When Variability Trumps Volatility: Optimal Control and Value of Reverse Logistics in Supply Chains with Multiple Flows of Product

Alexandar Angelus
aangelus@mays.tamu.edu
Mays School of Business
Texas A&M University
College Station, Texas 77845

Özalp Özer
oozer@utdallas.edu
Jindal School of Management
The University of Texas at Dallas
Richardson, Texas 75080

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Abstract

Reverse logistics has been gaining recognition in practice (and theory) for helping companies better match supply with demand, and thus reduce costs in their supply chains. In this paper, we study reverse logistics from the perspective of a supply chain in which each location can initiate multiple flows of product. Our first objective is to jointly optimize ordering decisions pertaining to regular, reverse and expedited flows of product in a stochastic, multi-stage inventory model of *a logistics supply chain*, in which the physical transformation of the product is completed at the most upstream location in the system. Due to those multiple flows of product, the feasible region for the problem acquires multi-dimensional boundaries that lead to the curse of dimensionality. To address this challenge, we develop a different solution method that allows us to reduce the dimensionality of the feasible region and, subsequently, identify the structure of the optimal policy. We refer to this policy as a *nested echelon basestock policy*, as decisions for different product flows are sequentially nested within each other. We show that this policy renders the model analytically and numerically tractable. Our results provide actionable policies for firms to jointly manage three different product flows in their supply chains, and allow us arrive at insights regarding the main drivers of the value of reverse logistics. One of our key findings is that, when it comes to the value generated by reverse logistics, demand variability (i.e., demand uncertainty across periods) matters more than demand volatility (i.e., demand uncertainty within each period). This is because, in the absence of demand variability, it is effectively never optimal to return product upstream, regardless of the level of inherent demand volatility. Our second objective is to extend our analysis to *product transforming-supply chains*, in which product transformation is allowed to occur at each location. In such a system, it becomes necessary to keep track of both the location and stage of completion of each unit of inventory, so that the number of state and decisions variables increases with the square of the number of locations in the system. To analyze such a supply chain, we first identify a policy that provides a lower bound on the total cost. Then, we establish a special decomposition of the objective cost function that allows us to propose a novel heuristic policy. We find that the performance gap of our heuristic policy relative to the lower-bounding policy averages less than 5% across a range of parameters and supply chain lengths.

Keywords: Reverse Logistics, Multiechelon Inventory, Optimal Policy, Demand Variability.

1. Introduction

In contrast to forward logistics that deals with the flow of goods and services from the point of origin to the point of consumption, reverse logistics refers to principles and practices for managing the flow of surplus inventory in the form of material, goods, or equipment back through the supply chain (for reuse, resale, recycling, or disposal). In 2002, the value of reverse logistics was estimated at $100 billion (Stock et al., 2002), while that estimate, as provided by the Reverse Logistics Association, grew to between $150 billion and $200 billion by 2013 (Rogers et al., 2013). As a result, companies have become more cognizant of the benefits of reverse logistics, and many specialized firms, such as Liquidity Services with the current market capitalization of $200 million, have been established to provide third-party reverse logistics services to supply chains. In an Aberdeen Group survey in 2010, 87% of respondents said that effective management of the reverse supply chain was either "extremely" or "very important" to their organization's operational and financial performance, up from 74% in 2008 and 61% in 2007 (Ball 2010).

The growing recognition of the importance of reverse logistics for managing the mismatch between supply and demand is also evidenced by a large number of trade-press publications and articles in practitioner journals (e.g., Carter and Ellram 1998; Jedd 1999; Rogers and Tiben-Lembke 2001; Rogers et al. 2013). In spite of those, there remains a lack of clarity in both practice and the research literature regarding precisely what in reverse logistics is so important, exactly how reverse logistics creates value for companies, and what the drivers of that value are. Without that clarity, attempts to improve reverse logistics performance or maximize its value have proven difficult. Further, if supply chains can generate significant value from reverse logistics without managers understanding its true drivers, then it seems worthwhile to identify those drivers and optimize their effect on the bottom line. This state of affairs and the resulting potential for companies to unlock additional profits have motivated us to explore one essential aspect of the reverse logistics capability having to do with jointly managing multiple flows of product in the supply chain, as reverse logistics necessarily involves more than just the regular flow of orders.

For that purpose, we build a model of a multi-stage supply chain with stochastic demand and the following three, commonly-encountered, product flows at each stage: regular and expedited flow downstream, and the reverse flow of product upstream. The regular flow of product in our model represents standard, regularly scheduled orders by which each stage in a supply chain replenishes its inventory, period by period, from the stock available at the stage immediately upstream. The expedited flow allows the product to move downstream through the entire supply chain (or any portion thereof) faster, possibly within the same period. Expediting of inventory is a common practice in industry. Dell, for example, is explicit in its annual 10-K, that "...our business model generally gives us flexibility to manage backlog at any point in time by expediting ... customer orders" (Dell, 2013, page 11). In a recent survey, a majority of managers

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from European divisions of medium- to large-sized manufacturing companies responded that they resort to expediting to avoid back orders (Özsen and Thonemann 2014). In contrast to regular and expedited orders that flow downstream, the reverse flow of product takes place through returning (excess) inventory upstream at each stage. Inventory returns were originally used in the distribution of books, magazines, and newspapers. More recently, reverse logistics in the form of upstream flow of (returned) products has become a widely used supply-chain capability across a range of other industries such as apparel, the high tech, and automobile manufacturing (Ball 2010). In practice, this reverse flow of orders tends to be managed independently of other product flows in the system, leading to suboptimal outcomes.

Even though each of the product flows considered in this paper has been found to create significant savings on its own, we are not aware of any research regarding savings that can be generated by simultaneously managing all three product flows in the system. As a result, the true value of reverse logistics has remained unclear. The present paper takes a step in the direction of providing the means to jointly and optimally manage different product flows in the supply chain and quantify the resulting value of the reverse logistics capability. Further, the insights obtained from our analysis regarding what impacts the value of reverse logistics (and what does not) are, to the best of our knowledge, new to the research literature.

The basic setting of our paper is a supply chain in which the physical transformation of the product is completed at the most upstream location in the system. The finished product is then moved downstream, from one location to another to bring it closer to the customer, without physically transforming the product any further in the process. Hence, in the first part of our paper, we are essentially studying *logistics supply chains* that abound, for instance, in the apparel industry or the pharmaceutical industry, in which the product is often fully manufactured at the uppermost location, and downstream locations represent points where inventory is held and shipments are coordinated. In such a supply chain, items that arrive by upstream, reverse-order flow into any location are physically indistinguishable from items that arrive into that location by downstream flow of regular and expedited orders. One supply chain that partially motivated our study is that of International Bearings, a Singapore-based firm, founded in 1971, which is in the business of providing automotive and other bearings to industrial customers in the Southeast Asia.

The supply chain for International Bearings (IB) starts out with the manufacturer, based in China, who produces bearings designed by IB. Manufactured bearings are shipped by boat to IB's warehouse in Singapore, where they are repackaged and then shipped again to the company's distribution centers throughout Southeast Asia. Those distribution centers serve the demand from local industrial customers. In addition to this regular flow of product, the company also uses air freight to expedite orders when demand at one of its distribution increases unexpectedly. At the same time, excess inventory of products at distributors that is not being turned over is shipped back to Singapore, where it is consolidated, repackaged and then shipped back to the manufacturer in China. The manufacturer can then hold the returned inventory in expectation of potential orders in the future, or resell it to a local distributor. Given those different product flows, some of the most important decisions made each week by International Bearings in coordinating its supply chain involve placing orders for different products flows up and down the system.

To address the problem faced by IB and other companies with multiple product flows in their systems, in this paper, we formulate and solve a multiechelon inventory model with multiple product flows including the reverse flow of items. In order to isolate the benefit of jointly managing all product flows in the system from other sources of value in the supply chain, we focus on: (1) reverse orders that originate *within* the supply chain itself and represent a strategy for managing excess inventory (referred to as "overstock inventory" in industry), rather than on those necessitated by customers' returns of product *into* the supply chain; and (2) essential characteristics and dynamics of those product flows, without unnecessarily complicating the model with all other features that might be present in a reverse-logistics supply chain.

Multiechelon models are generally difficult to analyze due to the curse of dimensionality inherent in the multi-dimensional nature of their objective cost functions. Only a small number of multiechelon problems has been solved, usually by the standard method of change of variables from installation quantities to echelon quantities, first applied in Clark and Scarf (1960). The model considered in our paper has an important dimension of complexity not encountered in other multiechelon systems studied in the literature – namely, the joint management of three different product flows, each with its own corresponding decision at each location in the supply chain. The combination of three product flows in our model leads to the formation of multi-dimensional boundaries of the feasible region even under the Clark-Scarf echelon reformulation of the problem. Consequently, optimal decisions remain dependent on the entire set of state space dimensions, and the resulting curse of dimensionality defies conventional approaches.

To deal with this challenge, we develop a novel solution approach, in which we allow orders to go *long*; that is, we allow regular orders to be placed for inventory that is not necessarily available. In that manner, we obtain a relaxed version of the problem with reduced dimensionality of the feasible region. Our first contribution is to show that the optimal policy for this relaxed version of the problem is identical to the optimal policy for the original reverse-logistics model. The second contribution of our paper is to derive the structure of that optimal policy and show that it achieves the decomposition of the multidimensional objective cost function for the problem into a nested sequence of single-dimensional subproblems.

Our third contribution is to establish additional characteristics of the optimal policy with implications for theory and practice of reverse logistics. For that purpose, it turns out to be useful to distinguish between *demand volatility* that represents uncertainty arising from the random nature of demand in each period, and *demand variability* that represents the uncertainty arising from the changing nature of demand across periods. A key insight we establish analytically is that, under the optimal policy, reverse logistics generate savings in a supply chain *only* in the presence of demand variability; in its absence, reverse logistics has no practical impact on the system, regardless of the level of inherent demand volatility. We also develop an efficient algorithm to compute optimal policies and quantify the value of reverse logistics under a variety of supply chain structures. Thus, we find, for example, that, while reverse logistics on its own results in an average of 3.74% cost savings for the three-location supply chain explored in our numerical studies, it can generate close to an additional 3% when integrated with product expediting, due to the complementarity effect from having both of those capabilities in the system. Further, having all three product flows in the system is found to generate an average of 16.2% in cost savings across all model parameters and supply chain lengths explored in our numerical studies, with the maximum of 26.4%.

Our fourth contribution is to provide an analysis of product-transforming supply chains in which a unit of product is allowed to be physically transformed at each location, as it moves downstream in the system. In such supply chains, an item that has reached a certain location will generally have different physical characteristics (and be more valuable) than an item that has not yet reached that location. As a result, in a product-transforming supply chain with reverse logistics, it becomes necessary to keep track, in each period, of *both* the location *and* stage of completion of each unit of inventory. This is because some of the items at a particular location might have arrived there by the reverse flow from downstream locations, in which case their completion will be further along relative to those items that have arrived there by regular or expedited orders flowing downstream. Hence, the number of state and decisions variables in a product-transforming supply chain increases with the square of the number of locations in the system. Further, in contrast to logistics supply systems, in a product-transforming supply chain, there exist multiple replenishment opportunities at each location, for both regular and reverse orders, each with its own distinct unit cost and impact on the inventory state. To the best of our knowledge, a supply chain of such complexity has not been considered in the literature before, despite its common occurrence in practice. To analyze such a system, we make use of our results for the logistics supply chain (i.e., the supply chain in which the physical transformation of the product is completed at the most upstream location) to obtain a policy that provides a low bound on the total cost. Then, we establish a special decomposition of the objective cost function that allows us to propose a novel heuristic policy. The performance gap of our heuristic policy, relative to our low-bounding policy, averages less than 5% across a range of model parameters.

Literature Review

The origins of the multiechelon inventory theory are in the classic paper by Clark and Scarf (1960) who show how to reformulate a multistage inventory model in terms of echelon variables to allow additive decomposition of a multi-dimensional objective cost function, and thus ensure tractability. Federgruen and

Zipkin (1984) extend those results to the infinite horizon case. Dong and Lee (2003) prove the optimal policy of Clark and Scarf also holds under time-correlated demand processes. Song and Zipkin (2013) generalize Clark and Scarf results to settings in which inventory can be held at any point along a continuum, not just at discrete stages. Goh and Porteus (2016) establish the structure of the optimal policy for a multiechelon model with take-or-pay contracts. Tong et al. (2018) consider payment timing in multiechelon systems.

Some papers in the multiechelon literature address problems in reverse logistics. DeCroix et al. (2005) introduce the idea of negative demand, representing customer returns, and establish the optimality of echelon basestock policies. DeCroix and Zipkin (2005) introduce customer returns to assembly systems, and describe an item-recovery pattern and restrictions on the inventory policy under which an equivalent series system is shown to exist. Both of those papers consider logistics supply chains where returns originate with the customers, and represent a random variable whose realization is observed each period. Our model can be extended to accommodate customer returns in the form of negative demand; in that case, all of our results concerning the structure of the optimal policy and the resulting decomposition of the objective cost function continue to hold. However, our focus in this paper is on reverse orders that originate within the supply chain itself and represent *decision variables* by means of which each stage manages its excess inventory, rather than *random variables* whose realizations signify customers returns into the supply chain.

Of particular interest to our work are Fukuda (1961) and Lawson and Porteus (2000). Fukuda (1961) extends the multiechelon model of Clark and Scarf (1960) to allow returns of stock upstream at each stage. He assumes stationary cost parameters and customer demands, and establishes the form of the optimal policy and the Clark-Scarf decomposition for the problem. While in recent years there have appeared many papers that deal with the reverse flow of product in multi-stage supply chains, to the best of our knowledge, Fukuda (1961) is the only paper to identify the *structure of the optimal policy* for a such a system under *demand uncertainty.* Other papers that consider the reverse flow of items in the system are either deterministic in nature (e.g., Cruz-Rivera and Ertel 2009, Salema et al. 2010), use decision tree analysis (e.g., Cardosa et al., 2013) or heuristic approaches (e.g., Min et al. 2006, Lee and Dong, 2008) to numerically solve the resulting models. In our paper, we extend Fukuda (1961) to allow for expediting of inventory. We also prove that it is not the uncertainty of demand in each period that drives upstream product returns but rather the variability of demand across periods. One implication of our results is that, although Fukuda (1961) allows for upstream returns of stock, due to the stationary setting of his model (i.e., lack of demand variability), those returns do not actually occur in his model under the optimal policy.

Lawson and Porteus (2000) extend the Clark and Scarf (1960) model to include inventory expediting. They prove the optimality of a top-down echelon basestock policy and establish the Clark-Scarf decomposition of the objective cost function. In contrast to Lawson and Porteus (2000), our basic model of

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the logistics supply chain includes not only expedited orders flowing downstream but also reverse orders flowing upstream in the system. Having the reverse flow of product in the model fundamentally alters the difficulty of the problem due to the resulting multi-dimensional boundary of the feasible region. As a result, the method prevalent in the multiechelon literature, based on the result of Karush (1959), does not suffice in our model to arrive at the structure of the optimal policy. Further, Lawson and Porteus (2000) allow only demand volatility in the system and do not explore the impact of demand variability on the structure of the optimal policy. Finally, we apply our results to the analysis of reverse logistics in product-transforming supply chains, which are not considered in Lawson and Porteus (2000).

More generally, our paper extends the existing multiechelon literature along three main dimensions of significance. First, we address three different product flows in the system, in contrast to other papers that allow only one or two of those flows. Second, we provide an analysis of reverse logistics in product-transforming supply chains, in which the number of state and decisions variables grows with the square of the number of locations. This is in contrast other multiechelon inventory problems studied in the literature in which that number grows linearly with the number of locations in the system. Having three different product flows necessitates the resolution of multi-dimensional boundaries of the feasible region that persist even under the echelon formulation, while the analysis of product-transforming supply chains requires the development of a heuristic policy capable of selecting successive replenishment opportunities among a large number of available ones at each location. Finally, our paper is the first one to delineate the impact of demand variability versus demand volatility on the optimal reverse logistics inventory policy.

There are aspects of reverse logistics not addressed in our work, such as the practice of remanufacturing. Noteworthy papers on multi-period remanufacturing models include Zhou et al., (2011), who consider multiple types of product returns differentiated by remanufacturing and storage costs, and DeCroix (2006) who allows for a recovery facility that receives a stochastic amount of used products, where they can be stored, disposed, or remanufactured. While those papers make important contributions to the theory of remanufacturing, they allow only for regular product orders and do not consider expedited and reverse flow of orders. In our model, we do not consider remanufacturing; instead, our focus is on the multiple product flows aspect of reverse logistics, and the corresponding control of those flows in the supply chain.

2. Model Formulation: The Logistics Supply Chain

We consider a centralized supply system with N locations arranged in series. Units ordered from an outside supplier arrive at location N, the most upstream location in the system. Stochastic customer demand D_t occurs at location 1 in each period t. Every period, three decisions are made at each location, regarding how many units to: (1) order by expedited flow *from* the next location upstream; (2) order by regular flow *from* the next location upstream; and (3) send by reverse flow *into* the next location upstream.

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Expedited orders are executed and received in the same period. Regular orders and reverse orders have a one-period delivery leadtime between adjacent locations. Having single-period leadtimes between locations is common to the literature dealing with multiple flows of product in the system (i.e., Fukuda 1961, Lawson and Porteus 2000). This assumption covers a variety of settings in which the review period is longer than the shipment/replenishment period. A multi-stage supply system with a single-period leadtime is effectively equivalent to any model in which the replenishment leadtime is shorter than the review/reorder period, which, in the continental US, covers most systems that have a weekly review/reorder cycle. (In Appendix B, we discuss settings in which our model can be extended to include multi-period leadtimes.)

We use the following notation for state and decision variables in the system:

x_{jt} : on-hand inventory observed at location j if $j > 1$ (the net inventory if $j = 1$), at the beginning of period t, prior to making any decisions;

X_{jt}^E : number of units expedited into location j from location $j+1$, in period t;

X_{jt}^R : number of units reverse-ordered into location $j+1$ from location j in period t;

X_{jt} : number of units regular-ordered into location j from location $j+1$ in period t.

The vector $\mathbf{x}_t$ of all on-hand inventories $\{x_{jt}\}$ is *the inventory state*, being the state of the system in period t; vector $\mathbf{X}_t^E$ of all expedited orders $\{X_{jt}^E\}$ is *the expedited order schedule*; vector $\mathbf{X}_t^R$ of all reverse orders $\{X_{jt}^R\}$ is *the reverse order schedule*; and vector $\mathbf{X}_t$ of all regular orders $\{X_{jt}\}$ is *the regular order schedule*. Vectors $\mathbf{X}_t^E$, $\mathbf{X}_t^R$, and $\mathbf{X}_t$ represent the decisions in the system in each period t. Figure 1 displays those states, decisions, and product flow directions in the system.

In our model, we allow customer demands to be modulated by an exogenous Markov chain. Markov-modulated demand has been used in inventory theory to model the influence of an uncertain external environment (e.g., Song and Zipkin 1993, Sethi and Cheng 1997; and Chen and Song 2001). Atali and Özer (2012) describe various dynamic changes in demand, such as obsolescence and cyclic demand, that can be captured with a distribution function that is dependent on the state of an exogenous Markov chain. Accordingly, we assume that there exists a countable Markov chain $\{\omega_t\}$ such that an exogenous state ω_t, determined independently of any decisions, can impact the demand distribution in each period t. We make this assumption to capture the non-stationary nature of customer demand. As revealed later in our paper, such demand variability turns out to be a key driver of the value of reverse logistics.

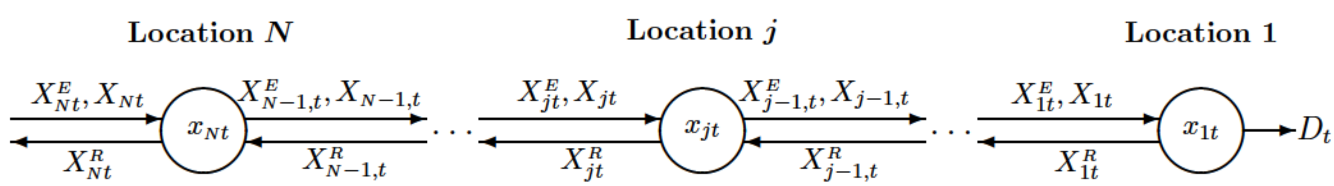

Figure 1. States, Decisions, and Product Flow Directions in the System

The sequence of events in each period t is: (1) Units regular-ordered and reverse-ordered in the previous period arrive; (2) Inventory state $\mathbf{x}_t$ and Markov state ω_t are observed; (3) Expedited orders $\mathbf{X}_t^E$ are placed, starting at the most upstream location and moving downstream, in the manner that a received expedited order at each location becomes available to be expedited further down the supply chain in the same period; (4) Reverse orders $\mathbf{X}_t^R$ are placed and they depart their locations of origin; (5) Regular orders $\mathbf{X}_t$ are placed; (6) Customer demand is realized and satisfied from available inventory; (7) Costs are incurred; and (8) Regular orders depart their locations of origin. This sequence of events defines the feasible set $\mathbb{X}(\mathbf{x}_t)$ (where $[x]^+ := \max(x, 0)$, and $[x]^- := \min(x, 0)$):

$$\mathbb{X}(\mathbf{x}_t) = \left\{ \mathbf{X}_t^E, \mathbf{X}_t^R, \mathbf{X}_t \;\middle|\; \begin{array}{ll} X_{jt}^E \in \left[0, x_{j+1,t} + X_{j+1,t}^E\right] & \text{for } j \in [1, N-1]; \\ X_{1t}^R \in \left[0, [x_{1t} + X_{1t}^E]^+\right] & \text{for } j = 1; \\ X_{jt}^R \in \left[0, x_{jt} + X_{jt}^E - X_{j-1,t}^E\right] & \text{for } j \in [2, N]; \\ X_{jt} \in \left[0, x_{j+1,t} - X_{j+1,t}^R + X_{j+1,t}^E - X_{jt}^E\right] & \text{for } j \in [1, N-1]. \end{array} \right. \tag{1}$$

The expediting decision at each location is limited to the inventory available at the next location upstream, after that upstream location has received its expedited order. Because x_{1t} represents the on-hand inventory *net* of the backlogged demand, it is allowed to be negative, and thus so is $x_{1t} + X_{1t}^E$, while the number of units returned upstream is always a positive number. Hence, X_{1t}^R,is bound from above by $\left[x_{1t} + X_{1t}^E\right]^+$. The number of units returned upstream from any other location is constrained by the on-hand inventory at that location after all expedited orders have been received. The regular order into each location is subject to the stock available immediately upstream, following departures and arrivals of all expedited orders, and departures (but not arrivals) of all reverse flow orders. We use $D_t(\omega)$ to highlight that demand D_t depends on the underlying Markov state ω. The state transition equations thus become

$$x_{j,t+1} = \begin{cases} x_{1t} - X_{1t}^R + X_{1t}^E + X_{1t} - D_t(\omega) & \text{if } j = 1, \\ x_{jt} - X_{jt}^R + X_{j-1,t}^R + X_{jt}^E - X_{j-1,t}^E + X_{jt} - X_{j-1,t} & \text{if } j > 1. \end{cases} \tag{2}$$

Inventory at the most downstream location is depleted by returned orders and customer demands. Inventory at all other locations is drawn down by the downstream flow of regular and expedited orders, and upstream flow of reverse orders. Inventory at each location is replenished through the downstream flow of regular and expedited orders and upstream flow of returned products.

Costs

A unit holding cost H_j is charged on inventory at location j in each period. In a logistics supply chain, items that arrive by upstream, reverse-order flow into any location are physically indistinguishable from items that arrive into that location by downstream flow of regular and expedited orders. Hence, the holding cost is the same for each unit of product at a particular location, regardless of whether that unit arrived there by expedited, reverse, or regular flow. This is because the unit holding cost is generally composed of the financial holding cost and the physical holding cost (e.g., Zipkin 2000 and Cachon and Terwiesch 2013). The financial holding cost represents the opportunity cost, exogenously given and related to the finished product itself, whereas the physical holding cost is associated with the cost of maintaining a unit of inventory (labor, warehousing, refrigeration, etc.), and is therefore location-specific. Without any physical transformation of the product occurring in the supply chain (other than possibly at the most upstream location), a unit holding cost will depend mostly on the location of an item, and not on how that item arrived to that location. As a result, the unit holding cost is identical for all items at a particular location. Other papers on reverse logistics in multi-stage systems (Fukuda 1961, DeCroix et al. 2005, and DeCroix and Zipkin 2005) similarly assume a logistics supply chain, so that unit holding costs in those papers are also the same for all items at a particular location, regardless of their prior trajectory through the system.

Unsatisfied demand is fully backlogged at unit cost p in each period. Further, each unit regular-ordered into location j incurs positive cost k_j, and each unit expedited into location j incurs positive cost k_j^E. Expedited orders cost more than regular ones; thus $k_j^E \geq k_j$ for all j. With regard to the expediting function, our formulation captures a variety of supply chain settings: from those in which inventory can be expedited from the outside supplier all the way to the most downstream location within a single period, (i.e., the practice of "drop-shipping", observed in a number of industries), to those in which an expedited product may take several periods (though still fewer than through regular ordering) to do the same, to those in which no expediting can take place whatsoever, and all other supply chain settings in between.

Each unit reverse-ordered into location $j+1$ from location j incurs cost k_j^R, which is positive at locations 1 through $N-1$. To capture the fact that, in IB's supply chain, excess inventory at location N can be resold to a local distributor, we allow excess inventory that exits supply chain at location N to generate revenue. Thus, k_N^R is allowed to be negative to reflect the unit revenue earned through reselling of excess inventory out of location N. At the same time, in order to prohibit the bringing of inventory into the supply chain at location N solely for the purpose of reselling it back upstream in the next period, rather than to satisfy downstream customer demand, we assume that $k_N \geq -k_N^R$. Otherwise, a supply chain could make unlimited profits by ordering inventory into location N and then immediately reselling it back. This assumption also implies that $k_N^E \geq -k_N^R$, since $k_N^E \geq k_N$. The one-period cost function is

$$p\,\mathbb{E}_\omega\Big[\big(D_t(\omega) - (x_{1t} + X_{1t}^E - X_{1t}^R)\big)^+\Big] + H_1\,\mathbb{E}_\omega\Big[\big(x_{1t} + X_{1t}^E - X_{1t}^R - D_t(\omega)\big)^+\Big]$$
$$+ \sum_{j=1}^{N}\big(k_j^E X_{jt}^E + k_j^R X_{jt}^R + k_j X_{jt}\big) + \sum_{j=2}^{N} H_j\big(x_{jt} - X_{j-1,t}^E + X_{jt}^E - X_{jt}^R + X_{j-1,t}^R\big),$$

where the expectation $\mathbb{E}_\omega$ is taken over $D_t(\omega)$, given ω. Let T be the time horizon for the problem and $F_t(\omega, \mathrm{x}_t)$ denote the minimum expected net present value of the costs over periods t through T, as of the beginning of period t, as a function of $\mathbf{x}_t$ and ω in period t. Let α be the single-period discount rate (with $\alpha < 1$), and $h_j := H_j - H_{j+1}$. Thus, we obtain the following optimality equations for $F_t(\omega, \mathbf{x}_t)$:

$$F_t(\omega, \mathbf{x}_t) = \min_{\mathbf{X}_t^E, \mathbf{X}_t^R, \mathbf{X}_t \in \mathbb{X}(\mathbf{x}_t)} V_t\big(\omega, \mathbf{x}_t, \mathbf{X}_t^E, \mathbf{X}_t^R, \mathbf{X}_t\big) \tag{3}$$

$$\begin{aligned} V_t\big(\omega, \mathbf{x}_t, \mathbf{X}_t^E, \mathbf{X}_t^R, \mathbf{X}_t\big) = \sum_{j=1}^{N} \Big[\big(k_j^E + h_j\big) X_{jt}^E + \big(k_j^R - h_j\big) X_{jt}^R + k_j X_{jt} + H_j x_{jt} \Big] & \\ + \gamma_t\big(\omega, x_{1t} + X_{1t}^E - X_{1t}^R\big) + \alpha\, \mathbb{E}_\omega \big[F_{t+1}(\omega_{t+1}, \mathbf{x}_{t+1})\big], & \end{aligned} \tag{4}$$

where $\gamma_t(\omega, x) := (p+H_1)\mathbb{E}_\omega[(D_t(\omega) - x)^+] - H_1 \mathbb{E}_\omega[D_t(\omega)]$. The expectation $\mathbb{E}_\omega$ in (4) is over both $D_t(\omega)$ and ω_{t+1}, given ω. Because the dynamic program in (3) and (4) employs installation quantities, we refer to it as *the installation formulation* of the reverse logistics model. Note that the inventory state space has N dimensions and that optimal decision vectors $\mathbf{X}_t^E$, $\mathbf{X}_t^R$, and $\mathbf{X}_t$ all depend on all the variables in the state space, as well as on each other. In addition, the boundary of the feasible region is multi-dimensional. This renders the problem practically impossible to solve numerically, even for systems with very few locations.

Echelon Formulation

We now reformulate the problem using the following change of variables:

$y_{jt} := x_{1t} + \cdots + x_{jt}$	referred to as *echelon j inventory*;
$Y_{jt}^E := y_{jt} + X_{jt}^E$	referred to as *post-expedite echelon j position*;
$Y_{jt}^R := Y_{jt}^E - X_{jt}^R$	referred to as *post-reverse order echelon j position*;
$Y_{jt} := Y_{jt}^R + X_{jt}$	referred to as *post-regular order echelon j position*.

Vector $\mathbf{y}_t$ of all echelon inventories is *the echelon (inventory) state*; vector $\mathbf{Y}_t^E$ of all post-expedite echelon positions is *the post-expedite echelon schedule*; vector $\mathbf{Y}_t^R$ of all post-reverse order echelon positions is *the post-reverse echelon schedule*; and vector $\mathbf{Y}_t$ of all post-regular order echelon positions is *the post-regular echelon schedule*. Thus, $\mathbf{Y}_t^E$, $\mathbf{Y}_t^R$, and $\mathbf{Y}_t$ are the new decision variables of the model. Transition equations become $y_{j,t+1} = Y_{jt} - D_t$. The one-period cost function becomes

$$\gamma_t(\omega_t, Y_{1t}^R) + \sum_{j=1}^{N} \Big[\big(k_j^E + k_j^R\big) Y_j^E + k_{jt} Y_{jt} - \big(k_j^R + k_j - h_j\big) Y_{jt}^R \Big] - \sum_{j=1}^{N} k_j^E\, y_{jt}, \tag{5}$$

where $h_j = H_j - H_{j+1}$ can now be interpreted as the unit echelon inventory holding cost. Let $\mathbb{Y}(\mathbf{y}_t)$ be the feasible set given $\mathbf{y}_t$. Using $Y_{N+1,t}^E := \infty$ and $Y_{N+1,t}^R := \infty$ for notational convenience,

$$\mathbb{Y}(\mathbf{y}_t) = \left\{ \mathbf{Y}_t^E, \mathbf{Y}_t^R, \mathbf{Y}_t \;\middle|\; \begin{array}{ll} Y_{jt}^E \in \big[y_{jt}, Y_{j+1,t}^E\big] & \text{for } j \in [1, N]; \\ Y_{1t}^R \in \big[[Y_{1t}^E]^-, Y_{1t}^E\big], \;\; Y_{jt}^R \in \big[Y_{j-1,t}^E, Y_{jt}^E\big] & \text{for } j \in [2, N]; \\ Y_{jt} \in \big[Y_{jt}^R, Y_{j+1,t}^R - (Y_{jt}^E - Y_{jt}^R)\big] & \text{for } j \in [1, N]. \end{array} \right. \tag{6}$$

Let $f_t(\omega, \mathbf{y}_t)$ denote the minimum discounted expected present value of the costs over periods t through

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T, as of the beginning of period t, as a function of ω and $\mathbf{y}_t$. Define $c_j^E := k_j^E + k_j^R$, $c_j^R := k_j^R + k_j - h_j$, and $c_j := k_j$. Let $\mathbf{Y}_t - D_t(\omega)$ be the vector $\{Y_{jt} - D_t(\omega)\}$. The optimality equations become

$$f_t(\omega, \mathbf{y}_t) = -\sum_{j=1}^{N} k_j^E\, y_{jt} + \min_{\mathbf{Y}_t^E, \mathbf{Y}_t^R, \mathbf{Y}_t \in \mathbb{Y}(\mathbf{y}_t)} v_t(\omega, \mathbf{Y}_t^E, \mathbf{Y}_t^R, \mathbf{Y}_t) \tag{7}$$

$$v_t(\omega, \mathbf{Y}_t^E, \mathbf{Y}_t^R, \mathbf{Y}_t) := \gamma_t(\omega, Y_{1t}^R) + \sum_{j=1}^{N}\Big(c_j^E Y_{jt}^E + c_j Y_{jt} - c_j^R Y_{jt}^R\Big) + \alpha\, \mathbb{E}_\omega\big[f_{t+1}\big(\omega_{t+1}, \mathbf{Y}_t - D_t(\omega)\big)\big] \tag{8}$$

We will refer to this dynamic program as *the echelon formulation* of the reverse logistics problem. Next, we say that a function $f : \mathbb{R}^N \to \mathbb{R}$ is *additively convex* if there exist convex functions $f_1, f_2, \ldots, f_N : \mathbb{R} \to \mathbb{R}$ such that $f(\mathbf{y}) = \sum_{j=1}^{N} f_j(y_j)$ for all $\mathbf{y} = \{y_1, y_2, \ldots, y_N\}$.

