{O}_t, z \mid k^E_m). \tag{57}$$

Fix j and m. Then,

$$\begin{aligned} &D_zU_{jt}(\tilde{O}_t, Z \mid \chi^E_m) - D_zU_{jt}(\tilde{O}_t, Z \mid k^E_m) \\ &\quad = D_zg^+_{jt}(\tilde{O}_t, Z \mid \chi^E_m) + D_zg^-_{j-1,t}(\tilde{O}_t, Z \mid \chi^E_m) \\ &\quad\quad - D_zg^+_{jt}(\tilde{O}_t, Z \mid k^E_m) - D_zg^-_{j-1,t}(\tilde{O}_t, Z \mid k^E_m) \leqslant 0, \end{aligned}$$

because $D_zg^+_{jt}(\tilde{O}_t, Z \mid \chi^E_m) \leqslant D_zg^+_{jt}(\tilde{O}_t, Z \mid k^E_m)$ by (56), and $D_zg^-_{j-1,t}(\tilde{O}_t, Z \mid \chi^E_m) \leqslant D_zg^-_{j-1,t}(\tilde{O}_t, Z \mid k^E_m)$ by Lemma 11(ii) if $m = j-2$, or by (57) if $m < j-2$. Thus, $D_zU_{jt}(\tilde{O}_t, Z \mid \chi^E_m) \leqslant D_zU_{jt}(\tilde{O}_t, Z \mid k^E_m)$.

Next, we use induction on j to prove that $D_zV_{jt}(\tilde{O}_t, Z \mid \chi^E_m) - D_zV_{jt}(\tilde{O}_t, Z \mid k^E_m) \leqslant 0$. We are considering any j such that $0 < m < j-1$, so our base case is $j = 3$, which allows only $m = 1$. We get

$$\begin{aligned} &D_zV_{3t}(\tilde{O}_t, Z \mid \chi^E_1) - D_zV_{3t}(\tilde{O}_t, Z \mid k^E_1) \\ &\quad = D_zU_{3t}(\tilde{O}_t, Z \mid \chi^E_1) + D_zV^-_{2t}(\tilde{O}_t, Z \mid \chi^E_1) \\ &\quad\quad - D_zU_{3t}(\tilde{O}_t, Z \mid k^E_1) - D_zV^-_{2t}(\tilde{O}_t, Z \mid k^E_1) \leqslant 0, \end{aligned}$$

since $D_zU_{3t}(\tilde{O}_t, Z \mid \chi^E_1) \leqslant D_zU_{3t}(\tilde{O}_t, Z \mid k^E_1)$, and $D_zV^-_{2t}(\tilde{O}_t, Z \mid \chi^E_1) \leqslant D_zV^-_{2t}(\tilde{O}_t, Z \mid k^E_1)$ follows directly from Lemma 11 (iii). Thus, $D_zV_{3t}(\tilde{O}_t, Z \mid \chi^E_1) \leqslant D_zV_{3t}(\tilde{O}_t, Z \mid k^E_1)$. Assume inductively that we have $D_zV_{qt}(\tilde{O}_t, Z \mid \chi^E_m) - D_zV_{qt}(\tilde{O}_t, Z \mid k^E_m) \leqslant 0$ for all $q = 3, \ldots, j-1$, and $m < q-1$. Then, we get

$$\begin{aligned} &D_zV_{jt}(\tilde{O}_t, Z \mid \chi^E_m) - D_zV_{jt}(\tilde{O}_t, Z \mid k^E_m) \\ &\quad = D_zU_{jt}(\tilde{O}_t, Z \mid \chi^E_m) + D_zV^-_{j-1,t}(\tilde{O}_t, Z \mid \chi^E_m) \\ &\quad\quad - D_zU_{jt}(\tilde{O}_t, Z \mid k^E_m) - D_zV^-_{j-1,t}(\tilde{O}_t, Z \mid k^E_m) \\ &\quad \leqslant D_zV^-_{j-1,t}(\tilde{O}_t, Z \mid \chi^E_m) - D_zV^-_{j-1,t}(\tilde{O}_t, Z \mid k^E_m) \\ &\quad\quad \text{[since } D_zU_{jt}(\tilde{O}_t, Z \mid \chi^E_m) \leqslant D_zU_{jt}(\tilde{O}_t, Z \mid k^E_m)] \\ &\quad \leqslant 0. \quad \text{[by inductive assumption on } D_zV_{qt}(\tilde{O}_t, Z)] \end{aligned}$$

By Lemma 8, $D_zV_{jt}(\tilde{O}_t, Z \mid \chi^E_m) \leqslant D_zV_{jt}(\tilde{O}_t, Z \mid k^E_m)$ implies that $S^E_{jt}(\tilde{O}_t \mid \chi^E_m) \geqslant S^E_{jt}(\tilde{O}_t \mid k^E_m)$. Furthermore, by steps used before, $D_zV_{jt}(\tilde{O}_t, Z \mid \chi^E_m) \leqslant D_zV_{jt}(\tilde{O}_t, Z \mid k^E_m)$ yields $D_zV^+_{jt}(\tilde{O}_t, Z \mid \chi^E_m) \leqslant D_zV^+_{jt}(\tilde{O}_t, Z \mid k^E_m)$. Consequently, we get $D_zf_{jt}(\tilde{O}_t, Z \mid \chi^E_m) \leqslant D_zf_{jt}(\tilde{O}_t, Z \mid k^E_m)$, which completes the proof.

