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Achieving Social Optimum in Dynamic Weight Adaptation for Virus Mitigation: A Potential Differential Game Approach

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Abstract: In this paper, a differential game framework is proposed to provide a theoretic underpinning for decentralized mitigation of virus spreading in which each node determines its own control control policy based on local information. To reduce the inefficiency of the Nash equilibrium and allow the decentralized policy to achieve social welfare, we propose a mechanism through a penalty scheme for a class of potential differential games over networks. The differential game under the penalty scheme turns out to be a potential differential game. To investigate the long term behaviors of the weight adaptation scheme, we study their turnpike properties. Numerical experiments are used to corroborate the results and demonstrate how the weight adapts to mitigate virus spreading and turnpike properties of the potential differential game.

Keywords: Epidemics, Differential Game, Mechanism Design, Turnpike Property, Potential Game.

1. INTRODUCTION

With an increasing number of wide-spreading cyber-attacks on networks, protection against malware and virus spreading in cyber networks is central to the security of network systems. However, designing a protection scheme for cyber networks is challenging due to the fact that cyber networks are often formed by a large number of self-interested agents or decision-makers. The noncooperation among the agents makes it almost impossible for the system to be coordinated as a whole to defend against wide-spreading cyber-attacks. Also, it is a challenge to figure out how the effect of the individual selfish behaviors and their interdependence in the network applies to the network system.

The most fundamental reason that virus and malware can go viral is the inherent property of networks: connectivity. Weight adaptation is a mechanism that addresses this issue as changing the network weights means changing the connectivity among agents. Reduction of weights at appropriate times can mitigate the virus and malware spreading while maintaining essential connectivity of the network. Compared with quarantining and link removal Khouzani et al. (2012), weight adaptation does not need to completely disconnect nodes from others but rather adjust weights to connect more loosely with nodes with a higher likelihood of infection. Instead of fixing the weights for the whole spreading process, in the weight adaptation scheme, each agent dynamically updates their weight in response to the state of the neighboring nodes. The idea of weight adaptation is from Feng et al. (2016) where the weight between agents is adapted based on the information regarding the global infection level. In this work, the agent adapts the weight of all its links no matter the neighbor each link pointing to is infected or not which is not an efficient weight adaptation. And the work did not consider the selfishness of the agents among the network.

We consider a directed weighted network. The original weight is pre-designed by multilateral agreement among agents to achieve certain goals or to optimize the system performance when there is no infection. When there are wide-spreading virus or cyber attacks, each agent can choose to deviate from the original weight to avoid being infected. Infected agents may not function normally. The agents and the network system will suffer losses. Thus, it is essential to consider the trade-off between malfunction cost caused by infection and inefficiency or performance degradation cost caused by weight deviation. The dynamical feature of virus spreading and the trade-off each agent needs to balance makes differential game a perfect framework.

In this paper, an N-person nonzero-sum differential game-based model is proposed to model the virus spreading and their selfish weight adaptation behavior. This model captures the non-cooperative behaviors among agents, dynamic properties of spreading process, and the complexity of the local interactions. We characterize the Nash equilibrium (NE) for the game and use a centralized optimal control problem to serve as a benchmark problem to characterize the inefficiency of NE. To address the inefficiency, we propose a dynamic penalty approach by designing a mechanism in which each agent pays a additional penalty for other agents' infection. The differential game under the penalty approach turns out to be a potential differential game. To investigate its long term behavior, we resort to turnpike properties of potential differential game. We show both in theory and simulation that the NE trajectory of

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the potential differential game remain in most of the time close to the solution of a static game problem which also turns out to be a potential game.

The paper is organized as follows. In Section 2, preliminaries are presented and the N-person nonzero-sum differential game framework is formulated. In Section 3, we characterize the open-loop NE of the differential game and the weight adaptation scheme. Sect. 4 discusses attaining the social optimum in a decentralized setting and studies the long term behavior of the potential differential differential game.

2. PRELIMINARIES AND PROBLEM FORMULATIONS

2.1 Graph Theory

A weighted, directed graph can be defined by a triple $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E}, \mathcal{W})$. $\mathcal{V} \triangleq \{v_1, v_2, ..., v_N\}$ represents a set of N nodes. Define $\mathcal{N} \triangleq \{1, ..., N\}$. A set of directed edges is denoted by $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The set of in-neighbors of node i is defined as $\mathcal{N}_i^{in} \triangleq \{j | j \in \mathcal{V}, (j,i) \in \mathcal{E}\}$. Denote by $|\cdot|$ the cardinality of a set. The weight adjacency matrix $\mathcal{G}$ is denoted by an $N \times N$ matrix $\mathcal{W} \triangleq [w_{ij}]$ where w_{ij} refers to the weight of the edge from node i to j. We assume that graph $\mathcal{G}$ has no self-loops. We denote the original weight adjacency matrix by $\mathcal{W}^o = [w_{ij}^o] \in \mathbb{R}^{N\times N}$. Let $\mathcal{N}_{i,o}^{out}(\mathcal{N}_{i,o}^{in})$ be the set of out-neighbors (in-neighbors) under the original optimal weight pattern $\mathcal{W}^o$. In the weight adaptation setting, the weight is time-varying. The time-varying weight is described by $\mathcal{U}(t) \triangleq [u_{ij}(t)]$ where $u_{ij}(t)$ is the weight of edge (i, j) at time t.

