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Asymptotics of the eigenmodes and stability of an elastic structure with general feedback matrix

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The distribution of natural frequencies of the Euler-Bernoulli beam subject to fully non-dissipative boundary conditions is investigated. The beam is clamped at the left end and equipped with a 4-parameter $(\alpha, \beta, k_1, k_2)$ linear boundary feedback law at the right end. The 2×2 boundary feedback matrix relates the control input (a vector of velocity and its spatial derivative at the right end), to the output (a vector of shear and moment at the right end). The role of the control parameters is examined and the following results are proven: (i) when $\beta \neq 0$, the set of vibrational modes is asymptotically close to a straight line parallel to the imaginary axis; (ii) when $\beta = 0$ and the parameter $K = (1 - k_1 k_2)/(k_1 + k_2) \neq \pm 1$ the rate of energy decay (or gain) is proportional to $|\Re \lambda_n|$, where λ_n is a vibrational mode and $|\Re \lambda_n| \sim \sqrt{|\Im \lambda_n|} \sim |n|$; (iii) when $\beta = 0$, $|K| = 1$, and $\alpha = 0$, the rate of energy decay/gain is proportional to $|\Re \lambda_n| \sim |\Im \lambda_n|^2 \sim n^2$; (iv) when $\beta = 0$, $|K| = 1$, and $\alpha > 0$, the rate of energy decay/gain is proportional to $|\Re \lambda_n| \sim \ln |\Im \lambda_n| \sim \ln |n|$.

Keywords: matrix differential operator, discrete spectrum, eigenvalues, Kantorovich's Theorem, asymptotic approximation

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1. Introduction

The present paper is concerned with the asymptotic properties of the eigenmodes of the Euler-Bernoulli beam model subject to a special type of non-conservative boundary conditions. At the left end the beam is assumed to be clamped, while at the right end to be subject to linear feedback-type conditions with a non-dissipative feedback matrix depending on four control parameters: α, β, k_1, and k_2. The boundary conditions are non-dissipative in the following sense: the energy of the beam is not a decreasing function of time. In our approach, the initial boundary value problem describing the dynamics of the beam is reduced to an evolution equation in the Hilbert state space, $\mathcal{H}$, of the system equipped with the energy norm. This evolution equation is completely determined by its dynamics generator, $i\mathcal{L}$, which is an unbounded non-skew-selfadjoint matrix differential operator in $\mathcal{H}$. The eigenmodes (vibrational modes) and mode shapes of the system are defined as the eigenvalues and the generalized eigenvectors of the operator $i\mathcal{L}$.

In what follows it is technically convenient to investigate spectral and asymptotic properties of the operator $\mathcal{L}$ (rather than $i\mathcal{L}$); the reason for eliminating the multiplier "i" is explained in Remark 2.1 below. Clearly, the dynamics generator, $i\mathcal{L}$, and the operator $\mathcal{L}$ have the same set of generalized eigenvectors.

The main object of interest in the present paper is the asymptotic distribution of vibrational modes for different combinations of the boundary controls. At this moment we would like to emphasize that there exists an extensive literature devoted to different aspects of the Euler-Bernoulli model dynamics, such as asymptotic, spectral, and stability analysis of the model (both linear and nonlinear versions) and control of this distributed parameter system. (For a discussion on a number of recent works, see the Introductions of our works (Shubov & Shubov, 2016; Shubov & Kindrat, 2018) and references therein.)

This paper is a continuation of the study initiated in our works (Shubov & Shubov, 2016; Shubov & Kindrat, 2018). In paper (Shubov & Shubov, 2016), we have considered the model whose boundary feedback matrix contained only two non-trivial parameters, k_1 and k_2 (with $\alpha = \beta = 0$). One of the main results of (Shubov & Shubov, 2016) is related to the stability of the model. Namely, it is shown in (Shubov & Shubov, 2016) that even though the model is not dissipative, for the case when one of the control parameters is positive and the other is sufficiently small, the set of the eigenmodes is located in the left half-plane of the complex plane. To prove such a result, we have derived "the main identity", which might be of interest in its own right. This identity establishes a relation between the eigenmodes and mode shapes of the non-conservative model corresponding to the case $(k_1,k_2) \neq (0,0)$ and the eigenmodes and mode shapes of the clamped-free conservative model corresponding to the case $(k_1,k_2) = (0,0)$. We suggest a hypothesis that a similar stability result can be proven for the multiparameter case ($|\alpha|+|\beta|+|k_1|+|k_2| > 0$) as well.