Assumption 1. *The terminal value function $f_{T+1}(\omega, \cdot)$ is additively convex for each ω.*

Assumption 1 holds, for example, when $f_{T+1}(\omega, \cdot)$ is zero, or linear in echelon inventories $y_{j,T+1}$. (Note that for the latter case, it suffices that $F_t(\omega, \mathbf{x}_{T+1})$ be linear in on-hand inventories x_{jt}.)

The Curse of Dimensionality

In general, the Clark-Scarf decomposition cannot be expected to hold for the objective cost function $f_t(\omega, \mathbf{y}_t)$ defined by the dynamic program in (7) and (8). This is because the upper boundary of the feasible set $\mathbb{Y}(\mathbf{y}_t)$ that constrains post-regular order echelon positions at each location j depends on three echelon quantities, $Y_{j+1,t}^R$, Y_{jt}^E, and Y_{jt}^R. (By comparison, each boundary in the feasible regions of Lawson and Porteus (2000) and Fukuda (1961), as well as most other multiechelon models found in the literature, depends only on a single echelon variable). In particular, in our model, each optimal post-expedite echelon position Y_{jt}^E will, in general, be a function of y_{jt}. Hence, so will each post-reverse order echelon position Y_{jt}^R. Further, since post-regular order echelon positions at each location j are constrained from above by echelon quantities $Y_{j+1,t}^R$, Y_{jt}^E, and Y_{jt}^R, then each Y_{jt} will, in general, be a function of both y_{jt}, and $y_{j+1,t}$. This dependence of each optimal post-regular order echelon position on two echelon inventories cascades from upper locations downward and renders each optimal decision dependent on the entire echelon state $\mathbf{y}_t$, instead of just a single echelon variable, and this results in the curse of dimensionality.

3. The Structure of the Optimal Policy

A Transformed Version of the Problem

Consider a problem identical to the original reverse logistics problem, but with a revised feasible set $\widetilde{\mathbb{Y}}(y_t)$:

$$\widetilde{\mathbb{Y}}(\mathbf{y}_t) = \left\{ \mathbf{Y}_t^E, \mathbf{Y}_t^R, \mathbf{Y}_t \;\middle|\; \begin{array}{ll} Y_{jt}^E \in \big[y_{jt}, Y_{j+1,t}^E\big] & \text{for } j \in [1, N]; \\ Y_{1t}^R \in \big[[Y_{1t}^E]^-, Y_{1t}^E\big],\;\; Y_{jt}^R \in \big[Y_{j-1,t}^E, Y_{jt}^E\big] & \text{for } j \in [2, N]; \\ Y_{jt} \in \big[Y_{jt}^R, Y_{j+1,t}^R\big] & \text{for } j \in [1, N]. \end{array} \right. \tag{9}$$

The feasible region $\widetilde{\mathbb{Y}}(\mathbf{y}_t)$ in the transformed problem represents a relaxation of the original feasible region $\mathbb{Y}(\mathbf{y}_t)$ because the upper constraint on Y_{jt} is relaxed from $Y^R_{j+1,t} - (Y^E_{jt} - Y^R_{jt})$ to just $Y^R_{j+1,t}$. (By definitions of $\mathbb{Y}(\mathbf{y}_t)$ and $\widetilde{\mathbb{Y}}(\mathbf{y}_t)$, $Y^R_{jt} \leq Y^E_{jt}$; hence, $Y^R_{j+1,t} \geq Y^R_{j+1,t} - (Y^E_{jt} - Y^R_{jt})$). This relaxation implies that we allow regular orders at each location j to go *long*, as regular orders can be placed even when the required inventory may not necessarily be available at location $j+1$, but rather in transit to that location.

Let $\tilde{f}_t(\omega, \mathbf{y}_t)$ be the minimum expected present value of the costs over periods t through T, as a function of ω and y_t, for this *the relaxed reverse logistics problem.* The new dynamic program becomes

$$\tilde{f}_t(\omega, \mathbf{y}_t) = -\sum_{j=1}^{N} k^E_j\, y_{jt} + \min_{\mathbf{Y}^E_t, \mathbf{Y}^R_t, \mathbf{Y}_t \in \widetilde{\mathbb{Y}}(\mathbf{y}_t)} \tilde{v}_t(\omega, \mathbf{Y}^E_t, \mathbf{Y}^R_t, \mathbf{Y}_t) \tag{10}$$

$$\tilde{v}_t(\mathbf{Y}^E_t, \mathbf{Y}^R_t, \mathbf{Y}_t) := \gamma_t(\omega, Y^R_{1t}) + \sum_{j=1}^{N}\Big(c^E_j Y^E_{jt} + c_j Y_{jt} - c^R_j Y^R_{jt}\Big) + \alpha\, \mathbb{E}_\omega\big[\tilde{f}_{t+1}\big(\omega_{t+1}, \mathbf{Y}_t - D_t(\omega)\big)\big].$$

The Unidirectional Property of the Optimal Policy

A policy $(\mathbf{X}^E_t, \mathbf{X}^R_t, \mathbf{X}_t)$ is said to be *unidirectional* in period t, if, at each location j, either $X^E_{jt} = X_{jt} = 0$ or $X^R_{jt} = 0$ (or both). Thus, under a unidirectional policy, the product can only flow either upstream or downstream at each location, but not in both directions in the same period. Our first result pertains to this unidirectional property of the optimal policy. All proofs are deferred to Appendix A.

Theorem 1. *For each ω and t, an optimal policy for the relaxed reverse logistics problem is unidirectional.*

Corollary 1. *If $\big(\mathbf{Y}^E_t, \mathbf{Y}^R_t, \mathbf{Y}_t\big)$ are optimal echelon schedules for the relaxed reverse logistics problem in period t, then either $Y^E_{jt} = y_{jt}$ and $Y_{jt} = Y^R_{jt}$, or $Y^R_{jt} = Y^E_{jt}$ at each location j, and for every $\mathbf{y}_t$ and ω.*

Thus, at each location j, the product optimally flows either (i) downstream by regular or expedited order, or (ii) upstream by reverse order, but not both in the same period. While the idea behind Theorem 1 might seems intuitive, to analytically demonstrate the optimality of a unidirectional policy under any set of cost parameters and possible sequence of Markov states is far from straightforward. The proof of Theorem 1 relies on a series of intermediate results, presented in Propositions 1 and 2 in Appendix A.

Implications for the Solution of the Reverse Logistics Problem

We now demonstrate how the relaxed reverse logistics problem leads to the solution for the original problem.

Theorem 2. *Let $\big(\mathbf{Y}^E_t, \mathbf{Y}^R_t, \mathbf{Y}_t\big)$ be optimal echelon schedules for the original reverse logistics problem and let $\big(\widetilde{\mathbf{Y}}^E_t, \widetilde{\mathbf{Y}}^R_t, \widetilde{\mathbf{Y}}_t\big)$ be optimal echelon schedules for the relaxed reverse logistics problem in period t. If $f_{T+1}(\omega_{T+1}, \cdot) = \tilde{f}_{T+1}(\omega_{T+1}, \cdot)$ for all ω_{T+1}, then the following hold in each period t:*
(a) $\big(\mathbf{Y}^E_t, \mathbf{Y}^R_t, \mathbf{Y}_t\big) = \big(\widetilde{\mathbf{Y}}^E_t, \widetilde{\mathbf{Y}}^R_t, \widetilde{\mathbf{Y}}_t\big)$;
(b) $f_t(\omega, \mathbf{y}_t) = \tilde{f}_t(\omega, \mathbf{y}_t)$, *for all ω.*

When terminal value functions are the same, the optimal policy for the relaxed reverse logistics problem is identical to that for the original reverse logistics problem, and the objective cost functions are equal in each period. Thus, in order to find the solution to the original reverse logistics problem, it suffices to solve the problem with the feasible set $\widetilde{\mathbb{Y}}(\mathbf{y}_t)$ given in (9). Due to the equivalence established in Theorem 2, the solution for the reverse logistics problem will be exactly the same as that for the for original reverse logistics problem. In that manner, we have reduced an optimization problem with multidimensional boundaries of the feasible region to an optimization problem with single-dimensional boundaries of the feasible region.

Preservation of Additive Convexity

Next, we establish the preservation of additive convexity of the objective cost function and derive explicit expressions for its constituent component cost functions. In doing so, we make use the following notation: for any convex function $f : \mathbb{R} \to \mathbb{R}$ with a finite minimizer S, we define f^+ and f^- as:

$$f^+(x) := \begin{cases} f(S) & \text{if } x \leq S; \\ f(x) & \text{otherwise;} \end{cases} \qquad \text{and} \qquad f^-(y) := \begin{cases} f(y) - f(S) & \text{if } y \leq S; \\ 0 & \text{otherwise.} \end{cases}$$

The functions f^+ and f^- defined in this manner are thus convex increasing and convex decreasing.

Theorem 3. (Preservation) *The following hold in every period t.*

(a) $\tilde{f}_t(\omega, \cdot)$ is additively convex for each ω, so that there exist convex functions $\{\tilde{f}_{1t}, \ldots, \tilde{f}_{Nt}\}$ such that $\tilde{f}_t(\omega, \mathbf{y}_t) = \sum_{j=1}^{N} \tilde{f}_{jt}(\omega, y_{jt})$ for each ω.

(b) Define functions $\{g_{1t}, \ldots, g_{Nt}\}$, for each ω, as

$$g_{jt}(\omega, Y) := c_j Y + \alpha\, \mathbb{E}_\omega\big[\tilde{f}_{j,t+1}\big(\omega_{t+1}, Y - D_t(\omega)\big)\big]. \tag{11}$$

Given $g_{1t}, \ldots, g_{Nt}$, let functions $\{u_{1t}, \ldots, u_{Nt}\}$ be defined as

$$u_{jt}(\omega, Y) := \begin{cases} \gamma_t(\omega, Y) - c_1^R Y + g_{1,t}^+(\omega, Y) & \text{if } j = 1; \\ -c_j^R Y + g_{jt}^+(\omega, Y) + g_{j-1,t}^-(\omega, Y) & \text{if } j \in [2, N]. \end{cases} \tag{12}$$

Given $u_{1t}, \ldots, u_{Nt}$, let functions $v_{1t}(\omega, \cdot), \ldots, v_{Nt}(\omega, \cdot)$ be defined as

$$v_{jt}(\omega, Y) := \begin{cases} c_1^E Y + u_{1t}^+(\omega, [Y]^-) + u_{1t}^-(\omega, Y) + u_{2t}^+(\omega, Y) & \text{if } j = 1; \\ c_j^E Y + u_{jt}^-(\omega, Y) + u_{j+1,t}^+(\omega, Y) & \text{if } j \in [2, N-1]; \\ c_N^E Y + u_{Nt}^-(\omega, Y) & \text{if } j = N. \end{cases} \tag{13}$$

Given $v_{1t}, \ldots, v_{Nt}$, let functions $w_{1t}(\omega, \cdot), \ldots, w_{Nt}(\omega, \cdot)$ be defined recursively as

$$w_{jt}(\omega, Y) := v_{jt}(\omega, Y) + w_{j-1,t}^-(\omega, Y), \tag{14}$$

with $w_{0t}(\omega, Y) := 0$. Then, $g_{jt}(\omega, \cdot)$, $u_{jt}(\omega, \cdot)$, $v_{jt}(\omega, \cdot)$, and $w_{jt}(\omega, \cdot)$ are convex for each j and ω. Further, *for each echelon j and state ω, $\tilde{f}_{jt}(\omega, y_{jt}) := -k_j^E y_{jt} + w_{jt}^+(\omega, y_{jt})$.*

Theorem 3 completes the process by which we have eliminated the curse of dimensionality inherent in the original reverse logistics problem by reducing the multi-variable objective cost function for this problem to a sum of single-variable, convex, component cost functions. Further, Theorem 3 provides explicit expressions for those component cost functions, so that they become straightforward to calculate using the recursive relationship given in Expressions (11) - (14). We have thus rendered the reverse logistics problem for logistics supply chains numerically (and analytically) tractable.

It is worth noting that the proof of Theorem 3 relies a new, multi-dimensional generalization (see Lemma 1 in Appendix A) of the original result due to Karush (1959) that is typically used to establish the Clark-Scarf decomposition in the multiechelon inventory literature (e.g., Lawson and Porteus 2000).

Structure of the Optimal Policy

The following theorem identifies the structure of the optimal policy for the reverse logistics problem in period t, assuming that the objective function in period $t+1$ is additively convex.

Theorem 4. (Optimality) *Let $g_{jt}(\omega, \cdot)$, $u_{jt}(\omega, \cdot)$, $v_{jt}(\omega, \cdot)$, and $w_{jt}(\omega, \cdot)$ be as defined as in Theorem 3.*

(a) Let $S_{jt}^E(\omega) := \inf \arg\min_Y w_{jt}(\omega, Y)$ for each ω and $j = 1, \ldots, N$. The optimal post-expedite echelon schedule $\widehat{\mathbf{Y}}_t^E$ in period t is then given recursively, starting with $j = N$, by

$$\widehat{Y}_{jt}^E(\omega) = \begin{cases} \max\left[y_{jt}, \min\left(S_{jt}^E(\omega), \widehat{Y}_{j+1,t}^E\right)\right] & \text{if } j \in [1, N-1]; \\ \max\left[y_{Nt}, S_{Nt}^E(\omega)\right] & \text{if } j = N. \end{cases} \tag{15}$$

(b) Let $S_{jt}^R(\omega) := \sup \arg\min_Y u_{jt}(\omega, Y)$ for each ω and $j = 1, \ldots, N$. Then, given $\widehat{\mathbf{Y}}_t^E$, the optimal post-reverse order echelon schedule $\widehat{\mathbf{Y}}_t^R$ in period t is given by

$$\widehat{Y}_{jt}^R(\omega) = \begin{cases} \max\left[[\widehat{Y}_{1t}^E]^-, \min\left(S_{1t}^R(\omega), \widehat{Y}_{1t}^E\right)\right] & \text{if } j = 1; \\ \max\left[\widehat{Y}_{j-1,t}^E, \min\left(S_{jt}^R(\omega), \widehat{Y}_{jt}^E\right)\right] & \text{if } j \in [2, N]. \end{cases} \tag{16}$$

(c) Let $S_{jt}(\omega) := \sup \arg\min_Y g_{jt}(\omega, Y)$ for each ω and $j = 1, \ldots, N$. Given $\widehat{\mathbf{Y}}_t^R$, $\widehat{\mathbf{Y}}_{jt}$ is given by

$$\widehat{Y}_{jt}(\omega) = \begin{cases} \max\left[\widehat{Y}_{jt}^R, \min\left(S_{jt}(\omega), \widehat{Y}_{j+1,t}^R\right)\right] & \text{if } j \in [1, N-1]; \\ \max\left[\widehat{Y}_{Nt}^R, S_{Nt}(\omega)\right] & \text{if } j = N. \end{cases} \tag{17}$$

Because, by Theorem 4, the optimal decisions for each product flow are sequentially nested within each other, we refer to the optimal policy given in Theorem 4 as *a Nested Echelon Basestock (NEBS)* policy.

In general, in multiechelon models, an ordering of basestock level is difficult to establish because any such ordering can be expected to vary with non-stationarity characteristics of model parameters.

Neverthess, for our reverse logistics problem, we are able to identify an explicit ordering of basestock levels for the three product flows. This ordering of optimal basestock levels is a key result of our paper, as it serves as a building block for the results that follow, including those about the impact of demand variability on the value of reverse logistics and those pertaining to our analysis of product-transforming supply chains.

Theorem 5. *At every location j in each period t, $S^E_{jt}(\omega) \leq S_{jt}(\omega) \leq S^R_{jt}(\omega)$ for each ω.*

To provide intuition for Theorem 5, suppose, for the sake of expositional ease, that it is not optimal to place any expedited orders into some location j in period t for some state ω. The second inequality, $S_{jt}(\omega) \leq S^R_{jt}(\omega)$, implies that there exists a *nonempty* closed interval $\left[S_{jt}(\omega), S^R_{jt}(\omega)\right]$ that can be viewed as a "target interval". In particular, it follows from Theorem 4 that, starting with echelon inventory y_{jt} at the beginning of period t, it is optimal to make the smallest inventory change that puts that echelon inventory (after the change, at the end of period t) into the target interval $\left[S_{jt}(\omega), S^R_{jt}(\omega)\right]$, or as close to it as possible given the feasibility constraints (i.e., the ending inventory has to remain between $y_{j-1,t}$ and $y_{j+1,t}$). That is, if echelon inventory y_{jt} is below S_{jt} then it is optimal to bring that echelon inventory level up (as close as possible) to S_{jt}; if y_{jt} is above $S^R_{jt}(\omega)$, then it is optimal to bring it down (as close as possible) to $S^R_{jt}(\omega)$; and if y_{jt} is between S_{jt} and $S^R_{jt}(\omega)$, it is optimal to leave it unchanged. Hence, Theorems 4 and 5 imply that the optimal inventory policy represents a *target interval policy*. Those theorems therefore generalize the equivalent results obtained in the literature for single-stage systems with reverse orders (Morton 1978).

The target interval policy implied by Theorems 4 and 5 is made possible by the particular ordering of $S_{jt}(\omega)$ and $S^R_{jt}(\omega)$ (i.e., $S_{jt}(\omega) \leq S^R_{jt}(\omega)$). If $S^R_{jt}(\omega) < S_{jt}(\omega)$ were true for some j and t, then the interval $\left[S^R_{jt}(\omega), S_{jt}(\omega)\right]$ would not lead to the target interval policy. Here is why: in that case, any y_{jt} such that $y_{jt} \geq S_{jt}(\omega)$ would lead to $y_{jt} > S^R_{jt}(\omega)$; thus, it would be optimal to bring y_{jt} down to $S^R_{jt}(\omega)$ while incurring a reverse order cost. But, bringing y_{jt} down to $S^R_{jt}(\omega)$ would lead to $y_{jt} < S_{jt}(\omega)$, so that it would then be necessary to bring y_{jt} back up to $S_{jt}(\omega)$, while incurring a regular order cost. Such order placement violates the definition of the target interval policy. Further placing such mutually-cancelling orders clearly incurs excess costs while resulting in the same echelon inventory state at the end of period t.

4. Impact of Demand Variability on Reverse Logistics

We now demonstrate that the presence of demand variability matters more for reverse logistics than the presence of demand volatility. We begin by establishing the existence of a stationary optimal policy.

Assumption 2. *For any given Markov state ω, demands are IID across all time periods.*

A Stationary Optimal Policy

Let $\pi := \left\{\left(\mathbf{Y}^E_1, \mathbf{Y}^R_1, \mathbf{Y}_1\right), \left(\mathbf{Y}^E_2, \mathbf{Y}^R_2, \mathbf{Y}_2\right), \ldots\right\}$ be an arbitrary infinite horizon policy for the original reverse logistics problem, and $\mathcal{V}_t(\pi)$ be the expected discounted present value of costs of implementing that policy, starting in period t. Let $\mathcal{F}_t$ be the smallest of those over the feasible space $\mathbb{Y}$. Then, we have

$$\mathcal{V}_t(\omega_t,\mathbf{y}_t,\pi)=\mathbb{E}_{\omega_t}\big[(p+H_1)(D_t(\omega_t)-Y^R_{1t})^+\big]+\sum_{j=1}^{N}\big(c^E_jY^E_{jt}+c_jY_{jt}-c^R_jY^R_{jt}-k^E_j\,y_{jt}\big)$$
$$+\lim_{T\to\infty}\sum_{\tau=t+1}^{T}\alpha^{\tau-t}\,\mathbb{E}_{\omega_\tau,\omega_{\tau-1},\ldots,\omega_t}\Big[(p+H_1)(D_\tau(\omega_\tau)-Y^R_{1\tau})^++\sum_{j=1}^{N}\big(c^E_jY^E_{j\tau}+c_jY_{j\tau}-c^R_jY^R_{j\tau}-k^E_j(Y_{j,\tau-1}-D_\tau(\omega_\tau))\big)\Big]$$
$$\mathcal{F}_t(\omega_t,\mathbf{y}_t)=\inf_{\pi\,\in\,\{\pi\,|\,\mathbf{Y}^E_\tau,\mathbf{Y}^R_\tau,\mathbf{Y}_\tau\in\mathbb{Y}(\mathbf{y}_\tau),\,\tau=t,t+1,\ldots\}}\mathcal{V}_t(\omega_t,\mathbf{y}_t,\pi). \tag{18}$$

Theorem 6. *There exist: (1) stationary, additively convex function $\mathcal{F}$ such that $\mathcal{F}_t(\omega,\cdot)=\mathcal{F}(\omega,\cdot)$ for all t and ω; and (2) stationary, NEBS policy optimal for $\mathcal{F}$, such that $S^E_j(\omega)\le S_j(\omega)\le S^R_j(\omega)$ for all j and ω.*

Thus, for each Markov state ω, the NEBS policy optimal for the finite-horizon reverse logistics problem converges to a stationary NEBS policy that is optimal for the infinite-horizon, reverse logistics problem, and whose basestock levels satisfy $S^E_j(\omega)\le S_j(\omega)\le S^R_j(\omega)$ for all j and ω. We now extend those stationarity results to the original, finite-horizon, reverse-logistics problem.

Theorem 7. *Let $\big\{S^E_j(\omega),S^R_j(\omega),S_j(\omega)\big\}_{j,\omega}$ be the basestock levels of a stationary NEBS policy $\big(\widetilde{\mathbf{Y}}^E_j,\widetilde{\mathbf{Y}}^R_j,\widetilde{\mathbf{Y}}_j\big\}$ optimal for the infinite-horizon reverse logistics problem. Then, for any finite time horizon T, there exists a terminal value function $F_{T+1}(\omega,\cdot)$ such that $\mathbf{Y}^E_{jt}=\widetilde{\mathbf{Y}}^E_j$, $\mathbf{Y}^R_{jt}=\widetilde{\mathbf{Y}}^R_j$, and $\mathbf{Y}_{jt}=\widetilde{\mathbf{Y}}_j$ is optimal and $F_t(\omega,\cdot)$ is additively convex in each period t for the finite-horizon reverse logistics problem with time horizon T.*

Hence, under IID demand, there exists a *stationary*, NEBS policy that is optimal policy for the original, finite-horizon, reverse logistics problem considered in this paper. The existence of an optimal stationary NEBS policy turns out to have important implications for the value of reverse logistics.

Implications for the Value of Reverse Logistics

Under the stationary optimal policy given in Theorem 7, when it comes to the value of reverse logistics, it turns out to be meaningful to distinguish between demand uncertainty arising from the random nature of demand in each period (i.e., volatility of demand) and the uncertainty arising from the changing nature of demand across periods (i.e., variability of demand). The reason is found in the following theorem.

Theorem 8. *Let $\big\{S^E_j(\omega),S^R_j(\omega),S_j(\omega)\big\}_{j,\omega}$ be the basestock levels of a stationary, optimal nested echelon basestock policy given in Theorem 7. Then, in the absence of Markov-modulation, the following hold:*
(a) If the initial echelon inventory state $\mathbf{y}_1$ in period 1 is such that $y_{j1}\le S^R_j$ at every location j, then it is never optimal to reverse order the product at any location in the system;
(b) Suppose there exists a location i such that $y_{i1}>S^R_i$. Let j be the smallest such location i. Then, after period $N-j+1$, it is never optimal to reverse-order the product at any location in the system.

Corollary 2. *Under a stationary optimal policy, the value of the option to reverse-order product in the supply chain (after at most the first N periods) is zero.*

Thus, in the absence of demand variability, it is effectively never optimal to return inventory upstream. In that case, reverse logistics creates no value for the supply chain, regardless of the level of demand volatility. This finding stands in contrast to the conventional wisdom found in practitioner journals and the academic literature, according to which reverse logistics creates value under high demand volatility (e.g., Jaaron and Backhouse, 2016, Lieckens and Vandaele 2007). One reason for this is that our paper, to the best of our knowledge, is the only one so far to jointly optimize multiple flows of product in the presence of reverse logistics, and our conclusions derive from that optimal policy. By comparison, other papers focus on different elements of reverse logistics such as stochastic leadtimes and customer involvement.

Theorem 8 also highlights the role and importance of Markov modulation in our model: it is used to introduce demand variability across time periods, without which reverse logistics creates effectively no value in the supply chain. Further, as illustrated in our numerical studies in the next section, the higher this demand variability, the higher the value of reverse logistics in the supply chain. Note that demand *volatility* (i.e. the uncertainty of demand in each period) is always present in our model regardless of demand variability. Even in the absence of demand variability, demand volatility continues to influence the optimal policy through its impact on cost functions, basestock levels, and state transition equations.

It is worth noting that the structure of the optimal policy given in Theorem 4 is not sufficient, by itself, to establish Theorem 8. What is also needed is the ordering of the optimal basestock levels given in Theorem 5, which is key to proving Theorem 8. Consider, for example, Fukuda (1961), which considers a special case of our logistics supply chain (he addresses a stationary system without expedited orders). Fukuda (1961) proves the optimality of a simpler echelon basestock policy than we have in our paper (the term 'disposal' in his paper is exactly equivalent to our 'reverse order'). However, because he does not establish the ordering of the optimal basestock levels, he does not arrive at the conclusion that reverse orders never actually happen. As a result, Fukuda (1961) discusses reverse orders as if they were occurring in the system, and the reader is left with the impression that reverse orders in his paper do actually happen. In fact, as shown in our Theorem 5, they never really do under the optimal policy.

In summary, when it comes to reverse logistics and companies seeking to grow this capability in their supply chains, it becomes crucial to: (1) differentiate a single-period's demand volatility from demand variability across time; (2) recognize that, in the absence of the latter, such capability may not be worth investing in; and (3) develop the means to quantify the value of reverse logistics in the presence of demand variability. Our results so far highlight the importance of tasks (1) and (2). We now address task (3).

5. Quantifying the Value of Reverse Logistics

In this section, we quantify the value of reverse logistics for the supply chain with multiple flows of product. For each set of model parameters, we solve four models: the classical Clark and Scarf setting with the regular flow only (the CS system), with regular and reverse flows (the $CS+R$ system), with regular and expedited flows (the $CS+E$ system), and with all three flows: regular, expedited, and reverse(the $CS+E+R$ system). All cost savings are reported in percentage terms relative to the CS system.

Demand Volatility and Variability

We use three Markov states that correspond to the low-, normal-, and high-demand scenarios. Normal-demand scenario in each period is Poisson-distributed with mean $\mu = 6$. To distinguish between demand volatility (i.e., demand uncertainty in each period,) and demand variability (i.e., demand uncertainty across time periods) we first identify the variance of this normal-demand scenario as our demand volatility. Thus, given our Poisson distribution, demand volatility in each period is equal to 6, which is what the variance of the underlying demand distribution would be, in each period, in the absence of Markov modulation. In our numerical studies, Markov states impact the demand distributions through the multiplier of the mean demand: given a Markov state ω, the demand in each period is Poisson-distributed with mean of $\beta(\omega)\mu$, where $\mu = 6$. Transition probabilities and the resulting steady-state probabilities are shown in Table 1.

State ω	Transition Probs. 1	2	3	Steady-State Probs.	Markov Mult. $\beta(\omega)$
1	0.85	0.10	0.05	0.4167	0.33
2	0.25	0.50	0.25	0.1667	1.0
3	0.05	0.10	0.85	0.4167	3.0

Table 1. Markov States, Transition and Steady-state Probabilities, and Demand Multipliers

In the numerical studies that follow, we keep this transition probability matrix, and the resulting stead-state probabilities, as fixed. Also shown in Table 1 are multipliers $\beta(\omega)$ for our base case model. One way to represent demand variability is by the average variance of the underlying demand distribution based on the steady-state probabilities for the three Markov states in Table 1. Thus, for the base case model with Markov demand multipliers of 0.33, 1, and 3, demand variability is equal to 9.33 ($= 0.4167 \cdot 0.33 \cdot 6 + 0.1667 \cdot 1.0 \cdot 6 + 0.4167 \cdot 3.0 \cdot 6$). We make use of this measure of demand variability and explore the impact of increasing demand variability by varying Markov-multipliers for the three Markov-states.

Three-location System

We start with a three-location supply chain whose unit costs are: $(k_1, k_2, k_3) = (1, 1, 1)$; $(k_1^R, k_2^R, k_3^R) = (1, 1, -0.8)$; $(k_1^E, k_2^E, k_3^E) = (4, 4, 4)$; $(H_1, H_2, H_3) = (1.5, 1.0, 0.5)$ (thus, $h_1 = h_2 = h_3 = 0.5$); and $p = 15$. Our unit backlogging and holding costs are thus in line with numerical studies for multi-stage inventory systems found in the literature (e.g, Chao et al., 2014). The discount factor is 0.95. We calculate expected costs for each Markov state, and compute the weighted average based on their steady-state probabilities.

In Study 1 (Tables 2 and 3), we vary unit reverse-order costs against the Markov demand multiplier. Unit reverse order costs at locations 1 and 2 are kept equal and varied from 0.5 to 1.75 in increments of 0.25. The unit reverse order cost at location 3 is kept fixed at -0.8 throughout the study. This negative value reflects the revenue generated with a reverse order out of location 3. The Markov multiplier $\beta(3)$ for the high-demand state ranges from 2 to 4.5, in increments of 0.5, with the normal-demand state multiplier $\beta(2)$ fixed at 1. The low-demand Markov multiplier $\beta(1)$ is the reciprocal of the high-demand Markov multiplier, and hence ranges from 0.5 to 0.22. Table 2 shows cost savings for the $CS+R$ system; Table 3 displays cost savings for the $CS+E+R$ system (all are relative to the CS system).

