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On the Dynamics of Unsteady Lift and Circulation and the Circulatory-Non-circulatory Classification

Haithem Taha* and Amir Rezaei†
University of California, Irvine, CA 92697

In this paper, we emphasize the dynamical-systems aspect of unsteady aerodynamics. That is, we consider the unsteady aerodynamics problem of a two-dimensional airfoil as a dynamical system whose input is the angle of attack (or airfoil motion) and output is the lift force. Based on this view, we discuss the lift evolution from a purely dynamical perspective through the step response, frequency response, transfer function, etc. In particular, we point to the relation between the high-frequency gain and the physics of circulatory lift and circulation. Based on this view, we show the circulatory lift dynamics is different from the circulation dynamics. That is, we show that the circulatory lift is not lift due to circulation. In fact, we show that the circulatory-non-circulatory classification is arbitrary. By comparing the steady and unsteady thin airfoil theory, we show that the circulatory lift possesses some added-mass contributions. Finally, we perform high-fidelity simulations of Navier Stokes equations to show that a non-circulatory maneuver in the absence of a free stream induces *viscous* circulation over the airfoil.

I. Introduction

Interests in unsteady aerodynamics have been continuously increasing over the last century with a recent research flurry because of the modern applications of rapidly maneuvering fighters, highly flexible airplanes, bio-inspired flight, etc. The classical theory of unsteady aerodynamics, whose foundation was laid down by Prandtl[1] and Birnbaum[2] in 1924, was mainly set for incompressible, slightly viscous flows around thin airfoils with sharp trailing edges. The key concept is that the flow non-uniformity leads to vorticity generation that emanates at the sharp trailing edge and freely shed behind the airfoil. In addition, the flow outside these sheets can be considered inviscid (e.g., circulation is conserved). These concepts are not sufficient to determine a unique solution for the wing and wake circulation distribution. Then, the Kutta condition (smooth flow off the sharp trailing edge) comes to play an essential role in the problem closure. That is, no flow around the sharp edge and hence, even within the framework of potential flow, the velocity has to be finite at the edge. Finally, in order to obtain an explicit analytical solution of the governing equation (the Laplace's equation in the velocity potential in this case), one more assumption is usually adopted. Assuming small disturbance to the mean flow (i.e., the vorticity sheet convects by the mean flow velocity: flat wake assumption) completes the framework of the classical theory. In summary, the classical theory of unsteady aerodynamics is based on replacing the airfoil and the wake by vorticity distributions (singularities) that satisfy the Laplace's equation everywhere in the flow field except at the surface of singularities. Three main conditions are applied: (1) no-penetration boundary condition (fluid velocity is parallel to the wing surface), (2) The Kutta condition (smooth flow off the sharp trailing edge), and (3) the conservation of total circulation. This formulation along with the flat wake assumption constitute the classical theory of unsteady aerodynamics.

The above formulation of the classical theory of unsteady aerodynamics was extensively used throughout the years. In 1925, Wagner[3] used this formulation to solve the indicial problem (lift response due to a step change in the angle of attack). In 1935, Theodorsen[4] used the same formulation to solve the frequency response problem (steady state lift response due to harmonic oscillation in the angle of attack). In 1938, Von Karman and Sears[5] provided a more general and elaborate representation of the classical formulation.

*Assistant Professor, Mechanical and Aerospace Engineering, AIAA Member.
†PhD Student, Mechanical and Aerospace Engineering, AIAA Student Member.

Also, the works of Kussner[6] for the sharp edged gust problem, Schwarz[7] for the frequency response problem, and Lowey[8] for the returning wake problem worth mentioning. It should be noted that while there may be different approaches within this framework, these efforts are essentially equivalent. For example, the lift frequency response functions derived by Von Karman and Sears[5] and Schwarz[7] are exactly the Theodorsen's function.[4] In addition, Garrick[9] showed that the Theodorsen function and the Wagner function form a Fourier transform pair.

It should be noted that the problem formulation of the classical theory of unsteady aerodynamics is infinite dimensional. The need for calculating the aerodynamic loads due to arbitrary time variations of the wing motion along with the need for structural and/or dynamic coupling (to analyze aeroelastic and/or flight dynamic stability problems[10, 11]) invokes more compact representations of the lift dynamics than Theodorsen or Wagner functions. Consequently, a number of finite-state approximations of these response functions were developed. Jones[12] and Jones[13] provided a two-state approximation of the Wagner function in the time-domain. Vepa[14] introduced the method of Pade approximants to determine a finite-state representation of the Theodorsen function in the frequency domain. Of particular interest to the aeroelasticity and flight dynamics community is the state space representation developed by Beddoes[15, 16] using the convolution integral with Jones approximation of the Wagner's step response function. In contrast to these finite-state models that are based on approximating the Theodorsen function in the frequency domain or the Wagner function in the time domain, Peters and his colleagues derived state space models from the basic governing principles using Glauert expansion[17] or the expansion of potential functions.[18, 19] That is, the internal states are of physical meaning in his formulation (they are the inflow distributions). Although the formulation of Peters is quite neat, it necessitates a relatively large number (eight) of inflow states to provide a good accuracy; two or three states were already shown to be sufficient for this simple linearized problem.

One of the well-known outcomes from Theodorsen's seminal effort[4] is the classification of lift into circulatory and non-circulatory components. The latter is due to the instantaneous acceleration of the airfoil. It represents the force required to accelerate the fluid surrounding the airfoil. Therefore, it is as if one has to accelerate not only the mass of the airfoil, but also the mass of the surrounding fluid; hence, the name *added mass* or *virtual mass*. This "lift" force, which is more of a resistance force, is characterized as non-circulatory because the associated flow field has no net circulation around the airfoil. Note that this non-circulatory lift force exists only in an unsteady flow as lift is solely due to circulation in steady flow according to the Kutta-Joukowsky lift theorem.[20] It should be noted that, according to Theodorsen's model,[4] the non-circulatory flow satisfies the no-penetration boundary condition but does not satisfy the Kutta condition; i.e., the velocity is not finite at the trailing edge. As such, Theodorsen added vortices in the wake that (i) do not disturb the no-penetration boundary condition (i.e., maintain the cylinder as a stream line), (ii) satisfy the conservation of circulation, and (iii) together with the non-circulatory flow satisfy the Kutta condition at the trailing edge. This last condition was used to write the governing integral equation for wake circulation. The other lift component (circulatory component) is the lift contribution from the second flow field that was added to satisfy the Kutta condition. The name and Theodorsen's formulation give the impression that the circulatory lift is due to circulation. One of the objectives of this paper is to emphasize that this is not true: the circulatory lift is not solely due to circulation in an unsteady flow. In fact, we show that such a classification is, in fact, arbitrary. Moreover, we show that the common definition of the circulatory lift, as posed by the pioneers Wagner[3] and Theodorsen,[4] is not lift due to circulation but it has some contribution from the airfoil's acceleration; i.e., it enjoys some non-circulatory aspects. Likewise, we also show that a non-circulatory motion may induce circulation over the airfoil. This latter observation is demonstrated via numerical simulation of the unsteady Reynolds-averaged Navier Stokes (URANS) equations on a dynamic mesh around NACA 0012 undergoing pitching motion in a quiescent fluid (no free stream). The spirit and illustrations will be imbued with the dynamical-systems aspect of unsteady aerodynamics.