In our second paper (Shubov & Kindrat, 2018), we have considered the case of a four-parameter feedback control matrix and have shown a number of results on the general spectral properties of the non-selfadjoint operator $\mathcal{L}$. In particular, it was shown that for any combination of the boundary parameters the corresponding operator, $\mathcal{L}$, is a finite-rank perturbation of one and the same selfadjoint operator, $\mathcal{L}_0$, where $i\mathcal{L}_0$ is the dynamics generator for a cantilever beam model. Moreover, it is shown that the non-selfadjoint operator, $\mathcal{L}$, shares a number of spectral properties specific to its selfadjoint counterparts. (i) Namely, we have introduced four selfadjoint operators (corresponding to the clamped-free, clamped-hinged, clamped-sliding, and clamped-clamped beam models) and derived important inequalities that describe the boundary behavior of the eigenfunctions of these operators. We have obtained the generalization of the aforementioned results for the non-selfadjoint operator $\mathcal{L}$. That is, we proved that for some combinations of the boundary parameters, the boundary inequalities for the eigenfunctions of $\mathcal{L}$ are similar to the boundary inequalities for the aforementioned selfadjoint operators, while for other combinations they are quite different. (ii) We have shown that each selfadjoint problem has a simple spectrum, and a similar result holds for the non-selfadjoint operator $\mathcal{L}$, i.e. the geometric multiplicity of any eigenvalue of $\mathcal{L}$ is one. However, since $\mathcal{L}$ is non-selfadjoint, the algebraic multiplicity of each eigenvalues is finite but not necessarily one. Thus, for each eigenvalue there could exist a finite chain of associate functions. (iii) Finally it is shown in (Shubov & Kindrat, 2018), that if exactly one control parameter is not equal to zero, the operator $\mathcal{L}$ does not have real eigenvalues. On the other hand, when there are two or more non-zero boundary parameters, there could be real eigenvalues depending on which parameters are non-zero.

In the present paper we derive explicit formulae describing the asymptotic distribution of the eigenvalues of the operator $\mathcal{L}$, as the number of an eigenvalue tends to infinity. As expected, the asymptotic results strongly depend on the boundary control parameters. Our goal is to obtain such asymptotic formulae that contain all four parameters. To this end, in some cases it is not enough to derive the main leading asymptotic terms, but also the next order terms to identify the role of all control parameters.

Concerning this, we provide more details in the description below. Now we are in a position to outline the organization of the paper.

In Section 2 we give the operator form of the initial boundary value problem given by (2.1), (2.2), (2.6), (2.9) and define its dynamics generator $i\mathcal{L}$. For the reader's convenience we reproduce one result from (Shubov & Kindrat, 2018) (see Theorem 2.1), which states that the inverse operator, $\mathcal{L}^{-1}$, exists and is a compact operator in $\mathcal{H}$. This means that the spectrum of $\mathcal{L}$ consists of a countable set of *normal* eigenvalues, i.e. each eigenvalue is an isolated point of the spectrum whose algebraic multiplicity is finite (Gohberg & Krein, 1996). We conclude Section 2 by providing the spectral equation for $\mathcal{L}$, as derived in (Shubov & Kindrat, 2018).

In Section 3–5 we derive asymptotic approximations for the eigenvalues (and hence for the vibrational modes) when all boundary control parameters are non-negative (practically, the most important case). However, technically all the steps of the derivation can be done for any complex values of the controls.

In Section 3 we derive an asymptotic approximation for the set of the eigenvalues when one of the parameters, β, is equal to zero. It turns out that in this case the spectral asymptotics depend on the parameter $K = (1-k_1k_2)/(k_1+k_2)$. One of the main results of the section is Theorem 3.2, where the asymptotic approximation for the eigenvalues is obtained for the case $|K| \neq 1$. It is shown that for $K > 0$, $K \neq 1$, the model is essentially stable, i.e. there exists an infinite countable set of stable vibrational modes and, at most, a finite set of unstable modes (see Theorem 3.2 cases 1) and 3)). The rate of energy decay of a stable mode depends on the number of a mode, i.e. if λ_n is the n-th vibrational mode, then the energy decay rate, which is proportional to $|\Re\lambda_n|$ behaves as $|\Re\lambda_n| \sim \sqrt{|\Im\lambda_n|} \sim |n|$. If $K < 0$, $K \neq -1$ (see Theorem 3.2 cases 2) and 4)), there exists an infinite set of unstable vibrational modes and at most a finite number of stable modes; the rate of energy gain is proportional to $\Re\lambda_n \sim |n|$.

The final result of the section, Theorem 3.3 below, is concerned with the spectral asymptotics when $K = \pm 1$ and $\alpha = 0$ (the case excluded in Theorem 3.2). If $K = 1$ (see formula (3.44) below) then the set of vibrational modes is essentially stable (with possible existence of a finite set of unstable modes). The rate of energy decay is proportional to $|\Re\lambda_n|$ and is such that $|\Re\lambda_n| \sim (\Im\lambda_n)^2 \sim n^2$. If $K = -1$, the set of vibrational modes is mostly unstable (see formula (3.47) below) and the rate of energy gain is proportional to $\Re\lambda_n \sim n^2$.