M. Mult.	Unit Reverse Order Costs $(k_1^R = k_2^R)$					
$\beta(1)$, $\beta(3)$	0.50	0.75	1.00	1.25	1.5	1.75
(0.50, 2.0)	3.35%	3.34%	3.34%	3.34%	3.34%	3.34%
(0.40, 2.5)	3.64%	3.61%	3.59%	3.58%	3.58%	3.58%
(0,33, 3.0)	3.86%	3.80%	3.76%	3.74%	3.72%	3.72%
(0.29, 3.5)	4.02%	3.94%	3.89%	3.86%	3.83%	3.82%
(0.25, 4.0)	4.14%	4.04%	3.97%	3.92%	3.89%	3.87%
(0.22, 4.5)	4.23%	4.11%	4.03%	3.98%	3.94%	3.92%

Table 2: Study 1 Cost Savings - *the* $CS + R$ *System*

M. Mult.	Unit Reverse Order Cost $(k_1^R = k_2^R)$					
$\beta(1)$, $\beta(3)$	0.50	0.75	1.00	1.25	1.5	1.75
(0.50, 2.0)	15.8%	15.7%	15.7%	15.7%	15.7%	15.7%
(0.40, 2.5)	16.9%	16.7%	16.6%	16.5%	16.4%	16.3%
(0.33, 3.0)	17.3%	17.0%	16.7%	16.5%	16.3%	16.1%
(0.29, 3.5)	17.4%	17.0%	16.6%	16.3%	16.0%	15.8%
(0.25, 4.0)	17.6%	17.0%	16.4%	16.0%	15.7%	15.4%
(0.22, 4.5)	17.7%	17.0%	16.3%	15.8%	15.3%	15.0%

Table 3: Study 1 Cost Savings - *the* $CS + R + E$ *System*

Thus, when only regular and reverse flows of product (Table 2), the value of reverse logistics: ranges from 3.3% to 4.2%, is increasing in demand variability, and is only barely decreasing in unit reverse order cost at locations 1 and 2. Hence, most of the value created from reverse logistics in Table 2 derives from the revenue generated by selling excess inventory at location 3. Table 3 shows that cost savings are not monotonic in demand variability. This is because the value of reverse logistics is increasing in demand variability, while the value of expediting is decreasing in demand variability. (Cost savings in Study 1 for the system with expediting only, the $CS + E$ system, are 14.35%, 14.29%, 13.32%, 12.31%, 11.48%, and 10.71%, in the order of increasing demand variability, as they do not vary with the reverse order cost.)

In Study 2 shown in Tables 4 and 5, we vary the unit expediting cost (from 4 to 9 in increments of 1) against the unit backlogging cost (from 10 to 35 in increments of 5). Table 4 presents cost savings for the $CS+E$ system, while Table 5 shows cost savings for the $CS+E+R$ system (relative to the CS system).

Backlog.	Unit Expediting Costs $(k_1^E = k_2^E = k_3^E)$					
Cost	4	5	6	7	8	9
10	12.8%	8.71%	6.08%	4.23%	2.84%	1.77%
15	13.2%	9.52%	7.10%	5.39%	4.10%	3.10%
20	13.7%	10.1%	7.70%	6.06%	4.78%	3.84%
25	14.1%	10.5%	8.16%	6.55%	5.30%	4.34%
30	14.3%	10.8%	8.49%	6.86%	5.67%	4.68%
35	14.5%	11.0%	8.73%	7.16%	5.93%	5.00%

Table 4: Study 2 Cost Savings - *the* $CS + E$ *System*

Backlog.	Unit Expediting Costs $(k_1^E = k_2^E = k_3^E)$					
Cost	4	5	6	7	8	9
10	15.7%	11.1%	8.02%	5.83%	4.17%	2.89%
15	16.7%	12.2%	9.21%	7.13%	5.53%	4.25%
20	17.3%	12.9%	9.89%	7.84%	6.25%	5.02%
25	17.8%	13.4%	10.4%	8.35%	6.79%	5.56%
30	18.0%	13.7%	10.8%	8.68%	7.18%	5.91%
35	18.3%	14.0%	11.0%	9.00%	7.46%	6.26%

Table 5: Study 2 Cost Savings - *the* $CS + R + E$ *System*

Tables 4 and 5 demonstrate that cost savings with only the expediting option (CS+E system), and with both product flow options ($CS+E+R$ system) are increasing in the unit backlogging cost and decreasing in unit expediting costs. (Cost savings from the system with reverse logistics only, the $CS+R$ system, in the order of increasing unit backlogging costs, are 4.01%, 3.76%, 3.69%, 3.64%, 3.61%, and 3.60%).

It is of interest to understand if reverse logistics and expediting behave as complements or substitutes with regard to the savings generated in the system. Let *synergy value* refer to the difference between percentage cost savings with both product flow options (i.e. reverse logistics and expediting) and the sum of the cost savings from using each flow option by itself. Thus, if F_{CS} is the optimal cost with the regular flow only, F_{CS+E} is the optimal cost with the option to expedite stock, F_{CS+R} is the optimal cost with the reverse order option, and F_{CS+E+R} is the optimal cost with *both* options, then the synergy value becomes

$$\frac{\left(F_{CS}-F_{CS+E+R}\right)-\left(F_{CS}-F_{CS+E}\right)-\left(F_{CS}-F_{CS+R}\right)}{F_{CS}}. \tag{19}$$

Tables 6 and 7 present this synergy value for model parameters used in Studies 1 and 2, respectively. Positive synergy values indicate complementarity, while negative ones imply substitution. As shown in Tables 6 and 7, with higher demand variability, or higher unit backlogging costs and lower unit expediting costs, reverse logistics and expediting are complementary. They become substitutes when demand variability or unit backlogging cost decrease, or when the unit expediting cost increases.

M. Mult.	Unit Reverse Order Costs ($k_1^R = k_2^R$)					
$\beta(1)$, $\beta(3)$	0.50	0.75	1.00	1.25	1.5	1.75
(0.50, 2.0)	-1.94%	-1.97%	-1.99%	-1.99%	-2.00%	-2.00%
(0.40, 2.5)	-1.00%	-1.16%	-1.29%	-1.40%	-1.48%	-1.53%
(0,33, 3.0)	0.10%	-0.15%	-0.37%	-0.57%	-0.75%	-0.89%
(0.29, 3.5)	1.10%	0.71%	0.36%	0.08%	-0.15%	-0.36%
(0.25, 4.0)	1.94%	1.45%	0.99%	0.59%	0.28%	0.04%
(0.22, 4.5)	2.71%	2.13%	1.58%	1.07%	0.66%	0.37%

Table 6: Study 1 – *The Synergy Value*

Backlog.	Unit Expediting Costs ($k_1^E = k_2^E = k_3^E$)					
Cost	4	5	6	7	8	9
10	-1.08%	-1.63%	-2.07%	-2.41%	-2.67%	-2.89%
15	-0.37%	-1.09%	-1.65%	-2.03%	-2.33%	-2.61%
20	-0.12%	-0.89%	-1.50%	-1.91%	-2.22%	-2.50%
25	0.02%	-0.78%	-1.41%	-1.84%	-2.15%	-2.42%
30	0.14%	-0.69%	-1.34%	-1.78%	-2.09%	-2.37%
35	0.20%	-0.64%	-1.31%	-1.76%	-2.07%	-2.34%

Table 7: Study 2 – *The Synergy Value*

Four-location System

We now consider a four-location system, and conduct the equivalent of Studies 1, 2 and 3 described above. Our basic version of the four-location system has the same demand characteristics and parameters choices as the three-echelon system studied above: $(k_1, k_2, k_3, k_4) = (1,1,1,1)$; $(k_1^R, k_2^R, k_3^R, k_4^R) = (1,1,1,-0.8)$; $(k_1^E, k_2^E, k_3^E, k_4^E) = (4,4,4,4)$; and $(H_1, H_2, H_3, H_4) = (2.0, 1.5, 1.0, 0.5)$ (thus, $h_j = 0.5$). The backlogging cost is unchanged at $p = 15$. In Study 3, shown in Tables 8 and 9, we again vary k_j^R from 0.5 to 1.75 in increments of 0.25, and the Markov multiplier $\beta(3)$ from 2 to 4.5, in increments of 0.5.

Table 8 presents cost savings with only the reverse-order option, while Table 9 shows cost savings for with both reverse logistics and expediting. The value of reverse logistics is lower for the four-location system in Table 8 than for the three-location system in Table 2. With a longer supply chain, an item returned upstream has to travel through more locations and incur higher costs before generating revenue through a resale at the most upstream location. In spite of that, cost savings generated by having both regular and expedited flows in the system are higher for the four-location system in Table 9 than for the three-location system in Table 3. This is because, in longer supply chains: (i) the option to expedite stock tends to be more valuable; and (ii) there is a greater synergy between expediting and reverse logistics.

M. Mult.	Unit Reverse Order Costs $(k_1^R = k_2^R)$					
$\beta(1)$, $\beta(3)$	0.50	0.75	1.00	1.25	1.5	1.75
(0.50, 2.0)	2.20%	2.19%	2.19%	2.18%	2.18%	2.18%
(0.40, 2.5)	2.43%	2.39%	2.37%	2.36%	2.35%	2.35%
(0,33, 3.0)	2.62%	2.55%	2.51%	2.48%	2.46%	2.45%
(0.29, 3.5)	2.75%	2.66%	2.59%	2.55%	2.52%	2.50%
(0.25, 4.0)	2.87%	2.76%	2.67%	2.61%	2.57%	2.55%
(0.22, 4.5)	2.95%	2.82%	2.72%	2.65%	2.60%	2.58%

Table 8: Study 3 Cost Savings - *the $CS + R$ System*

M. Mult.	Unit Reverse Order Cost $(k_1^R = k_2^R)$					
$\beta(1)$, $\beta(3)$	0.50	0.75	1.00	1.25	1.5	1.75
(0.50, 2.0)	19.6%	19.5%	19.5%	19.4%	19.4%	19.4%
(0.40, 2.5)	20.8%	20.6%	20.4%	20.3%	20.2%	20.1%
(0.33, 3.0)	21.2%	20.8%	20.6%	20.4%	20.2%	20.1%
(0.29, 3.5)	21.3%	20.9%	20.5%	20.2%	19.9%	19.7%
(0.25, 4.0)	21.4%	20.8%	20.4%	20.0%	19.7%	19.4%
(0.22, 4.5)	21.4%	20.8%	20.2%	19.7%	19.4%	19.1%

Table 9: Study 3 Cost Savings - *the $CS + R + E$ System*

In Study 4, we vary unit expediting costs against unit backlogging costs for this four-locations system. Table 10 presents cost savings for the $CS+E$ system, while Table 11 displays cost savings for the $CS+E+R$ system. Both tables show higher cost savings than those for the three-location system (Tables 4 and 5).

Backlog.	Unit Expediting Costs $(k_1^E = k_2^E = k_3^E)$					
Cost	4	5	6	7	8	9
10	17.3%	11.7%	8.14%	5.70%	3.90%	2.53%
15	17.8%	12.6%	9.27%	7.04%	5.38%	4.13%
20	18.2%	13.1%	9.93%	7.78%	6.18%	4.98%
25	18.5%	13.5%	10.4%	8.30%	6.73%	5.55%
30	18.6%	13.8%	10.7%	8.64%	7.14%	5.93%
35	18.8%	14.0%	11.0%	8.95%	7.41%	6.27%

Table 10: Study 4 Cost Savings - *the $CS + E$ System*

Backlog.	Unit Expediting Costs $(k_1^E = k_2^E = k_3^E)$					
Cost	4	5	6	7	8	9
10	19.6%	13.7%	9.76%	7.06%	5.04%	3.51%
15	20.6%	14.9%	11.1%	8.53%	6.62%	5.17%
20	21.2%	15.6%	11.9%	9.37%	7.48%	6.07%
25	21.7%	16.1%	12.5%	9.94%	8.06%	6.67%
30	21.9%	16.4%	12.9%	10.3%	8.50%	7.08%
35	22.1%	16.7%	13.1%	10.6%	8.78%	7.42%

Table 11: Study 4 Cost Savings - *the $CS + R + E$ System*

Study 4 confirms the observation established in Study 3 that longer supply chains increase the value of: (1) the expediting option; and (2) having expediting and reverse logistics jointly optimized in the system.

M. Mult.	Unit Reverse Order Costs $(k_1^R = k_2^R)$					
$\beta(1)$, $\beta(3)$	0.50	0.75	1.00	1.25	1.5	1.75
(0.50, 2.0)	-0.86%	-0.93%	-0.98%	-1.02%	-1.04%	-1.05%
(0.40, 2.5)	-0.08%	-0.24%	-0.37%	-0.47%	-0.56%	-0.65%
(0,33, 3.0)	0.76%	0.51%	0.28%	0.09%	-0.06%	-0.19%
(0.29, 3.5)	1.54%	1.18%	0.89%	0.62%	0.40%	0.21%
(0.25, 4.0)	2.14%	1.70%	1.31%	0.99%	0.71%	0.48%
(0.22, 4.5)	2.69%	2.18%	1.71%	1.31%	1.00%	0.72%

Table 12: Study 3 – *The Synergy Value*

Backlog.	Unit Expediting Costs $(k_1^E = k_2^E = k_3^E)$					
Cost	4	5	6	7	8	9
10	-0.47%	-0.75%	-1.09%	-1.35%	-1.57%	-1.74%
15	0.28%	-0.19%	-0.65%	-1.02%	-1.27%	-1.47%
20	0.62%	0.07%	-0.44%	-0.85%	-1.14%	-1.34%
25	0.81%	0.20%	-0.32%	-0.76%	-1.06%	-1.27%
30	0.93%	0.30%	-0.25%	-0.69%	-1.01%	-1.22%
35	1.02%	0.36%	-0.19%	-0.65%	-0.97%	-1.20%

Table 13: Study 4 – *The Synergy Value*

The synergy value for Studies 3 and 4 is shown in Tables 12 and 13. When compared to the Tables 6 and 7, this synergy value is increasing in the length of the supply chain. (This finding is confirmed in our numerical studies for a five-location system described in Appendix C, Section 1). The presence of positive synergy values (i.e., complementarity) between reverse logistics and expediting is exactly the reason that those product flows need to be optimized jointly in a supply chain, rather than analyzed as separate flows.

6. Reverse Logistics in a Product-Transforming Supply Chain

In this section, we generalize our model and show how our previous results on logistics supply chains can be used to assess, for the first time in the literature, the value of reverse logistics in product-transforming supply chains. As mentioned in the Introduction, in a product transforming supply chain, as a particular unit of product moves downstream, it is physically transformed at each location in the supply chain. Consequently, a unit of product that has reached location j in the supply chain will generally have different physical characteristics (and be more valuable than) a unit of product that has not yet reached location j.

Hence, in a product-transforming supply chain, items at any location j may be at different stages of completion. This is because some of those items might have arrived there by the reverse flow from locations downstream of j, in which case their completion would be further along relative to those items that have arrived there by regular or expedited flow downstream. Let index i denote an item's stage of completion, which represents *the most downstream location* reached up to that point. Thus, the inventory at location j may consist of items at any stage of completion i, for $i \in [1, j]$, since a product from any location i, $i < j$, where it reached the stage of completion i, may have been returned upstream into location j.

Therefore, in a product transforming supply chain, it is necessary to keep track of both an item's stage of completion and its location in the supply chain. This is because the financial holding cost for an item will vary with its stage of completion, while its physical holding cost will generally vary with both its stage of completion (e.g., due to a potential size difference), and its location (e.g., the cost of physical space). In addition, unit costs for regular and reverse order can also be expected to vary with both an item's stage of completion and its location in the supply chain. While expediting is an established industry practice for logistics supply chains in which the product is not physically transformed as it flows through the system, in product-transforming supply chains, expediting physical processes needed to transform raw materials into finished goods may not be feasible due to the required physical transformation of the product that necessarily takes time. Hence, in the discussion that follows, we do not consider expedited orders. As a result, to correctly account for different unit holding and ordering costs for items at different stages of completion at the same location, in this section, we use the following state and decision variables:

x_{ijt} : on-hand inventory of items at location j (the net inventory if $j = 1$) that are at stage of completion i at the beginning of period t, where $j \in [1, \ldots, N]$ and $i \in [1, \ldots, j]$;

X_{ijt}^R : number of units currently at stage of completion i at location j that are reverse-ordered into location $j+1$ in period t, where $j \in [1, \ldots, N]$ and $i \in [1, \ldots, j]$;

X_{ijt} : number of units currently at stage of completion i at location $j+1$ that are regular-ordered into location j in period t, where $j \in [1, \ldots, N]$ and $i \in [1, \ldots, j+1]$ for $j < N$, and $i = N$ for $j = N$.

At the most upstream location, only completely untransformed items (i.e., $i = N$) are brought *into* the supply chain; thus, when $j = N$, only regular orders with $i = N$ are permitted into the system.

The collection $\mathbf{x}_t := \{x_{ijt}\}$ is the inventory state for the problem; $\mathbf{X}_t^R := \{X_{ijt}^R\}$ is the reverse order schedule; and $\mathbf{X}_t := \{X_{ijt}\}$ is the regular order schedule. Thus, in a product-transforming supply chain with reverse orders, the number of state variables and decision variables is of order N^2. In addition, what makes this problem exceedingly difficult is that, at each location j, inventory can be replenished in $j+1$ different ways by regular flow alone. Due to this large number of alternative modes of replenishment, existing methods fail to provide an optimal replenishment policy for such a supply chain.

Moreover, the development of mathematical tools to optimally select which of the available j replenishment options to use at each location j in such a supply chain is (significantly) beyond the current scope of knowledge in the field. This is because, even for single-stage systems with only two different replenishment options (i.e., the so-called "dual-sourcing problem"), the structure of the optimal policy is not known (other than for some special cases). As a result, solving the reverse logistics problem in a product-transforming supply chain has as a prerequisite the solution of the dual-sourcing problem, which has remained open despite a number of attempts at solution (see Xin and Goldberg, 2017).

Nevertheless, the results developed in our paper so far make it possible to: 1) derive a lower bound on the cost; 2) propose a novel heuristic policy; and 3) evaluate the cost performance of that heuristic policy. Next, we formulate the optimality equations for the product-transforming supply chain with reverse logistics, then describe our lower-bounding policy and our heuristic policy. Finally, we present the results of numerical studies in which we use simulation to evaluate the cost performance of the heuristic policy.

Model Formulation: Product Transforming Supply Chains

Let H_{ij} be the unit inventory holding cost for an item at stage of completion i at location j, and k_{ij}^R be the unit cost of returning an item at stage of completion i from location j to location $j+1$. Let k_{ij} be the unit cost of regular-ordering an item currently at stage of completion i from location $j+1$ into location j. The one-period cost function for the product-transforming supply chain with reverse logistics becomes

$$\gamma_t(\omega, x_{11t} - X_{11t}^R) + \sum_{j=1}^{N} \sum_{i=1}^{j} \left(k_{ij}^R X_{ijt}^R + H_{ij} x_{jt}\right) + \sum_{j=1}^{N-1} \sum_{i=1}^{j+1} k_{ij} X_{ijt} + k_{NN} X_{NNt}. \tag{20}$$

The optimality equations for the product-transforming supply chain with reverse logistics thus become:

$$\overline{F}_t(\omega, \mathbf{x}_t) = \min_{\mathbf{X}_t^R, \mathbf{X}_t \in \overline{\mathbb{X}}(\mathbf{x}_t)} \overline{V}_t(\omega, \mathbf{x}_t \mathbf{X}_t^R, \mathbf{X}_t)$$

$$\overline{V}_t(\omega, \mathbf{x}_t, \mathbf{X}_t^R, \mathbf{X}_t) = \gamma_t(\omega, x_{11t} - X_{11t}^R) + \sum_{j=1}^{N}\sum_{i=1}^{j}(k_{ij}^R X_{ijt}^R + H_{ij} x_{ijt}) + \sum_{j=1}^{N-1}\sum_{i=1}^{j+1} k_{ij} X_{ijt} + k_{NN} X_{NNt} + \alpha \mathbb{E}_\omega\left[\overline{F}_{t+1}(\omega_{t+1}, \mathbf{x}_{t+1})\right], \tag{21}$$

where the state transition equations that generate $\mathbf{x}_{t+1}$, the inventory state next period, are now as follows

$$x_{ij,t+1} = \begin{cases} x_{11t} + X_{11t} + X_{21t} - X_{11t}^R - D_t(\omega) & \text{for } j = 1, \\ x_{ijt} + X_{ijt} - X_{i,j-1,t} - X_{ijt}^R + X_{i,j-1,t}^R & \text{for } j \in [2, N-1] \text{ and } i \in [1, j-1]; \\ x_{jjt} + X_{jjt} + X_{j+1,jt} - X_{j,j-1,t} - X_{jjt}^R & \text{for } j \in [2, N-1] \text{ and } i = j; \\ x_{iNt} - X_{i,N-1,t} - X_{iNt}^R + X_{i,N-1,t}^R & \text{for } j = N \text{ and } i \in [1, j-1]; \\ x_{NNt} + X_{NNt} - X_{N,N-1,t} - X_{NNt}^R & \text{for } j = N \text{ and } i = N. \end{cases} \tag{22}$$

The feasible state space $\overline{\mathbb{X}}(\mathbf{x}_t)$ for this product-transforming supply chain problem is given by

$$\overline{\mathbb{X}}(\mathbf{x}_t) = \left\{ \mathbf{X}_t^R, \mathbf{X}_t \;\middle|\; \begin{array}{ll} X_{11t}^R \in \left[0, [x_{11t}]^+\right],\ X_{ijt}^R \in \left[0, x_{ijt}\right] & \text{for } j \in [2, N] \text{ and } i \in [1, j]; \\ X_{ijt} \in \left[0, x_{i,j+1,t} - X_{i,j+1,t}^R\right] & \text{for } j \in [1, N-1], \text{ and } i \in [1, j+1]; \end{array} \right.$$

Lower-Bounding Policy

We now make use of our results for the logistics supply chain to arrive a policy that provides a lower bound on the optimal cost for the product-transforming supply chain. For each $j \in [1, N]$, let $\tilde{k}_j^R := \min_{i, i \in [1,j]} k_{ij}^R$ and $\widetilde{H}_j := \min_{i, i\in[1,j]} H_{ij}$; for each $j \in [1, N-1]$, let $\tilde{k}_j := \min_{i, i \in [1, j+1]} k_{ij}$ with $\tilde{k}_N := k_{NN}$. Let $\overline{F}_t^L(\omega, \mathbf{x}_t)$ be defined as

$$\overline{F}_t^L(\omega, \mathbf{x}_t) = \min_{\mathbf{X}_t^R, \mathbf{X}_t \in \overline{\mathbb{X}}(\mathbf{x}_t)} \left\{ \gamma_t(\omega, x_{11t} - X_{11t}^R) + \sum_{j=1}^{N}\sum_{i=1}^{j}(\tilde{k}_j^R X_{ijt}^R + \widetilde{H}_j x_{ijt}) + \sum_{j=1}^{N-1}\sum_{i=1}^{j+1} \tilde{k}_j X_{ijt} + \tilde{k}_N X_{NNt} + \alpha \mathbb{E}_\omega\left[\overline{F}_{t+1}^L(\omega_{t+1}, \mathbf{x}_{t+1})\right] \right\}$$

$$= \min_{\mathbf{X}_t^R, \mathbf{X}_t \in \overline{\mathbb{X}}(\mathbf{x}_t)} \left\{ \gamma_t(\omega, x_{11t} - X_{11t}^R) + \sum_{j=1}^{N}\left[\tilde{k}_j^R\left(\sum_{i=1}^{j} X_{ijt}^R\right) + \widetilde{H}_j\left(\sum_{j=1}^{N} x_{ijt}\right)\right] + \sum_{j=1}^{N-1}\tilde{k}_j\left(\sum_{i=1}^{j+1} X_{ijt}\right) + \tilde{k}_N X_{NNt} + \alpha \mathbb{E}_\omega\left[\overline{F}_{t+1}^L(\omega_{t+1}, \mathbf{x}_{t+1})\right] \right\}$$

Given the above optimality equations for $\overline{F}_t^L$, the optimality equations for $\overline{F}_t$, and the definitions of $\tilde{k}_j^R$, $\widetilde{H}_j$, and $\tilde{k}_j$, it follows that $\overline{F}_t^L(\omega, \mathbf{x}_t) \le \overline{F}_t(\omega, \mathbf{x}_t)$ for any given ω and $\mathbf{x}_t$. Thus, $\overline{F}_t^L$ is a lower bound for $\overline{F}_t$. Define new state and decision variables, for each location $j \in [1, N]$ and period t as:

$$\tilde{x}_{jt} := \sum_{j=1}^{N} x_{ijt}; \quad \widetilde{X}_{ijt}^R := \sum_{i=1}^{j} X_{ijt}^R; \quad \widetilde{X}_{Nt} := X_{NNt} \text{ and } \widetilde{X}_{jt} := \sum_{i=1}^{j+1} X_{ijt} \text{ for } j \in [1, N-1].$$

Using those new state and decisions variables, the optimality equations for $\overline{F}_t^L$ now become

$$\overline{F}_t^L(\omega, \tilde{\mathbf{x}}_t) = \min_{\widetilde{\mathbf{X}}_t^R, \widetilde{\mathbf{X}}_t \in \overline{\mathbb{X}}^L(\tilde{\mathbf{x}}_t)} \left\{ \gamma_t(\omega, \tilde{x}_{1t} - \widetilde{X}_{1t}^R) + \sum_{j=1}^{N}(\tilde{k}_j^R \widetilde{X}_{jt}^R + \tilde{k}_j \widetilde{X}_{jt} + \widetilde{H}_j \tilde{x}_{jt}) + \alpha \mathbb{E}_\omega\left[\overline{F}_{t+1}^L(\omega_{t+1}, \tilde{\mathbf{x}}_{t+1})\right] \right\}, \tag{23}$$

where the feasible decision set $\overline{\mathbb{X}}^L(\tilde{\mathbf{x}}_t)$ becomes

$$\overline{\mathbb{X}}(\tilde{\mathbf{x}}_t) = \left\{ \widetilde{\mathbf{X}}_t^R, \widetilde{\mathbf{X}}_t \;\middle|\; \begin{array}{ll} \widetilde{X}_{1t}^R \in \big[0, [x_{1t}]^+\big],\ \widetilde{X}_{jt}^R \in \big[0, x_{jt}\big] & \text{for } j \in [2, N]; \\ \widetilde{X}_{jt} \in \big[0, x_{j+1,t} - X_{j+1,t}^R\big] & \text{for } j \in [1, N-1]. \end{array} \right. \tag{24}$$

Next, we introduce a change of variables: $\tilde{y}_{jt} := \tilde{x}_{1t} + \cdots + \tilde{x}_{jt}$; $\widetilde{Y}_{jt}^R := \tilde{y}_{jt} - \widetilde{X}_{jt}^R$, and $\widetilde{Y}_{jt} := \widetilde{Y}_{jt}^R + \widetilde{X}_{jt}$. This change of variables transforms the dynamic program given in (23) to the following:

$$\tilde{f}_t^L(\omega, \tilde{\mathbf{y}}_t) = \sum_{j=1}^{N} \big(\tilde{k}_j^R + \tilde{k}_j\big)\, y_{jt} + \min_{\widetilde{\mathbf{Y}}_t^R, \widetilde{\mathbf{Y}}_t \in \overline{\mathbb{Y}}^L(\tilde{\mathbf{y}}_t)} \left\{ \gamma_t(\omega, \widetilde{Y}_{1t}^R) + \sum_{j=1}^{N} \Big(\tilde{k}_j \widetilde{Y}_{jt} - \big(\tilde{k}_j^R + \tilde{k}_j\big) \widetilde{Y}_{jt}^R \Big) + \alpha\, \mathbb{E}_\omega \big[\tilde{f}_{t+1}^L\big(\omega_{t+1}, \widetilde{\mathbf{Y}}_t - D_t(\omega)\big) \big] \right\},$$

where the feasible decision set is given by

$$\overline{\mathbb{Y}}^L(\tilde{\mathbf{y}}_t) = \left\{ \widetilde{\mathbf{Y}}_t^R, \widetilde{\mathbf{Y}}_t \;\middle|\; \begin{array}{ll} \widetilde{Y}_{1t}^R \in \big[[\tilde{y}_{1t}]^-, \tilde{y}_{1t}\big],\ \ \widetilde{Y}_{jt}^R \in \big[\tilde{y}_{j-1,t}, \tilde{y}_{jt}\big] & \text{for } j \in [2, N]; \\ \widetilde{Y}_{jt} \in \big[\widetilde{Y}_{jt}^R, \widetilde{Y}_{j+1,t}^R - \big(\tilde{y}_{jt} - \widetilde{Y}_{jt}^R\big)\big] & \text{for } j \in [1, N]. \end{array} \right.$$

The solution of the dynamic program for $\tilde{f}_t^L(\omega, \tilde{\mathbf{y}}_t)$ is arrived at with the same steps as those presented in Theorems 1 - 5. The final results of this analysis are summarized in the following theorem.

Theorem 9. *The following hold in every period t.*

(a) $\tilde{f}_t^L(\omega, \cdot)$ is additively convex for each ω;

(b) For each period t, state ω, and echelon j there exist basestock levels $\widetilde{S}_{jt}^R(\omega)$ and $\widetilde{S}_{jt}(\omega)$ such that $\widetilde{S}_{jt}^R(\omega) \geq \widetilde{S}_{jt}(\omega)$, and the optimal decisions for the lower bounding problem defined in (6) are given by

$$\widetilde{Y}_{jt}^{*R}(\omega) = \begin{cases} \max\Big[[\tilde{y}_{1t}]^-, \min\big(\widetilde{S}_{1t}^R(\omega), \tilde{y}_{1t}\big)\Big] & \text{if } j = 1; \\ \max\Big[y_{\tilde{y}-1,t}, \min\big(\widetilde{S}_{jt}^R(\omega), \tilde{y}_{jt}\big)\Big] & \text{if } j \in [2, N]; \end{cases} \qquad \widetilde{Y}_{jt}^{*}(\omega) = \begin{cases} \max\Big[\widetilde{Y}_{jt}^{*R}, \min\big(\widetilde{S}_{jt}(\omega), \widetilde{Y}_{j+1,t}^{*R}\big)\Big] & \text{if } j \in [1, N-1]; \\ \max\Big[\widetilde{Y}_{Nt}^{*R}, \widetilde{S}_{Nt}(\omega)\Big] & \text{if } j = N. \end{cases}$$

Thus, $\tilde{f}_t^L$: (1) provides a lower bound on $\overline{F}_t$; (2) is additively convex (and, hence, computationally fast); and (3) generates an optimal policy determined by two sets of mutually ordered basestock levels.

Heuristic Policy

The following theorem provides the motivation for our proposed heuristic. For $j \in [1, N]$ and $i \in [1, j]$, define $h_{ij} := H_{ij} - H_{i,j+1}$ (with $h_{NN} := H_{NN}$); for $j \in [1, N-1]$, let $\Delta H_j := H_{jj} - H_{j+1,j+1}$.

