Algebraic Approach to Chaos Induced by Snapback Repeller*

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Abstract

In this poster we present the results of [1]. We consider the problem of detecting chaotic behaviors in discrete dynamical systems. We propose an algebraic criterion for determining whether all the zeros of a given polynomial are outside the unit circle in the complex plane. This criterion is used to deduce critical algebraic conditions for the occurrence of chaos in multi-dimensional discrete systems based on Marotto's theorem. Using these algebraic conditions we reduce the problem of analyzing chaos induced by snapback repeller to an algebraic problem, and introduce an algorithmic approach to solve this problem by means of symbolic computation. The proposed approach is effective as shown by several examples and can be used to determine the possibility that all the fixed points are snapback repellers.

1 Introduction

We focus on the following class of n-dimensional discrete dynamical systems

$$\boldsymbol{x}(k+1) = \boldsymbol{f}(\boldsymbol{\mu}, \boldsymbol{x}(k)), \tag{1}$$

where $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a $\mathcal{C}^1$ nonlinear map with parameters $\boldsymbol{\mu}$ from $\mathbb{R}$, the real number field. Let $\boldsymbol{f}^t$ denote the t times of compositions of $\boldsymbol{f}$ with itself. A point $\boldsymbol{x}$, is said to be a p-periodic point of $\boldsymbol{f}$ if $\boldsymbol{f}^p(\boldsymbol{x}) = \boldsymbol{x}$ but $\boldsymbol{f}^t(\boldsymbol{x}) \neq \boldsymbol{x}$ for $p > t \geq 1$. If $p = 1$, i.e., $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$, then $\boldsymbol{x}$ is called a fixed point. Let $\boldsymbol{f}'(\boldsymbol{x})$ and $|\boldsymbol{f}'(\boldsymbol{x})|$ be the Jacobian matrix of $\boldsymbol{f}$ at the point $\boldsymbol{x}$ and its determinant respectively.

In what follows, we will describe the notion of snapback repeller and Marotto's theorem. We consider a $\mathcal{C}^1$ nonlinear map $\boldsymbol{f}$ of (1). Define $B_r(\boldsymbol{x})$ as the closed ball of radius r centered at $\boldsymbol{x}$ under certain norm $||\cdot||$ in $\mathbb{R}^n$. We say that a fixed $\bar{\boldsymbol{x}}$ is a *repelling fixed point* of $\boldsymbol{f}$ with respect to the norm $||\cdot||$ if there exists a constant $s > 1$ such that $||\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})|| > s \cdot ||\boldsymbol{x} - \boldsymbol{y}||$ for any $\boldsymbol{x}, \boldsymbol{y} \in B_r(\bar{\boldsymbol{x}})$ with $\boldsymbol{x} \neq \boldsymbol{y}$, where $B_r(\bar{\boldsymbol{x}})$ is defined on this norm $||\cdot||$, called a repelling neighborhood of $\bar{\boldsymbol{x}}$.

Definition 1 *Let $\bar{\boldsymbol{x}}$ be a repelling fixed point of $\boldsymbol{f}$ in $B_r(\bar{\boldsymbol{x}})$ for some $r > 0$. We say that $\bar{\boldsymbol{x}}$ is a snapback repeller of $\boldsymbol{f}$ if there exist a point $\boldsymbol{x}_0 \in B_r(\bar{\boldsymbol{x}})$ with $\boldsymbol{x}_0 \neq \bar{\boldsymbol{x}}$ and an integer $m > 1$, such that $\boldsymbol{x}_m = \bar{\boldsymbol{x}}$ and $|\boldsymbol{f}'(\boldsymbol{x}_k)| \neq 0$ for $1 \leq k \leq m$, where $\boldsymbol{x}_k = \boldsymbol{f}^k(\boldsymbol{x}_0)$.*

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The point $\boldsymbol{x}_0$ in this definition is called a *snapback point* of $\boldsymbol{f}$. Under this definition, the following theorem by Marotto holds [5, 6].