In Section 4, we derive the asymptotic approximation for the set of the eigenvalues of the operator $\mathcal{L}$ for the cases $|K| = 1$ and $\alpha > 0$ (see Theorem 4.2). It turns out that the spectral asymptotics for $\alpha > 0$ ($|K| = 1$) are totally different compared to the asymptotics in the case $\alpha = 0$. Namely, if $K = 1$, there exists a countable set of stable vibrational modes (see formula (4.20) below) and at most a finite set of unstable modes. The rate of energy decay is proportional to $|\Re\lambda_n|$ and behaves as $|\Re\lambda_n| \sim \ln(|\Im\lambda_n|) \sim \ln|n|$. If $K = -1$ (see formula (4.22) below), there exists a countable set of unstable vibrational modes and at most a finite set of stable modes. The rate of energy gain in this case is $\Re\lambda_n \sim \ln(|\Im\lambda_n|) \sim \ln|n|$.

In Section 5 we derive the spectral asymptotics for the case $\beta > 0$ (see Theorem 5.1 below). Formula (5.1) shows that distant vibrational modes have about the same rate of energy decay if $k_1k_2 < 1+\alpha\beta$ and about the same rate of energy gain if $k_1k_2 > 1+\alpha\beta$. In addition, if $k_1k_2 = 1$ and $\alpha = 0$, the spectrum is asymptotically close to the real axis which can be an indication that the corresponding non-sefladjoint operator $\mathcal{L}$ is "close" to its sefladjoint counterpart. The notion of "closeness" will be clarified in our forthcoming paper on the geometric properties of the set of the root vectors of $\mathcal{L}$ (such as completeness, minimality, and the Riesz basis property in the state space of the system).

2. Problem statement and auxiliary results

We consider the Euler-Bernoulli beam model subject to the general four-parameter family of non-conservative linear boundary conditions. The transverse displacement of the beam, $h(x,t)$, at position x and time t is governed by the hyperbolic partial differential equation

$$\rho(x)\,\frac{\partial^2 h(x,t)}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(EI(x)\,\frac{\partial^2 h(x,t)}{\partial x^2}\right)=0, \qquad 0\leqslant x\leqslant L, \qquad t\geqslant 0. \tag{2.1}$$

This equation represents a commonly used model for the motion of a straight beam of length L, linear density $\rho(x)$, modulus of elasticity of the beam material $E(x)$, and cross-sectional moment of inertia $I(x)$ ($EI(x)$ is the bending stiffness). The model is obtained by using Hooke's law and the simplifying assumptions that the thickness and width of the beam are small compared to the length, and the cross-sections of the beam remain plane during deformation (Shubov & Shubov, 2016; Benaroya, 1998; Gladwell, 2005). We assume that the beam is clamped at the left end, i.e.

$$h(0,t)=h_x(0,t)=0. \tag{2.2}$$

To describe the right-end conditions, we use the moment $M(x,t)$, and the shear $Q(x,t)$ defined by (Benaroya, 1998; Gladwell, 2005):

$$M(x,t)=EI(x)h_{xx}(x,t), \qquad Q(x,t)=\big(EI(x)h_{xx}(x,t)\big)_x. \tag{2.3}$$

Let the input, $U(t)$, and output, $Y(t)$, be given as $\mathbb{R}^2$-vectors

$$U(t)=[-Q(L,t),M(L,t)]^T \qquad \text{and} \qquad Y(t)=[h_t(L,t),h_{xt}(L,t)]^T, \tag{2.4}$$

where the superscript "T" stands for transposition. The feedback control law is given by

$$U(t)=\mathbb{K}Y(t), \qquad \text{and} \qquad \mathbb{K}=\begin{bmatrix}-\alpha & -k_2\\ -k_1 & -\beta\end{bmatrix}, \tag{2.5}$$

where α,β,k_1,k_2 are the control parameters. The feedback (2.5) can be written in the form

$$EI(L)h_{xx}(L,t)=-k_1h_t(L,t)-\beta h_{xt}(L,t), \qquad \big(EI(x)h_{xx}(x,t)\big)_x\big|_{x=L}=\alpha h_t(L,t)+k_2h_{xt}(L,t). \tag{2.6}$$

If all parameters are equal to zero, then the right-end conditions become $h_{xx}(L,t)=h_{xxx}(L,t)=0$, and the problem corresponds to the clamped-free beam.