Theorem 10. *Let $\overline{V}_t(\omega, \mathbf{x}_t, \mathbf{X}_t^R, \mathbf{X}_t)$ be as defined in (21). Then, $\overline{V}_t(\omega, \mathbf{x}_t, \mathbf{X}_t^R, \mathbf{X}_t)$ can be expressed as*

$$\overline{V}_t(\omega, \mathbf{x}_t, \mathbf{X}_t^R, \mathbf{X}_t) = \sum_{j=1}^{N} \sum_{i=1}^{j} (1+\alpha) H_{ij} x_{ijt} + \overline{U}_t(\omega, \mathbf{x}_t, \mathbf{X}_t^R, \mathbf{X}_t) + \mathbb{E}_\omega \Big[\min_{\mathbf{X}_{t+1}^R, \mathbf{X}_{t+1} \in \overline{\mathbb{X}}(\mathbf{x}_{t+1})} \alpha \overline{G}_{t+2}(\omega, \mathbf{x}_{t+1}, \mathbf{X}_{t+1}^R, \mathbf{X}_{t+1}) \Big],$$

where (25)

$$\overline{U}_t(\omega, \mathbf{x}_t, \mathbf{X}_t^R, \mathbf{X}_t) = \gamma_t\big(\omega, x_{11t} - X_{11t}^R\big) + \sum_{j=1}^{N} \sum_{i=1}^{j} \big(k_{ij}^R - \alpha h_{ij}\big) X_{ijt}^R + \sum_{j=1}^{N-1} \sum_{i=1}^{j} \big(k_{ij} + \alpha h_{ij}\big) X_{ijt} + \sum_{j=1}^{N-1} \big(k_{j+1,j} + \alpha \Delta H_j\big) X_{j+1,jt}$$
$$+ \big(k_{NN} + \alpha H_{NN}\big) X_{NNt} - \alpha \mathbb{E}_\omega \big[D_t\big] \tag{26}$$

and

$$\overline{G}_{t+2}(\omega,\mathbf{x}_{t+1},\mathbf{X}^R_{t+1},\mathbf{X}_{t+1})=\gamma_{t+1}\left(\omega_{t+1},x_{11,t+1}-X^R_{11,t+1}\right)+\sum_{j=1}^{N}\sum_{i=1}^{j}k^R_{ij}X^R_{ij,t+1}+\sum_{j=1}^{N-1}\sum_{i=1}^{j+1}k_{ij}X_{ij,t+1}+k_{NN}X_{NN,t+1}$$
$$+\alpha\mathbb{E}_{\omega_{t+1}}\left[\overline{F}_{t+2}(\omega_{t+2},\mathbf{x}_{t+2})\right]. \tag{27}$$

Thus, Theorem 10 decomposes the cost function $\overline{V}_t(\omega,\mathbf{x}_t,\mathbf{X}^R_t,\mathbf{X}_t)$ into two cost component functions plus a holding cost term. Observe that $\overline{U}_t$ includes all the cost terms that depend explicitly on either a regular order X_{ijt} or a reverse order X^R_{ijt} in period t. Hence, $\overline{U}_t$ captures all the direct costs incurred by any decision made in period t (where those costs are incurred in either period t or period $t+1$). The function $\overline{G}_{t+2}$, on the other hand, contains no terms that depend on any decisions made in period t; the decisions $\mathbf{X}^R_t$ and $\mathbf{X}_t$ enter $\overline{G}_{t+2}(\omega,\mathbf{x}_{t+1},\mathbf{X}^R_{t+1},\mathbf{X}_{t+1})$ only through their impact on the inventory state $\mathbf{x}_{t+1}$.

Note that the impact of X^R_{ijt} on $\overline{U}_t$ is $\delta^R_{ij}X^R_{ijt}$, where $\delta^R_{ij}:=k^R_{ij}-\alpha h_{ij}$ for $j\in[1,N]$ and $i\in[1,j]$. The impact of X_{ijt} on $\overline{U}_t$ is $\delta_{ij}X_{ijt}$, where $\delta_{ij}:=k_{ij}+\alpha h_{ij}$ for $j\in[1,N-1]$ and $i\in[1,j]$, and $\delta_{ij}:=k_{ij}+\alpha\Delta H_j$ if $i=j+1$. Thus, δ^R_{ij} and δ_{ij} represent the cost impact on $\overline{U}_t$ of unit reverse and regular orders, respectively.

At each location $j\in[1,N]$, define recursively auxiliary echelon variables in each period t as: $\hat{y}_{1t}=x_{11t}$; $\hat{y}_{jt}=\hat{y}_{j-1,t}+\sum_{i=1}^{j}x_{ijt}$. Thus, $\hat{y}_{jt}$ represents the (net) sum of all on-hand inventory at locations 1 through j.

Definition 1. *(a) For each location $j\in[2,N]$, let indices $i_1(j),i_2(j),\dots i_j(j)$ be defined recursively as:*

$$i_1(j):=\operatorname*{arg\,min}_{i,\,i\in[1,j]}\delta^R_{ij}\quad\text{and}\quad i_2(j):=\operatorname*{arg\,min}_{i,\,i\in[1,j],\,i\notin\{i_1(j)\}}\delta^R_{ij};\qquad i_k(j)=\operatorname*{arg\,min}_{i,\,i\in[1,j],\,i\notin\{i_1(j),\dots i_{k-1}(j)\}}\delta^R_{ij}.$$

(b) For each location $j\in[1,N-1]$, let indices $\iota_1(j),\iota_2(j),\dots\iota_{j+1}(j)$ be defined recursively as:

$$\iota_1(j)=\operatorname*{arg\,min}_{i,\,i\in[1,j+1]}\delta_{ij}\quad\text{and}\quad\iota_2(j)=\operatorname*{arg\,min}_{i,\,i\in[1,j+1],\,i\notin\{\iota_1(j)\}}\delta_{ij};\qquad\iota_k(j)=\operatorname*{arg\,min}_{i,\,i\in[1,j+1],\,i\notin\{\iota_1(j),\dots\iota_{k-1}(j)\}}\delta_{ij}.$$

Thus, indices $i_1(j),\dots,i_j(j)$ represent the ordering of reverse-order cost impact factors δ^R_{ij} from the smallest to the largest at each location j, while indices $\iota_1(j),\dots,\iota_{j+1}(j)$ denote the equivalent ordering of regular-order cost impact factors δ_{ij}. Our proposed heuristic policy will seek minimize the impact of ordering decisions on $\overline{U}_t$. Because $\overline{U}_t$ captures the impact of the decisions made in period t on the costs incurred in periods t and $t+1$, we will refer to this policy as the *2-Period Look Ahead* (2-PLA) Policy.

Definition 2. (2-PLA Policy) *Let $\{\widetilde{S}^R_{jt}(\omega)\}$ and $\{\widetilde{S}_{jt}(\omega)\}$ be the basestock levels of the low-bounding policy from Theorem 9. Then, the 2-PLA Policy specifies ordering decisions $\{X^R_{ijt}\}$ and $\{X_{ijt}\}$ as follows.*
(a) If $\hat{y}_{jt}>\widetilde{S}^R_{jt}(\omega)$ then: $X^R_{11t}=\hat{y}_{1t}-\widetilde{S}^R_{1t}(\omega)$; and, X^R_{ijt} for $j\in[2,N]$ and $i\in[1,j]$ are given recursively by:

$$X^R_{i_1(j),jt}=\min\left[x_{i_1(j),jt},\hat{y}_{jt}-\widetilde{S}^R_{jt}(\omega)\right]\quad\text{and}\quad X^R_{i_2(j),jt}=\min\left[x_{i_2(j),jt},\hat{y}_{jt}-\widetilde{S}^R_{jt}(\omega)-X^R_{i_1(j),jt}\right];$$
$$X^R_{i_k(j),jt}=\min\left[x_{i_k(j),jt},\hat{y}_{jt}-\widetilde{S}^R_{jt}(\omega)-\sum_{q=1}^{k-1}X^R_{i_q(j),jt}\right].$$

(b) If $\hat{y}_{jt} < \widetilde{S}_{jt}(\omega)$ *then:* $X_{NN,t} = \widetilde{S}_{Nt}(\omega) - \hat{y}_{Nt}$*; and* X_{ijt} *for* $j < N$ *and* $i \in [1, j+1]$ *are given recursively by:*

$$X_{\iota_1(j),jt} = \min\left[x_{\iota_1(j),j+1,t}, \widetilde{S}_{jt}(\omega) - \hat{y}_{jt}\right] \quad \text{and} \quad X_{\iota_2(j),jt} = \min\left[x_{\iota_2(j),j+1,t}, \widetilde{S}_{jt}(\omega) - \hat{y}_{jt} - X_{\iota_1(j),jt}\right];$$

$$X_{\iota_k(j),jt} = \min\left[x_{\iota_k(j),j+1,t}, \widetilde{S}_{jt}(\omega) - \hat{y}_{jt} - \sum_{q=1}^{k-1} X_{\iota_q(j),jt}\right].$$

(c) If $\widetilde{S}^R_{jt}(\omega) \le \hat{y}_{jt} \le \widetilde{S}_{jt}(\omega)$*, then* $X^R_{ijt} = 0$*, for all* $i \in [1, j]$ *and* $X_{ijt} = 0$ *for all* $i \in [1, j+1]$*.*

Thus, at each location j, the 2-PLA policy: (1) determines if either a regular order or reverse order (or neither) is required by looking at the difference between $\widetilde{S}_{jt}(\omega)$ and $\hat{y}_{jt}$ for regular orders, and between $\hat{y}_{jt}$ and $\widetilde{S}^R_{jt}(\omega)$ for reverse orders; (2) fulfills each such order, starting with the one at the stage of completion i that has the least cost impact on $\overline{U}_t$; and (3) if needed, continues to complete the order by resorting to items at the stage of completion with the next least cost impact on $\overline{U}_t$, and so on. In that manner, the 2-PLA policy can access, at each location j, every feasible replenishment option for both reverse and regular orders, and determine the most desirable ones among them. Note that the 2-PLA policy depends in a crucial way on the ordering of regular and reverse order basestock levels established first in Theorem 5, then Theorem 9; without that ordering it would not be possible to construct the 2-PLA heuristic policy.

Performance of the Heuristic

We now evaluate the cost performance of the 2-PLA heuristic policy for three-location and four-location product-transforming supply chains (Tables 14 and 15, respectively), over a range of Markov multipliers and unit backlogging costs. (Other model parameters used and results for a five-location product-transforming system are in Appendix C, Section 2). We first determine basestock levels $\{\widetilde{S}^R_{jt}(\omega)\}$ and $\{\widetilde{S}_{jt}(\omega)\}$ optimal for $\tilde{f}^L_t(\omega, \tilde{\mathbf{y}}_t)$ by solving the dynamic program given in (6). We then make use of simulation to calculate the difference (i.e., the performance gap) between $\overline{F}_t(\omega, \tilde{\mathbf{x}}_t)$ and $\overline{F}^L_t(\omega, \tilde{\mathbf{x}}_t)$, as a percentage of $\overline{F}^L_t(\omega, \tilde{\mathbf{x}}_t)$.

M. Mult.	Unit Backlogging Cost					
$\beta(1)$, $\beta(3)$	10	15	20	25	30	35
(0.50, 2.0)	6.80%	6.36%	6.13%	5.80%	4.86%	4.68%
(0.40, 2.5)	6.09%	5.83%	5.19%	4.93%	4.81%	4.30%
(0,33, 3.0)	5.78%	5.62%	5.65%	5.61%	3.84%	4.98%
(0.29, 3.5)	5.94%	4.67%	5.91%	4.90%	5.99%	4.35%
(0.25, 4.0)	5.28%	5.52%	5.23%	4.18%	4.07%	4.66%
(0.22, 4.5)	5.28%	5.23%	4.40%	4.72%	4.94%	3.90%

Table 14: Heuristic Performance, 3-Stage System

M. Mult.	Unit Backlogging Cost					
$\beta(1)$, $\beta(3)$	10	15	20	25	30	35
(0.50, 2.0)	6.76%	6.31%	5.72%	5.87%	3.96%	4.87%
(0.40, 2.5)	6.20%	6.08%	4.61%	5.05%	3.90%	3.78%
(0.33, 3.0)	6.36%	5.59%	4.54%	5.55%	4.40%	4.51%
(0.29, 3.5)	5.88%	5.14%	5.75%	4.49%	3.26%	2.53%
(0.25, 4.0)	5.66%	5.48%	4.67%	4.47%	3.20%	3.41%
(0.22, 4.5)	6.09%	4.60%	4.67%	5.02%	3.37%	4.20%

Table 15: Heuristic Performance, 4-Stage System

The average performance gap is 5.18% for the three-location product-transforming supply chain (Table 14) and 4.86% for the four-location one (Table 15), with the maximum value of 6.76%. Thus, our heuristic policy performs very well under model parameters considered, given its relatively small performance gap relative to our lower-bounding policy that ignores all product-transforming features of such supply chains.

7. Concluding Remarks

In this paper, we formulated and solved a model of reverse logistics in supply chains with expedited, reverse and regular flow of orders. Our motivation for doing so was: (1) the growing presence in industry of those product flows, which are commonly managed independently from one another; (2) the potential for creating additional value in supply chains by developing methods to manage those product flows jointly in an optimal manner; and (3) the absence of such methods in both research literature and industry practice.

For our basic model of the logistics supply chain with reverse logistics, we identified the form of the optimal policy that achieves the decomposition of the multi-variable objective cost function into single-variable component functions, in spite of the multidimensional boundary of the feasible region. Our solution approach may also be applicable to other inventory problems presently deemed intractable due to the curse of dimensionality inherent in multidimensional boundaries of their feasible regions.

One of our key findings is that, when it comes to the value of reverse logistics, demand variability matters more than demand volatility. This is because, without demand variability, it is effectively never optimal to return product upstream, regardless of the level of demand volatility. In such a setting, reverse logistics does not create any cost savings, and companies involved in the reverse logistics practice, such as International Bearings, are well advised to take a close look at the exact nature of their demand uncertainty.

Using the algorithms developed in our paper, we quantify the resulting savings and identify most favorable conditions for investing in the reverse logistics capability. In particular, we find that, in the 3-location supply chain explored in Section 5, cost saving from having only reverse logistics in the supply chain average 3.74% (averaged across our Studies 1 and 2), while cost savings from using both reverse logistics and expediting average 13.09%. In the 4-location supply chain, reverse logistics on its own generates cost savings that average 3.49%, while having both reverse logistics and expediting in the supply chain results in cost savings that average 16.04%. Further, reverse logistics is found to create most value in longer supply chains characterized by higher demand variability, and lower reverse and expedited order costs. This is because in such systems returning products upstream and expediting products downstream act as complements with regard to the cost savings generated. This complementary interaction between reverse logistics and expediting in the system is exactly why those product flows need to be managed together, rather be optimized individually, as has customarily been the case in both research and practice.

With shorter supply chains, on the other hand, low demand variability, and high unit order costs, reverse logistics and expediting become substitutes, so that, in those settings, companies like International Bearings should consider eliminating one of those product flow options or managing them independently.

Finally, we provide an analysis of product-transforming supply chains with reverse logistics. For such systems with high state-space and decisions-space dimensionality, we develop: (1) a lower-bounding policy

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on the cost; and (2) an innovative heuristic policy for managing multiple regular and reverse flow options available at each location. The proposed heuristic policy is found to perform very well, as its performance gap relative to our low-bounding policy averages 4.72%, across all model parameters and supply chain lengths studied, with its maximum value at 6.80% and its minimum value at 1.76%.

Our formulation of the expediting function allows for dynamic leadtime management in which the leadtime of each unit of product is determined dynamically based on the structure of the optimal policy and each period's demand realization. This formulation of expediting is standard in the multiechelon inventory literature (e.g., Lawson and Porteus 2000, Muharremoglu and Tsitsiklis 2003, Mamani and Moinzadeh 2014, Angelus and Özer 2016, Berling and Martinez-de-Albeniz 2016). While this formulation is relevant to a variety of supply chains in which product expediting can be observed in practice, it does not capture those settings, also encountered in practice, in which the expediting leadtime is fixed and given from the start. There is a growing literature on such 'dual-scourcing options' in which two different sources for inventory replenishment have different fixed leatimes. (For a recent review of this literature, see Xin and Goldberg, 2017). The resulting models are known to present technically challenging problems, so that, even for single-stage systems, and unless the two replenishment leadtimes are consecutive, the structure of the optimal policy has not yet been established. As a result, incorporating fixed expediting leadtimes into a multiechelon setting remains an open problem, outside the scope of our work, whose solution can be expected to require, as a prerequisite, the solution of the single-stage dual-sourcing problem.

Our focus in this paper is on reverse orders that originate within the supply chain itself as a strategy to manage overstock inventory, rather than on reverse orders necessitated by customers' returns of product. However, all of our results concerning the structure of the optimal policy and the resulting decomposition of the objective cost function continue to hold when customer returns, in the form of negative demand, are allowed in the system. We also chose to address models of serial supply chains with reverse logistics. This is because serial systems represent the key building block of all other supply chain structures, such as assembly or distribution systems. Our results for serial supply chains with reverse logistics can be generalized to assembly systems with reverse logistics using standard methods, such as those found in Angelus and Porteus (2007) and Angelus and Özer (2016). As for distribution systems, Clark and Scarf (1960) pointed out that an optimal policy for a distribution system, if it exists, is complex. Despite decades of active research since then, the existence of an optimal policy for such a system, even with only regular orders, has not yet been established. Research on distribution systems has therefore been focused on the identification of heuristics. All of those heuristic approaches first collapse a given distribution system to a (more relaxed) serial system. Therefore, a solution of the resulting serial system provides the backbone of all heuristics proposed to manage such distribution systems (see, for example, Federgruen 1993, Aviv and Federgruen 2001, Özer 2003). Hence, developing such heuristics for distribution systems with regular, reverse and expedited order flows would first require the use of the results in our current paper.

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Appendix A: Proofs

The following Propositions 1 and 2 provide intermediate results needed for the proof of Theorem 1. Their proofs make use of the installation formulation of the relaxed reverse logistics problem given by

$$\widetilde{F}_t(\omega,\mathbf{x}_t) = \min_{\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t\,\in\,\widetilde{\mathbb{X}}(\mathbf{x}_t)} \widetilde{V}_t\big(\omega,\mathbf{x}_t,\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t\big) \tag{28}$$

$$\begin{aligned}\widetilde{V}_t\big(\omega,\mathbf{x}_t,\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t\big) = &\sum_{j=1}^{N}\Big[\big(k_{jt}^E+h_{jt}\big)X_{jt}^E+\big(k_{jt}^R-h_{jt}\big)X_{jt}^R+\big(k_{jt}+h_{jt}\big)X_{jt}+H_{jt}x_{jt}\Big]\\ &+\gamma_t\big(\omega,x_{1t}+X_{1t}^E-X_{1t}^R\big)+\alpha\,\mathbb{E}_{D_t,\omega_{t+1}\,|\,\omega}\big[\widetilde{F}_{t+1}(\omega_{t+1},\mathbf{x}_{t+1})\big],\end{aligned} \tag{29}$$

where the feasible set $\widetilde{\mathbb{X}}(\mathbf{x}_t)$ for the relaxed reverse logistics problem becomes

$$\widetilde{\mathbb{X}}(\mathbf{x}_t) = \left\{\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t\;\middle|\;\begin{array}{ll} X_{jt}^E\in\big[0,x_{j+1,t}+X_{j+1,t}^E\big] & \text{for } j\in[1,N-1];\\ X_{1t}^R\in\big[0,[x_{1t}+X_{1t}^E]^+\big];\,X_{jt}^R\in\big[0,x_{jt}+X_{jt}^E-X_{j-1,t}^E\big] & \text{for } j\in[2,N];\\ X_{jt}\in\big[0,x_{j+1,t}-X_{j+1,t}^R+X_{jt}^R+X_{j+1,t}^E-X_{jt}^E\big] & \text{for } j\in[1,N-1].\end{array}\right.$$

We say that a policy $\big(\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t\big)$ is *semi-unidirectional* in period t, if, at each location j, either $X_{jt}^E=0$ or $X_{jt}^R=0$ (or both). Hence, under a semi-unidirectional policy, the product is either expedited or reverse ordered at each location in the system, but not both in the same period.

Proposition 1. *Let $\big(\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t\big)$ be a policy feasible for $\mathbf{x}_t$ in period t. Let a policy $\big(\widehat{\mathbf{X}}_t^E,\widehat{\mathbf{X}}_t^R,\widehat{\mathbf{X}}_t\big)$ in period t be defined as $\widehat{X}_{jt}^E=\big[X_{jt}^E-X_{jt}^R\big]^+$; $\widehat{X}_{jt}^R=\big[X_{jt}^R-X_{jt}^E\big]^+$; and $\widehat{X}_{jt}=X_{jt}$ for each j. Let $\mathbf{x}_{t+1}$ and $\hat{\mathbf{x}}_{t+1}$ be inventory states in period $t+1$ obtained by starting with $\mathbf{x}_t$ in period t and applying policies $\big(\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t\big)$ and $\big(\widehat{\mathbf{X}}_t^E,\widehat{\mathbf{X}}_t^R,\widehat{\mathbf{X}}_t\big)$, respectively. Then, the following hold.*

(a) $\widehat{X}_{jt}^E\ge 0$, $\widehat{X}_{jt}^R\ge 0$, and $\widehat{X}_{jt}\ge 0$ for all j;

(b) $\widehat{X}_{jt}^E=0$ or $\widehat{X}_{jt}^R=0$ (or both) for all j;

(c) $\widehat{X}_{jt}^E\le X_{jt}^E$ and $\widehat{X}_{jt}^R\le X_{jt}^R$ for all j;

(d) $\big(\widehat{\mathbf{X}}_t^E,\widehat{\mathbf{X}}_t^R,\widehat{\mathbf{X}}_t\big)\in\mathbb{X}(\mathbf{x}_t)$;

(e) $\hat{\mathbf{x}}_{t+1}=\mathbf{x}_{t+1}$;

(f) $\widetilde{V}_t\big(\omega,\mathbf{x}_t,\widehat{\mathbf{X}}_t^E,\widehat{\mathbf{X}}_t^R,\widehat{\mathbf{X}}_t\big)\le\widetilde{V}_t\big(\omega,\mathbf{x}_t,\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t\big)$;

(g) $\widetilde{V}_t\big(\omega,\mathbf{x}_t,\widehat{\mathbf{X}}_t^E,\widehat{\mathbf{X}}_t^R,\widehat{\mathbf{X}}_t\big)<\widetilde{V}_t\big(\omega,\mathbf{x}_t,\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t\big)$, if $(\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t)$ is not semi-unidirectional.

Therefore, for any given policy $(\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t)$ feasible for x_t, the policy $(\widehat{\mathbf{X}}_t^E,\widehat{\mathbf{X}}_t^R,\widehat{\mathbf{X}}_t)$, constructed in Proposition 1, is semi-unidirectional (part b), reduces the size of both the expedited and reverse order decisions (part c), is feasible for $\mathbf{x}_t$ (part d), results in the same on-hand inventory state next period as the original policy (part e), and generates lower total expected cost than the original policy. We will refer to the policy $(\widehat{\mathbf{X}}_t^E,\widehat{\mathbf{X}}_t^R,\widehat{\mathbf{X}}_t)$ thus constructed as *the semi-unidirectional reduction* of $(\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t)$.

Proof of Proposition 1: Parts (a), (b), and (c) follow directly from the construction of $(\widehat{\mathbf{X}}_t^E, \widehat{\mathbf{X}}_t^R, \widehat{\mathbf{X}}_t)$.

To prove (d), we first show that $\widehat{X}_{jt}^E \le x_{j+1,t} + \widehat{X}_{j+1,t}^E$. Fix j. If $\widehat{X}_{jt}^E = 0$, this inequality holds directly, since $\widehat{X}_{j+1,t}^E \ge 0$ by part (a). If $\widehat{X}_{jt}^E > 0$, then $\widehat{X}_{jt}^E = X_{jt}^E - X_{jt}^R > 0$ and we will show that

$$X_{jt}^E - X_{jt}^R \le x_{j+1,t} + \widehat{X}_{j+1,t}^E = x_{j+1,t} + \left[X_{j+1,t}^E - X_{j+1,t}^R\right]^+. \tag{30}$$

Note that the feasibility of $X_{j+1,t}^R$, by the definition of the feasible set $\mathbb{X}(\mathbf{x}_t)$, implies that

$$X_{j+1,t}^R \le x_{j+1,t} + X_{j+1,t}^E - X_{jt}^E. \tag{31}$$

If $X_{j+1,t}^E \ge X_{j+1,t}^R$, (30) then follows from (31) since $X_{jt}^R \ge 0$. If $X_{j+1,t}^E < X_{j+1,t}^R$, then (30) reduces to $X_{jt}^E - X_{jt}^R \le x_{j+1,t}$, while (31) implies $x_{j+1,t} - X_{jt}^E > 0$, and (30) follows from (31). Thus, $\widehat{\mathbf{X}}_t^E$ is feasible.

Next, we establish the feasibility of $\widehat{\mathbf{X}}_t^R$. If $\widehat{X}_{jt}^R = 0$, then it is clearly feasible. Suppose $\widehat{X}_{1t}^R > 0$. Thus, $\widehat{X}_{1t}^E = 0$ by part (b), and so $\widehat{X}_{1t}^R$ is feasible if $\widehat{X}_{1t}^R \le x_{1t}$. Since $\widehat{X}_{1t}^R = X_{1t}^R - X_{1t}^E$ that inequality follows directly from the feasibility of X_{1t}^R. Suppose $\widehat{X}_{jt}^R > 0$ for some $j > 1$. Then,

$$\begin{aligned}
\widehat{X}_{jt}^R &= X_{jt}^R - X_{jt}^E && [\text{definition of } \widehat{X}_{jt}^R] \\
&\le \left(x_{jt} + X_{jt}^E - X_{j-1,t}^E\right) - X_{jt}^E && [\mathbf{X}_t^R \in \mathbb{X}(\mathbf{x}_t)] \\
&= x_{jt} + \widehat{X}_{jt}^E - X_{j-1,t}^E && [\text{Since } \widehat{X}_{1t}^R > 0, \text{ then } \widehat{X}_{jt}^E = 0 \text{ by part (b)}] \\
&\le x_{jt} + \widehat{X}_{jt}^E - \widehat{X}_{j-1,t}^E. && [\widehat{X}_{j-1,t}^E \le X_{j-1,t}^E \text{ by part (c)}]
\end{aligned}$$

Therefore, $\widehat{X}_{jt}^R$ is feasible for each j. Finally, using the definitions of $\widehat{\mathbf{X}}_t^R$ and $\widehat{\mathbf{X}}_t^E$, we get

$$\begin{aligned}
x_{j+1,t} - \widehat{X}_{j+1,t}^R + \widehat{X}_{j+1,t}^E - \widehat{X}_{jt}^E + \widehat{X}_{jt}^R &= x_{j+1,t} - \left[X_{j+1,t}^R - X_{j+1,t}^E\right]^+ + \left[X_{j+1,t}^E - X_{j+1,t}^R\right]^+ - \left[X_{jt}^E - X_{jt}^R\right]^+ + \left[X_{jt}^R - X_{jt}^E\right]^+ \\
&= x_{j+1,t} - X_{j+1,t}^R + X_{j+1,t}^E - X_{jt}^E + X_{jt}^R \\
&\ge X_{jt} && [\mathbf{X}_t \in \mathbb{X}(\mathbf{x}_t)] \\
&= \widehat{X}_{jt}. && [\text{definition of } \widehat{\mathbf{X}}_t]
\end{aligned}$$

Since $\widehat{X}_{jt}$ satisfies the required upper bound, it is feasible for x_t, which completes the proof of part (d).

To prove part (e), first fix ω. Next, we use state transition equations given in (2) to get

$$\begin{aligned}
\hat{x}_{1,t+1} &= x_{1t} - \widehat{X}_{1t}^R + \widehat{X}_{1t}^E + \widehat{X}_{1t} - D_t(\omega) \\
&= x_{1t} - \left[X_{1t}^R - X_{1t}^E\right]^+ + \left[X_{1t}^E - X_{1t}^R\right]^+ + X_{1t} - D_t(\omega) && [\text{definitions of } \widehat{\mathbf{X}}_t^E,\ \widehat{\mathbf{X}}_t^E \text{ and } \widehat{\mathbf{X}}_t] \\
&= x_{1t} - X_{1t}^R + X_{1t}^E + X_{1t} - D_t(\omega) = x_{1,t+1}; \\
\hat{x}_{j,t+1} &= x_{jt} - \widehat{X}_{jt}^R + \widehat{X}_{j-1,t}^R + \widehat{X}_{jt}^E - \widehat{X}_{j-1,t}^E + \widehat{X}_{jt} - \widehat{X}_{j-1,t} \\
&= x_{jt} - \left[X_{jt}^R - X_{jt}^E\right]^+ + \left[X_{j-1,t}^R - X_{j-1,t}^E\right]^+ + \left[X_{jt}^E - X_{jt}^R\right]^+ - \left[X_{j-1,t}^E - X_{j-,1t}^R\right]^+ + X_{jt} - X_{j-1,t} \\
&= x_{jt} - X_{jt}^R + X_{j-1,t}^R + X_{jt}^E - X_{j-1,t}^E + X_{jt} - X_{j-1,t} = x_{j,t+1}.
\end{aligned}$$

Consequently, $\hat{\mathbf{x}}_{t+1} = \mathbf{x}_{t+1}$, which proves part (e). To prove part (f), let $\Delta := \widetilde{V}_t\big(\omega, \mathbf{x}_t, \widehat{\mathbf{X}}_t^E, \widehat{\mathbf{X}}_t^R, \widehat{\mathbf{X}}_t\big) - \widetilde{V}_t\big(\omega, \mathbf{x}_t, \mathbf{X}_t^E, \mathbf{X}_t^R, \mathbf{X}_t\big)$. Since $\hat{\mathbf{x}}_{t+1} = \mathbf{x}_{t+1}$ by part (e), and $\widehat{\mathbf{X}}_t = \mathbf{X}_t$, it follows from expression (29) that

$$\Delta = \gamma_t\big(\omega, x_{1t} - \widehat{X}_{1t}^R + \widehat{X}_{1t}^E\big) - \gamma_t\big(\omega, x_{1t} - X_{1t}^R + X_{1t}^E\big) + \sum_{j=1}^{N}\Big[\big(k_j^E + h_j\big)\big(\widehat{X}_{jt}^E - X_{jt}^E\big) + \big(k_j^R - h_j\big)\big(\widehat{X}_{jt}^R - X_{jt}^R\big)\Big]. \tag{32}$$

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Fix j, and consider the term $\big(k_j^E + h_j\big)\big(\widehat{X}_{jt}^E - X_{jt}^E\big) + \big(k_j^R - h_j\big)\big(\widehat{X}_{jt}^R - X_{jt}^R\big)$ in the summation in (32). By part (b), either $\widehat{X}_{jt}^E = 0$, or $\widehat{X}_{jt}^R = 0$. If $\widehat{X}_{jt}^E = 0$, then $\widehat{X}_{jt}^R = X_{jt}^R - X_{jt}^E$ and we get

$$\big(k_j^E + h_j\big)\big(\widehat{X}_{jt}^E - X_{jt}^E\big) + \big(k_j^R - h_j\big)\big(\widehat{X}_{jt}^R - X_{jt}^R\big) = -\big(k_j^E + k_j^R\big)X_{jt}^E \le 0.$$

Similarly, if $\widehat{X}_{jt}^R = 0$, then $\widehat{X}_{jt}^E = X_{jt}^E - X_{jt}^R$ and we get

$$\big(k_j^E + h_j\big)\big(\widehat{X}_{jt}^E - X_{jt}^E\big) + \big(k_j^R - h_j\big)\big(\widehat{X}_{jt}^R - X_{jt}^R\big) = -\big(k_j^E + k_j^R\big)X_{jt}^R \le 0.$$

By applying these results to Expression (32), we get

$$\begin{aligned}
\Delta &\le \gamma_t\big(\omega, x_{1t} - \widehat{X}_{1t}^R + \widehat{X}_{1t}^E\big) - \gamma_t\big(\omega, x_{1t} - X_{1t}^R + X_{1t}^E\big) - \sum_{j=1}^N \big(k_j^E + k_j^R\big)\max\big[X_{jt}^E, X_{jt}^R\big] \\
&= \gamma_t\big(\omega, x_{1t} - [X_{1t}^R - X_{1t}^E]^+ + [X_{1t}^E - X_{1t}^R]^+\big) - \gamma_t\big(\omega, x_{1t} - X_{1t}^R + X_{1t}^E\big) - \sum_{j=1}^N \big(k_j^E + k_j^R\big)\max\big[X_{jt}^E, X_{jt}^R\big] \\
&= \gamma_t\big(\omega, x_{1t} - X_{1t}^R + X_{1t}^E\big) - \gamma_t\big(\omega, x_{1t} - X_{1t}^R + X_{1t}^E\big) - \sum_{j=1}^N \big(k_j^E + k_j^R\big)\max\big[X_{jt}^E, X_{jt}^R\big] \\
&= -\sum_{j=1}^N \big(k_j^E + k_j^R\big)\max\big[X_{jt}^E, X_{jt}^R\big] \le 0,
\end{aligned}$$

which completes the proof of part (f). Finally, if $(\mathbf{X}_t^E, \mathbf{X}_t^R, \mathbf{X}_t)$ is not semi-unidirectional, then there exists a location j such that $X_{jt}^E, X_{jt}^R > 0$. In that case, it follows from the above expression that $\Delta < 0$. □

Proposition 2. *Let $(\mathbf{X}_t^E, \mathbf{X}_t^R, \mathbf{X}_t)$ be a policy feasible for $\mathbf{x}_t$ in period t. Let $(\widehat{\mathbf{X}}_t^E, \widehat{\mathbf{X}}_t^R, \widehat{\mathbf{X}}_t)$ be its semi-unidirectional reduction. Define a policy $(\widetilde{\mathbf{X}}_t^E, \widetilde{\mathbf{X}}_t^R, \widetilde{\mathbf{X}}_t)$ as $\widetilde{X}_{jt}^E = \widehat{X}_{jt}^E$, $\widetilde{X}_{jt}^R = \big[\widehat{X}_{jt}^R - \widehat{X}_{jt}\big]^+$, and $\widetilde{X}_{jt} = \big[\widehat{X}_{jt} - \widehat{X}_{jt}^R\big]^+$ for each j. Let $\hat{\mathbf{x}}_{t+1}$ and $\tilde{\mathbf{x}}_{t+1}$ be inventory states in period $t+1$ obtained by starting with $\mathbf{x}_t$ in period t and applying policies $(\widehat{\mathbf{X}}_t^E, \widehat{\mathbf{X}}_t^R, \widehat{\mathbf{X}}_t)$ and $(\widetilde{\mathbf{X}}_t^E, \widetilde{\mathbf{X}}_t^R, \widetilde{\mathbf{X}}_t)$, respectively. Then, the following hold.*

(a) $\widetilde{X}_{jt}^E \ge 0$, $\widetilde{X}_{jt}^R \ge 0$ and $\widetilde{X}_{jt} \ge 0$ for all j;

(b) $\widetilde{X}_{jt}^E = \widetilde{X}_{jt} = 0$ or $\widetilde{X}_{jt}^R = 0$ (or both) for all j;

(c) $\widetilde{X}_{jt}^R \le \widehat{X}_{jt}^R \le X_{jt}^R$ and $\widetilde{X}_{jt} \le \widehat{X}_{jt} \le X_{jt}$ for all j;

(d) $(\widetilde{\mathbf{X}}_t^E, \widetilde{\mathbf{X}}_t^R, \widetilde{\mathbf{X}}_t) \in \mathbb{X}(\mathbf{x}_t)$;

(e) $\tilde{\mathbf{x}}_{t+1} = \hat{\mathbf{x}}_{t+1}$;

(f) $\widetilde{V}_t\big(\omega, \mathbf{x}_t, \widetilde{\mathbf{X}}_t^E, \widetilde{\mathbf{X}}_t^R, \widetilde{\mathbf{X}}_t\big) \le \widetilde{V}_t\big(\omega, \mathbf{x}_t, \widehat{\mathbf{X}}_t^E, \widehat{\mathbf{X}}_t^R, \widehat{\mathbf{X}}_t\big)$;

(g) $\widetilde{V}_t\big(\omega, \mathbf{x}_t, \widetilde{\mathbf{X}}_t^E, \widetilde{\mathbf{X}}_t^R, \widetilde{\mathbf{X}}_t\big) < \widetilde{V}_t\big(\omega, \mathbf{x}_t, \widehat{\mathbf{X}}_t^E, \widehat{\mathbf{X}}_t^R, \widehat{\mathbf{X}}_t\big)$, if $(\widehat{\mathbf{X}}_t^E, \widehat{\mathbf{X}}_t^R, \widehat{\mathbf{X}}_t)$ is not unidirectional.