Theorem 1 *If $\boldsymbol{f}$ possesses a snapback repeller, then $\boldsymbol{f}$ is chaotic in the following sense: There exist (i) a positive integer N such that for each integer $p \geq N$, $\boldsymbol{f}$ has a periodic point of period p; (ii) a "scrambled set" of $\boldsymbol{f}$, i.e., an uncountable set S containing no periodic points of $\boldsymbol{f}$ such that*

(a) $\boldsymbol{f}(S) \subset S$,

(b) $\limsup_{k\to\infty} \parallel \boldsymbol{f}^k(\boldsymbol{u}) - \boldsymbol{f}^k(\boldsymbol{v}) \parallel > 0$, *for all* $\boldsymbol{u}, \boldsymbol{v} \in S$ *with* $\boldsymbol{u} \neq \boldsymbol{v}$,

(c) $\limsup_{k\to\infty} \parallel \boldsymbol{f}^k(\boldsymbol{u}) - \boldsymbol{f}^k(\boldsymbol{v}_p) \parallel > 0$, *for all* $\boldsymbol{u} \in S$ *and any periodic point* $\boldsymbol{v}_p$ *of* $\boldsymbol{f}$;

(iii) an uncountable subset S_0 *of* S *such that* $\liminf_{k\to\infty} \parallel \boldsymbol{f}^k(\boldsymbol{u}) - \boldsymbol{f}^k(\boldsymbol{v}) \parallel = 0$, *for every* $\boldsymbol{u}, \boldsymbol{v} \in S_0$.

Marotto's theorem is significant in extending the analytic theory of chaos from one-dimension to multi-dimension. It is also effective in applications, for example, in finding the chaotic regimes (parameter ranges) for dynamical systems. Focusing on practical applications, there exist two directions to confirm that a repelling fixed point is a snapback repeller for multi-dimensional maps. The first one is to find the repelling neighborhood U of the repeller $\bar{\boldsymbol{x}}$ and a preimage point $\bar{\boldsymbol{x}}_0$ of $\bar{\boldsymbol{x}}$ lying in U, i.e., with $\boldsymbol{f}^m(\bar{\boldsymbol{x}}_0) = \bar{\boldsymbol{x}}$, $\boldsymbol{x}_0 \in U$ and $\boldsymbol{x}_0 \neq \bar{\boldsymbol{x}}$, for some $m > 1$. Therefore, deriving an estimation of the repelling neighborhood for a repeller becomes the key part in utilizing this theorem. Moreover, a computable norm is needed for practical application. The second direction is to construct the preimages $\{\bar{\boldsymbol{x}}^{-k}\}_{k=1}^{\infty}$ of $\bar{\boldsymbol{x}}$, such that $\boldsymbol{f}(\bar{\boldsymbol{x}}^{-k}) = \bar{\boldsymbol{x}}^{-k+1}$, $k \geq 2$, $\boldsymbol{f}(\bar{\boldsymbol{x}}^{-1}) = \bar{\boldsymbol{x}}$, $\lim_{k\to\infty} \boldsymbol{f}(\bar{\boldsymbol{x}}^{-k}) = \bar{\boldsymbol{x}}$. We call such an orbit $\{\bar{\boldsymbol{x}}^{-k}\}_{k=1}^{\infty}$ a (degenerate) homoclinic orbit for the repeller $\bar{\boldsymbol{x}}$. The existence of such a homoclinic orbit guarantees the existence of a snapback point in the repeller neighborhood of repeller $\bar{\boldsymbol{x}}$. Marotto's theorem thus holds without knowing the repelling region of the fixed point.

In this work we focus on the first direction on the study of snapback repeller by quoting the following lemma from [2] which can be used to determine a repelling fixed point of $\boldsymbol{f}$ under the Euclidean norm.

Lemma 1 *Let $\bar{\boldsymbol{x}}$ be a fixed point of $\boldsymbol{f}$ which is continuously differentiable in $B_r(\bar{\boldsymbol{x}})$. If*

$$\lambda > 1, \quad \text{for all eigenvalues } \lambda \text{ of } \Big(\boldsymbol{f}'(\bar{\boldsymbol{x}})\Big)^T \boldsymbol{f}'(\bar{\boldsymbol{x}}), \tag{2}$$

then there exist $s > 1$ and $r' \in (0, r]$ such that $||\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})||_2 > s \cdot ||\boldsymbol{x} - \boldsymbol{y}||_2$, for all $\boldsymbol{x}, \boldsymbol{y} \in B_{r'}(\bar{\boldsymbol{x}})$ with $\boldsymbol{x} \neq \boldsymbol{y}$, and all eigenvalues of $\Big(\boldsymbol{f}'(\boldsymbol{x})\Big)^T \boldsymbol{f}'(\boldsymbol{x})$ exceed one for all $\boldsymbol{x} \in B_{r'}(\bar{\boldsymbol{x}})$.