Proof of Theorem 7. If $m > j$, then by Lemma 9, $\hat{Z}^R_{jt}(\tilde{O}_t, z_t \mid \chi^E_m) = \hat{Z}^R_{jt}(\tilde{O}^1_t, z_t \mid k^E_m)$ and so the result holds. Thus, assume $m \leqslant j$. Consider first $j = L$. In that case, $\hat{Z}^E_{Lt}(\tilde{O}_t) = z_{Lt} \vee S^E_{Lt}(\tilde{O}_t)$, and we get

$$\begin{aligned} &\hat{Z}^R_{Lt}(\tilde{O}_t \mid \chi^E_m) - \hat{Z}^R_{Lt}(\tilde{O}_t \mid k^E_m) \\ &\quad = \hat{Z}^E_{Lt}(\tilde{O}_t \mid \chi^E_m) \vee S^R_{Lt}(\tilde{O}_t \mid \chi^E_m) - \hat{Z}^E_{Lt}(\tilde{O}_t \mid k^E_m) \vee S^R_{Lt}(\tilde{O}_t \mid k^E_m) \\ &\quad\quad \text{[by Theorem 4(ii)]} \\ &\quad = z_{Lt} \vee S^E_{Lt}(\tilde{O}_t \mid \chi^E_m) \vee S^R_{Lt}(\tilde{O}_t \mid \chi^E_m) \\ &\quad\quad - z_{Lt} \vee S^E_{Lt}(\tilde{O}_t \mid k^E_m) \vee S^R_{Lt}(\tilde{O}_t \mid k^E_m) \\ &\quad = z_{Lt} \vee S^R_{Lt}(\tilde{O}_t \mid \chi^E_m) - z_{Lt} \vee S^R_{Lt}(\tilde{O}_t \mid k^E_m). \quad \text{[by Lemma 7]} \end{aligned} \tag{58}$$

If $m = j = L$, then $S^R_{Lt}(\tilde{O}_t \mid \chi^E_m) \geqslant S^R_{Lt}(\tilde{O}_t \mid k^E_m)$ by Lemma 10(iv); if $m < L$, then $S^R_{Lt}(\tilde{O}_t \mid \chi^E_m) \geqslant S^R_{Lt}(\tilde{O}_t \mid k^E_m)$ by Lemma 12(ii). In either case, (58) implies that $\hat{Z}^R_{Lt}(\tilde{O}_t \mid \chi^E_m) \geqslant \hat{Z}^R_{Lt}(\tilde{O}_t \mid k^E_m)$.

Now suppose $j < L$. Then, we address two distinct cases.

Case 1: $m < j$. In that case, by Lemma 12(ii), $S_{it}^E(\tilde{O}_t \mid \chi_m^E) \geqslant S_{it}^E(\tilde{O}_t \mid k_m^E)$ for every $i = j, \ldots, L$. Since $\hat{Z}_{jt}^E(\tilde{O}_t) = \bigwedge_{i=j}^{L}[z_{it} \vee S_{it}^E(\tilde{O}_t)]$ by Theorem 4(i), it follows that

$$\hat{Z}_{jt}^E(\tilde{O}_t \mid \chi_m^E) \geqslant \hat{Z}_{jt}^E(\tilde{O}_t \mid k_m^E) \tag{59}$$

$$\hat{Z}_{j+1,t}^E(\tilde{O}_t \mid \chi_m^E) \geqslant \hat{Z}_{j+1,t}^E(\tilde{O}_t \mid k_m^E) \tag{60}$$

Applying (59) and (60), we get

$$\begin{aligned}\hat{Z}_{jt}^R(\tilde{O}_t \mid \chi_m^E) - \hat{Z}_{jt}^R(\tilde{O}_t \mid k_m^E) &= [\hat{Z}_{jt}^E(\tilde{O}_t \mid \chi_m^E) \vee S_{jt}^R(\tilde{O}_t \mid \chi_m^E)] \\ &\quad \wedge \hat{Z}_{j+1,t}^E(\tilde{O}_t \mid \chi_m^E) \\ &\quad - [\hat{Z}_{jt}^E(\tilde{O}_t \mid k_m^E) \vee S_{jt}^R(\tilde{O}_t \mid k_m^E)] \wedge \hat{Z}_{j+1,t}^E(\tilde{O}_t \mid k_m^E) \geqslant 0,\end{aligned}$$

since $S_{jt}^R(\tilde{O}_t \mid \chi_m^E) \geqslant S_{jt}^R(\tilde{O}_t \mid k_m^E)$ by Lemma 12(ii).

