# Methods for Simplifying Differential Equations*

Shayea Aldossari
Department of Mathematics
Florida State University
swa15@my.fsu.edu

Mark van Hoeij
Department of Mathematics
Florida State University
hoeij@math.fsu.edu

## 1 Example

**A special case of simplification: Closed form solutions**

Given

$$L_{input} = \partial^2 - \frac{(32x^4 - 92x^3 + 105x^2 - 141x + 168)}{4(x-1)(4x^3 - 9x^2 + 9x - 12)}\partial + \frac{(16x^6 - 72x^5 + 141x^4 - 258x^3 + 552x^2 - 630x + 315)}{4(4x^3 - 9x^2 + 9x - 12)(x-1)^3},$$

in $\mathbb{C}(x)[\partial]$, where $\partial = \frac{d}{dx}$. A closed form solution for $L_{input}(Y(x)) = 0$ is

$$Y(x) = \exp(x)\left(xy(\frac{2}{x-1}) + (x-1)(y(\frac{2}{x-1}))'\right)$$

where

$$y(x) = {}_0F_1\left(-; \frac{3}{4}; x\right).$$

We can interpret this solution as a simplification from $L_{input}$ to the minimum operator for $y(x)$ which is

$$L_{\text{simpler}} = \partial^2 + \frac{3}{4x}\partial - \frac{1}{x},$$

combined with transformations: (where $f = \frac{2}{x-1}$, $G = x + (x-1)\partial$, and $r = 1$)

(i) $y(x) \mapsto y(f)$, it is called change of variables transformation and denoted by $L_1 \xrightarrow{(i)} L_2$.

(ii) $y(x) \mapsto G(y(x))$, it is called gauge transformation and denoted by $L_1 \xrightarrow{(ii)} L_2$.

(iii) $y(x) \mapsto \exp(\int r dx)y(x)$, it is called exponential product and denoted by $L_1 \xrightarrow{(iii)} L_2$.

These are the transformations known to preserve the order.

*This work was supported by the grants NSF 1618657, and NSF CCF-1708884.

## 2 Goal

Our goal is to simplify differential operators, in $\mathbb{C}(x)[\partial]$, as much as possible whether they have closed form solutions or not.

**Example 2.1** *To produce an example, $L_{input}$, lets start with*

$$L_{simple} = \partial^3 + \frac{1}{x}\partial^2 + (x-1)\partial - \frac{1}{x-2},$$

*and then apply transformations (i),(ii),(iii) with $f = \frac{x^3+1}{x}$, $G = x + x^2\partial^2$ and $r = x + 1/x$.*

Clearly this $L_{input}$ (too large to print here) comes from a much smaller operator under the order-preserving transformations (i),(ii),(iii). That is how it was constructed.

**Main Question:** What if we are given another $L_{\text{input}}$? How can we decide if it comes from a simpler operator under order preserving transformations (i),(ii),(iii), and if so, how to find such a simplification?

## 3 $(ii)$-simplifier

For regular singular differential operators, Imamoglu and Van Hoeij [7] implemented an algorithm which tries to reduce $L_{input}$ to a simple operator under transformation $(ii)$. This works well for regular singular operators of order 2.

To extend to irregular singular operator we generalized a key tool: Integral basis for irregular singular operators. This tool produces a finite dimension vector space $V$ of candidate gauge transformations. To extend [7] to order $> 2$, we are developing a new way to find the "best" element of $V$.

## 4 $(i) + (iii)$-simplifier

A $(i)+(iii)$ transformation $L_1 \xrightarrow{(i)+(iii)} L_2$ sends any solution $y(x)$ of $L_1$ to a solution $\exp(\int r) \cdot y(f)$ of $L_2$ for some rational functions $r, f$. Given $L_2$ but not $L_1$, we want to find a nontrivial simplification $L_1$ if it exists. Here, nontrivial means $\mathbb{C}(f) \subsetneq \mathbb{C}(x)$ (i.e. $f$ is not a Möbius transformation).

A differential formula in terms of coefficients of differential operators is called an invariant if it does not change under transformation $(i)$. Chalkley [3] gave special invariants whose quotients $I_1, I_2, ...,$ and $I_n$ satisfy

$$I_i(L_2) = I_i(L_1)_{\big|x=f}$$

Recall that $L_2$ is the input and $L_1$ is the simplification we aim to find. To find $L_1$, we need to find $f$ which we find by computing a common composition factor of $I_1(L_2), I_2(L_2), ....$ Such invariants exist only for differential operators of order $\geq 3$. This was used in [4] to compute hypergeometric solutions for order 3, but the same works more generally for any $(i) + (iii)$ simplification, and for any order $\geq 3$.

## 5 $(i) + (ii) + (iii)$-simplifier

The $(i) + (iii)$-simplifier only works if transformation $(ii)$ is not needed. This raises this question how to find (if it exists) an element $G \in V$ ($V$ as in section 3) that sends $L_{input}$ to an operator that has a non-trivial $(i) + (iii)$ simplification. (work in progress, implementation seems to work well).

We tested it on irregular singular differential operators of order 3 with closed form solutions (solvable by [8]). So far, our simplifier effectively solved all such examples.

**Example** Maple does not solve this operator

$$L = \partial^4 + 10\partial^2 - 2\partial + 9x^2,$$

but it can solve an operator which is gauge equivalent to $L$ using [9].

Our simplifier finds the smallest operators that are gauge equivalent to $L$. Among them is:

$$\partial^4 - \frac{1}{x}\partial^3 + 10x\partial^2 + 5\partial + \frac{9x^3 - 5}{x},$$

which is solvable by Maple because it is a symmetric product of

$$L_1 = \partial^2 + x,$$

and

$$L_2 = \partial^2 + 4x.$$

## References


[1] R. Chalkley. Relative invariants for homogeneous linear differential equations. *Journal of differential equations*, 80(1):107–153, 1989.

[2] R. Chalkley. Semi-invariants and relative invariants for homogeneous linear differential equations. *Journal of mathematical analysis and applications*, 176(1):49–75, 1993.

[3] R. Chalkley. *Basic global relative invariants for homogeneous linear differential equations.* Number 744. American Mathematical Soc, 2002.

[4] E.S Cheb-Terrab and A.D Roche. Hypergeometric solutions for third order linear odes, 2008. *arXiv preprint arXiv:0803.3474*, 5.

[5] S. Chen, M. van Hoeij, M. Kauers, and C. Koutschan. Reduction-based creative telescoping for fuchsian d-finite functions. *arXiv preprint arXiv:1611.07421*, 2016.

[6] E. Imamoglu. *Algorithms for solving linear differential equations with rational function coefficients.* Florida State University, 2017.

[7] E. Imamoglu and M. van Hoeij. Computing hypergeometric solutions of second order linear differential equations using quotients of formal solutions and integral bases. *Journal of Symbolic Computation*, 83:254–271, 2017.

[8] M. Mouafo Wouodjie. *On the solutions of holonomic third-order linear irreducible differential equations in terms of hypergeometric functions.* PhD thesis, 2018.

[9] M. van Hoeij. Decomposing a 4th order linear differential equation as a symmetric product. *Banach Center Publications*, 58:89–96, 2002.