Our objective in this paper is to present a symbolic computation approach to detect the chaotic behavior of system (1) by using Marotto's theorem. In Sections 2-3, we present the main results of [1]. Section 4 shows some experiments. In conclusion, we summarize the results.

2 Zeros Distribution with Respect to the Unit Circle

Our aim is to derive the algebraic criterion for all zeros of the characteristic polynomial of $\Big(\boldsymbol{f}'(\boldsymbol{x})\Big)^T \boldsymbol{f}'(\boldsymbol{x})$ to be outside the unit circle (OUC). To this end, we will use a sequence of symmetric polynomials of descending degrees for the characteristic polynomial, see [9].

Let

$$D(\lambda) = d_0 + d_1\lambda + \ldots + d_n\lambda^n \tag{3}$$

be this characteristic polynomial, where $d_i = d_i(\boldsymbol{\mu}, \boldsymbol{x})$, $i = 0, \ldots, n$. Then denote by $D^*(\lambda)$ the reciprocated polynomial of $D(\lambda)$, namely, $D^*(\lambda) = \lambda^n D(\lambda^{-1}) = d_n + d_{n-1}\lambda + \cdots + d_0\lambda^n$.

Given the polynomial $D(\lambda)$, we assign to it a sequence of $n+1$ polynomials $T_n(\lambda), T_{n-1}(\lambda), \cdots, T_0(\lambda)$ according to the following formal definition:

$$T_n(\lambda) = D(\lambda) + D^*(\lambda), \quad T_{n-1}(\lambda) = [D(\lambda) - D^*(\lambda)]/(\lambda - 1),$$
$$T_{k-2}(\lambda) = \lambda^{-1}[\delta_k(\lambda+1)T_{k-1}(\lambda) - T_k(\lambda)],\ k = n, n-1, \ldots, 2,$$

where $\delta_k = T_k(0)/T_{k-1}(0)$. The recursion requires the normal conditions $T_{n-i}(0) \neq 0,\ i = 0, 1, \ldots, n$. The construction is interrupted when a $T_k(0) = 0$ occurs, and in [9], such singular cases can be classified into two types. The following theorem is the main result of [1] and it shows that there is no need to consider such singular cases.

Theorem 2 *All zeros of $D(\lambda)$ are OUC if and only if the normal conditions $T_{n-i}(0) \neq 0,\ i = 0, 1, \ldots, n$ hold and $\nu_n = Var\{T_n(1), \ldots, T_0(1)\} = n$.*

3 Semi-Algebraic Systems for Marotto's theorem

It should be noticed that the repelling neighbourhood $B_{r'}(\bar{\boldsymbol{x}})$ in Lemma 1 can be found by a series of inequalities and inequations automatically. In fact, we take $B_{r'}(\bar{\boldsymbol{x}})$ as $B_{r'}(\bar{\boldsymbol{x}}) = \{Var\{T_i(1)|_{\bar{\boldsymbol{x}}}\} = n, T_{n-i}(0)|_{\bar{\boldsymbol{x}}} \neq 0, i = 0, \ldots, n\}$ (ie., the solution space of the inequality set $\{Var\{T_i(1)|_{\bar{\boldsymbol{x}}}\} = n, T_{n-i}(0)|_{\bar{\boldsymbol{x}}} \neq 0, i = 0, \ldots, n\}$ consists of a repelling neighbourhood). Note that there may have four cases for the sign variation pattern of the polynomials in the sequence $\{T_k(1)\}_{k=0}^n$ at two different points. The following result of [1] tells us that only two cases may occur for any $\boldsymbol{x} \in B_{r'}(\bar{\boldsymbol{x}})$ for sufficiently small $r' > 0$.