II. Dynamical Systems Background

One of the main characteristics that distinguishes unsteady aerodynamics from steady aerodynamics is the dynamical system aspect of the former. For example, it is well-known that the steady lift coefficient of a flat plate at a small angle of attack is given by $C_L = 2\pi\alpha$. If the angle of attack is time-varying, this relation is written as $C_L(t) = 2\pi\alpha(t)$, which implies an instantaneous lift development. That is the angle of attack $\alpha(t)$ at the instant t dictates a lift force at the same instant, which is not physically possible. There must be a finite time (i.e., lag) for the lift to build up, no matter how small this time is. This type of instantaneous behavior is called *quasi-steady* which follows the same time-variation of the angle of attack, as shown in Fig. 1 for a 5° step change in the angle of attack. In reality, the lift will take some time to build up and reach the steady value corresponding to the given angle of attack, as shown in the schematic of Fig. 1.

From the above discussion, one can infer that the angle of attack (or airfoil motion) is the input to the aerodynamic system and the lift is one of its main outputs. For an unsteady aerodynamic model, the C_L-α relation cannot be algebraic (e.g., $C_L(t) = f(\alpha(t))$ for any function f). It also cannot be in the form $C_L(t) = f(\alpha(t), \dot{\alpha}(t), \ddot{\alpha}(t))$. Rather, it has to be a dynamical relation described by a differential equation for C_L (e.g., $\dot{C}_L = f(\alpha(t), \dot{\alpha}(t), \ddot{\alpha}(t))$), transfer function $\frac{C_L}{\alpha}$, state space model, frequency response, etc. In other words, the C_L-α relation is no longer a simple gain (2π), but rather a dynamical system. It should be noted that while this view is quite natural and intuitive, it was not explicitly described by the efforts of the early pioneers,[3–5] obviously because the dynamical-systems theory was not mature in their period as it currently is. Some of the efforts that adopted this dynamical-systems perspective includes Beddoes,[15,16] Leishman,[21] Peters et al.,[18] Hemati et al.,[22] Brunton et al.,[23] Zakaria et al.,[24,25] and Shehata et al.[26]

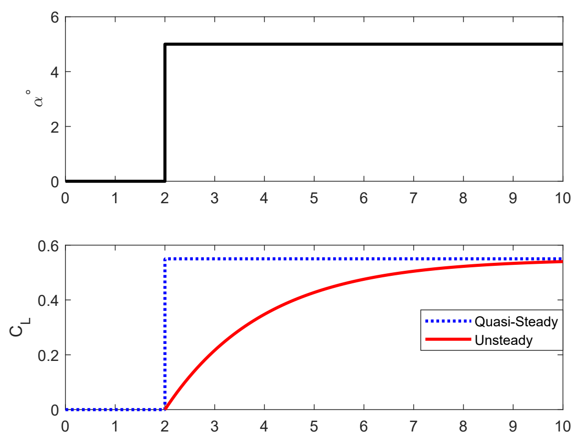


Figure 1: Illustration of the steady (quasi-steady) and unsteady behavior of lift due to a step change in the angle of attack α

Having the above description in mind, we believe that it would be better to provide a quick review of some of the standard dynamical-systems tools that will be used throughout the paper to describe the unsteady lift and circulation dynamics. There are two main mathematical representations of linear dynamical systems:[27] the classical representation via transfer function and the modern representation after Kalman[28] in a state space form. This latter representation is written as

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= [\boldsymbol{A}]\boldsymbol{x}(t) + [\boldsymbol{B}]\boldsymbol{u}, \\ \boldsymbol{y}(t) &= [\boldsymbol{C}]\boldsymbol{x}(t) + [\boldsymbol{D}]\boldsymbol{u}(t), \end{aligned}$$

where $\boldsymbol{x}$ is a vector of state variables (that define the system), $\boldsymbol{u}$ is the input vector, $\boldsymbol{y}$ is the output vector, and the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}, \boldsymbol{D}$ are constant matrices that define the system. On the other hand, the transfer function is defined as the Laplace transform of the output divided by the Laplace transform of the input at zero initial conditions[27]

$$G(s) = \left.\frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{u(t)\}}\right|_{y(0)=0, \dot{y}(0)=0, \ldots} = \frac{Y(s)}{U(s)},$$

where $\mathcal{L}$ is the Laplace transform and s is its variable. Also, there are two main response functions that define the characteristics of a linear dynamical system: the step response (transient response due to a step input) and the frequency response (the steady state response due to a harmonically oscillating input). It should be noted that any of the two mathematical representations or any of the two response functions completely define the dynamical characteristics of a linear system. Given any of the four, one can determine the other three. For example, given a state space model ($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$), the transfer function can be obtained as

$$G(s) = \boldsymbol{C}\left[s\boldsymbol{I} - \boldsymbol{A}\right]^{-1}\boldsymbol{B} + \boldsymbol{D},$$

where $\boldsymbol{I}$ is the identity matrix, from which the step response can be determined as $y(t) = \mathcal{L}^{-1}\{G(s).\frac{1}{s}\}$ and the frequency response can be determined by substituting $s = j\omega$.