Thus, the policy $(\widetilde{\mathbf{X}}_t^E, \widetilde{\mathbf{X}}_t^R, \widetilde{\mathbf{X}}_t)$ constructed in Proposition 2 is unidirectional (part b), reduces regular and reverse order decisions relative to $(\widehat{\mathbf{X}}_t^E, \widehat{\mathbf{X}}_t^R, \widehat{\mathbf{X}}_t)$ (part c), is feasible for $\mathbf{x}_t$ (part d), results in the same inventory state next period as the original policy (part e), and has lower total expected cost than the original policy. The policy $(\widetilde{\mathbf{X}}_t^E, \widetilde{\mathbf{X}}_t^R, \widetilde{\mathbf{X}}_t)$ is *the unidirectional reduction* of $(\mathbf{X}_t^E, \mathbf{X}_t^R, \mathbf{X}_t)$.

Proof of Proposition 2: Parts (a), (b), and (c) follow directly from the construction of $(\widetilde{\mathbf{X}}_t^E, \widetilde{\mathbf{X}}_t^R, \widetilde{\mathbf{X}}_t)$. Since $\widetilde{X}_{jt}^E = \widehat{X}_{jt}^E$ for each j, then $\widetilde{\mathbf{X}}_t^E$ is feasible because $\widehat{X}_{jt}^E$ is feasible by Proposition 1 (d). Next,

$$\begin{aligned}
\widetilde{X}_{1t}^R &= \big[\widehat{X}_{1t}^R - \widehat{X}_{1t}\big]^+ && [\text{definition of } \widetilde{X}_{jt}^R] \\
&\le \widehat{X}_{1t}^R \\
&\le \big[x_{1t} + \widehat{X}_{1t}^E\big]^+ && [\text{By Proposition 1(d), } \widehat{\mathbf{X}}_t^R \in \mathbb{X}(\mathbf{x}_t)] \\
&= \big[x_{1t} + \widetilde{X}_{1t}^E\big]^+. && [\text{definition of } \widetilde{X}_{jt}^E]
\end{aligned}$$

Therefore, $\widetilde{X}_{1t}^R$ is feasible for $\mathbf{x}_t$. Similarly, for $j > 1$, we get

$$\begin{aligned}
\widetilde{X}_{jt}^R &= \big[\widehat{X}_{jt}^R - \widehat{X}_{jt}\big]^+ && [\text{definition of } \widetilde{X}_{jt}^R] \\
&\le \widehat{X}_{jt}^R \\
&\le x_{jt} + \widehat{X}_{jt}^E - \widehat{X}_{j-1,t}^E && [\text{By Proposition 1(d), } \widehat{\mathbf{X}}_t^R \in \mathbb{X}(\mathbf{x}_t)] \\
&= x_{jt} + \widetilde{X}_{jt}^E - \widetilde{X}_{j-1,t}^E. && [\text{definition of } \widetilde{X}_{jt}^E]
\end{aligned}$$

Hence, $\widetilde{\mathbf{X}}_t^R$ is feasible for $\mathbf{x}_t$. Next, if $\widetilde{X}_{jt} = 0$, then $\widetilde{X}_{jt}$ is clearly feasible. Suppose $\widetilde{X}_{jt} > 0$. In that case, $\widetilde{X}_{jt} = \widehat{X}_{jt} - \widehat{X}_{jt}^R$, and, by Part (b), $\widetilde{X}_{jt}^R = 0$. It follows that

$$\begin{aligned}
x_{j+1,t} - \widetilde{X}_{j+1,t}^R + \widetilde{X}_{jt}^R + \widetilde{X}_{j+1,t}^E - \widetilde{X}_{jt}^E &= x_{j+1,t} - \widetilde{X}_{j+1,t}^R + \widetilde{X}_{j+1,t}^E - \widetilde{X}_{jt}^E \\
&= x_{j+1,t} - \big[\widehat{X}_{j+1,t}^R - \widehat{X}_{j+1,t}\big]^+ + \widehat{X}_{j+1,t}^E - \widehat{X}_{jt}^E && [\text{definitions of } \widetilde{\mathbf{X}}_t^E \text{ and } \widetilde{\mathbf{X}}_t^R] \\
&\ge x_{j+1,t} - \widehat{X}_{j+1,t}^R + \widehat{X}_{j+1,t}^E - \widehat{X}_{jt}^E \\
&\ge \widehat{X}_{jt} - \widehat{X}_{jt}^R && [\text{By Proposition 1(d), } \widehat{\mathbf{X}}_t \in \mathbb{X}(\mathbf{x}_t)] \\
&= \widetilde{X}_{jt}.
\end{aligned}$$

Thus, $\widetilde{\mathbf{X}}_t$ is feasible for $\mathbf{x}_t$, which proves (d). To prove (e), fix ω. By state transition equations in (2),

$$\begin{aligned}
\tilde{x}_{1,t+1} &= x_{1t} - \widetilde{X}_{1t}^R + \widetilde{X}_{1t}^E + \widetilde{X}_{1t} - D_t(\omega) \\
&= x_{1t} - \big[\widehat{X}_{1t}^R - \widehat{X}_{1t}\big]^+ + \big[\widehat{X}_{1t} - \widehat{X}_{1t}^R\big]^+ + \widehat{X}_{1t}^E - D_t(\omega) && [\text{definitions of } \widetilde{\mathbf{X}}_t^E,\ \widetilde{\mathbf{X}}_t^R \text{ and } \widetilde{\mathbf{X}}_t] \\
&= x_{1t} - \widehat{X}_{1t}^R + \widehat{X}_{1t}^E + \widehat{X}_{1t} - D_t(\omega) = \hat{x}_{1,t+1}; \\
\tilde{x}_{j,t+1} &= x_{jt} - \widetilde{X}_{jt}^R + \widetilde{X}_{j-1,t}^R + \widetilde{X}_{jt}^E - \widetilde{X}_{j-1,t}^E + \widetilde{X}_{jt} - \widetilde{X}_{j-1,t} \\
&= x_{jt} - \big[\widehat{X}_{jt}^R - \widehat{X}_{jt}\big]^+ + \big[\widehat{X}_{j-1,t}^R - \widehat{X}_{j-1,t}\big]^+ + \widehat{X}_{jt}^E - \widehat{X}_{j-1,t}^E + \big[\widehat{X}_{jt} - \widehat{X}_{jt}^R\big]^+ - \big[\widehat{X}_{j-1,t} - \widehat{X}_{j-1,t}^R\big]^+ \\
&= x_{jt} - \widehat{X}_{jt}^R + \widehat{X}_{j-1,t}^R + \widehat{X}_{jt}^E - \widehat{X}_{j-1,t}^E + \widehat{X}_{jt} - \widehat{X}_{j-1,t} = \hat{x}_{j,t+1}.
\end{aligned}$$

Thus, $\tilde{\mathbf{x}}_{t+1} = \hat{\mathbf{x}}_{t+1}$, which proves part (e). To prove part (f), let $\Delta := \widetilde{V}_t(\omega, \mathbf{x}_t, \widetilde{\mathbf{X}}_t^E, \widetilde{\mathbf{X}}_t^R, \widetilde{\mathbf{X}}_t) - \widetilde{V}_t(\omega, \mathbf{x}_t, \widehat{\mathbf{X}}_t^E, \widehat{\mathbf{X}}_t^R, \widehat{\mathbf{X}}_t)$. Because $\tilde{\mathbf{x}}_{t+1} = \hat{\mathbf{x}}_{t+1}$ by part (e), and $\widetilde{\mathbf{X}}_t^E = \widehat{\mathbf{X}}_t^E$, it follows from (29) that

$$\Delta = \gamma_t\big(\omega, x_{1t} - \widetilde{X}_{1t}^R + \widetilde{X}_{1t}^E\big) - \gamma_t\big(\omega, x_{1t} - \widehat{X}_{1t}^R + \widehat{X}_{1t}^E\big) + \sum_{j=1}^{N} \Big[\big(k_j + h_j\big)\big(\widetilde{X}_{jt} - \widehat{X}_{jt}\big) + \big(k_j^R - h_j\big)\big(\widetilde{X}_{jt}^R - \widehat{X}_{jt}^R\big)\Big]. \quad (33)$$

Fix j, and consider the term $\big(k_j + h_j\big)\big(\widetilde{X}_{jt} - \widehat{X}_{jt}\big) + \big(k_j^R - h_j\big)\big(\widetilde{X}_{jt}^R - \widehat{X}_{jt}^R\big)$ in the summation in (33). By part (b), either $\widetilde{X}_{jt} = 0$, or $\widetilde{X}_{jt}^R = 0$. If $\widetilde{X}_{jt} = 0$, then $\widetilde{X}_{jt}^R = \widehat{X}_{jt}^R - \widehat{X}_{jt}$ and we get

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$$\big(k_j+h_j\big)\big(\widetilde{X}_{jt}-\widehat{X}_{jt}\big)+\big(k_j^R-h_j\big)\big(\widetilde{X}_{jt}^R-\widehat{X}_{jt}^R\big)=-\big(k_j+k_j^R\big)\widehat{X}_{jt}\le 0.$$

Similarly, if $\widetilde{X}_{jt}^R=0$, then $\widetilde{X}_{jt}=\widehat{X}_{jt}-\widehat{X}_{jt}^R$ and we get

$$\big(k_j^E+h_j\big)\big(k_j+h_j\big)\big(\widetilde{X}_{jt}-\widehat{X}_{jt}\big)+\big(k_j^R-h_j\big)\big(\widetilde{X}_{jt}^R-\widehat{X}_{jt}^R\big)=-\big(k_j+k_j^R\big)\widehat{X}_{jt}^R\le 0,$$

since $k_j+k_j^R\ge 0$ at location N by assumption, and for all other j because $k_j,k_j^R\ge 0$ for all $j<N$. By applying these results to Expression (33), we get

$$\begin{aligned}\Delta &\le \gamma_t\big(\omega,x_{1t}-\widetilde{X}_{1t}^R+\widetilde{X}_{1t}^E\big)-\gamma_t\big(\omega,x_{1t}-\widehat{X}_{1t}^R+\widehat{X}_{1t}^E\big)-\sum_{j=1}^{N}\big(k_j+k_j^R\big)\max\big[\widehat{X}_{jt},\widehat{X}_{jt}^R\big]\\ &=\gamma_t\big(\omega,x_{1t}+\widehat{X}_{1t}^E-[\widehat{X}_{1t}^R-\widehat{X}_{1t}]^+\big)-\gamma_t\big(\omega,x_{1t}+\widehat{X}_{1t}^E-\widehat{X}_{1t}^R\big)-\sum_{j=1}^{N}\big(k_j+k_j^R\big)\max\big[\widehat{X}_{jt},\widehat{X}_{jt}^R\big].\end{aligned}$$

By definition, $\gamma_t(\omega,x):=(p+H_1)\mathbb{E}_\omega[(D_t(\omega)-x)^+]-H_1\mathbb{E}_\omega[D_t(\omega)]$. Thus, since $\mathbb{E}_\omega[(D_t(\omega)-x)^+]$ is decreasing in x, then so is $\gamma_t(\omega,x)$. Since $\widehat{X}_{1t}^E-[\widehat{X}_{1t}^R-\widehat{X}_{1t}]^+\ge\widehat{X}_{1t}^E-\widehat{X}_{1t}^R$, it follows that $\gamma_t\big(\omega,x_{1t}+\widehat{X}_{1t}^E-[\widehat{X}_{1t}^R-\widehat{X}_{1t}]^+\big)\le\gamma_t\big(\omega,x_{1t}+\widehat{X}_{1t}^E-\widehat{X}_{1t}^R\big)$. Hence, $\Delta\le 0$, which proves part (f).

To prove part (g), suppose that $(\widehat{\mathbf{X}}_t^E,\widehat{\mathbf{X}}_t^R,\widehat{\mathbf{X}}_t)$ is not unidirectional. It that case, there exists a location j such that $\widehat{X}_{jt},\widehat{X}_{jt}^R>0$. It then follows from the above inequality for Δ that $\Delta<0$. □

We are now in a position to prove Theorem 1.

Proof of Theorem 1: Suppose that there exists an optimal policy $(\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t)$ that is not unidirectional for some state $\mathbf{x}_t$ and ω. Thus, there exists a location j such that either $X_{jt}^E>0$ and $X_{jt}^R>0$, or $X_{jt}>0$ and $X_{jt}^R>0$. If $X_{jt}^E>0$ and $X_{jt}^R>0$, then, by definition, $(\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t)$ is not semi-unidirectional. Suppose instead that $X_{jt}^E=0$ and $X_{jt},X_{jt}^R>0$. Let $(\widehat{\mathbf{X}}_t^E,\widehat{\mathbf{X}}_t^R,\widehat{\mathbf{X}}_t)$ be the semi-unidirectional reduction of $(\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t)$. In that case, by definition, $\widehat{X}_{jt}^R[X_{jt}^R-X_{jt}^E]^+=X_{jt}^R$ and $\widehat{X}_{jt}=X_{jt}$. Therefore, $\widehat{X}_{jt},\widehat{X}_{jt}^R>0$, and thus $(\widehat{\mathbf{X}}_t^E,\widehat{\mathbf{X}}_t^R,\widehat{\mathbf{X}}_t)$ is not unidirectional. Hence, when $(\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t)$ is not unidirectional, then either $(\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t)$ is not semi-unidirectional or $(\widehat{\mathbf{X}}_t^E,\widehat{\mathbf{X}}_t^R,\widehat{\mathbf{X}}_t)$ is not unidirectional.

By Proposition 1, we can construct a policy $(\widehat{\mathbf{X}}_t^E,\widehat{\mathbf{X}}_t^R,\widehat{\mathbf{X}}_t)$ that is the semi-unidirectional reduction of $(\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t)$. This policy is feasible for $\mathbf{x}_t$, semi-unidirectional, results in the same inventory state next period as the original policy, and generates a lower total cost than the original policy $(\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t)$. Further, their cost difference is strictly negative if $(\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t)$ is not semi-unidirectional.

Hence, we can construct a policy $(\widetilde{\mathbf{X}}_t^E,\widetilde{\mathbf{X}}_t^R,\widetilde{\mathbf{X}}_t)$ that is unidirectional, feasible for $\mathbf{x}_t$, results in the same inventory state as the original policy, and a generates lower total cost than the semi-unidirectional reduction $(\widehat{\mathbf{X}}_t^E,\widehat{\mathbf{X}}_t^R,\widehat{\mathbf{X}}_t)$. Further, their cost difference is strictly negative if $(\widehat{\mathbf{X}}_t^E,\widehat{\mathbf{X}}_t^R,\widehat{\mathbf{X}}_t)$ is not unidirectional. Finally, as established above, if $(\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t)$ is not unidirectional in period t, then either $(\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t)$ is not semi-unidirectional or $(\widehat{\mathbf{X}}_t^E,\widehat{\mathbf{X}}_t^R,\widehat{\mathbf{X}}_t)$ is not unidirectional. It follows that $(\widetilde{\mathbf{X}}_t^E,\widetilde{\mathbf{X}}_t^R,\widetilde{\mathbf{X}}_t)$ generates a strictly lower cost than any policy $(\mathbf{X}_t^E,\mathbf{X}_t^R,\mathbf{X}_t)$ that is not unidirectional for $\mathbf{x}_t$. Therefore, we conclude that any optimal

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policy must be unidirectional in each period t and and for each state ω. $\square$

Proof of Theorem 2: Let $(\widetilde{\mathbf{Y}}_t^E, \widetilde{\mathbf{Y}}_t^R, \widetilde{\mathbf{Y}}_t)$ be an optimal policy in period t for the relaxed reverse logistics problem. To prove (a), we first show that, for any given y_t and ω in period t, $\widetilde{\mathbf{Y}}_t^E, \widetilde{\mathbf{Y}}_t^R, \widetilde{\mathbf{Y}}_t \in \mathbb{Y}(\mathbf{y}_t)$, so that the schedules $(\widetilde{\mathbf{Y}}_t^E, \widetilde{\mathbf{Y}}_t^R, \widetilde{\mathbf{Y}}_t)$ *optimal* for the relaxed reverse logistics problem with $\mathbf{y}_t$ in period t are *feasible* for the original reverse logistics problem with $\mathbf{y}_t$ in period t.

Being optimal, schedules $(\widetilde{\mathbf{Y}}_t^E, \widetilde{\mathbf{Y}}_t^R, \widetilde{\mathbf{Y}}_t)$ are feasible for the relaxed reverse logistics problem. Thus, (i) $y_{jt} \le \widetilde{Y}_{jt}^E \le \widetilde{Y}_{j+1,t}^E$ for $1 \le j \le N$; (ii) $\left[\widetilde{Y}_{1t}^E\right]^- \le \widetilde{Y}_{1t}^R \le \widetilde{Y}_{1t}^E$ and $\widetilde{Y}_{j-1,t}^E \le \widetilde{Y}_{jt}^R \le \widetilde{Y}_{jt}^E$ for $2 \le j \le N$; and (iii) $\widetilde{Y}_{jt}^R \le \widetilde{Y}_{jt} \le \widetilde{Y}_{j+1,t}^R$ for $1 \le j \le N$. To prove that schedules $(\widetilde{\mathbf{Y}}_t^E, \widetilde{\mathbf{Y}}_t^R, \widetilde{\mathbf{Y}}_t)$ are feasible for the original reverse logistics problem with $\mathbf{y}_t$ in period t (i.e., $\widetilde{\mathbf{Y}}_t^E, \widetilde{\mathbf{Y}}_t^R, \widetilde{\mathbf{Y}}_t \in \mathbb{Y}(\mathbf{y}_t)$) it thus suffices, by the definition of $\mathbb{Y}(\mathbf{y}_t)$ given in (6), to show that, at each location j, $\widetilde{Y}_{jt} \le \widetilde{Y}_{j+1,t}^R - (\widetilde{Y}_{jt}^E - \widetilde{Y}_{jt}^R)$.

We now apply Corollary 1: thus, either $\widetilde{Y}_{jt}^E = y_{jt}$ and $\widetilde{Y}_{jt} = \widetilde{Y}_{jt}^R$, or $\widetilde{Y}_{jt}^R = \widetilde{Y}_{jt}^E$ for each j. Accordingly, suppose first that at a given location j, $\widetilde{Y}_{jt}^E = y_{jt}$ and $\widetilde{Y}_{jt} = \widetilde{Y}_{jt}^R$. In that case, the inequality $\widetilde{Y}_{jt} \le \widetilde{Y}_{j+1,t}^R - (\widetilde{Y}_{jt}^E - \widetilde{Y}_{jt}^R)$ reduces to $\widetilde{Y}_{jt}^R \le \widetilde{Y}_{j+1,t}^R - (\widetilde{Y}_{jt}^E - \widetilde{Y}_{jt}^R)$, which holds directly, because $\widetilde{Y}_{j+1,t}^R \ge \widetilde{Y}_{jt}^E$ since $\widetilde{\mathbf{Y}}_t^R \in \widetilde{\mathbb{Y}}(\mathbf{y}_t)$. Suppose next that, at location j, $\widetilde{Y}_{jt}^R = \widetilde{Y}_{jt}^E$. In that case, the inequality $\widetilde{Y}_{jt} \le \widetilde{Y}_{j+1,t}^R - (\widetilde{Y}_{jt}^E - \widetilde{Y}_{jt}^R)$ reduces to $\widetilde{Y}_{jt} \le \widetilde{Y}_{j+1,t}^R$ which again holds directly because $\widetilde{\mathbf{Y}}_t \in \widetilde{\mathbf{Y}}(\mathbf{y}_t)$. It follows that $\widetilde{\mathbf{Y}}_t^E, \widetilde{\mathbf{Y}}_t^R, \widetilde{\mathbf{Y}}_t \in \mathbb{Y}(\mathbf{y}_t)$.

Since the relaxed reverse logistics problem is a relaxation of the original reverse logistics problem, then, because the optimal policy $(\widetilde{\mathbf{Y}}_t^E, \widetilde{\mathbf{Y}}_t^R, \widetilde{\mathbf{Y}}_t)$ for the relaxed reverse logistics problem is feasible in period t for the corresponding original reverse logistics problem for any given $\mathbf{y}_t$, it follows that $(\widetilde{\mathbf{Y}}_t^E, \widetilde{\mathbf{Y}}_t^R, \widetilde{\mathbf{Y}}_t)$ is also optimal for the original reverse logistics problem. Hence, $(\mathbf{Y}_t^E, \mathbf{Y}_t^R, \mathbf{Y}_t) = (\widetilde{\mathbf{Y}}_t^E, \widetilde{\mathbf{Y}}_t^R, \widetilde{\mathbf{Y}}_t)$ in every period t.

We prove part (b) by induction. By assumption, $f_{T+1}(\omega_{T+1}, \cdot) = \tilde{f}_{T+1}(\omega_{T+1}, \cdot)$, for all ω_{T+1}. Assume inductively that part (b) holds for period $t+1$ so that $f_{t+1}(\omega_{t+1}, \mathbf{y}_{t+1}) = \tilde{f}_{t+1}(\omega_{t+1}, \mathbf{y}_{t+1})$ for all ω_{t+1} and $\mathbf{y}_{t+1}$. Fix $\mathbf{y}_t$. Let $(\widetilde{\mathbf{Y}}_t^E, \widetilde{\mathbf{Y}}_t^R, \widetilde{\mathbf{Y}}_t)$ be optimal for $\tilde{f}_t(\omega, \mathbf{y}_t)$, and $(\overline{\mathbf{Y}}_t^E, \overline{\mathbf{Y}}_t^R, \overline{\mathbf{Y}}_t)$ be optimal for $f_t(\omega, \mathbf{y}_t)$. It then follows from the optimality equation in (10) that

$$
\begin{aligned}
\tilde{f}_t(\omega, \mathbf{y}_t) &= -\sum_{j=1}^N k_j^E\, y_{jt} + \min_{\mathbf{Y}_t^E, \mathbf{Y}_t^R, \mathbf{Y}_t \in \widetilde{\mathbb{Y}}(\mathbf{y}_t)} \left\{ \gamma_t(\omega, Y_{1t}^R) + \sum_{j=1}^N \left(c_j^E Y_{jt}^E - c_j^R Y_{jt}^R + c_j Y_{jt} \right) + \alpha \mathbb{E}_\omega \left[\tilde{f}_{t+1}\left(\omega_{t+1}, \widetilde{\mathbf{Y}}_t - D_t(\omega)\right)\right] \right\} \\
&= -\sum_{j=1}^N k_j^E\, y_{jt} + \gamma_t(\omega, \widetilde{Y}_{1t}^R) + \sum_{j=1}^N \left(c_j^E \widetilde{Y}_{jt}^E - c_j^R \widetilde{Y}_{jt}^R + c_j \widetilde{Y}_{jt} \right) + \alpha \mathbb{E}_\omega \left[\tilde{f}_{t+1}\left(\omega_{t+1}, \widetilde{\mathbf{Y}}_t - D_t(\omega)\right)\right] \quad \left[\text{dfn. of } (\widetilde{\mathbf{Y}}_t^E, \widetilde{\mathbf{Y}}_t^R, \widetilde{\mathbf{Y}}_t)\right] \\
&= -\sum_{j=1}^N k_j^E\, y_{jt} + \gamma_t(\omega, \widetilde{Y}_{1t}^R) + \sum_{j=1}^N \left(c_j^E \widetilde{Y}_{jt}^E - c_j^R \widetilde{Y}_{jt}^R + c_j \widetilde{Y}_{jt} \right) + \alpha \mathbb{E}_\omega \left[f_{t+1}\left(\omega_{t+1}, \widetilde{\mathbf{Y}}_t - D_t(\omega)\right)\right] \quad [\text{induct. assumption}] \\
&= -\sum_{j=1}^N k_j^E\, y_{jt} + \gamma_t(\omega, \overline{Y}_{1t}^R) + \sum_{j=1}^N \left(c_j^E \overline{Y}_{jt}^E - c_j^R \overline{Y}_{jt}^R + c_j \overline{Y}_{jt} \right) + \alpha \mathbb{E}_\omega \left[f_{t+1}\left(\omega_{t+1}, \overline{\mathbf{Y}}_t - D_t(\omega)\right)\right] \quad [\text{part (a)}] \\
&= f_t(\omega, \mathbf{y}_t). \qquad \left[(\overline{\mathbf{Y}}_t^E, \overline{\mathbf{Y}}_t^R, \overline{\mathbf{Y}}_t) \text{ is optimal for } f_t(\omega, \mathbf{y}_t)\right]
\end{aligned}
$$

In what follows, we make use of the following lemma, which, as far as we know, is new to the literature.