Case 2: $m = j$. In this case, by Lemma 12, $S_{qt}^E(\tilde{O}_t \mid \chi_j^E) \geqslant S_{qt}^E(\tilde{O}_t \mid k_j^E)$ for every $q > j$ and, consequently, by Theorem 4(i), $\hat{Z}_{j+1,t}^E(\tilde{O}_t \mid \chi_j^E) \geqslant \hat{Z}_{j+1,t}^E(\tilde{O}_t \mid k_j^E)$.

Now suppose first that $S_{jt}^R(\tilde{O}_t \mid \chi_j^E) \geqslant \hat{Z}_{j+1,t}^E(\tilde{O}_t \mid \chi_j^E)$. In that case, $\hat{Z}_{jt}^R(\tilde{O}_t \mid \chi_j^E) = \hat{Z}_{j+1,t}^E(\tilde{O}_t \mid \chi_j^E)$. By Theorem 4(ii), $\hat{Z}_{jt}^R(\tilde{O}_t \mid k_j^E) \leqslant \hat{Z}_{j+1,t}^E(\tilde{O}_t \mid k_j^E)$. Since $\hat{Z}_{j+1,t}^E(\tilde{O}_t \mid \chi_j^E) \geqslant \hat{Z}_{j+1,t}^E(\tilde{O}_t \mid k_j^E)$, it follows that $\hat{Z}_{jt}^R(\tilde{O}_t \mid \chi_m^E) \geqslant \hat{Z}_{jt}^R(\tilde{O}_t \mid k_m^E)$. Next suppose that $S_{jt}^R(\tilde{O}_t \mid \chi_j^E) < \hat{Z}_{j+1,t}^E(\tilde{O}_t \mid \chi_j^E)$. Then, we get

$$\begin{aligned}&\hat{Z}_{jt}^R(\tilde{O}_t \mid \chi_j^E) \\ &= \hat{Z}_{jt}^E(\tilde{O}_t, \mid \chi_j^E) \vee S_{jt}^R(\tilde{O}_t \mid \chi_j^E) \quad [\text{by } (13)] \\ &= [\hat{Z}_{j+1,t}^E(\tilde{O}_t, \mid \chi_j^E) \wedge (z_{jt} \vee S_{jt}^E(\tilde{O}_t, \mid \chi_j^E))] \vee S_{jt}^R(\tilde{O}_t \mid \chi_j^E) \quad [\text{by } (12)] \\ &= [\hat{Z}_{j+1,t}^E(\tilde{O}_t, \mid \chi_j^E) \vee S_{jt}^R(\tilde{O}_t \mid \chi_j^E)] \\ &\quad \wedge [z_{jt} \vee S_{jt}^E(\tilde{O}_t, \mid \chi_j^E) \vee S_{jt}^R(\tilde{O}_t \mid \chi_j^E)] \\ &= \hat{Z}_{j+1,t}^E(\tilde{O}_t, \mid \chi_j^E) \wedge [z_{jt} \vee S_{jt}^E(\tilde{O}_t, \mid \chi_j^E) \vee S_{jt}^R(\tilde{O}_t \mid \chi_j^E)] \\ &\qquad [S_{jt}^R(\tilde{O}_t \mid \chi_j^E) < \hat{Z}_{j+1,t}^E(\tilde{O}_t \mid \chi_j^E)] \\ &= \hat{Z}_{j+1,t}^E(\tilde{O}_t, \mid \chi_j^E) \wedge [z_{jt} \vee S_{jt}^R(\tilde{O}_t \mid \chi_j^E)]. \quad [\text{by Lemma } 7]\end{aligned} \tag{61}$$