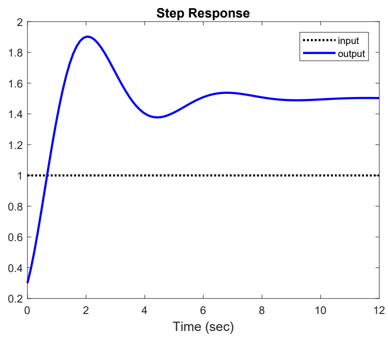
Table 1 shows an illustration for the step and frequency response functions. The following transfer function was used to generate the responses in Table 1

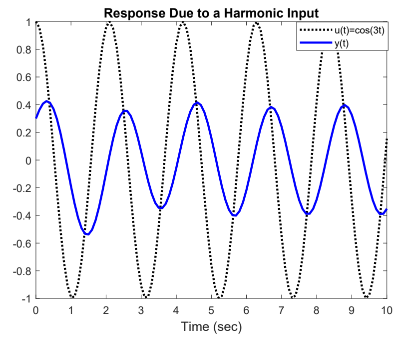
$$G(s) = \frac{0.3s^2 + s + 3}{s^2 + s + 2} \quad \Rightarrow \quad G(s = j\omega) = \frac{3 - 0.3\omega^2 + j\omega}{2 - \omega^2 + j\omega}$$

whose state space matrices can be written as

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \\ y &= \begin{bmatrix} 2.4 & 0.7 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 0.3u. \end{aligned}$$

Step Response	Frequency Response
$u(t) = 1$ for all $t \geq 0$	$u(t) = A\cos\omega t$
$y(t) = \mathcal{L}^{-1}\{G(s).\frac{1}{s}\}$	$y_{ss}(t) = A\lvert G(j\omega)\rvert \cos(\omega t + \angle G(j\omega))$





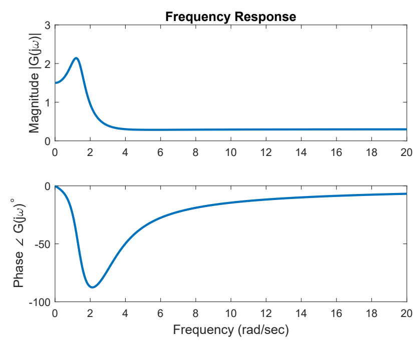


Table 1: The two main response functions describing a dynamical system.

The focus of this review is to recall the two concepts of the *dc gain* k_{dc} and the *high-frequency gain* k_{hf}. The dc gain simply represents the steady amplification. That is, it is the steady state value of the output due to a unit step input ($k_{dc} = \lim_{t\to\infty} y(t)$). From the step response in Table 1, one can deduce its value of 1.5. It can be determined from the transfer function as $k_{dc} = \lim_{s\to 0} G(s)$, which results in 1.5 when substituting $s = 0$ in the $G(s)$ above. It can also be determined from the frequency response in Table 1: the magnitude at zero frequency which gives the same value of 1.5.

On the other hand, the high-frequency gain represents the other end of the spectrum (at infinite frequency and zero time). That is, it is defined as $k_{hf} = \lim_{\omega\to\infty} |G(j\omega)|$: the infinite-frequency amplification, whose value is found to be 0.3 from the frequency response in Table 1. It can also be determined from the transfer function as $k_{dc} = \lim_{s\to\infty} G(s)$. As such, its value is zero for a *strictly proper* transfer function (degree of numerator is strictly lower than the degree of denominator). If the two polynomials have the same degree, then k_{hf} is simply equal to the ratio of the coefficients of the highest power. This ratio is also the value of $\boldsymbol{D}$ of the corresponding state space model. From the definition of k_{hf} (the gain at infinite-frequency) and $\boldsymbol{D}$, it is understood that a system with nonzero $k_{hf} = \boldsymbol{D}$ will have some instantaneous response. In particular, k_{hf} (or $\boldsymbol{D}$) represents the instantaneous initial response in the time domain due to a unit step input, as can be seen from the step response in Table 1. In this case of non-zero k_{hf}, the high-frequency phase is either 0 or 180°. In general, the high-frequency limit of the phase angle is $-90°$ multiples of the relative degree between the numerator and denominator provided that the system is minimum phase.

In conclusion, the numerator's degree of a transfer function cannot be higher than the denominator's degree for causality (i.e., the current output depends on the current and past input). If there is dependence on the current input (i.e., non-zero $\boldsymbol{D}$ or k_{hf}), then it implies that there is some immediate response from the system, which is rare to find in mechanical systems. The most common case is for a strictly proper transfer function where $k_{hf} = \boldsymbol{D} = 0$ for which the input $u(t)$ at an instant t cannot dictate the output y at the same instant or even part of it. Rather, it may dictate its rate of change at this instant, affecting its value in the future.

III. Unsteady-Lift Step and Frequency Responses

In the previous section, we reviewed the two main response functions describing a linear dynamical system: the step response and the frequency response. For the aerodynamic system whose input is the airfoil motion (e.g., angle of attack) and output is the unsteady lift, these two fundamental response functions have been determined in the first half of the past century, assuming potential flow. Wagner[3] solved the indicial problem (lift response due to a step change in the angle of attack) and Theodorsen[4] solved the frequency response problem (steady state lift response due to harmonic oscillation in the angle of attack). Therefore, if the aerodynamic transfer function is denoted by $G_a(s)$, then the Wagner function $\phi(t)$ will be given by $\phi(t) = \mathcal{L}^{-1}\{G_a(s).\frac{1}{s}\}$ and the Theodorsen function $C(\omega)$ will be given by $C(\omega) = G_a(s = j\omega)$. It should be noted that both Wagner and Theodorsen non-dimensionalized their solutions by the free-stream velocity U and the semi-chord length b. That is, the Wagner function is given in terms of the non-dimensional time $\tau = \frac{Ut}{b}$ (traveled distance in semi-chord lengths) and the Theodorsen function is given explicitly in terms of the non-dimensional frequency $k = \frac{\omega b}{U}$, called the *reduced frequency*, as

$$C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)} = \frac{K_1(jk)}{K_1(jk) + K_0(jk)}, \tag{1}$$

where $H_n^{(m)}$ is the Hankel function of m^{th} kind of order n and K_n is the modified Bessel function of second kind of order n. Consequently, taking the Laplace transform from the non-dimensional time τ-domain, one reaches the non-dimensional Laplace $\hat{s}$-domain and the corresponding frequency domain is in terms of the reduced frequency k. In other words, $\phi(\tau) = \mathcal{L}^{-1}\{G_a(\hat{s}).\frac{1}{\hat{s}}\}$ and $C(k) = G_a(\hat{s} = jk)$. Thus, from Eq. (1), one can write the aerodynamic transfer function in the Laplace domain as