2.2 Virus Spreading Model

In this paper, we study the so-called susceptible-infected-susceptible (SIS) model. Consider a population of N agents. Each agent can be either susceptible (S) or infected (I). Infected individuals infect others at rate $\beta_i \geq 0$. The intensity of interaction between v_i and v_j is described by the weight $u_{ij} \in \mathbb{R}$. The virus spreading process can be precisely modelled by an exact 2^N state Markov Chain. See Van Mieghem et al. (2009), Paré et al. (2018). But solving only the 2^N state Markov Chain is computationally challenging due to the exponential increasing state space especially for large-scale networks. Hence, we resort to the mean-field approximation of the Markov process, i.e., the N-intertwined model proposed by Van Mieghem et al. (2009), extended by Paré et al. (2018). Denote $x_i(t) \in [0,1]$ as the probability of agent i being infected at time t. The mean-field approximation then provides

$$\dot{x}_i(t) = (1 - x_i(t)) \sum_{j=1}^{N} u_{ij}(t)\beta_j x_j(t) - \sigma_i x_i(t), \tag{1}$$

for $i = 1, 2, ..., N$.

Remark 1. According to the discussions in Paré et al. (2018), the N-intertwined model (1) gives an upper-bound for the exact probability of infection, $x_i(t)$. However, the mean-field approximation consider herein, while it is an approximation, is legitimate as we focus on the cases where β/σ is above the outbreak threshold. Under these cases, the approximation is well behaved according to Van Mieghem et al. (2009).

Remark 2. Given an intial point, differential equation in (1) admits a unique solution since the right side of (1) satisfies the Lipschitz condition Khalil and Grizzle (2002). See Lemma 1 in Huang and Zhu (2019) for a proof of Lipschitz conditions.

2.3 Differential Game Over Networks

The self-interested agents aim to minimize their own cost. One cost arises from malfunction caused by infection, measured by $f_i : [0,1] \to \mathbb{R}^+$, a function of $x_i(t) \in [0,1]$. f_i is assumed to be monotonically increasing to capture the loss of being infected. Another cost for agent i is to describe inefficiency or degradation of system performance caused by deviation from the original weights w_{ij}^o for all $j \in \mathcal{N}$. Thus, a weight cost function for edge from i to j is given by $g_{ij}(u_{ij}(t) - w_{ij}^o)$ where $g_{ij} : \mathbb{R} \to \mathbb{R}^+$ is assumed to be convex. The function satisfies the property that $g_{ij}(w) = 0$ if and only if $w = 0$ for all $i, j \in \mathcal{N}$. Given a time duration from 0 to $T \geq 0$, the cost function of agent i during time interval $[0,T]$ is given by

$$J_i = \int_0^T f_i(x_i(t)) + \sum_{j=1}^{N} g_{ij}(u_{ij}(t) - w_{ij}^o)dt. \tag{2}$$

The graph evolution and the epidemic spreading process described by (1) can be viewed as physical constraints. The agents in the network are coupled by these constraints while trying to minimize their own cost. Such behaviors lead to a differential game over network defined as follows,

$$\begin{aligned} \min_{\mathbf{u}_i \in U_i} J_i &= \int_0^T f_i(x_i(t)) + \sum_{j\in\mathcal{N}_{i,o}^{out}} g_{ij}(u_{ij}(t) - w_{ij}^o)dt \\ \text{s.t. } \dot{x}_i(t) &= (1 - x_i(t)) \sum_{j\in\mathcal{N}_{i,o}^{out}} u_{ij}(t)\beta_j x_j(t) - \sigma_i x_i(t), \\ x_i(0) &= x_{i0},\ i = 1, 2, ..., N, \end{aligned} \tag{3}$$

where $\mathbf{u}_i = (u_{ij_1}, ..., u_{ij_{|\mathcal{N}_{i,o}^{out}|}})$ with $j_k \in \mathcal{N}_{i,o}^{out}$ for $k = 1, ..., |\mathcal{N}_{i,o}^{out}|$, and u_{ij} is a trajectory describing the weight between agent i and j over the time interval $[0,T]$. Here, U_i is the admissible control set of agent i defined as $U_i = \{0 \leq u_{ij}(t) \leq w_{ij}^o, j \in \mathcal{N}_{i,o}^{out}, t \in [0,T]\}$. Each player aims to find a control policy $\mu_i(x_{i0})$ to generate a weight trajectory $\mathbf{u}_i$. Such control policies are open-loop policies that only depend on the initial conditions.

Remark 3. Here, we only consider the adaptation of weights that are originally connected. This is because there is no incentive for agents to connect an agent who is not originally connected. Also, the adaptation of the weights are bounded by the original ones since there is no benefit for the agents to increase their weights more than the original weights. See Observation 1 in Huang and Zhu (2019) for more discussions on the choice of admissible control set.

3. NASH EQUILIBRIUM AND ITS INEFFICIENCY

The solution to the N-person nonzero-sum differential game (3) played with an open-loop information structure are open-loop NE. Define $\mathbf{u}_{-i} = \{\mathbf{u}_j, j \in \mathcal{N}\backslash\{i\}\}$.