Lemma 2 *The sign variation pattern of the polynomials in the sequence $\{T_k(1)\}_{k=0}^n$ for any point in the repelling neighbourhood $B_{r'}(\bar{\boldsymbol{x}})$ remains consistent for sufficiently small $r' > 0$.*

The following result is the main theorem for detecting the chaotic behavior of system (1).

Theorem 3 *For a general n-dimensional discrete system (1), the system is chaotic in the sense of Marotto if one of the following semi-algebraic systems to have at least one real solution:*

$$\Psi_j : \begin{cases} \bar{\boldsymbol{x}} - \boldsymbol{f}(\boldsymbol{\mu}, \bar{\boldsymbol{x}}) = 0,\ \boldsymbol{f}^m(\boldsymbol{x}_0) - \bar{\boldsymbol{x}} = 0, \\ (-1)^{i+j-1}T_{n-i}(1)|_{\bar{\boldsymbol{x}}} > 0,\ i = 0, \ldots, n, \\ (-1)^{i+j-1}T_{n-i}(1)|_{\boldsymbol{x}_0} > 0,\ i = 0, \ldots, n, \\ T_{n-i}(0)|_{\boldsymbol{x}_0} \neq 0,\ T_{n-i}(0)|_{\bar{\boldsymbol{x}}} \neq 0,\ i = 0, \ldots, n, \\ \boldsymbol{x}_0 \neq \bar{\boldsymbol{x}},\ |\boldsymbol{f}'(\boldsymbol{x}_k)| \neq 0,\ k = 1, \ldots, m, \end{cases} \tag{4}$$

where $j = 1, 2$, $\boldsymbol{\mu}$ and $\bar{\boldsymbol{x}}$ are respectively the parameters and fixed point of system (1) and m ($m \geq 2$) is a given positive integer number.

In Theorem 3, what we want to find are the conditions on the parameters $\boldsymbol{\mu}$ for each of the semi-algebraic system (4) to have at least one real solution. In [1], we propose a systematic approach for solving semi-algebraic systems and analyzing the chaotic behavior by using methods of symbolic computation. This approach is based on the one for solving semi-algebraic systems proposed by Wang and Xia [3] and then developed by Niu [8]. There are several packages or software for realization of certain steps in our approach. For example, the method of discriminant varieties of Lazard and Rouillier [4] (implemented as a Maple package DV by Moroz and Rouillier), and the Maple package DISCOVERER, developed by Xia, implements the methods of Yang and Xia [7] for real solution classification.

4 Experiments

We analyze the chaotic behavior for a concrete practical system by using symbolic computations in order to illustrate its feasibility. More experiments and remarks can be found in [1].

We consider an extension of Mira 2 map which takes the form

$$x_{n+1} = Ax_n + y_n, \quad y_{n+1} = x_n^2 + By_n, \tag{5}$$

where A and B are nonzero numbers. Solving the corresponding semi-algebraic systems Ψ_1 and Ψ_2 of (5) by taking $m = 2$ based on the methods of symbolic computation, we can obtain that the semi-algebraic system Ψ_1 or Ψ_2 has at least one real solution if and only if one of the following conditions holds:

$$C_1 = [0 < R_1, 0 < R_2], \quad C_2 = [R_3 < 0, 0 < R_4].$$

The explicit conditions of C_1 and C_2 can be found in [1]. Then map (5) is chaotic in the sense of Marotto if one of the conditions C_1 or C_2 holds.

5 Conclusion

We have presented an approach to analyze the occurrence of snapback repeller of systems (1) automatically by using symbolic computation. Theoretically, we give a necessary and sufficient algebraic condition for all the zeros of a given polynomial to be OUC. Then with the aid of this condition, we reduce the analysis of Marotto's theorem to the solution of some semi-algebraic systems and introduce a systematical approach for analyzing the conditions on the parameters under which a snapback repeller exists. Moreover, in our approach, the procedure for the iteration can be done by computer automatically (m times). Besides, our approach can be used to determine the possibility that all the fixed points are snapback repellers due to the inexplicit computation of the fixed points of (1).

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