Lemma 1. *Let functions $v_1, v_2, \ldots, v_N : \mathbb{R} \to \mathbb{R}$ be convex. Let $x_1 \le x_2 \le \cdots \le x_N$ be an ordered set of state variables and $y_1 \le y_2 \le \cdots \le y_N$ be an ordered set of decision variables. Then, for any $n \le N$,*

$$\min_{\substack{x_j \le y_j \le y_{j+1} \\ j=1,2,\ldots,N}} \sum_{j=1}^{N} v_j(y_j) = \sum_{j=1}^{n} w_j^+(x_j) + \min_{\substack{x_j \le y_j \le y_{j+1} \\ j=n+1,\ldots,N}} \Big\{ w_{n+1}(y_{n+1}) + \sum_{j=n+2}^{N} v_j(y_j) \Big\}, \tag{34}$$

where functions $w_1, w_2, \ldots, w_N$ are defined recursively as $w_1 := v_1$ and $w_j := v_j + w_{j-1}^-$. In particular,

$$\min_{\substack{x_j \le y_j \le y_{j+1} \\ j=1,2,\ldots,N}} \sum_{j=1}^{N} v_j(y_j) = \sum_{j=1}^{N} w_j^+(x_j).$$

Proof. Observe first that, if f is a convex function on $\mathbb{R}$, then, given $x \le y$,

$$\min_{x \le \theta \le y} f(\theta) = f^+(x) + f^-(y), \tag{35}$$

using the definition of f^+ and f^- given in the paper. Next, to prove the lemma, we start with $n = 1$. Using the definition of w_1 and w_2, we apply (35) to get

$$\min_{\substack{x_j \le y_j \le y_{j+1} \\ j=1,2,\ldots,N}} \sum_{j=1}^{N} v_j(y_j) = \min_{\substack{x_j \le y_j \le y_{j+1} \\ j=2,\ldots,N}} \Big[\min_{x_1 \le y_1 \le y_2} v_1(y_1) + \sum_{j=2}^{N} v_j(y_j) \Big] = \min_{\substack{x_j \le y_j \le y_{j+1} \\ j=2,\ldots,N}} \Big[v_1^+(x_1) + v_1^-(y_2) + \sum_{j=2}^{N} v_j(y_j) \Big]$$

$$= v_1^+(x_1) + \min_{\substack{x_j \le y_j \le y_{j+1} \\ j=2,3,\ldots,N}} \Big[v_1^-(y_2) + v_2(y_2) + \sum_{j=3}^{N} v_j(y_j) \Big] = w_1^+(x_1) + \min_{\substack{x_j \le y_j \le y_{j+1} \\ j=2,3,\ldots,N}} \Big[w_2(y_2) + \sum_{j=3}^{N} v_j(y_j) \Big].$$

Therefore, the expression (34) holds for $n = 1$. Assume inductively that (34) holds for n. Using the observation stated in (35) and the definition of w_{n+1}, for the RHS of (34) we obtain

$$\sum_{j=1}^{n} w_j^+(x_j) + \min_{\substack{x_j \le y_j \le y_{j+1} \\ j=n+1,\ldots,N}} \Big[w_n^-(y_{n+1}) + v_{n+1}(y_{n+1}) + \sum_{j=n+2}^{N} v_j(y_j) \Big] = \sum_{j=1}^{n} w_j^+(x_j) + \min_{\substack{x_j \le y_j \le y_{j+1} \\ j=n+1,\ldots,N}} \Big[w_{n+1}(y_{n+1}) + \sum_{j=n+2}^{N} v_j(y_j) \Big]$$

$$= \sum_{j=1}^{n} w_j^+(x_j) + \min_{\substack{x_j \le y_j \le y_{j+1} \\ j=n+2,\ldots,N}} \Big[w_{n+1}^+(x_{n+1}) + w_{n+1}^-(y_{n+2}) + \sum_{j=n+2}^{N} v_j(y_j) \Big]$$

$$= \sum_{j=1}^{n+1} w_j^+(x_j) + \min_{\substack{x_j \le y_j \le y_{j+1} \\ j=n+2,\ldots,N}} \Big[w_{n+1}^-(y_{n+2}) + v_{n+2}(y_{n+2}) + \sum_{j=n+3}^{N} v_j(y_j) \Big]$$

$$= \sum_{j=1}^{n+1} w_j^+(x_j) + \min_{\substack{x_j \le y_j \le y_{j+1} \\ j=n+2,\ldots,N}} \Big[w_{n+2}(y_{n+2}) + \sum_{j=n+3}^{N} v_j(y_j) \Big].$$

Thus, expression (34) also holds for $n+1$. Take $n = N-1$. The RHS of (34) becomes

$$\sum_{j=1}^{N-1} w_j^+(y_j) + \min_{x_N \le y_N} w_N(y_N) = \sum_{j=1}^{N} w_j^+(x_j).$$

Proof of Theorem 3: By Assumption 1, $\tilde{f}_{T+1}(\omega_{T+1}, \mathbf{y}_{T+1})$ is additively convex in $\mathbf{y}_{T+1}$ for each ω_{T+1}. Assume inductively that $f_{t+1}(\omega_{t+1}, \cdot)$ is additively convex for each ω_{t+1} in period $t+1 < T+1$. Thus, there exist convex functions $\{f_{1,t+1}, \ldots, f_{N,t+1}\}$ such that $f_{t+1}(\omega_{t+1}, y_{t+1}) = \sum_{j=1}^{N} f_{j,t+1}(\omega_{t+1}, y_{j,t+1})$. By the optimality equations in (10), we get

$$\tilde{f}_t(\omega, \mathbf{y}_t) = -\sum_{j=1}^{N} k_j^E\, y_{jt} + \min_{\mathbf{Y}_t^E, \mathbf{Y}_t^R, \mathbf{Y}_t \in \widetilde{\mathbb{Y}}(\mathbf{y}_t)} \left\{ \gamma_t(\omega, Y_{1t}^R) + \sum_{j=1}^{N} \Big(c_j^E Y_{jt}^E - c_j^R Y_{jt}^R + c_j Y_{jt} \Big) + \alpha\, \mathbb{E}_\omega \big[\tilde{f}_{t+1}\big(\omega_{t+1}, Y_t - D_t(\omega)\big) \big] \right\}.$$

Using the definition of $\widetilde{\mathbb{Y}}(y_t)$ given in (9), and the definition of g_{jt}, it follows that

$$\tilde{f}_t(\omega, y_t) = -\sum_{j=1}^{N} k_j^E\, y_{jt} + \min_{\substack{y_{jt} \le Y_{jt}^E \le Y_{j+1,t}^E \\ j=1,\ldots,N; \\ [Y_{1t}^E]^- \le Y_{1t}^R \le Y_{1t}^E \\ Y_{j-1,t}^E \le Y_{jt}^R \le Y_{jt}^E \\ j=2,\ldots,N.}} \left\{ \gamma_t(\omega, Y_{1t}^R) + \sum_{j=1}^{N} \Big[c_j^E\, Y_{jt}^E - c_j^R\, Y_{jt}^R + c_j\, Y_{jt} + \min_{Y_{jt}^R \le Y_{jt} \le Y_{j+1,t}^R} g_{jt}(\omega, Y_t) \Big] \right\}$$

$$= -\sum_{j=1}^{N} k_j^E\, y_{jt} + \min_{\substack{y_{jt} \le Y_{jt}^E \le Y_{j+1,t}^E \\ j=1,\ldots,N; \\ [Y_{1t}^E]^- \le Y_{1t}^R \le Y_{1t}^E \\ Y_{j-1,t}^E \le Y_{jt}^R \le Y_{jt}^E \\ j=2,\ldots,N.}} \left\{ \gamma_t(\omega, Y_{1t}^R) + \sum_{j=1}^{N} \Big[c_j^E\, Y_{jt}^E - c_j^R\, Y_{jt}^R + c_j\, Y_{jt} + g_{jt}^+(\omega, Y_{jt}^R) + g_{jt}^-(\omega, Y_{j+1,t}^R) \Big] \right\},$$

Consequently, we get

$$\tilde{f}_t(\omega, \mathbf{y}_t) = -\sum_{j=1}^{N} k_j^E\, y_{jt} + \min_{\substack{y_{jt} \le Y_{jt}^E \le Y_{j+1,t}^E \\ j=1,\ldots,N; \\ [Y_{1t}^E]^- \le Y_{1t}^R \le Y_{1t}^E \\ Y_{j-1,t}^E \le Y_{jt}^R \le Y_{jt}^E \\ j=2,\ldots,N.}} \left\{ \sum_{j=1}^{N} \Big[c_j^E\, Y_{jt}^E + u_{jt}(\omega, Y_{jt}^R) \Big] \right\} \qquad [\text{definition of } u_{jt}]$$

$$= -\sum_{j=1}^{N} k_j^E\, y_{jt} + \min_{\substack{y_{jt} \le Y_{jt}^E \le Y_{j+1,t}^E \\ j=1,\ldots,N.}} \left\{ \sum_{j=1}^{N} c_j^E\, Y_{jt}^E + \min_{\substack{[Y_{1t}^E]^- \le Y_{1t}^R \le Y_{1t}^E \\ Y_{j-1,t}^E \le Y_{jt}^R \le Y_{jt}^E \\ j=2,\ldots,N.}} \sum_{j=1}^{N} u_{jt}(\omega, Y_{jt}^R) \right\}$$

$$= -\sum_{j=1}^{N} k_j^E\, y_{jt} + \min_{\substack{y_{jt} \le Y_{jt}^E \le Y_{j+1,t}^E \\ j=1,\ldots,N.}} \left\{ \sum_{j=1}^{N} c_j^E\, Y_{jt}^E + u_{1t}^+(\omega, [Y_{1t}^E]^-) + u_{1t}^-(\omega, Y_{1t}^E) + u_{2t}^+(\omega, Y_{1t}^E) + \sum_{j=2}^{N-1} \Big(u_{jt}^-(\omega, Y_{jt}^E) + u_{j+1,t}^+(\omega, Y_{jt}^E) \Big) + u_{Nt}^-(\omega, Y_{Nt}) \right\}$$

$$= -\sum_{j=1}^{N} k_j^E\, y_{jt} + \min_{\substack{y_{jt} \le Y_{jt}^E \le Y_{j+1,t}^E \\ j=1,\ldots,N.}} \left\{ \sum_{j=1}^{N} v_{jt}(\omega, Y_{jt}^E) \right\} \qquad [\text{definition of } v_{jt}]$$

$$= -\sum_{j=1}^{N} k_j^E\, y_{jt} + \sum_{j=1}^{N} w_{jt}^+(\omega, y_{jt}). \qquad [\text{Lemma 1}]$$

Defining $f_{jt}(\omega, y_{jt})$ for each ω as $f_{jt}(\omega, y_{jt}) := -k_j^E y_{jt} + w_{jt}^+(\omega, y_{jt})$ completes the proof. □

Proof of Theorem 4: Following the same steps as in the proof of Theorem 3, we arrive at

$$\tilde{f}_t(\omega, \mathbf{y}_t) = -\sum_{j=1}^{N} k_j^E\, y_{jt} + \min_{\substack{y_{jt} \le Y_{jt}^E \le Y_{j+1,t}^E \\ j=1,\ldots,N.}} \left\{ \sum_{j=1}^{N} v_{jt}(\omega, Y_{jt}^E) \right\}. \tag{36}$$

Since $S_{jt}^E(\omega) := \inf \arg\min_Y w_{jt}(\omega, Y)$, then part (a) follows from (36) and Lemma 1.

To prove part (b), let the optimal post-expedite order schedule $\widehat{\mathbf{Y}}_t^E(\omega)$ be given. Then, starting with optimality equations in (10), and by the same steps used in the proof of Theorem 3, we obtain

$$\tilde{f}_t(\omega, \mathbf{y}_t) = -\sum_{j=1}^{N} k_j^E\, y_{jt} + \sum_{j=1}^{N} c_j^E\, \widehat{Y}_{jt}^E + \min_{\substack{[\widehat{Y}_{1t}^E]^- \le Y_{1t}^R \le \widehat{Y}_{1t}^E \\ \widehat{Y}_{j-1,t}^E \le Y_{jt}^R \le \widehat{Y}_{jt}^E \\ j=2,\ldots,N.}} \sum_{j=1}^{N} u_{jt}(\omega, Y_{jt}^R) \Big\}.$$

By definition of $S_{jt}^R(\omega)$ and convexity of $u_{jt}(\omega,\cdot)$, it follows that

$$\widehat{Y}_{jt}^R(\omega) = \begin{cases} \max\left[[\widehat{Y}_{1t}^E]^-, \min\big(S_{1t}^R(\omega), \widehat{Y}_{1t}^E\big) \right] & \text{if } j = 1; \\ \max\left[\widehat{Y}_{j-1,t}^E, \min\big(S_{jt}^R(\omega), \widehat{Y}_{jt}^E\big) \right] & \text{if } 2 \le j \le N. \end{cases}$$

To prove part (c), let optimal echelon schedules $\widehat{\mathbf{Y}}_t^R(\omega)$ and $\widehat{\mathbf{Y}}_t^E(\omega)$ be given. Then, the optimality equations given in (10) lead to

$$\tilde{f}_t(\omega, \mathbf{y}_t) = \gamma_t(\omega, \widehat{Y}_{1t}^R) + \sum_{j=1}^{N} \big(c_j^E\, \widehat{Y}_{jt}^E - c_j^R\, \widehat{Y}_{jt}^R\big) + \sum_{j=1}^{N} \Big[\min_{\widehat{Y}_{jt}^R \le Y_{jt} \le \widehat{Y}_{j+1,t}^R} g_{jt}(\omega, Y_t) \Big].$$

Let $S_{jt}(\omega) := \sup \arg\min_Y g_{jt}(\omega, Y)$ for each ω and $j = 1,\ldots,N$. Since $g_{jt}(\omega,\cdot)$ is convex, then (7) implies that the optimal echelon j position after regular ordering, $\widehat{Y}_{jt}(\omega)$ is exactly as given in (17). $\square$

Proof of Theorem 5: To prove that $S_{jt}(\omega) \le S_{jt}^R(\omega)$, let $S_t^E(\omega) := \big(S_{1t}^E(\omega), \ldots, S_{Nt}^E(\omega)\big)$, $S_t^R(\omega) := \big(S_{1t}^R(\omega), \ldots, S_{Nt}^R(\omega)\big)$, and $S_t(\omega) := \big(S_{1t}(\omega), \ldots, S_{Nt}(\omega)\big)$ be the optimal basestock levels in period t. Let $\big(\widehat{\mathbf{Y}}_t^E(\omega), \widehat{\mathbf{Y}}_t^R(\omega), \widehat{\mathbf{Y}}_t(\omega)\big)$ be the policy corresponding to $\big(S_t^E(\omega), S_t^R(\omega), S_t(\omega)\big)$. Suppose there exist ω and j such that $S_{jt}^R(\omega) < S_{jt}(\omega)$. Define basestock levels $\big(\overline{S}_t^R(\omega), \overline{S}_t^E(\omega), \overline{S}_t(\omega)\big)$ as $\overline{S}_{it}^E(\omega) = S_{it}^E(\omega)$ and $\overline{S}_{it}(\omega) = S_{it}(\omega)$ for all i; and $\overline{S}_{it}^R(\omega) = S_{it}^R(\omega)$ for all $i \ne j$ while $\overline{S}_{jt}^R(\omega) = S_{jt}^R(\omega) + \delta$ for any δ, $0 < \delta \le S_{jt}(\omega) - S_{jt}^R(\omega)$. Thus, $S_{jt}^R(\omega) < \overline{S}_{jt}^R(\omega) < S_{jt}(\omega)$. Let $\big(\overline{\mathbf{Y}}_t^E, \overline{\mathbf{Y}}_t^R, \overline{\mathbf{Y}}_t\big)$ be the policy corresponding to $\big(\overline{S}_t^E(\omega), \overline{S}_t^R(\omega), \overline{S}_t(\omega)\big)$. Suppose first that $j = N$. Using "$\vee$" for "max" and "$\wedge$" for "min", it follows that

$$\begin{aligned}
\widehat{Y}_{Nt} &= \widehat{Y}_{Nt}^R \vee S_{Nt}(\omega) && [\text{Expression (17), Theorem 4}] \\
&= S_{Nt}(\omega) \vee \widehat{Y}_{N-1,t}^E \vee \big[S_{Nt}^R(\omega) \wedge \widehat{Y}_{Nt}^E\big] && [\text{Expression (16), Theorem 4}] \\
&= \big[S_{Nt}(\omega) \vee \widehat{Y}_{N-1,t}^E \vee S_{Nt}^R(\omega)\big] \wedge \big[S_{Nt}(\omega) \vee \widehat{Y}_{N-1,t}^E \vee \widehat{Y}_{Nt}^E\big] && \\
&= \big[S_{Nt}(\omega) \vee \widehat{Y}_{N-1,t}^E\big] \wedge \big[S_{Nt}(\omega) \vee \widehat{Y}_{N-1,t}^E \vee \widehat{Y}_{Nt}^E\big] && [S_{Nt}^R(\omega) < S_{Nt}(\omega) \text{ by assumption}] \\
&= S_{Nt}(\omega) \vee \widehat{Y}_{N-1,t}^E. && [Y_{N-1,t}^E(\omega) \le Y_{Nt}^E(\omega)]
\end{aligned}$$

Next, because $\overline{S}^E_{jt}(\omega) = S^E_{jt}(\omega)$ for all j implies that $\overline{Y}^E_{jt} = \widehat{Y}^E_{jt}$ for all j, it follows that

$$\begin{aligned}
\overline{Y}_{Nt} &= \overline{Y}^R_{Nt} \vee \overline{S}_{Nt}(\omega) && [\text{Expression (17), Theorem 4}]\\
&= \overline{S}_{Nt}(\omega) \vee \overline{Y}^E_{N-1,t} \vee \left[\overline{S}^R_{Nt}(\omega) \wedge \overline{Y}^E_{Nt}\right] && [\text{Expression (16), Theorem 4}]\\
&= \left[\overline{S}_{Nt}(\omega) \vee \overline{Y}^E_{N-1,t} \vee \overline{S}^R_{Nt}(\omega)\right] \wedge \left[\overline{S}_{Nt}(\omega) \vee \overline{Y}^E_{N-1,t} \vee \overline{Y}^E_{Nt}\right] &&\\
&= \left[S_{Nt}(\omega) \vee \widehat{Y}^E_{N-1,t} \vee \overline{S}^R_{Nt}(\omega)\right] \wedge \left[S_{Nt}(\omega) \vee \widehat{Y}^E_{N-1,t} \vee \widehat{Y}^E_{Nt}\right] && [\overline{Y}^E_{jt}(\omega) = \widehat{Y}^E_{jt} \text{ for all } j]\\
&= \left[S_{Nt}(\omega) \vee \widehat{Y}^E_{N-1,t}\right] \wedge \left[S_{Nt}(\omega) \vee \widehat{Y}^E_{N-1,t} \vee \widehat{Y}^E_{Nt}\right] && [\overline{S}^R_{Nt}(\omega) < S_{Nt}(\omega) \text{ by assumption}]\\
&= S_{Nt}(\omega) \vee \widehat{Y}^E_{N-1,t},
\end{aligned}$$

It follows that $\widehat{Y}_{Nt} = \overline{Y}_{Nt}$. Further, $S^R_{Nt}(\omega) < \overline{S}^R_{Nt}(\omega)$ implies that $\widehat{Y}^R_{Nt}(\omega) \leq \overline{Y}^R_{Nt}(\omega)$. Therefore, based on all the results obtained so far, and using the definition of the cost function v_t given in (8), we get

$$\Delta := v_t(\omega, \widehat{\mathbf{Y}}^R_t, \widehat{\mathbf{Y}}^E_t, \widehat{\mathbf{Y}}_t) - v_t(\omega, \overline{\mathbf{Y}}^R_t, \overline{\mathbf{Y}}^E_t, \overline{\mathbf{Y}}_t) = -c^R_N\left[\widehat{Y}^R_{Nt} - \overline{Y}^R_{Nt}\right] \geq 0,$$

Suppose $\Delta > 0$. In that case, since $(\widehat{\mathbf{Y}}^R_t, \widehat{\mathbf{Y}}^E_t, \widehat{\mathbf{Y}}_t)$ generates a higher cost than $(\overline{\mathbf{Y}}^R_t, \overline{\mathbf{Y}}^E_t, \overline{\mathbf{Y}}_t)$, it follows that $(\widehat{\mathbf{Y}}^R_t, \widehat{\mathbf{Y}}^E_t, \widehat{\mathbf{Y}}_t)$ cannot be optimal, as that contradicts the initial assumption. If $\Delta = 0$, then $\overline{S}^R_{Nt}(\omega)$ is also an optimal basestock level, and since $\overline{S}^R_{Nt}(\omega) > S^R_{Nt}(\omega)$, this contradicts the definition of $S^R_{Nt}(\omega)$. Hence, we must have $S^R_{Nt}(\omega) \geq S_{Nt}(\omega)$. For $1 \leq j < N$, we get

$$\begin{aligned}
\widehat{Y}_{jt} &= \left(\widehat{Y}^R_{jt} \vee S_{jt}(\omega)\right) \wedge \widehat{Y}^R_{j+1,t} && [\text{Expression (17), Theorem 4}]\\
&= \left[S_{jt}(\omega) \vee \widehat{Y}^E_{j-1,t} \vee \left(S^R_{jt}(\omega) \wedge \widehat{Y}^E_{jt}\right)\right] \wedge \widehat{Y}^R_{j+1,t} && [\text{Expression (16), Theorem 4}]\\
&= \left(S_{jt}(\omega) \vee \widehat{Y}^E_{j-1,t} \vee S^R_{jt}(\omega)\right) \wedge \left(S_{jt}(\omega) \vee \widehat{Y}^E_{j-1,t} \vee \widehat{Y}^E_{jt}\right) \wedge \widehat{Y}^R_{j+1,t} &&\\
&= \left(S_{jt}(\omega) \vee \widehat{Y}^E_{j-1,t}\right) \wedge \left(S_{jt}(\omega) \vee \widehat{Y}^E_{j-1,t} \vee \widehat{Y}^E_{jt}\right) \wedge \widehat{Y}^R_{j+1,t} && [S^R_{jt}(\omega) < S_{jt}(\omega) \text{ by assumption}]\\
&= \left(S_{jt}(\omega) \vee \widehat{Y}^E_{j-1,t}\right) \wedge \widehat{Y}^R_{j+1,t}. && [Y^E_{j-1,t}(\omega) \leq Y^E_{jt}(\omega)]
\end{aligned}$$

Next, since $\overline{Y}^E_{it} = \widehat{Y}^E_{it}$ for all i and $\overline{S}^R_{it}(\omega) = S^R_{it}(\omega)$ for all $i \neq$, then $\overline{Y}^R_{it} = \widehat{Y}^R_{it}$ for all $i \neq j$. Then, since $\overline{S}^R_{jt}(\omega) < S_{jt}(\omega)$, it follows by similar reasoning that $\overline{Y}_{jt} = \left(S_{jt}(\omega) \vee \widehat{Y}^E_{j-1,t}\right) \wedge \widehat{Y}^R_{j+1,t}$. Thus, $\widehat{Y}_{jt} = \overline{Y}_{jt}$. Also, since $\overline{S}^R_{jt}(\omega) > S^R_{jt}(\omega)$, then $\overline{Y}^R_{jt} \geq \widehat{Y}^R_{jt}$. If $1 < j < N$, then using the results obtained above, we get

$$\Delta := v_t(\omega, \widehat{\mathbf{Y}}^R_t, \widehat{\mathbf{Y}}^E_t, \widehat{\mathbf{Y}}_t) - v_t(\omega, \overline{\mathbf{Y}}^R_t, \overline{\mathbf{Y}}^E_t, \overline{\mathbf{Y}}_t) = -c^R_j\left[\widehat{Y}^R_{jt} - \overline{Y}^R_{jt}\right] \geq 0.$$

If, on the other hand, $j = 1$, we get

$$\Delta := v_t(\omega, \widehat{\mathbf{Y}}^R_t, \widehat{\mathbf{Y}}^E_t, \widehat{\mathbf{Y}}_t) - v_t(\omega, \overline{\mathbf{Y}}^R_t, \overline{\mathbf{Y}}^E_t, \overline{\mathbf{Y}}_t) = \gamma_t(\omega, \widehat{Y}^R_{1t}) - \gamma_t(\omega, \overline{Y}^R_{1t}) - c^R_j\left[\widehat{Y}^R_{jt} - \overline{Y}^R_{jt}\right] > \gamma_t(\omega, \widehat{Y}^R_{1t}) - \gamma_t(\omega, \overline{Y}^R_{1t}).$$

By definition, $\gamma_t(\omega, x) := (p + H_1)\mathbb{E}_\omega[(D_t(\omega) - x)^+] - H_{1t}\mathbb{E}_\omega[D_t(\omega)]$. Thus, since $\mathbb{E}_\omega[(D_t(\omega) - x)^+]$ is decreasing in x, then so is $\gamma_t(\omega, x)$. Since $\overline{Y}^R_{1t} \geq \widehat{Y}^R_{1t}$, it follows that $\gamma_t(\omega, \widehat{Y}^R_{1t}) \geq \gamma_t(\omega, \overline{Y}^R_{1t})$. Hence, $\Delta \geq 0$. If $\Delta > 0$, then, since $(\widehat{\mathbf{Y}}^R_t, \widehat{\mathbf{Y}}^E_t, \widehat{\mathbf{Y}}_t)$ has a higher cost than $(\overline{\mathbf{Y}}^R_t, \overline{\mathbf{Y}}^E_t, \overline{\mathbf{Y}}_t)$, $(\widehat{\mathbf{Y}}^R_t, \widehat{\mathbf{Y}}^E_t, \widehat{\mathbf{Y}}_t)$ cannot be optimal, which contradicts the initial assumption. If $\Delta = 0$, then $\overline{S}^R_{jt}(\omega)$ is also an optimal basestock level at stage j. Since $\overline{S}^R_{jt}(\omega) > S^R_{jt}(\omega)$ this contradicts the definition of $S^R_{jt}(\omega)$. Thus, $S^R_{jt}(\omega) \geq S_{jt}(\omega)$ for all j and ω.

To show that $S^R_{jt}(\omega) \geq S^E_{jt}(\omega)$ for all j and ω, we follow a similar procedure. Let $\left(\widehat{\mathbf{Y}}^E_t(\omega), \widehat{\mathbf{Y}}^R_t(\omega), \widehat{\mathbf{Y}}_t(\omega)\right)$

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be the policy corresponding to optimal basestock levels $\left(S_t^E(\omega), S_t^R(\omega), S_t(\omega)\right)$. Suppose there exist ω and j such that $S_{jt}^R(\omega) < S_{jt}^E(\omega)$. Consider another set of basestock levels $\left(\overline{S}_t^R(\omega), \overline{S}_t^E(\omega), \overline{S}_t(\omega)\right)$ defined as $\overline{S}_{it}^R(\omega) = S_{it}^R(\omega)$ and $\overline{S}_{it}(\omega) = S_{it}(\omega)$ for all i; and $\overline{S}_{it}^E(\omega) = S_{it}^E(\omega)$ for all $i \neq j$ while $\overline{S}_{jt}^E(\omega) = S_{jt}^E(\omega) - \delta$ for any δ, $0 < \delta \leq S_{jt}^E(\omega) - S_{jt}^R(\omega)$. Thus, $S_{jt}^E(\omega) > \overline{S}_{jt}^E(\omega) > S_{jt}^R(\omega)$. Let $\left(\overline{\mathbf{Y}}_t^E(\omega), \overline{\mathbf{Y}}_t^R(\omega), \overline{\mathbf{Y}}_t(\omega)\right)$ be the optimal policy corresponding to $\left(\overline{S}_t^E(\omega), \overline{S}_t^R(\omega), \overline{S}_t(\omega)\right)$. Suppose first that $j = N$. It follows that

$$\begin{aligned}
\widehat{Y}_{Nt} &= \widehat{Y}_{Nt}^R \vee S_{Nt}(\omega) && [\text{Expression (17), Theorem 4}]\\
&= S_{Nt}(\omega) \vee \widehat{Y}_{N-1,t}^E \vee \left[S_{Nt}^R(\omega) \wedge \widehat{Y}_{Nt}^E\right] && [\text{Expression (16), Theorem 4}]\\
&= S_{Nt}(\omega) \vee \widehat{Y}_{N-1,t}^E \vee \left[S_{Nt}^R(\omega) \wedge \left(y_{Nt}(\omega) \vee S_{Nt}^E(\omega)\right)\right] && [\text{Expression (15), Theorem 4}]\\
&= S_{Nt}(\omega) \vee \widehat{Y}_{N-1,t}^E \vee \left[S_{Nt}^R(\omega) \wedge y_{Nt}\right] \vee \left[S_{Nt}^R(\omega) \wedge S_{Nt}^E(\omega)\right] &&\\
&= S_{Nt}(\omega) \vee \widehat{Y}_{N-1,t}^E \vee \left[S_{Nt}^R(\omega) \wedge y_{Nt}\right] \vee S_{Nt}^R(\omega) && [S_{Nt}^R(\omega) < S_{Nt}^E(\omega) \text{ by assumption}]\\
&= S_{Nt}(\omega) \vee \widehat{Y}_{N-1,t}^E \vee S_{Nt}^R(\omega) &&\\
&= \widehat{Y}_{N-1,t}^E \vee S_{Nt}^R(\omega), &&
\end{aligned}$$

since $S_{Nt}^R(\omega) \geq S_{Nt}(\omega)$ as established earlier in the proof. Similarly, because $S_{Nt}^R(\omega) < \overline{S}_{Nt}^E(\omega)$ by assumption, we get that $\overline{Y}_{Nt} = \widehat{Y}_{N-1,t}^E \vee S_{Nt}^R(\omega)$, and therefore $\widehat{Y}_{Nt} = \overline{Y}_{Nt}$. Further, $S_{Nt}^E(\omega) > \overline{S}_{Nt}^E(\omega)$ implies that $\widehat{Y}_{Nt}^E(\omega) \geq \overline{Y}_{Nt}^E(\omega)$. Therefore, based on all the results obtained so far, we get

$$\Delta := v_t(\omega, \widehat{\mathbf{Y}}_t^R, \widehat{\mathbf{Y}}_t^E, \widehat{\mathbf{Y}}_t) - v_t(\omega, \overline{\mathbf{Y}}_t^R, \overline{\mathbf{Y}}_t^E, \overline{\mathbf{Y}}_t) = c_N^E\left[\widehat{Y}_{Nt}^E - \overline{Y}_{Nt}^E\right] \geq 0.$$

If $\Delta > 0$, then, since $(\widehat{\mathbf{Y}}_t^E, \widehat{\mathbf{Y}}_t^R, \widehat{\mathbf{Y}}_t)$ generates a higher cost than $(\overline{\mathbf{Y}}_t^E, \overline{\mathbf{Y}}_t^R, \overline{\mathbf{Y}}_t)$, $(\widehat{\mathbf{Y}}_t^E, \widehat{\mathbf{Y}}_t^R, \widehat{\mathbf{Y}}_t)$ cannot be optimal, which contradicts the initial assumption. If $\Delta = 0$, then $\overline{S}_{Nt}^E(\omega)$ is also an optimal basestock level; since $\overline{S}_{Nt}^E(\omega) < S_{Nt}^E(\omega)$, this contradicts the definition of $S_{Nt}^E(\omega)$. Thus, $S_{Nt}^R(\omega) \geq S_{Nt}^E(\omega)$. For $j < N$, we get

$$\begin{aligned}
\widehat{Y}_{jt} &= \left(\widehat{Y}_{jt}^R \vee S_{jt}(\omega)\right) \wedge \widehat{Y}_{j+1,t}^R && [\text{Expression (17), Theorem 4}]\\
&= \left[S_{jt}(\omega) \vee \widehat{Y}_{j-1,t}^E \vee \left(S_{jt}^R(\omega) \wedge \widehat{Y}_{jt}^E\right)\right] \wedge \widehat{Y}_{j+1,t}^R && [\text{Expression (16), Theorem 4}]\\
&= \left\{S_{jt}(\omega) \vee \widehat{Y}_{j-1,t}^E \vee \left[S_{jt}^R(\omega) \wedge \left[y_{jt} \vee \left(S_{jt}^E(\omega) \wedge \widehat{Y}_{j+1,t}^E\right)\right]\right]\right\} \wedge \widehat{Y}_{j+1,t}^R && [\text{Expression (15), Theorem 4}]\\
&= \left\{S_{jt}(\omega) \vee \widehat{Y}_{j-1,t}^E \vee \left[S_{jt}^R(\omega) \wedge y_{jt}\right] \vee \left[S_{jt}^R(\omega) \wedge S_{jt}^E(\omega) \wedge \widehat{Y}_{j+1,t}^E\right]\right\} \wedge \widehat{Y}_{j+1,t}^R &&\\
&= \left\{S_{jt}(\omega) \vee \widehat{Y}_{j-1,t}^E \vee \left[S_{jt}^R(\omega) \wedge y_{jt}\right] \vee \left[S_{jt}^R(\omega) \wedge \widehat{Y}_{j+1,t}^E\right]\right\} \wedge \widehat{Y}_{j+1,t}^R && [S_{jt}^R(\omega) < S_{jt}^E(\omega) \text{ by assumption}]\\
&= \left\{S_{jt}(\omega) \vee \widehat{Y}_{j-1,t}^E \vee \left[S_{jt}^R(\omega) \wedge \widehat{Y}_{j+1,t}^E\right]\right\} \wedge \widehat{Y}_{j+1,t}^R. && [y_{jt} \leq \widehat{Y}_{j+1,t}^E(\omega)]
\end{aligned}$$

Because $S_{jt}^R(\omega) < \overline{S}_{jt}^E(\omega)$ by assumption, we also get $\overline{Y}_{jt} = \left\{S_{jt}(\omega) \vee \widehat{Y}_{j-1,t}^E \vee \left[S_{jt}^R(\omega) \wedge \widehat{Y}_{j+1,t}^E\right]\right\} \wedge \widehat{Y}_{j+1,t}^R$, and therefore $\widehat{Y}_{jt} = \overline{Y}_{jt}$. Further, $S_{jt}^E(\omega) > \overline{S}_{jt}^E(\omega)$ implies that $\widehat{Y}_{jt}^E(\omega) \geq \overline{Y}_{jt}^E(\omega)$. Therefore, based on all the results obtained so far, we get $\Delta := v_t(\omega, \widehat{\mathbf{Y}}_t^R, \widehat{\mathbf{Y}}_t^E, \widehat{\mathbf{Y}}_t) - v_t(\omega, \overline{\mathbf{Y}}_t^R, \overline{\mathbf{Y}}_t^E, \overline{\mathbf{Y}}_t) = c_j^E\left[\widehat{Y}_{jt}^E - \overline{Y}_{jt}^E\right] \geq 0$.