Definition 1. The weight adaptation trajectories or the control trajectories $\{\mathbf{u}_i^*,\ i \in \mathcal{N}\}$ constitute an open-loop NE solution of the differential game (3) if the inequalities

$$J_1(\mathbf{u}_i^*, \mathbf{u}_{-i}^*) \leq J_1(\mathbf{u}_i, \mathbf{u}_{-i}^*) \quad (4)$$

hold for all control trajectories $\mathbf{u}_i(t) \in S_i, t \in [0,T]$ and for all $i \in \mathcal{N}$. Here, denote $x_i^*(t), t \in [0,T]$ the associated state trajectory for $i \in \mathcal{N}$.

To obtain the necessary conditions for the open-loop NE, we impose two mild assumptions.

Assumption 1. For each $i \in \mathcal{N}$, the infection cost function $f_i(\cdot)$ is to be of C^1 class.

Assumption 2. For each $i, j \in \mathcal{N}$, the weight deviation cost function $g_{ij}(\cdot)$ is to be of C^1 class.

Following the techniques in optimal control theory Basar and Olsder (1999), we arrive at the following result.

Theorem 1. Consider the N-person differential game (3) under assumptions 1 and 2. Then, if $\{\mathbf{u}_i^*(t), i \in \mathcal{N}\}$ is an open-loop NE solution, and $\{\mathbf{x}^*(t), 0 \leq t \leq T\}$ is the corresponding state trajectory, there exist N costate functions $\mathbf{p}_i(\cdot) : [0,T] \to \mathbb{R}^N, i \in \mathcal{N}$, whose j-th component is denoted by $p_{ij}(\cdot)$, such that the following relations are satisfied:

$$\dot{x}_i^*(t) = (1 - x_i^*) \sum_{j \in \mathcal{N}_{i,o}^{out}} u_{ij}^*(t)\beta_j x_j^* - \sigma_i x_i^*(t), \quad x_i^*(0) = x_{i0}, \quad \forall i \in \mathcal{N}, \quad (5)$$

$$\mathbf{u}_i^*(t) = \arg\min_{\mathbf{u}_i \in U_i} H_i(t, \mathbf{p}_i(t), \mathbf{x}^*, \mathbf{u}_1^*(t), ..., \mathbf{u}_{i-1}^*, \mathbf{u}_i, \mathbf{u}_{i+1}^*(t), ..., \mathbf{u}_N^*(t)), \quad (6)$$

$$\dot{\mathbf{p}}_i(t) = \Gamma(t, \mathbf{x}^*, \mathbf{u}_1^*, ..., \mathbf{u}_N^*)\mathbf{p}_i(t) + \gamma_i(t), \quad \mathbf{p}_i(T) = 0, \quad (7)$$

where

$$H_i(t, \mathbf{p}_i, \mathbf{x}, \mathbf{u}_1, ..., \mathbf{u}_N) \triangleq f_i(x_i(t)) + \sum_{j \in \mathcal{N}_{i,o}^{out}} g_{ij}(u_{ij} - w_{ij}^o) + \sum_{j=1}^{N} p_{ij} \left\{ (1 - x_j) \sum_{k \in \mathcal{N}_{j,o}^{out}} u_{jk}\beta_k x_k(t) + \sigma_j x_j(t) \right\}, \quad (8)$$

and Γ is a matrix given by

$$\Gamma_{mn} = \begin{cases} \sum_{j \in \mathcal{N}_{m,o}^{out}} u_{mj}^*(t)\beta_j x_j^*(t) + \sigma_m & \text{if } n = m \\ -(1 - x_n^*(t))u_{nm}^*(t)\beta_m & \text{if } n \in \mathcal{N}_{m,o}^{in} \\ 0 & \text{otherwise,} \end{cases} \quad (9)$$

γ_i is a vector whose i-th component is $-df_i/dx_i$ and other components are zero, for $i \in \mathcal{N}$ and Γ_{mn} denote the element in the mth row and the nth column of Γ.

For the proof of Theorem 1, one can refer to Huang and Zhu (2019). To find the structure of the NE control trajectory, we solve the minimization problem given in (6), which gives the following.

Corollary 1. Define $\phi_{ij}(t) := p_{ii}(t)(1 - x_i^*(t))\beta_j x_j^*(t)$ where $p_{ii}(\cdot)$ is the ith component of the costate function $\mathbf{p}_i(\cdot)$. The basic structure of the NE-based optimal weight control, i.e., the solution to (6), can be written as:

$$u_{ij}^*(t) = \begin{cases} 0, & -\phi_{ij}(t) \leq g_{ij}'(-w_{ij}^o), \\ (g_{ij}')^{-1}(-\phi_{ij}(t)), & g_{ij}'(-w_{ij}^o) < -\phi_{ij}(t) < g_{ij}'(0), \\ w_{ij}^o, & -\phi_{ij}(t) \geq g_{ij}'(0), \end{cases} \quad (10)$$

for $i \in \mathcal{N}, j \in \mathcal{N}_{i,o}^{out}$.

Proof. Since H_i is convex over u_i, one can easily solve optimization problem Eq. (6) using first order necessary condition which gives Eq. (10).

From (10), one can notice that agents tend to decrease its connectivity to neighbors with high probability of infection. The information about the infection of the whole network is contained in $p_{ii}(t)$ whose dynamics is highly coupled with the states and the controls of other agents. The larger $p_{ii}(t)$ is, the lower the optimal weight $u_{ij}(t)$ should be. When the probability of agent i being infected $x_i(t)$ is high, it does not care much about the risk of connecting to an infected out-neighbor.