$$G_a(\hat{s}) = \frac{K_1(\hat{s})}{K_1(\hat{s}) + K_0(\hat{s})}. \tag{2}$$

Unlike Theodorsen function, the Wagner function $\phi(\tau)$ is not written explicitly in τ, but rather is given by the integral[3,5]

$$\phi(\tau) = \int_0^\tau \mu(\sigma) \frac{1 + \tau - \sigma}{\sqrt{(1 + \tau - \sigma)^2 - 1}} d\sigma, \tag{3}$$

where the function μ is governed by the integral equation

$$\int_0^\tau \mu(\sigma) \sqrt{\frac{2 + \tau - \sigma}{\tau - \sigma}} d\sigma = 1. \tag{4}$$

Wagner[3] provided tables for the functions μ and ϕ in terms of τ.

As discussed above, either $\phi(\tau)$ or $C(k)$ completely characterize the inviscid lift dynamics. For example, the circulatory lift response due to an arbitrary time-varying angle of attack (input) can be determined from the step response $\phi(\tau)$ using the Duhamel superposition principle[9,15,16,21,29–32]

$$\ell_C(\tau) = 2\pi\rho U^2 b \left[\alpha(0)\phi(\tau) + \int_0^\tau \frac{d\alpha(\sigma)}{d\sigma}\phi(\tau - \sigma) d\sigma\right], \tag{5}$$

where ρ is the fluid density. On the other hand, the same response due to an arbitrary input can be determined from the frequency response $C(k)$ using the Fourier transform[9,29]

$$\ell_C(\tau) = \rho U^2 b \int_{-\infty}^{\infty} \hat{\alpha}(k) C(k) e^{jk\tau} dk, \tag{6}$$

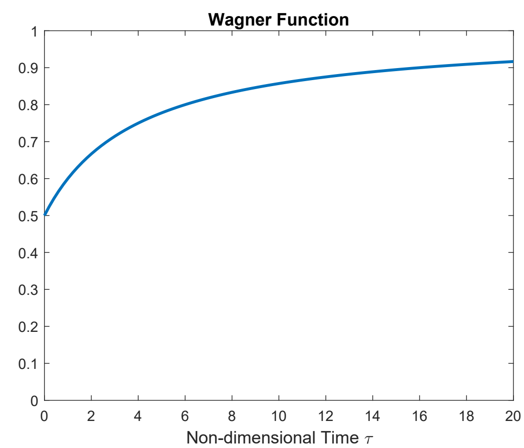
where $\hat{\alpha}(k) = \int_{-\infty}^{\infty} \alpha(\tau) e^{-jk\tau} d\tau$ is the Fourier transform of the angle of attack aerodynamic input. Using this analysis, and realizing that the Fourier transform of a unit step angle of attack is $\hat{\alpha}(k) = \frac{1}{jk}$, Garrick[9] showed that the Wagner function and the Theodorsen function form a Fourier Transform pair:

$$\phi(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{C(k)}{jk} e^{jk\tau} dk \text{ and } C(k) = jk \int_{-\infty}^{\infty} \phi(\tau) e^{-jk\tau} d\tau. \tag{7}$$

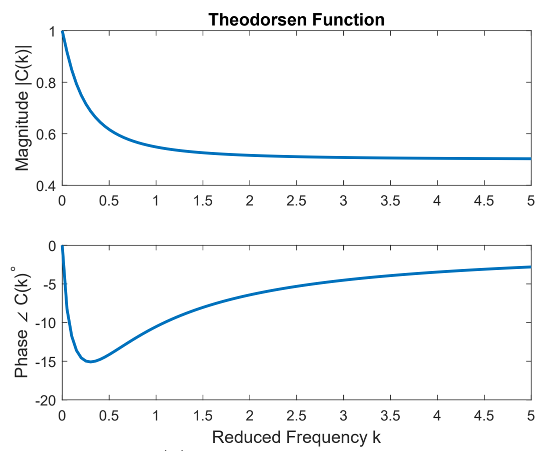
Figure 2 shows the inviscid circulatory lift step and frequency responses (i.e., Wagner and Theodorsen functions). While Theodorsen function is shown exactly according to Eq. (1), Fig. 2(a) shows Garrick's approximation $\phi(\tau) \simeq \frac{\tau+2}{\tau+4}$ of the Wagner function. Recalling the review provided in Sec. II, one can clearly see that the high-frequency gain associated with the inviscid circulatory lift is half: $\phi(\tau = 0) = \frac{1}{2} = \lim_{k\to\infty} |C(k)| = \lim_{\hat{s}\to\infty} G_a(\hat{s})$. Actually, any approximation of the Wagner function in the time domain will have to satisfy this property; e.g., the Garrick's approximation $\frac{\tau+2}{\tau+4}$[9] or the two-state approximations of Jones[12] and Jones[13]

$$\phi(\tau) \simeq 1 - A_1 e^{-b_1\tau} - A_2 e^{-b_2\tau}.$$

Therefore, $A_1 + A_2 = \frac{1}{2}$ in all of their approximations. On the other side, any finite-state approximation of the Theodorsen function in the frequency domain or its transfer function in the Laplace domain will also have to satisfy this property; e.g., the Pade approximations of Theodorsen function by Vepa[14] $\frac{\hat{s}+0.5}{2\hat{s}+0.5}$, $\frac{\hat{s}^2+1.5\hat{s}+0.375}{2\hat{s}^2+2.5\hat{s}+0.375}$, and the several ones by Dinyavari and Friedmann.[33] This fact implies that half of the steady lift is immediately generated (instantaneously developed) without any lag and the system dynamics apply to the other half. This behavior is not physically appealing since there must be a finite time for the lift to develop, particularly because we are concerned with the circulatory lift; it is understandable that the non-circulatory lift responds instantaneously based on the instantaneous acceleration, at least in the absence of viscosity and compressibility. However, this behavior cannot be true for the lift due to circulation even when assuming potential flow. This point will be discussed in more detail in the next section.



(a) Wagner Function.



(b) Theodorsen Function.