Continuing with the case when $S_{jt}^R(\tilde{O}_t \mid \chi_j^E) \geqslant \hat{Z}_{j+1,t}^E(\tilde{O}_t \mid \chi_j^E)$ yielding (61), suppose that $S_{jt}^R(\tilde{O}_t \mid k_j^E) \geqslant \hat{Z}_{j+1,t}^E(\tilde{O}_t \mid k_j^E)$ holds, so that we have $\hat{Z}_{jt}^R(\tilde{O}_t \mid k_j^E) = \hat{Z}_{j+1,t}^E(\tilde{O}_t \mid k_j^E)$. Since $S_{jt}^R(\tilde{O}_t \mid \chi_j^E) \geqslant S_{jt}^R(\tilde{O}_t \mid k_j^E)$ by Lemma 10 (iv), then $S_{jt}^R(\tilde{O}_t \mid \chi_j^E) \geqslant \hat{Z}_{j+1,t}^E(\tilde{O}_t \mid k_j^E)$. Since $\hat{Z}_{j+1,t}^E(\tilde{O}_t \mid \chi_j^E) \geqslant \hat{Z}_{j+1,t}^E(\tilde{O}_t \mid k_j^E)$, then (61) implies that $\hat{Z}_{jt}^R(\tilde{O}_t \mid \chi_j^E) \geqslant \hat{Z}_{jt}^R(\tilde{O}_t \mid k_j^E)$. If, on the other hand, we suppose that $S_{jt}^R(\tilde{O}_t \mid k_j^E) < \hat{Z}_{j+1,t}^E(\tilde{O}_t \mid k_j^E)$, then the identical steps to those used to derive (61) yield $\hat{Z}_{jt}^R(\tilde{O}_t \mid k_j^E) = \hat{Z}_{j+1,t}^E(\tilde{O}_t, \mid k_j^E) \wedge [z_{jt} \vee S_{jt}^R(\tilde{O}_t \mid k_j^E)]$. Thus,

$$\begin{aligned}\hat{Z}_{jt}^R(\tilde{O}_t \mid \chi_j^E) - \hat{Z}_{jt}^R(\tilde{O}_t \mid k_j^E) &= \hat{Z}_{j+1,t}^E(\tilde{O}_t, \mid \chi_j^E) \wedge [z_{jt} \vee S_{jt}^R(\tilde{O}_t \mid \chi_j^E)] \\ &\quad - \hat{Z}_{j+1,t}^E(\tilde{O}_t, \mid k_j^E) \wedge [z_{jt} \vee S_{jt}^R(\tilde{O}_t \mid k_j^E)] \geqslant 0,\end{aligned}$$

because $S_{jt}^R(\tilde{O}_t \mid \chi_j^E) \geqslant S_{jt}^R(\tilde{O}_t \mid k_j^E)$ and $\hat{Z}_{j+1,t}^E(\tilde{O}_t \mid \chi_j^E) \geqslant \hat{Z}_{j+1,t}^E(\tilde{O}_t \mid k_j^E)$. This completes the proof.

Proof of Theorem 8. It is straightforward to verify that Lemmas 1 and 2 continue to hold for a capacitated component system. To prove parts (i) and (ii), we follow steps identical to those in the proof of Lemma 3. The only additional step needed to establish the feasibility of $\bar{X}_t^E$ is to prove that the expedited order $\bar{X}_{ijt}^E$ satisfies the appropriate capacity constraint at every stage j. Using $\bar{X}_{ijt}^E = X_{ijt}^E - \delta$, we get

$$\begin{aligned}\bar{X}_{ijt}^E &= X_{ijt}^E - \delta \quad [\text{definition of } \bar{X}_t^E] \\ &< X_{ijt}^E \quad [\delta > 0] \\ &\leqslant K_{ij}. \quad [\bar{X}_t^E \text{ is feasible}]\end{aligned}$$

Thus, $\bar{X}_{ijt}^E$ satisfies the appropriate capacity constraint, and is thus feasible. Since, by Lemma 3, $\bar{X}_{ijt}^E$ also leads to lower cost, it is optimal, which proves (i) and (ii). To prove (iii), fix j. Let q be a component in $\mathbb{C}(j)$ such that $K_{qj} := \min_{i \in \mathbb{C}(j)} K_{ij}$, so that $K_j^* = K_{qj}$. Consider any component $i \in \mathbb{C}(j)$. Then, we get

$$\begin{aligned}X_{ijt}^E &= X_{qjt}^E \quad [\text{by part } (i), X_t^E \text{ is balanced}] \\ &\leqslant K_{qj} \quad [X_{qjt}^E \text{ is feasible}] \\ &= K_j^*. \quad [\text{definition of } K_j^*]\end{aligned}$$

Therefore, the expedited order for each component i relevant at stage satisfies the capacity constraint K_j^*, so that the capacity constraint for all such components i at each stage j can be replaced with K_j^* without affecting either the optimal policy or the optimal cost. This completes the proof.

Proof of Theorem 9. The proof of Theorem 9 follows exactly the same steps as the proof of Theorem 2 since relaxing the component-matching constraints of the general capacitated assembly system allows the resulting relaxed system to inherit capacity constraints from the original system. Thus, a general capacitated assembly system can be reduced to an equivalent capacitated component assembly system.

Proof of Theorem 10. The proof of Theorem 10 follows the exact same steps as those used in the proofs of Theorems 3 and 4, and therefore we omit the details.

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