It is well-known that the non-cooperative NE in nonzero-sum games is generally inefficient Dubey (1986). There is need to develop a mechanism to attain a higher social welfare or lower aggregate costs through cooperation behavior Başar and Zhu (2011). In the network, the social cost is the aggregate costs of all players. The social problem thus can be stated as an optimal control problem:

$$\min_{\mathbf{u} \in U_o} J_o = \int_0^T \sum_{i=1}^{N} f_i(x_i(t)) + \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_{i,o}^{out}} g_{ij}(u_{ij}(t) - w_{ij}^o)dt$$
$$s.t.\ \dot{x}_i(t) = (1 - x_i(t)) \sum_{j \in \mathcal{N}_{i,o}^{out}} u_{ij}(t)\beta_j x_j(t) - \sigma_i x_i(t),$$
$$x_i(0) = x_{i0}, i = 1, 2, ..., N. \quad (11)$$

Here, $\mathbf{u} = \{\mathbf{u}_1, ..., \mathbf{u}_N\}$ where $\mathbf{u}_i(t) \in \mathbb{R}^{|\mathcal{N}_{i,o}^{out}|}$ is the weight control variable for the whole network with admissible set $U_o = \{\mathbf{u} : u_{ij}(t) \in [0, \omega_{ij}^o], \forall i \in \mathcal{N}, j \in \mathcal{N}_{i,o}^{out}, t \in [0,T]\}$.

The social optimum can be attained by solving the optimal control problem. The optimal solution $\mathbf{u}^o(t)$ of problem (11) and corresponding trajectory $\mathbf{x}^o(t)$ must satisfy the following so-called canonical equations Basar and Olsder (1999):

$$\dot{x}_i^o(t) = (1 - x_i^o) \sum_{j \in \mathcal{N}_{i,o}^{out}} u_{ij}^o(t)\beta_j x_j^o - \sigma_i x_i^o(t), x_i^o(0) = x_{i0}, \quad (12)$$

$$\dot{\lambda}(t) = \Gamma(t, \mathbf{x}^o, \mathbf{u}_1^o, ..., \mathbf{u}_N^o)\lambda(t) + \gamma, \lambda(T) = 0, \quad (13)$$

$$\mathbf{u}^o(t) = arg\min_{\mathbf{u} \in U_o} H(t, \mathbf{x}^o(t), \lambda(t), \mathbf{u}(t)), \quad (14)$$

for all $i \in \mathcal{N}$, where $\Gamma(t)$ is the same with the one given in (9) and $\gamma(t) = [-f_1'(x_1(t)), -f_2'(x_2(t)), ..., -f_N'(x_N)]'$, the Hamiltonian of the optimal control problem is defined as

$$H(t, \mathbf{x}(t), \lambda(t), \mathbf{u}(t)) = \sum_{i=1}^{N} f_i(x_i(t)) + \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_{i,o}^{out}} g_{ij}(u_{ij}(t) - w_{ij}^o) + \sum_{i=1}^{N} \lambda_i(t) \left\{ (1 - x_i(t)) \sum_{j \in \mathcal{N}_{i,o}^{out}} u_{ij}(t)\beta_j x_j(t) - \sigma_i x_i(t) \right\}, \quad (15)$$

and $\lambda(\cdot) : [0,T] \to \mathbb{R}^N$ is the costate function, λ_i is its ith component. The inefficiency of NE solution then can be described by the quotient $(\sum_i^N J_i(\mathbf{u}_i^*))/J_o(\mathbf{u}_1^o, ..., \mathbf{u}_N^o)$ which also can be written as $J_o(\mathbf{u}_1^*, ..., \mathbf{u}_N^*)/J_o(\mathbf{u}_1^o, ..., \mathbf{u}_N^o)$.

4. SOCIAL OPTIMUM AND TURNPIKE PROPERTY

The social optimum can in principle be computed centrally by network operator. However, this will require the network operator to be omniscient and also not all the agents have incentives to adapt their connection weights based on the rule designed to minimize the aggregate costs. Also, for large-scale network/system, centralized solution gives rise to computational problems and implementability issues. Therefore, centralized optimal control mechanism is impractical and there is need for decentralized mechanism designs that can achieve the social optimum. To this end, we develop a mechanism that assigns a penalty $c_i : [0,T] \to \mathbb{R}_+$ to agent i. Hence the differential game under the penalties can be written as

$$\min_{\mathbf{u}_i \in U_i} \hat{J}_i = \int_0^T f_i(x_i(t)) + \sum_{j \in \mathcal{N}_{i,o}^{out}} g_{ij}(u_{ij}(t) - w_{ij}^o) + c_i(t), dt \quad (16)$$

subject to the same dynamics equation stated in (3). Define $\hat{l}_i(t, \mathbf{x}, \mathbf{u}) \triangleq f_i(x_i(t)) + \sum_{j \in \mathcal{N}_{i,o}^{out}} g_{ij}(u_{ij}(t) - w_{ij}^o) + c_i(t)$ such that $\hat{J}_i = \int_0^T \hat{l}_i(t)dt$.