Figure 2: The step and frequency response functions (Wagner and Theodorsen functions) of the inviscid circulatory lift.

IV. Dynamics of the Circulatory Lift and Bound Circulation

It is well-known that at the very first moment ($t = 0^+$) right after an airfoil impulsive start, the flow looks like an inviscid flow without circulation, as discussed in Prandtl's lectures,[34] pp. 158-168; Goldstein,[35] pp. 26-36; Milne-Thomson,[36] pp. 89-90; and Schlichting and Truckenbrodt,[20] pp. 33-35. That is, the initial response (high-frequency response/gain) of the bound circulation around the airfoil is zero. This behavior can be physically explained as follows. For an impulsive starting airfoil, at $t = 0^+$, the velocity everywhere is zero except on the airfoil. There is an infinitely thin boundary layer taking care of the no-slip, but at $t = 0^+$, it has to be infinitely thin so that if the airfoil is enclosed with any contour, no matter how close it gets to the airfoil, the contour will pass through points outside of this layer. On this contour, the diffusion term in Navier-Stokes equation is zero (i.e., the flow behaves like an inviscid flow temporarily). As such, the Kelvin's circulation theorem implies zero rate of change of circulation around that contour at $t = 0^+$. Since the bound circulation Γ was zero at $t=0$, it follows that the $\lim_{t\to 0}\Gamma(t)=0$, which is physically intuitive because circulation needs time to build up even in potential flow.

It is interesting to point out that the frequency response of circulation is not given by the Theodorsen function $C(k)$, as shown in Fig. 3. Following Schwarz,[7] one can write the frequency response of circulation as[7,29]

$$\frac{\Gamma}{\Gamma_0}(k)=\frac{-2e^{-jk}}{jk\pi\left(H_1^{(2)}(k)+jH_0^{(2)}(k)\right)}, \qquad (8)$$

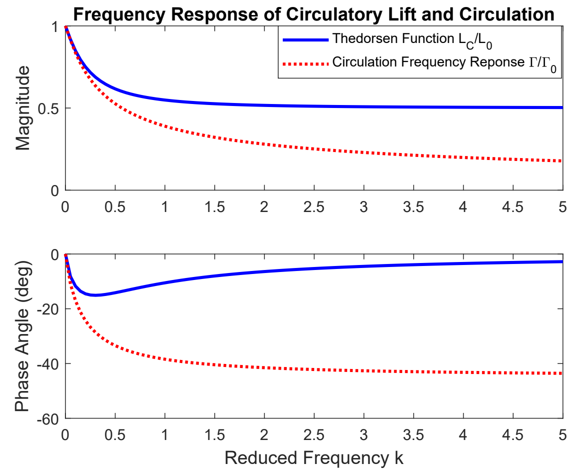


Figure 3: Comparison between the frequency response functions of the circulatory lift and circulation in potential flow.

where Γ_0 is the quasi-steady circulation. The potential-flow lift frequency response (Theodorsen function $C(k)$) does not represent a physical system in mechanics: from a dynamical system perspective, it does not act as a low pass filter. This characteristic is apparent from its high frequency limit $\lim_{k\to\infty}C(k)=\frac{1}{2}$: a non-zero high-frequency gain with zero phase lag. That is, there exists some component in the circulatory lift that responds instantaneously; i.e., it depends *algebraically* on the *current* input (airfoil motion) without dynamics (lag). This characteristic is apparent in the non-zero initial-time response due to a step input ($\phi(\tau=0)=\frac{1}{2}$).

On the other hand, the high-frequency limit of the circulation transfer function is given by

$$\lim_{k\to\infty}\left|\frac{\Gamma}{\Gamma_0}(k)\right|=0 \;\text{ and }\; \lim_{k\to\infty}\angle\frac{\Gamma}{\Gamma_0}(k)=-\pi/4,$$

which points to a strictly proper transfer function. That is, in contrast to the circulatory lift response, there is no instantaneous component of circulation. Indeed, the circulation development possesses lag similar to a mechanical system with a strictly proper transfer function even within the framework of potential flow. Therefore, $\lim_{t\to 0}\Gamma(t)=0$ due to a step input (i.e., an impulsive start), which conforms with the flow physics due to an impulsive start as discussed above.

We note that Eq. (8) can be re-written as

$$\frac{\Gamma}{\Gamma_0}(k)=\frac{-2}{\pi jk}\frac{e^{-jk}}{H_1^{(2)}(k)}C(k). \qquad (9)$$

As such, realizing that $K_1(jk)=-\frac{\pi}{2}H_1^{(2)}(k)$, we write the following relation between the circulatory lift transfer function $G_a(\hat{s})$ (i.e., corresponding to Theodorsen function: $G_a(ik)\equiv C(k)$) and the circulation transfer function, denoted by $G_\Gamma(\hat{s})$

$$G_\Gamma(\hat{s})=\frac{1}{\hat{s}}\frac{e^{-\hat{s}}}{K_1(\hat{s})}G_a(\hat{s}), \qquad (10)$$

Equation (10) is quite revealing for the relation between the dynamics of the circulatory lift and circulation in potential flow. Clearly, there is a time-integrator $\frac{1}{\hat{s}}$ and a time-delay $e^{-\hat{s}}$ between the circulatory lift development and circulation development. However, the term $\frac{1}{K_1(\hat{s})}$ acts as a differentiator in the low frequency range ($K_1(z)\sim\frac{1}{z}$ as $z\to 0$,[37] pp. 4). Therefore, it cancels the effect of the integrator, hence, the circulation development follows the lift evolution with a time-delay in the low-frequency range. On the other hand, the term $\frac{1}{K_1(\hat{s})}$ cancels the delay effect in the high frequency range ($K_1(z)\sim\sqrt{\frac{\pi}{2z}}e^{-z}$ as $z\to\infty$,[38] pp. 236) leading to a circulation transfer function of the form $\frac{1}{\sqrt{\hat{s}}}$ which explains the zero-high frequency gain and the $-45°$ high-frequency phase. Figure 4 shows the relation between the airfoil motion, circulatory lift, and circulation development in potential flow. It is legitimate to think that the circulation

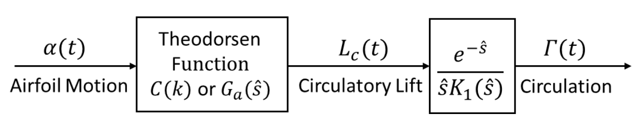


Figure 4: Relation between the airfoil motion, circulatory lift, and circulation development in potential flow.

leads the circulatory lift in development as the latter is mainly due to the former. That is, causality dictates a proper transfer function from Γ to L_C. However, in the potential-flow unsteady aerodynamics, L_C leads Γ; there is a delay from lift to circulation in the low-frequency range and a fractional integrator $\frac{1}{\sqrt{\hat{s}}}$ in the high-frequency range.