If $\Delta > 0$, then, since $(\widehat{\mathbf{Y}}_t^E, \widehat{\mathbf{Y}}_t^R, \widehat{\mathbf{Y}}_t)$ generates a higher cost than $(\overline{\mathbf{Y}}_t^E, \overline{\mathbf{Y}}_t^R, \overline{\mathbf{Y}}_t)$, $(\widehat{\mathbf{Y}}_t^E, \widehat{\mathbf{Y}}_t^R, \widehat{\mathbf{Y}}_t)$ cannot be optimal, which contradicts the initial assumption. If $\Delta = 0$, then $\overline{S}_{jt}^E(\omega)$ is also an optimal basestock level, and since $\overline{S}_{jt}^E(\omega) < S_{jt}^E(\omega)$, this contradicts the definition of $S_{jt}^E(\omega)$. Thus, $S_{jt}^R(\omega) \geq S_{jt}^E(\omega)$ for all j.

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Finally, we prove $S_{jt}(\omega) \geq S^E_{jt}(\omega)$. Let $\big(\widehat{\mathbf{Y}}^E_t(\omega), \widehat{\mathbf{Y}}^R_t(\omega), \widehat{\mathbf{Y}}_t(\omega)\big)$ be the policy corresponding to optimal basestock levels $\big(S^E_t(\omega), S^R_t(\omega), S_t(\omega)\big)$. Suppose there exist ω and j such that $S_{jt}(\omega) < S^E_{jt}(\omega)$. Consider basestock levels $\big(\overline{S}^R_t(\omega), \overline{S}^E_t(\omega), \overline{S}_t(\omega)\big)$ defined as $\overline{S}^E_{it}(\omega) = S^E_{it}(\omega)$, $\overline{S}^R_{it}(\omega) = S^R_{it}(\omega)$ and $\overline{S}_{it}(\omega) = S_{it}(\omega)$ for all $i \neq j$ while $\overline{S}_{jt}(\omega) = S_{jt}(\omega) + \delta$ for any δ, $0 < \delta \leq S^E_{jt}(\omega) - S_{jt}(\omega)$. Thus, $S^E_{jt}(\omega) > \overline{S}_{jt}(\omega) > S_{jt}(\omega)$. Let $\big(\overline{\mathbf{Y}}^E_t(\omega), \overline{\mathbf{Y}}^R_t(\omega), \overline{\mathbf{Y}}_t(\omega)\big)$ be the policy corresponding to $\big(\overline{S}^E_t(\omega), \overline{S}^R_t(\omega), \overline{S}_t(\omega)\big)$. Suppose $j = N$. Then,

$$
\begin{aligned}
\widehat{Y}_{Nt} &= \widehat{Y}^R_{Nt} \vee S_{Nt}(\omega) && [\text{Expression (17), Theorem 4}]\\
&= S_{Nt}(\omega) \vee \widehat{Y}^E_{N-1,t} \vee \big[S^R_{Nt}(\omega) \wedge \widehat{Y}^E_{Nt}\big] && [\text{Expression (16), Theorem 4}]\\
&= S_{Nt}(\omega) \vee \widehat{Y}^E_{N-1,t} \vee \big[S^R_{Nt}(\omega) \wedge \big(y_{Nt}(\omega) \vee S^E_{Nt}(\omega)\big)\big] && [\text{Expression (15), Theorem 4}]\\
&= S_{Nt}(\omega) \vee \widehat{Y}^E_{N-1,t} \vee \big[S^R_{Nt}(\omega) \wedge y_{Nt}\big] \vee \big[S^R_{Nt}(\omega) \wedge S^E_{Nt}(\omega)\big]\\
&= S_{Nt}(\omega) \vee \widehat{Y}^E_{N-1,t} \vee \big[S^R_{Nt}(\omega) \wedge y_{Nt}\big] \vee S^E_{Nt}(\omega) && [S^R_{Nt}(\omega) \geq S^E_{Nt}(\omega)]\\
&= \widehat{Y}^E_{N-1,t} \vee \big[S^R_{Nt}(\omega) \wedge y_{Nt}\big] \vee S^E_{Nt}(\omega). && [S^E_{Nt}(\omega) > S_{Nt}(\omega) \text{ by assumption}]
\end{aligned}
$$

Because $S^E_{Nt}(\omega) > \overline{S}_{Nt}(\omega)$ by assumption, we also get $\overline{Y}_{Nt} = \widehat{Y}^E_{N-1,t} \vee \big[S^R_{Nt}(\omega) \wedge y_{Nt}\big] \vee S^E_{Nt}(\omega)$, and therefore $\widehat{Y}_{Nt} = \overline{Y}_{Nt}$. Therefore, $v_t(\omega, \widehat{\mathbf{Y}}^R_t, \widehat{\mathbf{Y}}^E_t, \widehat{\mathbf{Y}}_t) - v_t(\omega, \overline{\mathbf{Y}}^R_t, \overline{\mathbf{Y}}^E_t, \overline{\mathbf{Y}}_t) = 0$. Thus, $\overline{S}_{Nt}(\omega)$ is also an optimal basestock level, and since $\overline{S}_{Nt}(\omega) > S_{Nt}(\omega)$, this contradicts the definition of $S_{Nt}(\omega)$. Thus, $S_{Nt}(\omega) \geq S^E_{Nt}(\omega)$. For $1 \leq j < N$, we get

$$
\begin{aligned}
\widehat{Y}_{jt} &= \big(\widehat{Y}^R_{jt} \vee S_{jt}(\omega)\big) \wedge \widehat{Y}^R_{j+1,t} && [\text{Expression (17), Theorem 4}]\\
&= \Big[S_{jt}(\omega) \vee \widehat{Y}^E_{j-1,t} \vee \big(S^R_{jt}(\omega) \wedge \widehat{Y}^E_{jt}\big)\Big] \wedge \widehat{Y}^R_{j+1,t} && [\text{Expression (16), Theorem 4}]\\
&= \Big\{S_{jt}(\omega) \vee \widehat{Y}^E_{j-1,t} \vee \Big[S^R_{jt}(\omega) \wedge \big[y_{jt} \vee \big(S^E_{jt}(\omega) \wedge \widehat{Y}^E_{j+1,t}\big)\big]\Big]\Big\} \wedge \widehat{Y}^R_{j+1,t} && [\text{Expression (15), Theorem 4}]\\
&= \Big\{S_{jt}(\omega) \vee \widehat{Y}^E_{j-1,t} \vee \big[S^R_{jt}(\omega) \wedge y_{jt}\big] \vee \big[S^R_{jt}(\omega) \wedge S^E_{jt}(\omega) \wedge \widehat{Y}^E_{j+1,t}\big]\Big\} \wedge \widehat{Y}^R_{j+1,t}\\
&= \Big\{S_{jt}(\omega) \vee \widehat{Y}^E_{j-1,t} \vee \big[S^R_{jt}(\omega) \wedge y_{jt}\big] \vee \big[S^E_{jt}(\omega) \wedge \widehat{Y}^E_{j+1,t}\big]\Big\} \wedge \widehat{Y}^R_{j+1,t}, && (37)
\end{aligned}
$$

since $S^E_{jt}(\omega) \leq S^R_{jt}(\omega)$, as established earlier in the proof. Now, we distinguish the following three cases.

Case 1: $\widehat{Y}^E_{j+1,t} \geq \overline{S}_{jt}(\omega) > S_{jt}(\omega)$. Then, since $S^E_{jt} \geq \overline{S}_{jt}(\omega) > S_{jt}(\omega)$ by assumption, (37) reduces to

$$\widehat{Y}_{jt} = \Big\{\widehat{Y}^E_{j-1,t} \vee \big[S^R_{jt}(\omega) \wedge y_{jt}\big] \vee \big[S^E_{jt}(\omega) \wedge \widehat{Y}^E_{j+1,t}\big]\Big\} \wedge \widehat{Y}^R_{j+1,t}.$$

Using a similar derivation, we also get $\overline{Y}_{jt} = \Big\{\widehat{Y}^E_{j-1,t} \vee \big[S^R_{jt}(\omega) \wedge y_{jt}\big] \vee \big[S^E_{jt}(\omega) \wedge \widehat{Y}^E_{j+1,t}\big]\Big\} \wedge \widehat{Y}^R_{j+1,t}$, and, therefore, $\widehat{Y}_{jt} = \overline{Y}_{jt}$. As a result, it follows that $\Delta := v_t(\omega, \widehat{\mathbf{Y}}^R_t, \widehat{\mathbf{Y}}^E_t, \widehat{\mathbf{Y}}_t) - v_t(\omega, \overline{\mathbf{Y}}^R_t, \overline{\mathbf{Y}}^E_t, \overline{\mathbf{Y}}_t) = 0$.

Case 2: $\overline{S}_{jt}(\omega) \geq \widehat{Y}^E_{j+1,t} > S_{jt}(\omega)$. Since $\widehat{Y}^E_{j+1,t} > S_{jt}(\omega)$, then (37) reduces to

$$\widehat{Y}_{jt} = \Big\{\widehat{Y}^E_{j-1,t} \vee \big[S^R_{jt}(\omega) \wedge y_{jt}\big] \vee \big[S^E_{jt}(\omega) \wedge \widehat{Y}^E_{j+1,t}\big]\Big\} \wedge \widehat{Y}^R_{j+1,t} = \Big\{\widehat{Y}^E_{jt} \vee \big[S^R_{jt}(\omega) \wedge y_{jt}\big] \vee \widehat{Y}^E_{j+1,t}\big]\Big\} \wedge \widehat{Y}^R_{j+1,t},$$

because $S^E_{jt}(\omega) > \overline{S}_{jt}(\omega)$ by the original assumption, and $\overline{S}_{jt}(\omega) \geq \widehat{Y}^E_{j+1,t} > S_{jt}(\omega)$ by the current assumption. Thus, $S^E_{jt}(\omega) > \widehat{Y}^E_{j+1,t}$. Because $\widehat{Y}^E_{j+1,t} \geq \widehat{Y}^E_{jt}$ and $\widehat{Y}^E_{j+1,t} \geq \widehat{Y}^R_{j+1,t}$, we get $\widehat{Y}_{jt} = \Big\{\big[S^R_{jt}(\omega) \wedge y_{jt}\big] \vee$

$\widehat{Y}^E_{j+1,t}]\Big\} \wedge \widehat{Y}^R_{j+1,t} = \widehat{Y}^R_{j+1,t}$. Also, since $\overline{S}_{jt}(\omega) \geq \widehat{Y}^E_{j+1,t}$, then $\overline{S}_{jt}(\omega) \geq \big[\widehat{Y}^E_{j+1,t} \wedge S^E_{jt}(\omega)\big]$ and (37) becomes

$$\overline{Y}_{jt} = \Big\{\overline{S}_{jt}(\omega) \vee \widehat{Y}^E_{j-1,t} \vee \big[S^R_{jt}(\omega) \wedge y_{jt}\big]\Big\} \wedge \widehat{Y}^R_{j+1,t} = \widehat{Y}^R_{j+1,t}, \tag{38}$$

because, given $\overline{S}_{jt}(\omega) \geq \widehat{Y}^E_{j+1,t}$, $\widehat{Y}^E_{j+1,t} \geq \widehat{Y}^R_{j+1,t}$ implies that $\overline{S}_{jt}(\omega) \geq \widehat{Y}^R_{j+1,t}$.

Therefore $\widehat{Y}_{jt} = \overline{Y}_{jt}$. As a result, $\Delta := v_t(\omega, \widehat{\mathbf{Y}}^R_t, \widehat{\mathbf{Y}}^E_t, \widehat{\mathbf{Y}}_t) - v_t(\omega, \overline{\mathbf{Y}}^R_t, \overline{\mathbf{Y}}^E_t, \overline{\mathbf{Y}}_t) = 0$.

Case 3: $\overline{S}_{jt}(\omega) > S_{jt}(\omega) \geq \widehat{Y}^E_{j+1,t}$. Hence, $\overline{S}_{jt}(\omega) > S_{jt}(\omega) \geq \big[\widehat{Y}^E_{j+1,t} \wedge S^E_{jt}(\omega)\big]$ and (37) reduces to

$$\widehat{Y}_{jt} = \Big\{S_{jt}(\omega) \vee \widehat{Y}^E_{j-1,t} \vee \big[S^R_{jt}(\omega) \wedge y_{jt}\big]\Big\} \wedge \widehat{Y}^R_{j+1,t} = \widehat{Y}^R_{j+1,t},$$

since, by assumption, $\overline{S}_{jt}(\omega) > S_{jt}(\omega) \geq \widehat{Y}^E_{j+1,t}$ and $\widehat{Y}^E_{j+1,t} \geq \widehat{Y}^R_{j+1,t}$, from which it follows that $\overline{S}_{jt}(\omega) > S_{jt}(\omega) \geq \widehat{Y}^R_{j+1,t}$. Similarly, we obtain $\overline{Y}_{jt} = \widehat{Y}^R_{j+1,t}$. Thus, $\widehat{Y}_{jt} = \overline{Y}_{jt}$. It follows that $\Delta := v_t(\omega, \widehat{\mathbf{Y}}^R_t, \widehat{\mathbf{Y}}^E_t, \widehat{\mathbf{Y}}_t) - v_t(\omega, \overline{\mathbf{Y}}^R_t, \overline{\mathbf{Y}}^E_t, \overline{\mathbf{Y}}_t) = 0$. Since $\Delta = 0$ in all three cases, then $\overline{S}_{jt}(\omega)$ is also an optimal basestock level, and since $\overline{S}_{jt}(\omega) > S_{jt}(\omega)$, this contradicts the definition of $S_{jt}(\omega)$. Thus, $S_{jt}(\omega) \geq S^E_{jt}(\omega)$ for all j. □

Proof of Theorem 6: First, we need to show that $\mathcal{F}_t(\omega, \cdot)$ is uniformly bounded from above. Using a myopic basestock echelon policy given by: $S^R_1(\omega) = \inf \arg\min_Y \big[\gamma(\omega, Y) - c^R_1 Y\big]$; $S^R_j(\omega) = S^R_1(\omega)$ for all j; and $S_j(\omega) = S^E_j(\omega) = S^R_j(\omega)$, in every period yields a trivial upper bound on $\mathcal{F}_t(\omega, \cdot)$. Since $\mathcal{F}_t(\omega, \cdot)$ is positive, increasing in time horizon T and bounded from above, by the uniform converges theorem, $\mathcal{F}_t(\omega, \cdot)$ converges to its infinite-horizon limit. The rest of the proof follows the steps identical to those in Federgruen and Zipkin (1984). In particular, given $\alpha < 1$, the finite horizon optimal policy and the resulting objective cost function converge, as the number of period goes to infinity, to their infinite-horizon counterparts, so that the structure of the optimal policy is preserved under this limit-taking operation. Alternatively, the existence of an unstructured stationary optimal policy for the discounted-cost infinite-horizon setting can be established using contraction mapping results (eg., Bertsekas and Shreve, 1976, Chapter 4). In that case, the optimality of a nested echelon basestock policy for the infinite-horizon, discounted cost, reverse logistics problem may be established using structured policy results such as those in Porteus (1982).

Proof of Theorem 7. Consider an infinite-horizon, reverse logistics problem under Assumption 2. By Theorem 6, there exists for this problem an additively convex, stationary, objective cost function $\mathcal{F}$ for which a stationary, nested echelon basestock policy is optimal with $S^E_j(\omega) \leq S_j(\omega) \leq S^R_j(\omega)$ for all j and ω. Let $\big(\widetilde{\mathbf{Y}}^E_j, \widetilde{\mathbf{Y}}^R_j, \widetilde{\mathbf{Y}}_j\big\}$ be that optimal policy. The infinite-horizon problem in (18) can then be written as

$$\mathcal{F}(\omega, \mathbf{y}_t) = -\sum_{j=1}^{N} k^E_j\, y_{jt} + \min_{\mathbf{Y}^E_t, \mathbf{Y}^R_t, \mathbf{Y}_t \in \mathbb{Y}(\mathbf{y}_t)} \Big\{\gamma(\omega, Y^R_{1t}) + \sum_{j=1}^{N}\big(c^E_j Y^E_{jt} + c_j Y_{jt} - c^R_j Y^R_{jt}\big) + \alpha\, \mathbb{E}_{\omega_t}\big[\mathcal{F}\big(\omega_{t+1}, \mathbf{Y}_t - D_t(\omega)\big)\big]\Big\} \tag{39}$$

$$= -\sum_{j=1}^{N} k^E_j\, y_{jt} + \gamma(\omega, \widetilde{Y}^R_1) + \sum_{j=1}^{N}\big(c^E_j \widetilde{Y}^E_j + c_j \widetilde{Y}_j - c^R_j \widetilde{Y}^R_j\big) + \alpha\, \mathbb{E}_{\omega}\big[\mathcal{F}\big(\omega_{t+1}, \widetilde{\mathbf{Y}} - D_t(\omega)\big)\big]\Big\}, \tag{40}$$

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by definition of $\big(\widetilde{\mathbf{Y}}_j^E, \widetilde{\mathbf{Y}}_j^R, \widetilde{\mathbf{Y}}_j\big\}$. Consider a corresponding finite horizon problem with same model parameters and demand distributions, and time horizon T. Let the salvage value function in period $T+1$ be given by $f_{T+1}(\omega, \mathbf{y}_{T+1}) = \mathcal{F}(\omega, \mathbf{y}_{T+1})$. For period T, the optimality equations in (7) and (8) become

$$f_T(\omega, \mathbf{y}_T) = -\sum_{j=1}^N k_j^E\, y_{jT} + \min_{\mathbf{Y}_T^E, \mathbf{Y}_T^R, \mathbf{Y}_T \in \widetilde{\mathbb{Y}}(\mathbf{y}_T)} \Big\{ \gamma(\omega, Y_{1,T}^R) + \sum_{j=1}^N \big(c_j^E Y_{jT}^E + c_j Y_{jT} - c_j^R Y_{jT}^R\big) + \alpha\, \mathbb{E}_\omega \big[\mathcal{F}\big(\omega_{T+1}, \mathbf{Y}_T - D_T(\omega)\big)\big] \Big\}.$$

Next, observe that this minimization problem is identical to the one given in (39). Consequently, a solution to the latter must be a solution to the former. Thus, $\big(\widetilde{\mathbf{Y}}_j^E, \widetilde{\mathbf{Y}}_j^R, \widetilde{\mathbf{Y}}_j\big\}$ is the optimal policy for this finite-horizon problem in period T, and it then follows from the above that $f_T(\omega, \mathbf{y}_T) = \mathcal{F}(\omega, \mathbf{y}_T)$ for all ω.

Next, assume inductively that $f_t(\omega, \mathbf{y}_t) = \mathcal{F}(\omega, \mathbf{y}_t)$ for all ω and some period $t < T$. Our optimality equations in period t then become

$$f_t(\omega, \mathbf{y}_T) = -\sum_{j=1}^N k_j^E\, y_{jt} + \min_{\mathbf{Y}_t^E, \mathbf{Y}_t^R, \mathbf{Y}_t \in \widetilde{\mathbb{Y}}(\mathbf{y}_t)} \Big\{ \gamma(\omega, Y_{1,t}^R) + \sum_{j=1}^N \big(c_j^E Y_{jt}^E + c_j Y_{jt} - c_j^R Y_{jt}^R\big) + \alpha\, \mathbb{E}_\omega \big[\mathcal{F}\big(\omega_{t+1}, \mathbf{Y}_t - D_t(\omega)\big)\big] \Big\}.$$

Using reasoning identical to that used for period T, we conclude that $\big(\widetilde{\mathbf{Y}}_j^E, \widetilde{\mathbf{Y}}_j^R, \widetilde{\mathbf{Y}}_j\big\}$ must be an optimal policy for this finite-horizon problem in period t, and that $f_T(\omega, \mathbf{y}_t) = \mathcal{F}(\omega, \mathbf{y}_t)$ for all ω. Since $\mathcal{F}(\omega, \cdot)$ is additively convex for all ω, this concludes the proof. □

Proof of Theorem 8: We first prove the optimal post-expedite echelon j position can be written as

$$\widehat{Y}_{jt}^E(\omega) = \bigwedge_{i=j}^N \big[y_{it} \vee S_{it}^E(\omega)\big]. \tag{41}$$

By Theorem 4 (a), $\widehat{Y}_{Nt}^E(\omega) = y_{Nt} \vee S_{Nt}^E(\omega)$. Thus expression (41) holds for $j = N$. Assume inductively that expression (41) holds for $j+1, \ldots, N$, so that

$$\widehat{Y}_{j+1,t}^E(\omega) = \bigwedge_{i=j+1}^N \big[y_{it} \vee S_{it}^E(\omega)\big]. \tag{42}$$

It follows that

$$\begin{aligned}
\widehat{Y}_{jt}^E(\omega) &= y_{jt} \vee \big[S_{jt}^E(\omega) \wedge \widehat{Y}_{j+1,t}^E(\omega)\big] && \big[\,\widehat{Y}_{jt}^E = \widehat{Y}_{jt}^R \vee \big(S_{jt}^E(\omega) \wedge Y_{j+1,t}^E\big)\,\big] \\
&= \big[y_{jt}(\omega) \vee S_{jt}^E(\omega)\big] \wedge \big[y_{jt}(\omega) \vee \widehat{Y}_{j+1,t}^E(\omega)\big] && \big[\,\text{distributive property of “}\vee\text{”}\,\big] \\
&= \big[y_{jt}(\omega) \vee S_{jt}^E(\omega)\big] \wedge \widehat{Y}_{j+1,t}^E(\omega) && \big[\,\text{since } \widehat{Y}_{j+1,t}^E(\omega) \geq y_{j+1,t} \geq y_{jt}\,\big] \\
&= \big[y_{jt} \vee S_{jt}^E(\omega)\big] \wedge \bigwedge_{i=j+1}^N \big[y_{it} \vee S_{it}^E(\omega)\big] && \big[\,\text{inductive hypothesis in (42)}\,\big] \\
&= \bigwedge_{i=j}^N \big[y_{it} \vee S_{it}^E(\omega)\big].
\end{aligned}$$

Next, by assumption, $S_{jt}^R = S_j^R$, $S_{jt}^E = S_j^E$, and $S_{jt} = S_j$ for each t. Suppose $y_{i1} \leq S_i^R$ for every i. Consider some location j. By Theorem 1, either $Y_{j1}^R = Y_{j1}^E$ and there are no returns at location j in period

1 (in which case part (a) holds), or $Y_{j1}^E = y_{j1}$ and there is no expediting at location j. In the latter case, by Theorem 4 (b), we have $\widehat{Y}_{jt}^R = \widehat{Y}_{j-1,t}^E \vee \big(S_j^R \wedge y_{j1}\big)$. Since $y_{j1} \le S_j^R$, it follows that $\widehat{Y}_{j1}^R = y_{j1}$, and it is not optimal to reverse order product in period 1 at location j. Assume inductively that $y_{jt} \le S_j^R$ for every j in some period t. Then, by the same logic used above, there are no reverse orders in period t. Fix j. First we evaluate $\widehat{Y}_{jt}^E$. We get

$$\begin{aligned}
\widehat{Y}_{jt}^E &= \bigwedge_{i=j}^N [y_{it} \vee S_i^E] && [\text{expression (41)}]\\
&\le \bigwedge_{i=j}^N [S_i^R \vee S_i^E] && [\text{inductive assumption}]\\
&= \bigwedge_{i=j}^N S_i^R && [S_i^R \ge S_i^E \text{ by Theorem 5}]\\
&\le S_j^R. && (43)
\end{aligned}$$

Next, we evaluate the post-regular order echelon j position $\widehat{Y}_{jt}$. Because there are no reverse orders in period t, $\widehat{Y}_{jt}^R = \widehat{Y}_{jt}^E$ for all j. Therefore, we get

$$\begin{aligned}
\widehat{Y}_{jt} &= \max\Big[\widehat{Y}_{jt}^R, \min\big(S_j, \widehat{Y}_{j+1,t}^R\big)\Big] && [\text{expression (17), Theorem 4 (c)}]\\
&\le \max\big(\widehat{Y}_{jt}^E, S_j\big) && [\widehat{Y}_{jt}^R = \widehat{Y}_{jt}^E \text{ for all } j]\\
&\le \max\big(S_j^R, S_j\big) && [\text{expression (43)}]\\
&= S_j^R. && [S_i^R \ge S_i \text{ by Theorem 5}] \quad (44)
\end{aligned}$$

Since $y_{j,t+1} = Y_{jt} - D_t$, then expression (44) implies that $y_{j,t+1} \le S_j^R$ and consequently, it is not optimal to reverse order product in period $t+1$ at stage j. This concludes the proof of part (a).

To prove part (b), we make use of induction within induction. Let $i = N$, so that $y_{N1} > S_N^R$ and $y_{j1} \le S_j^R$ for every $j < N$. For $j < N$, we get

$$\begin{aligned}
\widehat{Y}_{j1}^E &= \bigwedge_{i=j}^N [y_{i1} \vee S_i^E] && [\text{expression (41)}]\\
&= \left(\bigwedge_{i=j}^{N-1} \big[y_{i1} \vee S_i^E\big]\right) \wedge \big(y_{N,1} \vee S_N^E\big)\\
&\le \left(\bigwedge_{i=j}^{N-1} \big[S_i^R \vee S_i^E\big]\right) \wedge \big(y_{N,1} \vee S_N^E\big) && [y_{i1} \le S_i^R \text{ for } i < N]\\
&= \left(\bigwedge_{i=j}^{N-1} S_i^R\right) \wedge \big(y_{N,1} \vee S_N^E\big) && [S_i^R \ge S_i^E \text{ by Theorem 5}]\\
&\le S_j^R. && (45)
\end{aligned}$$

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Therefore, $\widehat{Y}^R_{j1} = \widehat{Y}^E_{j-1,1} \vee \big(S^R_j \wedge \widehat{Y}^E_{j1}\big) = \widehat{Y}^E_{j-1,1} \vee \widehat{Y}^E_{j1} = \widehat{Y}^E_{j1}$, and it follows that for any $j < N$ there are no reverse orders in period 1. Further, by making use of (45) and Theorem 5, we get for any $j < N$ that

$$\widehat{Y}_{j1} = \max\Big[\widehat{Y}^R_{j1}, \min\big(S_j, \widehat{Y}^R_{j+1,1}\big)\Big] \le \max\big(\widehat{Y}^R_{j1}, S_j\big) = \big(\widehat{Y}^E_{j1}, S_j\big) \le \max\big(S^R_j, S_j\big) = S^R_j. \tag{46}$$

Since $y_{j2} = \widehat{Y}_{j1} - D_1$, then it follows from expression (46) that, for any $j < N$, $y_{j2} \le S^R_j - D_1$. Consequently, in period 2, it is not optimal to reverse product flow at any stage $j < N$. Assume inductively that $y_{jt} \le S^R_j$ for some period t. Then, following the exact same steps used to derive (45), we get $\widehat{Y}^E_{jt} \le S^R_j$. $\widehat{Y}^R_{jt} = \widehat{Y}^E_{j-1,t} \vee \big(S^R_j \wedge \widehat{Y}^E_{jt}\big) = \widehat{Y}^E_{j-1,t} \vee \widehat{Y}^E_{jt} = \widehat{Y}^E_{jt}$, and it follows that there are no reverse orders in period t for any $j < N$. Further, for any $j < N$,

$$\widehat{Y}_{jt} = \max\Big[\widehat{Y}^R_{j1}, \min\big(S_j, \widehat{Y}^R_{j+1,1}\big)\Big] \le \max\big(\widehat{Y}^R_{j1}, S_j\big) = \big(\widehat{Y}^E_{j1}, S_j\big) \le \max\big(S^R_j, S_j\big) = S^R_j.$$

Since $y_{j,t+1} = \widehat{Y}_{jt} - D_t$, it follows that, for any $j < N$, $y_{j,t+1} \le S^R_j - D_1 \le S^R_j$. Hence, in period t+1, it is not optimal to reverse product flow at any stage $j < N$.

Next, we consider stage N. For period 1, we get $\widehat{Y}^E_{N1} = y_{N1} \vee S^E_N = y_{N1}$, since $y_{N1} > S^R_N \ge S^E_N$. Further,

$$\widehat{Y}^R_{N1} = \widehat{Y}^E_{N-1,1} \vee \big(S^R_N \wedge \widehat{Y}^E_{Nt}\big) = \widehat{Y}^E_{N-1,1} \vee \big(S^R_N \wedge y_{Nt}\big) = \widehat{Y}^E_{N-1,1} \vee S^R_N.$$

Since $y_{N1} > S^R_N \ge S^E_N$, we also get

$$\begin{aligned}\widehat{Y}_{N1} &= \widehat{Y}^R_{N1} \vee S_N = \widehat{Y}^E_{N-1,1} \vee \big(S^R_N \wedge \widehat{Y}^E_{Nt}\big) \vee S_N \\ &= \widehat{Y}^E_{N-1,1} \vee \Big[S^R_N \wedge \big(y_{N1} \vee S^E_N\big)\Big] \vee S_N = \widehat{Y}^E_{N-1,1} \vee S^R_N \vee S_N = \widehat{Y}^E_{N-1,1} \vee S^R_N.\end{aligned}$$

If $\widehat{Y}^E_{N-1,1} < S^R_N$, then it follows that: (1) there is a reverse order at location N in period 1; and (2) $\widehat{Y}_{N1} = S^R_N$. Following the reasoning used for $j < N$, we therefore obtain that there are no reverse orders made in any period $t > 1$ at location N. If, on the other hand, $\widehat{Y}^E_{N-1,1} \ge S^R_N$, then it follows from above that there are no expedited, reverse, or regular orders made in period 1 at location N. Let τ be the first period in which $\widehat{Y}^E_{N-1,1} < S^R_N$. Prior to τ there are no reverse orders, by the same reasoning as the one used for period 1. Then, following the same reasoning used for $j < N$, there are no reverse orders in period τ, or in any period $t > \tau$. This proves the theorem for $j = N$.