Definition 2. The differential game defined in (16) is an potential differential game if there exists a function $\pi : \mathcal{X} \times U_1 \times \cdots \times U_N \times [0,T] \to \mathbb{R}$ satisfying the following condition: for every $i \in \mathcal{N}$,

$$\begin{aligned}\int_0^T & \hat{l}_i(t, x_i, \mathbf{x}_{-i}, \mathbf{u}_i, \mathbf{u}_{-i}) - \hat{l}_i(t, \hat{x}_i, \hat{\mathbf{x}}_{-i}, \mathbf{v}_i, \mathbf{u}_{-i})dt \\ &= \int_0^T \pi(t, x_i, \mathbf{x}_{-i}, \mathbf{u}_i, \mathbf{u}_{-i}) - \pi(t, \hat{x}_i, \hat{\mathbf{x}}_{-i}, \mathbf{v}_i, \mathbf{u}_{-i})dt,\end{aligned} \quad (17)$$

for all $\mathbf{u}_i, \mathbf{v}_i \in U_i$, where $\mathbf{x}, \hat{\mathbf{x}} \in \mathcal{X}$ are the corresponding states under controls $\{\mathbf{u}_i, \mathbf{u}_{-i}\}$ and $\{\mathbf{v}_i, \mathbf{u}_{-i}\}$, respectively.

For a more general definition of potential differential game, one can refer to Fonseca-Morales and Hernández-Lerma (2018). Note that the differential game (16) and the optimal control problem (11) share the same dynamical system constraints. If we can find $c_i(t)$ for every $i \in \mathcal{N}$ such that relation (17) holds for $\pi = \sum_{i=1}^N f_i(x_i(t)) - \sum_{i=1}^N \sum_{j \in \mathcal{N}_{i,o}^{out}} g_{ij}(u_{ij}(t) - w_{ij}^o)$, then the differential game (16) is a potential game corresponding to the optimal control problem (11).

Theorem 2. Let $c_i(t) = \sum_{j \neq i} f_j(x_j(t))$. the differential game (16) is a potential differential game corresponding to the optimal control problem (11). Moreover, if $\{\mathbf{u}_i^*(t), i \in \mathcal{N}\}$ is an open-loop NE solution for the differential game (16), and $\{\mathbf{x}^*(t), 0 \le t \le T\}$ is the corresponding state trajectory, the relations (12) (13) and (14) also hold for $\mathbf{u}^*$ and $\mathbf{x}^*$ with $\mathbf{u}^o$ replaced by $\{\mathbf{u}_i^*(t), i \in \mathcal{N}\}$ and $\mathbf{x}^o$ replaced by $\mathbf{x}^*$.

Proof. As $\mathbf{u}^o$ and $\mathbf{x}^o$ are optimal for the optimal control problem (11), then

$$\begin{aligned}&\int_0^T \sum_{i=1}^N f_i(x_i(t)) + \sum_{j \neq i} \sum_{k \in \mathcal{N}_{j,o}^{out}} g_{jk}(u_{jk}^o(t) - w_{jk}^o) \\ &\qquad + \sum_{k \in \mathcal{N}_{i,o}^{out}} g_{ik}(u_{ik}(t) - w_{ik}^o)dt \\ &\geq \int_0^T \sum_{i=1}^N f_i(x_i^o(t)) + \sum_{i=1}^N \sum_{j \in \mathcal{N}_{i,o}^{out}} g_{ij}(u_{ij}^o(t) - w_{ij}^o)dt.\end{aligned}$$

Adding to both sides of this this inequality the constant

$$-\int_0^T \sum_{j \neq i} \sum_{k \in \mathcal{N}_{j,o}^{out}} g_{jk}(u_{jk}^o(t) - w_{jk}^o)dt,$$

we obtain that $\hat{J}_i(\mathbf{u}_i, \mathbf{u}_{-i}^o) \ge \hat{J}_i(\mathbf{u}^o)$ for all $\mathbf{u}_i \in U_i$. According to the definition of open-loop NE for differential games in (1), we know $\mathbf{u}^o$ is also an open-loop NE for the differential game with penalties.

This theorem enables the network operator to attain social optimum under a decentralized weight adaptation control scheme. What we have developed so far in this section is for cases where the original weighted network is strongly connected. If the original network is not strongly connected, we need the following definition.

Definition 3. Given a graph, if there exists a directed path from vertex j to vertex i, we say i is reachable from j. Denote by $\mathcal{R}_i \subseteq \mathcal{N}$ the set of vertices that i can be reachable from.

Under Definition 3, we have assumed that graph $\mathcal{G}$ has no self-loops which implies $i \notin \mathcal{R}_i$. If the graph is strongly connected, for every $i \in \mathcal{N}$, $\mathcal{R}_i = \mathcal{N} \backslash \{i\}$. Denote $\mathcal{R}_{i,o}$ the counterpart of $\mathcal{R}_i$ under the original graph defined by $(\mathcal{V}, \mathcal{E}, \mathcal{W}^o)$.

Corollary 2. Consider the differential game defined by (16). Let $c_i(t) = \sum_{j \in \mathcal{R}_{i,o}} f_j(x_j)$. Then, if $\{\mathbf{u}_i^*(t), i \in \mathcal{N}\}$ is an open-loop NE solution for the new differential game, and $\{\mathbf{x}^*(t), 0 \le t \le T\}$ is the corresponding state trajectory, the relations (12), (13), and (14) hold for $\mathbf{u}^*$ and $\mathbf{x}^*$ with $\mathbf{u}^o$ replaced by $\{\mathbf{u}_i^*(t), i \in \mathcal{N}\}$ and $\mathbf{x}^o$ replaced by $\mathbf{x}^*$.