All of the above evidences support the fact that the circulatory lift is not the lift due to circulation ($L_C \neq \rho U \Gamma$). The pioneers[3–5] opted to define the circulatory lift in a different way. This point becomes clearer when the problem is formulated using thin airfoil theory, which is explained in the next section.

V. Relation Between Circulatory Lift and Circulation: Steady and Unsteady Thin Airfoil Theory Perspective

To show how the steady lift is solely due to circulation ($L = \rho U \Gamma$) whereas the unsteady "circulatory" lift is not lift due to circulation ($L_C \neq \rho U \Gamma$), we recall the thin airfoil theory. In such a theory, a thin airfoil is represented by a circulation distribution along the chord line (x-axis) and the no-penetration boundary condition results in the integral equation for circulation

$$\frac{1}{2\pi}\int_{-b}^{b}\frac{\gamma(\zeta)}{x-\zeta}d\zeta = w(x), \ \ \forall\, x \in [-b, b], \tag{11}$$

where γ is the circulation distribution over the airfoil (i.e., $\Gamma = \int_{-b}^{b}\gamma(x)dx$) and w is the airfoil velocity normal to the surface (positive downward). This integral equation is typically solved assuming the series solution

$$\gamma(\varphi) = 2U\left[A_0 \tan\varphi/2 + \sum_{n=1}^{\infty} A_n \sin n\varphi\right], \tag{12}$$

where $x = b\cos\varphi$. Then, the no-penetration boundary condition (11) implies that the A_n's are the coefficients of the Fourier cosine representation of the normal velocity; that is,

$$A_0 = \frac{1}{\pi}\int_0^{\pi} w(\varphi)d\varphi, \ \ \text{and} \ \ A_n = \frac{2}{\pi}\int_0^{\pi} w(\varphi)\cos n\varphi d\varphi, \ \forall\, n \geq 1.$$

After solving for the series coefficients, and consequently the circulation distribution γ, the normalized pressure difference can be determined from the *steady* Bernoulli equation as

$$\Delta C_P(x) = \frac{2}{U}\gamma(x), \tag{13}$$

which implies that the local pressure is dependent only on the local circulation distribution; that is, $\Delta C_P(x)$ and $\gamma(x)$ possess the same distribution over the airfoil resulting in

$$\hat{\Gamma} = C_L = 2\pi\left(A_0 + A_1/2\right), \tag{14}$$

where $\hat{\Gamma} = \frac{\Gamma}{Ub}$ is the non-dimensional bound circulation. Equation (14) is equivalent to $L = \rho U \Gamma$.

The unsteady thin airfoil theory has been introduced by Katz and Plotkin[39] and Peters.[18,19] In contrast to the steady thin airfoil theory discussed above, the wake effects must be taken into account for an unsteady analysis. That is, in addition to the circulation distribution γ bound the airfoil, there is a sheet of vorticity in the wake whose circulation distribution γ_w is also unknown. As such, the no-penetration boundary condition (11) becomes

$$\frac{1}{2\pi}\int_{-b}^{b}\frac{\gamma(\zeta,t)}{x-\zeta}d\zeta + \frac{1}{2\pi}\int_{b}^{\infty}\frac{\gamma_w(\zeta,t)}{x-\zeta}d\zeta = w(x,t), \ \ \forall\, x \in [-b, b], \tag{15}$$

Assuming a similar series solution to that of the steady case (12), but with time-varying coefficients

$$\gamma(\varphi,t) = 2U\left[A_0(t) \tan\varphi/2 + \sum_{n=1}^{\infty} A_n(t) \sin n\varphi\right] \tag{16}$$

and expanding the normal velocity w in a Fourier cosine series similar to the steady case

$$w(x,t) = U\left[B_0(t) + \sum_{n=1}^{\infty} B_n(t)\cos n\varphi\right],$$

the no-penetration boundary condition (15) implies

$$A_n + \lambda_n = B_n \;\; \forall\, n \geq 0, \tag{17}$$

where λ_n's are Peters inflow coefficients:[18,19] the coefficients of the Fourier cosine representation of the downwash $\lambda(x,t)$ on the plate due to wake vorticity. That is, λ is defined as

$$\lambda(x,t) = \frac{1}{2\pi}\int_b^\infty \frac{\gamma_w(\zeta,t)}{x-\zeta}d\zeta = U\left[\lambda_0(t) + \sum_{n=1}^{\infty}\lambda_n(t)\cos n\varphi\right].$$

Unlike the steady case where the no-penetration boundary condition was sufficient to determine γ (or its coefficients A_n's), Eq. (17) has two unknowns A_n, λ_n for each n and, hence, is not sufficient.

To achieve closure in the unsteady case, a relation must be established between the wake vorticity and bound circulation. Peters[18,19] utilized the fact that the wake convects freely

$$\frac{D\gamma_w}{Dt} = U\frac{\partial \gamma_w(x,t)}{\partial x} + \frac{\partial \gamma_w(x,t)}{\partial t} = 0$$

along with the Kutta condition (or conservation of total circulation) to derive the following equation

$$U\frac{\partial \lambda(x,t)}{\partial x} + \frac{\partial \lambda(x,t)}{\partial t} = \frac{1}{2\pi}\frac{\dot{\Gamma}}{b-x}, \tag{18}$$

which results in linear differential equations governing the inflow coefficients λ_n's (i.e., a state space model for λ_n's): Peters finite state model. Equations (17, 18) can be used to determine the coefficients A_n's and λ_n's after truncating the series representations at a certain n. Peters[18,19] showed that eight terms (states) may be sufficient to capture the inviscid unsteady lift dynamics.