Assume inductively that part (b) holds for some $j+1, \dots, N$, so that if i is the smallest location such that $y_{i1} > S^R_i$, and $j+1 \le i \le N$, then, after period $N - i + 1$, it is never optimal to reverse-order the product anywhere in the system. To prove that (b) also holds for j, where j is the smallest location q such that $y_{q1} > S^R_q$, consider any location i, $i < j$. Since $y_{i1} \le S^R_i$, then $\widehat{Y}^R_{i1} = y_{i1}$, and there are no reverse orders in period 1. By(41), we get

$$\widehat{Y}^E_{i1} = \bigwedge_{m=i}^{N} \big[y_{i1} \vee S^E_i\big] \le \left(\bigwedge_{m=i}^{j-1} \big[y_{m1} \vee S^E_m\big]\right) \le \left(\bigwedge_{m=i}^{j-1} \big[S^R_m \vee S^E_m\big]\right) = \left(\bigwedge_{m=i}^{j-1} S^R_m\right) \le S^R_i. \tag{47}$$

Next, using (47) for the optimal post-regular order echelon position, for any $i < j$, we get

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$$\widehat{Y}_{i1} = \max\Big[\widehat{Y}_{i1}^{E}, \min\big(S_i, \widehat{Y}_{i+1,1}^{E}\big)\Big] \le \max\big(\widehat{Y}_{i1}^{E}, S_i\big) \le \max\big(S_i^R, S_i\big) = S_i^R. \tag{48}$$

Therefore, $\widehat{Y}_{i1}^{R} = \widehat{Y}_{i-1,1}^{E} \vee \big(S_i^R \wedge \widehat{Y}_{i1}^{E}\big) = \widehat{Y}_{i-1,1}^{E} \vee \widehat{Y}_{i1}^{E} = \widehat{Y}_{i1}^{E}$, and it follows that for any i there are no reverse orders in period 1. Further, making use of (45) we get

$$\widehat{Y}_{i1} = \max\Big[\widehat{Y}_{i1}^{R}, \min\big(S_i, \widehat{Y}_{i+1,1}^{R}\big)\Big] \le \max\big(\widehat{Y}_{i1}^{R}, S_i\big) = \big(\widehat{Y}_{i1}^{E}, S_i\big) \le \max\big(S_i^R, S_i\big) = S_i^R. \tag{49}$$

Consequently, for every $i < j$, we then have $y_{i2} = \widehat{Y}_{i1} - D_1 \le S_i^R$, by expressions (48) and (49). Hence, for any such stage i it is never optimal to reverse order the product at any period during the time horizon.

We now consider optimal decisions at stage j in period 1. We get

$$\begin{aligned}
\widehat{Y}_{j1}^{E} &= y_{j1} \vee \big(S_j^E \wedge \widehat{Y}_{j+1,1}^{E}\big) && [\text{expression (15) of Theorem 4}]\\
&= y_{j1} && [\,y_{j1} > S_j^R \ge S_j^E \ge \min(S_j^E, \widehat{Y}_{j+1,1}^{E})\,]\\
\widehat{Y}_{N1}^{R} &= \widehat{Y}_{j-1,1}^{E} \vee \big(S_j^R \wedge \widehat{Y}_{j1}^{E}\big) = \widehat{Y}_{j-1,1}^{E} \vee \big(S_j^R \wedge y_{1t}\big) = \widehat{Y}_{j-1,1}^{E} \vee S_j^R &&\\
\widehat{Y}_{j1} &= \widehat{Y}_{j1}^{R} \vee \big(S_j \wedge \widehat{Y}_{j+1,1}^{R}\big) && [\text{expression (17) of Theorem 4}]\\
&= \widehat{Y}_{j-1,1}^{E} \vee S_j^R \vee \big(S_j \wedge \widehat{Y}_{j+1,1}^{R}\big) = \widehat{Y}_{j-1,1}^{E} \vee S_j^R && [\,S_j^R \ge S_j \ge (S_j \wedge \widehat{Y}_{j+1,1}^{R})\,]
\end{aligned}$$

If $\widehat{Y}_{j-1,1}^{E} < S_j^R$, then it follows that: (1) there is a reverse order at location j in period 1; (2) $\widehat{Y}_{j1} = S_j^R$; and (3) $\widehat{Y}_{j1} = S_j^R$. In that case, $y_{j2} = S_j^R - D_1 \le S_j^R$. Consequently, we start period 2 with the echelon state y_t such that $y_{i2} \le S_i^R$. So, in period 2 we can apply the inductive hypothesis to location $j+1$ to conclude that after another $N-j$ periods, it will never be optimal to reverse-order the product at any location. Consequently, starting in period 1, it follows that after period $N-j+1$, it is never optimal to reverse-order the product at any location. Now, suppose that $\widehat{Y}_{j-1,1}^{E} \ge S_j^R$. Then it follows from above that there are no expedited, reverse, or regular orders made in period 1 at location j. Let τ be the first period in which $\widehat{Y}_{j-1,1}^{E} < S_N^R$. Prior to τ there are no reverse orders, by the same reasoning used for period 1. Then, for any period $t > \tau$ there are also no reverse orders following the same reasoning used for $i < j$. This concludes the proof of the inductive hypothesis and therefore of part (b). □

Proof of Theorem 9: This proof follows identical steps to those used to prove Theorems 1 - 5 and is, therefore, omitted.

Proof of Theorem 10: The term $\alpha\mathbb{E}_{\omega}\big[\overline{F}_{t+1}(\omega_{t+1}, \mathbf{x}_{t+1})\big]$, from expression (21) can be written as

$$\alpha\mathbb{E}_{\omega}\big[\overline{F}_{t+1}(\omega_{t+1}, \mathbf{x}_{t+1})\big] = \alpha\mathbb{E}_{\omega}\Bigg\{\min_{\mathbf{X}_{t+1}^{R}, \mathbf{X}_{t+1} \in \overline{\mathbb{X}}(\mathbf{x}_{t+1})}\Big[\gamma_{t+1}\big(\omega, x_{11,t+1} - X_{11,t+1}^{R}\big) + \sum_{j=1}^{N}\sum_{i=1}^{j}\big(k_{ij}^{R}X_{ij,t+1}^{R} + H_{ij}x_{ij,t+1}\big)$$
$$+ \sum_{j=1}^{N-1}\sum_{i=1}^{j+1} k_{ij}X_{ij,t+1} + k_{NN}X_{NN,t+1} + \alpha\mathbb{E}_{\omega_{t+1}}\big[\overline{F}_{t+2}(\omega_{t+2}, \mathbf{x}_{t+2})\big],\Big]\Bigg\}.$$

By the state transition equations given in (22), the term $H_{ij}x_{ij,t+1}$ in the above expression does not

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depend on $\mathbf{X}^R_{t+1}$ or $\mathbf{X}_{t+1}$. Thus, using the definition of $\overline{G}_{t+2}(\omega, \mathbf{x}_{t+1}\mathbf{X}^R_{t+1}, \mathbf{X}_{t+1})$ given in (27), we get

$$\alpha\mathbb{E}_\omega\left[\overline{F}_{t+1}(\omega_{t+1}, \mathbf{x}_{t+1})\right] = \alpha\mathbb{E}_\omega\left[\sum_{j=1}^{N}\sum_{i=1}^{j} H_{ij}x_{ij,t+1}\right] + \alpha\mathbb{E}_\omega\left[\min_{\mathbf{X}^R_{t+1}, \mathbf{X}_{t+1} \in \overline{\mathbb{X}}(\mathbf{x}_{t+1})} \alpha\overline{G}_{t+2}(\omega, \mathbf{x}_{t+1}\mathbf{X}^R_{t+1}, \mathbf{X}_{t+1})\right].$$

Therefore, by comparing the expression for $\overline{V}_t$ given in (21) with the expression for $\overline{V}_t$ given in (25) and the expression for $\overline{U}_t$ given in (26), it follows that it only remains to show that

$$\mathbb{E}_\omega\left[\sum_{j=1}^{N}\sum_{i=1}^{j} H_{ij}x_{ij,t+1}\right] = \sum_{j=1}^{N}\sum_{i=1}^{j} H_{ij}x_{ijt} - \sum_{j=1}^{N}\sum_{i=1}^{j} h_{ij}X^R_{ijt} + \sum_{j=1}^{N-1}\sum_{i=1}^{j} h_{ij}X_{ijt} + \sum_{j=1}^{N-1}\Delta H_j X_{j+1,jt} + H_{NN}X_{NNt} - \mathbb{E}_\omega\left[D_t\right].$$

By the state transition equations given in (22), the above expression is equivalent to showing that

$$H_{11}X^R_{11t} + \sum_{j=2}^{N-1}\sum_{i=1}^{j-1} H_{ij}\left(X^R_{ijt} - X^R_{i,j-1,t}\right) + \sum_{j=2}^{N} H_{jj}X^R_{jjt} + \sum_{i=1}^{N-1} H_{iN}\left(X^R_{iNt} - X^R_{i,N-1,t}\right) = \sum_{j=1}^{N}\sum_{i=1}^{j} h_{ij}X^R_{ijt} \tag{50}$$

and

$$H_{11}\left(X_{11t} + X_{21t}\right) + \sum_{j=2}^{N-1}\sum_{i=1}^{j-1} H_{ij}\left(X_{ijt} - X_{i,j-1,t}\right) + \sum_{j=2}^{N-1} H_{jj}\left(X_{jjt} + X_{j+1,jt} - X_{j,j-1,t}\right) - \sum_{i=1}^{N-1} H_{iN}X_{i,N-1,t} - H_{NN}X_{N,N-1,t}$$
$$= \sum_{j=1}^{N-1}\sum_{i=1}^{j} h_{ij}X_{ijt} + \sum_{j=1}^{N-1}\Delta H_j X_{j+1,jt}. \tag{51}$$

To begin with, by changing the order of summation, we can write the left-hand side of (50) as

$$\sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}\left(H_{ij}X^R_{ijt} - H_{ij}X^R_{i,j-1,t}\right) + \sum_{j=1}^{N} H_{jj}X^R_{jjt} + \sum_{i=1}^{N-1} H_{iN}\left(X^R_{iNt} - X^R_{i,N-1,t}\right)$$
$$= \sum_{i=1}^{N-2}\left[\sum_{j=i+1}^{N-1}(H_{ij} - H_{i,j+1})X^R_{ijt} - H_{i,i+1}X^R_{iit} + H_{iN}X^R_{i,N-1,t}\right] + \sum_{j=1}^{N} H_{jj}X^R_{jjt} + \sum_{i=1}^{N-1} H_{iN}\left(X^R_{iNt} - X^R_{i,N-1,t}\right)$$
$$= \sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1} h_{ij}X^R_{ijt} - \sum_{i=1}^{N-2} H_{i,i+1}X^R_{iit} + \sum_{i=1}^{N-2} H_{iN}X^R_{i,N-1,t} + \sum_{j=1}^{N} H_{jj}X^R_{jjt} + \sum_{i=1}^{N-1} H_{iN}\left(X^R_{iNt} - X^R_{i,N-1,t}\right)$$
$$= \sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1} h_{ij}X^R_{ijt} + \sum_{i=1}^{N-2} H_{iN}X^R_{i,N-1,t} + \sum_{j=1}^{N-2}\left(H_{jj}X^R_{jjt} - H_{j,j+1}X^R_{jjt}\right) + \sum_{j=N-1}^{N} H_{jj}X^R_{jjt} + \sum_{i=1}^{N-1} H_{iN}\left(X^R_{iNt} - X^R_{i,N-1,t}\right)$$
$$= \sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1} h_{ij}X^R_{ijt} + \sum_{j=1}^{N-2} h_{jj}X^R_{jjt} + \sum_{j=N-1}^{N} H_{jj}X^R_{jjt} + \sum_{i=1}^{N-1} H_{iN}X^R_{iNt} - H_{N-1,N}X^R_{N-1,N-1,t}$$
$$= \sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1} h_{ij}X^R_{ijt} + \sum_{i=1}^{N-2} H_{iN}X^R_{iNt} + \sum_{j=1}^{N-2} h_{jj}X^R_{jjt} + \sum_{j=N-1}^{N} H_{jj}X^R_{jjt} + H_{N-1,N}X^R_{N-1,Nt} - H_{N-1,N}X^R_{N-1,N-1,t}$$
$$= \sum_{i=1}^{N-2}\sum_{j=i}^{N} h_{ij}X^R_{ijt} + \sum_{j=N-1}^{N} H_{jj}X^R_{jjt} + H_{N-1,N}X^R_{N-1,Nt} - H_{N-1,N}X^R_{N-1,N-1,t}.$$

Consequently, the left-hand side of (50) finally becomes

$$\sum_{i=1}^{N-2}\sum_{j=i}^{N} h_{ij}X_{ijt}^{R}+H_{N-1,N-1}X_{N-1,N-1,t}^{R}-H_{N-1,N}X_{N-1,N-1,t}^{R}+H_{N-1,N}X_{N-1,Nt}^{R}+H_{NN}X_{NNt}^{R}$$

$$=\sum_{i=1}^{N-2}\sum_{j=i}^{N} h_{ij}X_{ijt}^{R}+h_{N-1,N-1}X_{N-1,N-1,t}^{R}+h_{N-1,N}X_{N-1,Nt}^{R}+h_{NN}X_{NNt}^{R}=\sum_{i=1}^{N}\sum_{j=i}^{N} h_{ij}X_{ijt}^{R}=\sum_{j=1}^{N}\sum_{i=1}^{j} h_{ij}X_{ijt}^{R}.$$

Similarly, by changing the order of summation, and defining $X_{i0t} := 0$ for all i and t for notational convenience, we can write the left-hand side of (51) as

$$\sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1} H_{ij}\left(X_{ijt}-X_{i,j-1,t}\right)+\sum_{j=1}^{N-1} H_{jj}\left(X_{jjt}+X_{j+1,jt}-X_{j,j-1,t}\right)-\sum_{i=1}^{N-1} H_{iN}X_{i,N-1,t}-H_{NN}X_{N,N-1,t}$$

$$=\sum_{i=1}^{N-2}\left[\sum_{j=i+1}^{N-1}(H_{ij}-H_{i,j+1})X_{ijt}-H_{i,i+1}X_{iit}+H_{iN}X_{i,N-1,t}\right]+\sum_{j=1}^{N-1} H_{jj}\left(X_{jjt}+X_{j+1,jt}-X_{j,j-1,t}\right)$$
$$-\sum_{i=1}^{N-1} H_{iN}X_{i,N-1,t}-H_{NN}X_{N,N-1,t}$$

$$=\sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1} h_{ij}X_{ijt}-\sum_{i=1}^{N-2} H_{i,i+1}X_{iit}+\sum_{i=1}^{N-2} H_{iN}X_{i,N-1,t}+\sum_{j=1}^{N-1} H_{jj}\left(X_{jjt}+X_{j+1,jt}-X_{j,j-1,t}\right)$$
$$-\sum_{i=1}^{N-1} H_{iN}X_{i,N-1,t}-H_{NN}X_{N,N-1,t}$$

$$=\sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1} h_{ij}X_{ijt}-\sum_{i=1}^{N-2} H_{i,i+1}X_{iit}+\sum_{j=1}^{N-1} H_{jj}\left(X_{jjt}+X_{j+1,jt}-X_{j,j-1,t}\right)-H_{N-1,N}X_{N-1,N-1,t}-H_{NN}X_{N,N-1,t}$$

$$=\sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1} h_{ij}X_{ijt}+\sum_{i=1}^{N-2}(H_{jj}-H_{j,j+1})X_{jjt}+\sum_{j=1}^{N-1} H_{jj}\left(X_{j+1,jt}-X_{j,j-1,t}\right)$$
$$+H_{N-1,N-1}X_{N-1,N-1,t}-H_{N-1,N}X_{N-1,N-1,t}-H_{NN}X_{N,N-1,t}$$

$$=\sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1} h_{ij}X_{ijt}+\sum_{i=1}^{N-2} h_{jj}X_{jjt}+\sum_{j=1}^{N-1}(H_{jj}-H_{j+1,j+1})X_{j+1,jt}+h_{N-1,N-1}X_{N-1,N-1,t}+H_{NN}X_{N,N-1,t}-H_{NN}X_{N,N-1,t}$$

$$=\sum_{i=1}^{N-2}\sum_{j=i}^{N-1} h_{ij}X_{ijt}+h_{N-1,N-1}X_{N-1,N-1,t}+\sum_{j=1}^{N-1}\Delta H_{jj}X_{j+1,jt}$$

$$=\sum_{i=1}^{N-1}\sum_{j=i}^{N-1} h_{ij}X_{ijt}+\sum_{j=1}^{N-1}\Delta H_{jj}X_{j+1,jt}=\sum_{j=1}^{N-1}\sum_{i=1}^{j} h_{ij}X_{ijt}+\sum_{j=1}^{N-1}\Delta H_{jj}X_{j+1,jt},$$

which concludes the proof.

Appendix B: Multi-period Leadtimes

In the multiechelon literature, there exist two methods for dealing with multi-period leadtimes (i.e., when the replenisment leadtime in the system is longer than the review interval). The first approach, proposed in Clark and Scark (1960) for systems with only the regular flow of product, involves replacing echelon j inventory y_{jt} with the variable that is the sum of that echelon inventory and all the inventory in transit from the next location upstream. Suppose, for example, that the leadtime between locations 2 and 3 in the system is L periods, so that there are $z^2_{1t}, z^2_{2t}, \ldots, z^2_{Lt}$ amounts in transit from location 3 to location 2 in period t , where the amount z^2_{jt} is scheduled to be delivered j periods later. Then, the new state variable at location 2, observed at the beginning of period t but after the scheduled delivery ordered L periods ago has arrived, becomes $\hat{y}_{jt} = y_{jt} + z^2_{1t} + z^2_{2t} + \ldots + z^2_{L-1,t}$. The new decision variable in period t at location 2 becomes $\widehat{Y} = \hat{y}_{jt} + z_{Lt}$. Once costs are accounted for properly by allocating to location 2 in period t the discounted cost incurred at that location in period $t + L$, the optimal policy that achieves the decomposition of the objective cost function is to bring $\widehat{Y} = \hat{y}_{jt} + z^2_{Lt}$ up to an echelon basestock level S_{jt} (subject to the availability of stock at location 3). This approach was used successfully, for example, by Gallego and Özer (2003) and DeCroix (2006), who address important problems in multi-stage supply systems while dealing with only the regular flow of orders. Unfortunately, this approach for dealing with multi-period leadtimes cannot accommodate reverse orders of product in the supply chain. This is because, with reverse logistics, $\hat{y}_{jt}$ is increased through regular orders while echelon inventory y_{jt} itself is lowered through reverse orders. Thus, the state space cannot be collapsed to only $\hat{y}_{jt}$ at any location j that has a multi-period leadtime L. Instead, it becomes necessary to keep track of both y_{jt} and $\hat{y}_{jt}$, as well as all the reverse orders placed up to L periods ago, at each such location j. As a result, the dimensionality of the state space blows up and the decomposition of the objective function can no longer be obtained.

The second approach, introduced by Lawson and Porteus (2000), inserts additional locations in-between those with multiperiod leadtimes, so that the transformed system would then have only a single period leadtime period between any two adjacent locations. By choosing unit regular order, expediting, and holding costs in an appropriate way, Lawson and Porteus (2000) show that the transformed system can be made equivalent, with regard to product flows and associated costs, to the original system with leadtimes, while still being amenable to the methods provided in their paper. However, this approach does not work in our setting because reverse orders and regular orders with multi-period leadtimes can cross during transit, with both passing through an identical (inserted) location in the same period. As a result, any selection of unit order costs, for regular and reverse flows, that motivates the system to continue shipping the incoming product in a particular direction will, at any such location, either ship all inventory upstream or ship all inventory downstream, without regard to the original direction of each flow (since the two streams of orders are physically indistinguishable once they arrive at a particular location). Hence, leadtime constraints would be violated whenever reverse orders and regular orders associated with a multi-period leadtime cross at an inserted location. To manage multi-period leadtimes in a such a system, it again becomes necessary to increase the state space by keeping track of orders made in previous periods, so that downstream orders can be kept going downstream and upstream orders can be kept going upstream. The result is an explosion in the state space dimensionality.

There exists, however, a class of reverse logistics problems for which an application of the Lawson and Porteus (2000) approach for managing multi-period leadtimes bears fruit. This class of problems involves systems in which a multi-period leadtime occurs at only the most upstream location. This can occur, for example, when the manufacturer takes longer to complete the physical transformation of production inputs into the finished product than it takes for the finished product to be shipped from one downstream location to another, something that is encountered in practice more often than not. (With International Bearings for example, completing the manufacture of an order for bearings takes roughtly three times as long as it does to ship it by boat from manufacturer's location in China to Singapore.) Then, following Lawson and Porteus (2000), we insert additional locations that correspond to multiple periods in that multi-period leadtime, and allocate regular, expedited and holding costs appropriately. Given that reverse orders can be considered as having left the system completely at location N, there will not be any reverse orders going in the opposite direction of the regular and expedited orders at those additional locations upstream of location N. Consequently, this method will adequately capture the leadtime effect at the most upstream location, while allowing the problem to be solved using the results established in this paper.

Developing more general methods to solve reverse logistics problems with multi-period leadtimes represents a worthwhile extension of our work. Until those methods are developed, the results obtained in our paper can also be used to obtain a lower bound on the total cost for such a system. The relevant lower bound is obtained by: (1) transforming the system by adding locations in place of multi-period leadtimes; (2) allocating unit order costs along the lines prescribed by Lawson and Porteus (2000); and, (3) solving the resulting system with single-period leadtimes between locations by means of the methods presented in our paper. Because leadtime constraints are relaxed, the resulting system has fewer constraints, and having the same unit costs and product flows as the original system, it will generate a lower total cost.

Appendix C: Additional Numerical Studies

1. Logistics Supply Chain

Here we quantify the value of reverse logistics for a five-location, logistics supply chain with regular, reverse, and expedited flow of product. Our numerical studies in this section essentially replicate those from Section 5. In particular, Study 5 mirrors Study 1 and Study 3, while Study 6 mirrors Study 2 and Study 4. Our basic version of the five -location system has the same demand characteristics and parameters choices as the three- and four-echelon systems studied ain Section 5: $(k_1, k_2, k_3, k_4, k_5) = (1, 1, 1, 1, 1)$; $(k_1^R, k_2^R, k_3^R, k_4^R, k_5^R) = (1, 1, 1, 1, -0.8)$; $(k_1^E, k_2^E, k_3^E, k_4^E, k_5^E) = (4, 4, 4, 4, 4)$; and $(H_1, H_2, H_3, H_4, H_5) = (2.5, 2.0, 1.5, 1.0, 0.5)$ (thus, $h_j = 0.5$). The backlogging cost is unchanged at $p = 15$.

In Study 5, we again vary k_j^R from 0.5 to 1.75 in increments of 0.25, and the Markov multiplier $\beta(3)$ from 2 to 4.5, in increments of 0.5. In Study 6, we vary unit expediting costs against the unit backlogging cost. Tables 16 and 17 presents cost savings for the $CS + R + E$ system for Studies 5 and 6, respectively.

M. Mult.	Unit Reverse Order Costs $(k_1^R = k_2^R)$					
$\beta(1)$, $\beta(3)$	0.50	0.75	1.00	1.25	1.5	1.75
(0.50, 2.0)	25.6%	25.5%	25.4%	25.3%	25.3%	25.2%
(0.40, 2.5)	27.4%	27.1%	26.8%	26.6%	26.5%	26.4%
(0,33, 3.0)	28.0%	27.5%	27.1%	26.8%	26.6%	26.3%
(0.29, 3.5)	28.2%	27.6%	27.1%	26.7%	26.3%	26.1%
(0.25, 4.0)	28.4%	27.7%	27.0%	26.5%	26.1%	25.8%
(0.22, 4.5)	28.5%	27.6%	26.9%	26.3%	25.8%	25.4%

Table 16: Study 5 Cost Savings - *the $CS + R + E$ System*

Backlog.	Unit Expediting Costs $(k_1^E = k_2^E = k_3^E)$					
Cost	4	5	6	7	8	9
10	26.6%	19.6%	14.5%	10.7%	7.83%	5.52%
15	27.1%	20.5%	15.7%	11.9%	9.59%	7.61%
20	27.5%	21.0%	16.3%	12.0%	10.5%	8.54%
25	27.7%	21.3%	16.7%	13.5%	11.0%	9.15%
30	27.9%	21.6%	17.0%	13.9%	11.5%	9.62%
35	28.1%	21.8%	17.3%	14.2%	11.8%	10.0%

Table 17: Study 6 Cost Savings - *the $CS + R + E$ System*

Table 16 further confirms the behavior of cost savings for the $CS + R + E$ system observed for three-location and four-location systems; namely, that those cost savings are decreasing in unit reverse order costs and are non-monotonic in the Markov demand multiplier. Further, by comparing the values in Table 16 to those in Table 9, it can be observed that cost savings continue to increase with supply chain length, as we go from a four-location system to a five-location system. This behavior of cost saving increasing with supply chain length can also be observed by comparing Table 17 with Table 11. The values shown in Table 17 are also found to be decreasing in the unit expediting cost and increasing in the unit backlogging cost, exactly as was the case with three-location and four-location systems studied in Section 5.

Tables 18 and 19 display the synergy value for Studies 5 and 6, respectively, for the five-location logistics supply chain considered in this section. We can observe that the synergy value is increasing in the Markov demand multiplier and the unit backlogging cost, and decreasing in the unit reverse order cost and unit expediting cost. Finally, by comparing Table 18 with Table 12, and Table 19 with Table 13, it can be discerned that the synergy value continues to increase with the length of the supply chain. Thus, the

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conclusions drawn in Section 5 regarding the impact of supply chain length of synergy value are confirmed by our numerical studies for the five-location logistics supply chain considered in this Appendix.

M. Mult.	Unit Reverse Order Costs ($k_1^R = k_2^R$)					
$\beta(1)$, $\beta(3)$	0.50	0.75	1.00	1.25	1.5	1.75
(0.50, 2.0)	-0.28%	-0.48%	-0.50%	-0.58%	-0.65%	-0.72%
(0.40, 2.5)	0.72%	0.45%	0.21%	0.01%	-0.15%	-0.26%
(0,33, 3.0)	1.59%	1.18%	0.83%	0.55%	0.31%	0.08%
(0.29, 3.5)	2.41%	1.88%	1.43%	1.04%	0.72%	0.47%
(0.25, 4.0)	3.08%	2.44%	1.90%	1.45%	1.07%	0.76%
(0.22, 4.5)	3.59%	2.86%	2.21%	1.70%	1.26%	0.91%

Table 6: Study 1 – *The Synergy Value*

Backlog.	Unit Expediting Costs ($k_1^E = k_2^E = k_3^E$)					
Cost	4	5	6	7	8	9
10	-0.68%	-0.96%	-1.10%	-1.21%	-1.31%	-1.41%
15	0.83%	0.16%	-0.31%	-0.63%	-0.90%	-1.06%
20	1.66%	0.81%	0.17%	-0.25%	-0.60%	-0.84%
25	2.16%	1.21%	0.50%	0.00%	-0.38%	-0.67%
30	2.38%	1.38%	0.64%	0.12%	-0.28%	-0.59%
35	2.63%	1.59%	0.82%	0.25%	-0.16%	-0.49%

Table 7: Study 2 – *The Synergy Value*

2. Product-Transforming Supply Chain

Next, we present the performance error of the 2-PLA heuristic for a 5-location product-transforming supply chain with reverse logistics. To begin with, Tables 22 and 23 parameter present values for the 3-location and 4-location product transforming supply chains used for numerical studies in Tables 14 and 15.

Stg. of Compl.	Holding Cost (H_{ij})			Reg. Order Cost (k_{ij})			Reverse Order Cost (k_{ij}^R)		
(i)	$j=1$	$j=2$	$j=3$	$j=1$	$j=2$	$j=3$	$j=1$	$j=2$	$j=3$
1	0.80	0.60	0.40	0.50	0.50		0.30	0.30	-0.80
2		0.40	0.20	0.60	0.60			0.40	-0.60
3			0.10		0.70	0.95			-0.50

Table 22: Unit Costs for the 3-Location Product-Transforming Supply Chain

Stage of	Holding Cost (H_{ij})				Reg. Order Cost (k_{ij})				Reverse Order Cost (k_{ij}^R)			
Completion (i)	$j=1$	$j=2$	$j=3$	$j=4$	$j=1$	$j=2$	$j=3$	$j=4$	$j=1$	$j=2$	$j=3$	$j=4$
1	0.80	0.70	0.60	0.40	0.50	0.50	0.50		0.30	0.30	0.30	-0.80
2		0.60	0.50	0.30	0.60	0.60	0.60			0.40	0.40	-0.60
3			0.40	0.20		0.65	0.65				0.50	-0.50
4				0.10			0.70	0.95				-0.40

Table 23: Unit Costs for the 4-Location Product-Transforming Supply Chain

With regard to unit holding costs displayed in Tables 22 and 23, observe that, for any given stage of completion i, unit holding costs are increasing looking downstream (i.e, in any given row, unit holding costs are increasing from right to left). This is because, the closer one physically gets to the customer, the higher one can expect the physical holding costs to be. Because the financial holding costs can be expected to generally vary only with the stage of completion, rather than physical location, than the total unit holding cost is increasing looking downstream. At the same time, at any given location j, the closer the item is to full completion, the higher its financial holding cost, where completion progresses from stage

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N, at which the items enters the supply chain, to stage 1 at which it is ready to satisfy customer demand. Thus, in Tables 22 and 23, unit holding costs are higher the closer an item is to full completion.

When it comes to unit regular order costs, Tables 22 and 23 capture two key effects. First, at any location j, the unit regular cost $k_{j+1,j}$ exceeds k_{ij} for all $i \in [1, j]$. This because it is only when an item at stage of completion $j+1$ at location $j+1$ is ordered into location j that the product transformation takes place (i.e., for items at all other stages of completion ordered from location $j+1$ into location j there is no product transformation involved), so that we would expect $k_{j+1,j}$ to be higher than k_{ij} for all $i \in [1, j]$. Second, because, by an assumption of our model, at location N partially completed items (i.e., $i \in [1,])$ are not brought into the supply chain, it is only $k_{N+1,N}$ that enters into the single-period cost function.

The key assumption reflected in unit reverse order costs shown in Tables 22 and 23 is that, at location N, the closer an item is to completion the higher the unit revenue associated with its resale out of that location. Since revenues are negative costs, then an item at stage of completion 1 at location N can be expected to have higher unit revenue (i.e., lower negative cost) than an item at that location that is only at stage of completion 3, for example.

Those structural assumptions about unit costs observed in Tables 22 and 23 are also used in our numerical studies for a 5-location product transforming supply chain with reverse logistics. Model parameters for that system are presented in Table 24. Further, in order to have meaningful comparisons of the performance error for the 2-PLA heuristic across product-transforming supply chains of different lengths (i.e., number of locations), we have kept roughly the same range of unit costs (i.e. holding, regular order, and reverse order) across items at different stages of completion at the same location, for each of the three supply chain lengths studied.

Stage of	Holding Cost (H_{ij})					Reg. Order Cost (k_{ij})					Reverse Order Cost (k^R_{ij})				
Completion (i)	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
1	0.80	0.75	0.70	0.60	0.40	0.50	0.50	0.50	0.50		0.30	0.30	0.30	0.30	-0.80
2		0.70	0.65	0.50	0.30	0.55	0.55	0.55	0.55			0.40	0.40	0.40	-0.60
3			0.65	0.45	0.25		0.60	0.60	0.60				0.50	0.50	-0.50
4				0.40	0.20			0.65	0.65					0.60	-0.40
5					0.10				0.70	0.95					-0.30

Table 24: Unit Costs for the 5-Location Product-Transforming Supply Chain

Table 25 exhibits the cost performance of the 2-PLA heuristic policy for this five-location product-transforming supply chain over the same range of Markov multipliers and unit backlogging costs as those used in the main body of the paper for 3-location and 4-location product-transforming supply chains. In Table 25, the average heuristic performance error for the five-location product-transforming supply chain reverse logistics, relative to our low-bound heuristic, was 4.11%.

M. Mult.	Unit Backlogging Cost					
$\beta(1)$, $\beta(3)$	10	15	20	25	30	35
(0.50, 2.0)	5.78%	4.97%	4.96%	4.59%	4.68%	3.75%
(0.40, 2.5)	5.46%	5.14%	3.97%	4.78%	3.90%	4.12%
(0,33, 3.0)	5.69%	5.05%	3.83%	3.78%	2.98%	3.10%
(0.29, 3.5)	5..83%	3.52%	4.17%	3.27%	2.38%	3.84%
(0.25, 4.0)	5.60%	4.50%	3.95%	4.26%	1.67%	2.09%
(0.22, 4.5)	5.67%	3.90%	2.85%	5.33%	2.00%	2.51%

Table 25: Heuristic Performance, 5-Location System