Proof. The proof of Corollary 2 simply follows from Theorem 2 and the structure of the necessary condition (7).

Corollary 2 indicates that the choice of penalties $c_i(t)$ really depends on the topology of the graph, more specifically, the reachable set of vertex v_i.

To see the long term behavior of the potential differential game, we resort to turnpike properties. Turnpike properties have been established long time ago in finite-dimensional optimal control problems arising in econometrics. See Cass (1966). Turnpike properties Trélat and Zuazua (2015) are basically telling us the fact that, under some mild assumptions, the optimal solutions of a given optimal control problem settled in large time consist approximately of three pieces, the first and the last of which being transient short-time arcs, and the middle piece being a long-time arc staying exponentially close to the optimal steady-state solution of an associated static optimization problem. Thus, one can approximate the solution of a long term differential game/optimal control problem by simply looking into a static optimization problem and design distributed computation based on the static problem.

The static optimization problem associated with optimal control problem (11) can be given as

$$\begin{aligned} \min_{\mathbf{x},\mathbf{u}} \ & \sum_{i=1}^{N} f_i(x_i) + \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_{i,o}^{out}} g_{ij}(u_{ij} - w_{ij}^o) \\ s.t. \ & (\mathbf{x},\mathbf{u}) \in (\mathbb{R}^N, S) \\ & (1-x_i) \sum_{j\in\mathcal{N}_{i,o}^{out}} u_{ij}\beta_j x_j = \sigma_i x_i, i \in \mathcal{N}, \end{aligned} \tag{18}$$

where $S = \{1 \le u_{ij} \le w_{ij}^o, i \in \mathcal{N}, j \in \mathcal{N}_{i,o}^{out}\}$. Let $c_i(t) = \sum_{j\neq i} f_j(x_j(t))$. Consider the potential differential game (16). The associated static game problem is

$$\begin{aligned} \min_{x_i,\mathbf{u}_i} \ & \sum_{i=1}^{N} f_i(x_i) + \sum_{j\in\mathcal{N}_{i,o}^{out}} g_{ij}(u_{ij} - w_{ij}^o) \\ s.t. \ & (x_i,\mathbf{u}_i) \in (\mathbb{R}, S_i) \\ & (1-x_i) \sum_{j\in\mathcal{N}_{i,o}^{out}} u_{ij}\beta_j x_j = \sigma_i x_i, \end{aligned} \tag{19}$$

for $i \in \mathcal{N}$, where $S_i = \{0 \le u_{ij} \le w_{ij}^o\}$.

Theorem 3. The static game (19) is a potential game associated with a potential function which is the cost function of problem (18).

For the definition of static potential game, one should refer to Monderer and Shapley (1996). The proof of Theorem 3 follows immediately from the definition. The NE solutions of problem (19) is the optimal solutions of problem (18). We assume that the minimization problem (18) has a solution $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$. Consider a quadratic cost function $g_{ij}(u_{ij} - w_{ij}^o) = \frac{1}{2}(u_{ij} - w_{ij}^o)^2$. The optimal solution to the static optimization problem (18) can be characterized by:

$$\bar{x}_i = 1 - \frac{\sigma_i}{\sum_{j\in\mathcal{N}_{i,o}^{out}} \bar{u}_{ij}\beta_j\bar{x}_j + \sigma_i} \tag{20}$$

$$f'(\bar{x}_i) = \bar{\lambda}_i\left\{\frac{\sigma_i}{1-\bar{x}_i}\right\} - \sum_{j\in\mathcal{N}_{i,o}^{in}} \bar{\lambda}_j\{(1-\bar{x}_j)\bar{u}_{ji}\beta_i\} \tag{21}$$

$$\bar{u}_{ij} = \max\{w_{ij}^o - \bar{\lambda}_i(1-\bar{x}_i)\beta_j\bar{x}_j, 0\}, \tag{22}$$

for some $\bar{\lambda} \in \mathbb{R}^N$, for every $i \in \mathcal{N}$ and $j \in \mathcal{N}_{i,o}^{out}$.

Remark 4. The system of optimality conditions is obtained by letting the Hamiltonian $H(\mathbf{x},\mathbf{u},\lambda)$ defined in (15) satisfy $\partial H/\partial\lambda = 0$, $\partial H/\partial\mathbf{x} = 0$ and $\bar{\mathbf{u}} = \arg\min_{\mathbf{u}\in S} H(\bar{\mathbf{x}},\mathbf{u},\bar{\lambda})$ and re-arranging the algebraic expression.

Remark 5. From (20), we know $\bar{x}_i \in [0,1)$ which implies that the first term in the right hand side of (21) is well defined. Note that $\bar{x}_i = 0$ only if $\bar{u}_{ij}x_j = 0$ for all $j \in \mathcal{N}_{i,o}^{out}$ which tells that the probability of agent i being infected is 0 only if it does not connect to an agents with positive probability of being infected.

Remark 6. Relation (22) shows that the agent does not cut down his weight $\bar{u}_{ij}$ if and only if the corresponding neighbor has 0 probability of being infected, i.e, $\bar{x}_j = 0$.