Having solved for the circulation distribution γ (and its coefficients), the non-dimensional circulation will have the exact same form

$$\hat{\Gamma} = 2\pi\left(A_0 + A_1/2\right) = 2\pi\left(B_0 + B_1/2 - \lambda_0 - \lambda_1/2\right) \tag{19}$$

as in the steady case (14) because γ has the same series representation. However, the *unsteady* Bernoulli equation implies

$$\Delta C_P(x) = \frac{2}{U}\gamma(x) + \frac{2}{U^2}\frac{\partial}{\partial t}\int_{-b}^{x}\gamma(\zeta)d\zeta, \tag{20}$$

which is different from its steady counterpart (13); the second term is due to the time-derivative term in the unsteady Bernoulli equation. Consequently, in contrast to the steady case, Eq. (20) implies that the local pressure distribution may be different from the circlation distribution. Integrating the pressure distribution over the airfoil and utilizing Eq. (18), one obtains

$$C_L = 2\pi\left[\underbrace{B_0 + B_1/2 - \lambda_0}_{\text{Circulatory}} + \underbrace{\frac{1}{2}\left(\dot{B}_0 - \dot{B}_2/2\right)}_{\text{Non-Circulatory}}\right] \tag{21}$$

The common classification of the total unsteady lift into circulatory and non-circulatory components can be clearly seen from Eq. (21). First, it should be pointed out that such a classification is arbitrary. For example, one can add and subtract $\lambda_1/2$ to the terms inside the square bracket in Eq. (21) and collect the terms that produce $\hat{\Gamma}$ referring to them as circulatory lift (lift due to circulation in this case) and the rest are non-circulatory terms. This classification has been adopted by Fung.[40] In contrast, the most common classification, adopted by Wagner,[3] Theodorsen,[4] Von Karman and Sears,[5] among many others[29] is that the last two terms (acceleration terms) in Eq. (21) represent the non-circulatory loads and the first three terms (velocity terms) represent the circulatory loads. According to this common classification, the circulatory lift is not lift due to circulation (i.e., $L_C \neq \rho U \Gamma$). Rather, they are related as

$$L_C = \rho U \Gamma + \pi \rho U^2 b \lambda_1 \tag{22}$$

Moreover, Eq. (18) can be used to write λ_1 as

$$\lambda_1 = \frac{b}{U}\left[\frac{\dot{\hat{\Gamma}}}{\pi} - \dot{\lambda}_0 + \dot{\lambda}_2/2\right], \tag{23}$$

which implies that λ_1 is related to the rate of change of circulation. That is, the common definition of circulatory lift is not only lift due to instantaneous circulation but also due to its rate of change. Another view of Eqs. (22, 23) suggests that the circulatory lift L_C is obtained from the lift due to circulation $\rho U \Gamma$ after adding some acceleration terms. Therefore, the circulatory lift has indeed some "added mass" effects. This fact explains the reason behind the non-zero high-frequency gain of the circulatory lift frequency response ($\phi(0) = \lim_{k\to\infty} |C(k)| = \lim_{\hat{s}\to\infty} G_a(\hat{s}) = \frac{1}{2}$): half of the steady-state lift is immediately generated. This instantaneous component of L_C must be the added mass part of its definition. Because L_C is composed of lift due to circulation and some added-mass effects and since the latter is negligible at lower frequencies, Theodorsen function behaves like a low pass filter in the low frequency range (following the circulation transfer function) with a monotonically decreasing magnitude and phase, as shown in Fig. 3. On the other hand, since the added mass (acceleration) terms are dominant in the high frequency range, the Theodorsen function departs from the circulation transfer function and follows an algebraic-type relation with the input (with half of the magnitude and zero phase difference), similar to the added mass effects which are proportional to the instantaneous input acceleration.

VI. Circulation in a Non-circulatory Motion

As we showed that the circulatory-non-circulatory classification is arbitrary and that the circulatory lift actually possesses some added-mass contribution, it is interesting to investigate whether a non-circulatory maneuver induces loads due to circulation. To pursue such an investigation, we perform numerical simulations of the unsteady Reynolds-averaged Navier Stokes (URANS) equations on a dynamic mesh around NACA 0012 undergoing pitching motion in a quiescent fluid (no free stream). Details about the computational setup and the dynamic mesh can be found in our previous efforts.[41–43] According to the classical inviscid theory of unsteady aerodynamics,[3–5] the expected lift is purely due to added mass effects, which is given by

$$L(t) = L_{NC}(t) = \frac{1}{4}\pi\rho b^3 \ddot{\alpha}(t)$$

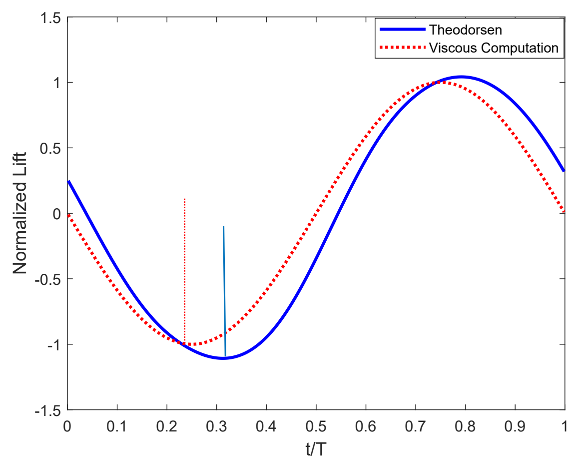
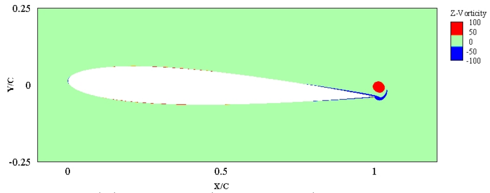


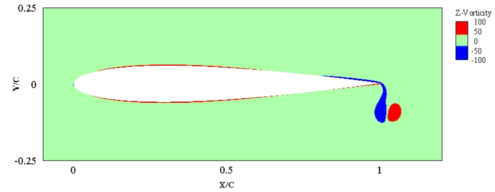
Figure 5: A comparison between the time history of the viscous added-mass lift and the potential flow one, both normalized by the added-mass $m_a = \pi\rho b^2$ and the maximum acceleration of the mid-chord point. This run is at $\frac{b}{\delta_{\text{stokes}}} = \sqrt{\frac{\omega b^2}{2\nu}} = 141.4$;

for pitching around the quarter-chord point. Therefore, the generated lift force is proportional to the instantaneous acceleration with no phase lag. Figure 5 shows a comparison between the computed lift time-history against the potential-flow one. Clearly, the computed *viscous* lift possesses a phase lag with respect to the acceleration $\ddot{\alpha}$ in contrast to the potential-flow lift. It is quite interesting to conclude that even the added mass-effects possess non-trivial dynamics (lag), induced by viscosity. In fact, the snapshots of the vorticity contours at some instants during the cycle, shown in Fig. 6, point to the shedding of strong vortices from the airfoil trailing edge, which implies the development of circulation over the airfoil by virtue of conservation of circulation. As such, it may be concluded that viscosity (and perhaps compressibility) induces bound circulation over the airfoil even in the absence of a free-stream. In other words, similar to the fact that the circulatory lift includes non-circulatory aspects (even in potential flow), the non-circulatory lift seems to also enjoy some *viscous* circulatory contributions.