The turnpike properties of potential differential game states that, when T is large, the NE solutions of the potential differential game (16), i.e., $(\mathbf{x}^*(\cdot),\mathbf{u}^*(\cdot),\mathbf{p}_i(\cdot))$, remains most of the time close to static optimization solution $(\bar{\mathbf{x}},\bar{\mathbf{u}},\bar{\lambda})$. Note that $\mathbf{p}_i = \mathbf{p}_j$ for any $i,j \in \mathcal{N}$ for the potential differential game (16). Define

$$H_{\mathbf{xx}} := \frac{\partial H}{\partial^2\mathbf{x}}(\bar{\mathbf{x}},\bar{\mathbf{u}},\bar{\lambda}), \quad H_{\mathbf{x}\lambda} := \frac{\partial H}{\partial^2\mathbf{x}\partial\lambda}(\bar{\mathbf{x}},\bar{\mathbf{u}},\bar{\lambda}),$$

which are of size $N\times N$, with $H_{\mathbf{x}\lambda} = H_{\lambda\mathbf{x}}^\dagger$ (where the upper dag stands for the transpose). Similarly, the matricies

$$H_{\mathbf{xu}} := \frac{\partial H}{\partial\mathbf{x}\partial\mathbf{u}}(\bar{\mathbf{x}},\bar{\mathbf{u}},\bar{\lambda}), \quad H_{\lambda\mathbf{u}} := \frac{\partial H}{\partial\lambda\partial\mathbf{u}}(\bar{\mathbf{x}},\bar{\mathbf{u}},\bar{\lambda}),$$

are of size $N\times M$, where $M := \sum_{i=1}^N |\mathcal{N}_{i,o}^{out}|$, with $H_{\mathbf{xu}} = H_{\mathbf{ux}}^\dagger$ and $H_{\lambda\mathbf{u}} = H_{\mathbf{u}\lambda}^\dagger$. We define the matrices

$$A = H_{\lambda\mathbf{x}} + H_{\lambda\mathbf{u}}H_{\mathbf{ux}}, \ B = H_{\lambda\mathbf{u}}, \ W = -H_{\mathbf{xx}} - H_{\mathbf{xu}}H_{\mathbf{ux}}.$$

The following theorem gives a formal statement of turnpike properties of the potential differential game.

Theorem 4. Assume that the matrix W is symmetric positive definite, and that the pair (A,B) satisfies the Kalman condition, i.e., the controllability matrix formed by (A,B) is of full rank, and that there exists $i \in \mathcal{N}$ $\bar{x}_i \neq x_{i0}$. Then, there exist constant ϵ, $C_1 > 0$, $C_2 > 0$ and a time $T_0 > 0$ such that, if

$$\bar{D} = \|\bar{\mathbf{x}} - \mathbf{x}_0\| + \left\| \begin{pmatrix} -\bar{\lambda} \\ \bar{\lambda} \end{pmatrix} - \begin{pmatrix} -\mathbf{p}_i(0) \\ 0 \end{pmatrix} \right\| \le \epsilon,$$

then, for any $T > T_0$, the potential game (16) has at least one NE solution $(\mathbf{x}^*(\cdot),\mathbf{u}^*(\cdot),\mathbf{p}_i(\cdot))$ satisfying

$$\begin{aligned} &\|\mathbf{x}^*(t) - \bar{\mathbf{x}}\| + \|\mathbf{p}_i(t) - \bar{\lambda}\| + \|\mathbf{u}^*(t) - \bar{\mathbf{u}}\| \\ \le & C_1(e^{-C_2 t} + e^{-C_2(T-t)}). \end{aligned} \tag{23}$$

Having established that the differential game (16) is a potential one given proper c_i, the remaining steps of the proof of Theorem 4 follows similar steps as the one in Trélat and Zuazua (2015) and thus are omitted.

Remark 7. The assumption that there exists $i \in \mathcal{N}$, $\bar{x}_i \neq x_{i0}$ is mild. Note that $\bar{\mathbf{x}} = 0$ and $\bar{u}_{ij} = w_{ij}^o, i \in \mathcal{N}, j \in \mathcal{N}_{i,o}^{out}$ is a global optimal solution of problem (18) and a trivial one (disease free). But at this optimal solution, the assumptions stated in Theorem 4 do not hold.

Remark 8. Here, C_1 depends linearly on $\bar{D}$ and e^{-C_2T}. To be more specific, C_1 is smaller as T is larger or $\bar{D}$ is smaller.

5. EXPERIMENTS

We study the graph stated in Fig. 1 with $N = 4$ agents. Consider a homogeneous linear infection cost $f_i(x_i) = \alpha x_i$ for $i \in \mathcal{N}$ and quadratic weight adaptation cost $g_{ij}(u_{ij} - w_{ij}^o) = \frac{1}{2}(u_{ij} - w_{ij}^o)^2$. Based on the graph $\mathcal{G}$ given in Fig. 1, let $w_{ij}^o = 1$ if $(i,j) \in \mathcal{E}$. Otherwise, $w_{ij}^o = 0$. So, we have $\mathcal{N}_{1,o}^{out} = \{2\}$, $\mathcal{N}_{2,o}^{out} = \{1,4\}$, $\mathcal{N}_{3,o}^{out} = \{1,2\}$, $\mathcal{N}_{4,o}^{out} = \{3\}$. Let $\beta_i = \beta = 1.6$, $\sigma_i = \sigma = 0.8$ for $i \in \mathcal{N}$. Arrange the weight adaptation control as $\mathbf{u} = (u_{12}\ u_{21}\ u_{24}\ u_{31}\ u_{32}\ u_{42})^\dagger$ and $\mathbf{x} = (x_1\ x_2\ x_3\ x_4)^\dagger$. Thus, we have

$$H_{\mathbf{xx}} = -\beta \begin{bmatrix} 0 & \bar{\lambda}_1\bar{u}_{12} + \bar{\lambda}_2\bar{u}_{21} & \bar{\lambda}_3\bar{u}_{31} & 0 \\ \bar{\lambda}_2\bar{u}_{21} + \bar{\lambda}_1\bar{u}_{12} & 0 & 0 & \bar{\lambda}_2\bar{u}_{24} \\ \bar{\lambda}_3\bar{u}_{31} & \bar{\lambda}_3\bar{u}_{32} & 0 & \bar{\lambda}_4\bar{u}_{43} \\ 0 & \bar{\lambda}_2\bar{u}_{24} & \bar{\lambda}_4\bar{u}_{43} & 0 \end{bmatrix}.$$

Similarly, we can write down $H_{\lambda\mathbf{x}}, H_{\mathbf{xu}}$ and $H_{\lambda\mathbf{u}}$. We choose $\alpha = 0.5$. Given the topology of the graph and the parameters we specified above, using Sequential Quadratic Programming (SQP) to solve problem (18), we obtain an local

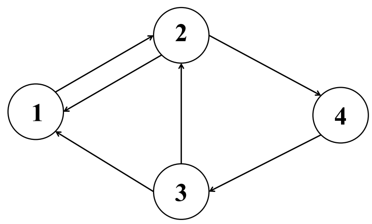

Fig. 1. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W}^o)$ with 4 agents which is strongly connected.

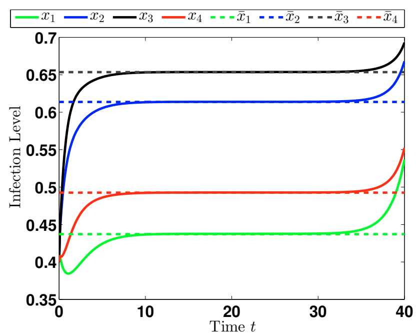

Fig. 2. The evolution of infection probability $x_i(t)$ corresponding to the potential differential game problem (16) over time interval $t \in [0, T]$ marked by solid lines. The solutions of static problem (18) marked by dash lines.

optimal solution $\bar{\mathbf{x}} = (0.4327, 0.6137, 0.6535, 0.4926)$ and $\bar{\mathbf{u}} = (0.6329, 0.8634, 0.8461, 0.9168, 0.8832, 0.7428)$ as well as the trivial optimal solution. Using the optimality system (21-22), we obtain $\bar{\lambda} = (0.6471, 0.5054, 0.3353, 0.4848)$. Under the optimal solution $(\bar{\mathbf{x}}, \bar{\mathbf{u}}, \bar{\lambda})$, $W(\bar{\mathbf{x}}, \bar{\mathbf{u}}, \bar{\lambda})$ is positive definite and the pair (A, B) satisfies the Kalman condition.

We solve the differential game (16) under the parameters we specified in this section with penalty function $c_i(t) = \sum_{j \neq i} \alpha x_j(t)$ using Distributed Resistance Algorithm in Huang and Zhu (2019), where we choose $T = 20$ and the initial point $x_0 = (0.4, 0.4, 0.4, 0.4)$. The turnpike property can be observed on Fig. 2-3. As expected, except transient initial and final arcs, the NE solution $\mathbf{u}^*$ of the corresponding potential differential game (16) and the corresponding state trajectory $(\mathbf{x}^*, \mathbf{u}^*)$ remain close to the steady-state $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$. During the transient time, as you can see, the agent adapts its weight with his neighbors to avoid being infected. For example, u_{12}, the weight between agent 1 and agent 2, has been cut down from 1 to less than 0.5 by agent 1. So, agent 1 can avoid being infected by agent 1 whose infection level is relatively high.

6. CONCLUSION

In this paper, we have established a differential game framework to develop decentralized virus-resistant mechanisms over complex. We have discussed the inefficiency of the open-loop Nash equilibrium and have proposed a penalty-based mechanism to achieve social optimum. We have studied the steady-state behavior of a long-term virus-resistance scheme where the duration of virus spreading is sufficiently long. We have discussed their turnpike properties under the decentralized socially optimal mechanism. One future direction would be to generalize the turnpike properties to a general class of potential differential games and study the application of turnpike properties in mechanism design problems for this class of games.

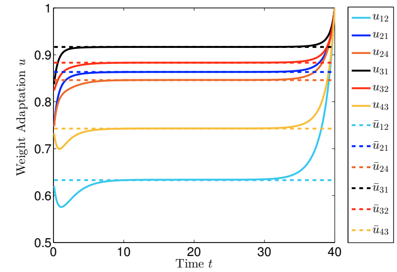

Fig. 3. The weight adaptation of $\mathbf{u}(t)$ corresponding to the potential differential game problem (16) over time interval $t \in [0, T]$ marked by solid lines. The solutions of static problem (18) marked by dash lines.

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