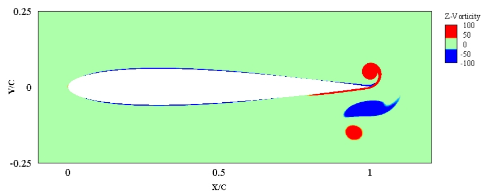
Running this computational experiment at different frequencies to construct the frequency response of the added mass contribution, we find that the potential flow theory predicts the magnitude (quadratic in frequency) quite well, as shown in Fig. 7. However, interestingly, we observe a phase difference from the potential flow results that is almost independent of frequency, quite similar to the flow behavior in the Stokes



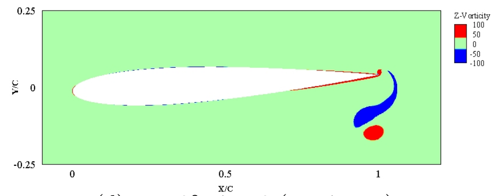
(a) $\alpha = 3^\circ$ (maximum), $\dot{\alpha} = 0$.



(b) $\alpha = 0^\circ$, $\dot{\alpha} < 0$ (minimum).



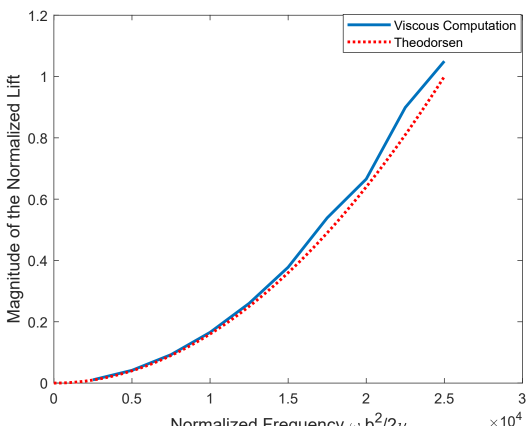
(c) $\alpha = -3^\circ$ (minimum), $\dot{\alpha} = 0$.



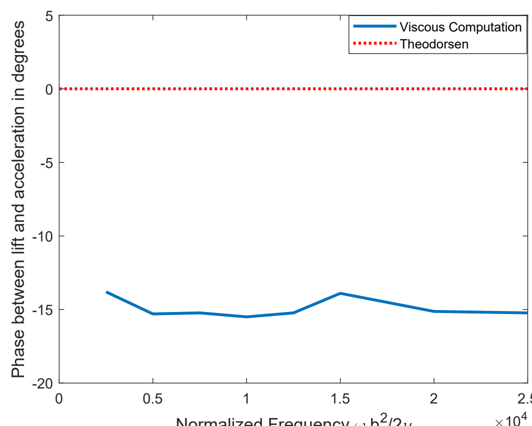
(d) $\alpha = 0^\circ$, $\dot{\alpha} > 0$ (maximum).

Figure 6: Vorticity contours at four time instants during a cycle of pure pitching in the absence of free stream at $\frac{b}{\delta_{\text{stokes}}} = \sqrt{\frac{\omega b^2}{2\nu}} = 141.4$.

second problem,[44] pp. 191-193;,[45] pp. 619-623;,[46] pp. 109-111. In fact, the only difference between the current problem and that of Stokes is the finite length of the considered airfoil in contrast to Stokes infinite plate. For more details about the viscosity-induced lag, the reader is referred to our recent effort[42,43] or the work of Othman et al.[47]



(a) Magnitude Variation.



(b) Phase Variation.

Figure 7: Computational results of the non-circulatory lift frequency response versus Theodorsen's potential flow results. Since the computation is performed at zero free-stream, the frequency is presented in terms of the ratio $\frac{b^2}{\delta^2_{\text{stokes}}} = \frac{\omega b^2}{2\nu}$ between the semi-chord and the Stokes layer.

VII. Conclusion

In this paper, we emphasize the dynamical-systems aspect of unsteady aerodynamics. That is, we consider the unsteady aerodynamics problem of a two-dimensional airfoil as a dynamical system whose input is the angle of attack (or airfoil motion) and output is the lift force. Based on this view, we discuss the lift evolution from a purely dynamical perspective through the step response (Wagner function), frequency response (Theodorsen function), transfer function, etc. In particular, we show that the circulatory lift dynamics has a non-zero high-frequency gain. Therefore its transfer function is not strictly proper, which points to the existence of a component in the circulatory lift that responds instantaneously without a lag; half of the steady lift is immediately generated (instantaneously developed). In contrast, the circulation transfer function is strictly proper with a zero high-frequency gain (i.e., acts as a low pass filter). We show the dynamical relation between the circulatory lift and circulation. While it may be thought that circulation leads lift, as the latter is mainly due to the former, we find that the circulation follows the circulatory lift with a delay in the low frequency range and a fractional integrator in the high frequency range. These observations confirm the fact that the common definition of circulatory lift is not lift due to circulation. Actually, by comparing the steady and unsteady versions of thin airfoil theory, we show that the circulatory-non-circulatory classification is arbitrary. We show that the circulatory lift includes some acceleration terms, which explain its high-frequency (instantaneous) response. On the other hand, we perform numerical simulation of the unsteady Reynolds-averaged Navier Stokes (URANS) equations on a dynamic mesh around NACA 0012 undergoing pitching motion in a quiescent fluid (no free stream) to show that such a non-circulatory maneuver induces *viscous* circulation over the airfoil.

Acknowledgments

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