Consider the energy functional for the beam

$$\mathcal{E}(t)=\frac{1}{2}\int_0^L\left[\rho(x)\,h_t^2(x,t)+EI(x)\,h_{xx}^2(x,t)\right]\mathrm{d}x. \tag{2.7}$$

Evaluating $\mathcal{E}_t(t)$ on the solutions of Eq.(2.1) satisfying the left-end conditions (2.2), we obtain that $\mathcal{E}_t(t)=Y(t)\cdot U(t)$, where "$\cdot$" denotes the dot-product in $\mathbb{R}^2$. If we use (2.5), then

$$\mathcal{E}_t(t)=Y(t)\cdot\mathbb{K}Y(t)=-\alpha h_t^2(L,t)-\beta h_{xt}^2(L,t)-(k_1+k_2)h_t(L,t)h_{xt}(L,t), \tag{2.8}$$

which means that if $k_1+k_2=0$ and $\alpha\geqslant 0$, $\beta\geqslant 0$, then $\mathcal{E}_t(t)\leqslant 0$ and the system is dissipative. Different combinations of the boundary parameters yield different energy dynamics for the structure. In the present paper we consider the initial boundary-value problem defined by Eq.(2.1), conditions (2.2) and (2.6), and a standard set of the initial conditions

$$h(x,0) = h_0(x), \qquad \frac{\partial h(x,0)}{\partial t} = h_1(x). \tag{2.9}$$

Let us rewrite problem (2.1), (2.2), and (2.6), as the first order in time evolution equation in the state space of the system (the energy space). Without loss of generality we assume that the spatial extent of the beam is $L = 1$. We also assume that EI and ρ are strictly positive functions and such that $EI, \rho \in C^2[0,1]$.

Let $\mathcal{H}$ be the Hilbert space of two-component complex vector-valued functions obtained as the closure of smooth functions $\Phi(x) = [\varphi_0(x), \varphi_1(x)]^T$, such that $\varphi_0(0) = \varphi_0'(0) = 0$, in the following norm:

$$\|\Phi\|^2_{\mathcal{H}} = \frac{1}{2}\int_0^1 \left[EI(x)\left|\varphi_0''(x)\right|^2 + \rho(x)\left|\varphi_1(x)\right|^2\right] \mathrm{d}x. \tag{2.10}$$

The energy space $\mathcal{H}$ is topologically equivalent to the space $\widetilde{H}^2_0(0,1) \times L^2(0,1)$, where

$$\widetilde{H}^2_0(0,1) = \left\{\varphi \in H^2(0,1) : \varphi(0) = \varphi'(0) = 0\right\}.$$

The problem (2.1), (2.2), (2.6), and (2.9) can be represented as an evolution problem

$$\Phi_t(x,t) = i(\mathcal{L}\Phi)(x,t) \qquad \text{and} \qquad \Phi(x,0) = [\varphi_0(x), \varphi_1(x)]^T, \quad 0 \leqslant x \leqslant 1, \quad t \geqslant 0. \tag{2.11}$$

The dynamics generator is $i\mathcal{L}$, where $\mathcal{L}$ is given by the following matrix differential expression:

$$\mathcal{L} = -i\begin{bmatrix} 0 & 1 \\ -\dfrac{1}{\rho(x)}\dfrac{\partial^2}{\partial x^2}\left(EI(x)\dfrac{\partial^2}{\partial x^2}\cdot\right) & 0 \end{bmatrix}, \tag{2.12}$$

defined on the domain

$$\begin{aligned} \mathcal{D}(\mathcal{L}) = \big\{&\Phi = (\varphi_0, \varphi_1)^T \in \mathcal{H} : \varphi_0 \in H^4(0,1),\ \varphi_1 \in H^2(0,1);\ \ \varphi_1(0) = \varphi_1'(0) = 0; \\ &EI(1)\varphi_0''(1) = -k_1\varphi_1(1) - \beta\varphi_1'(1),\ \ (EI(x)\varphi_0''(x))'_{x=1} = k_2\varphi_1'(1) + \alpha\varphi_1(1)\big\}. \end{aligned} \tag{2.13}$$

REMARK 2.1 *1) We introduce the factor "i" in the definition* (2.12) *and into Eq.*(2.11) *for convenience. As is shown in (Shubov & Kindrat, 2018) (see Theorem 2.1 below), the operator $\mathcal{L}$ is a finite-rank perturbation of the selfadjoint operator corresponding to the cantilever beam (the model with clamped-free end conditions). So owing to this factor we deal with a selfadjoint (or symmetric) operator rather than with a skew-selfadjoint (or skew-symmetric) operator.*

2) The problem (2.1), (2.2), *and* (2.6) *defines a C_0-semigroup in $\mathcal{H}$ and the operator $i\mathcal{L}$ is an infinitesimal generator of this semigroup. This fact follows, e.g. from the Riesz-basis property of the generalized eigenfunctions of $\mathcal{L}$, which will be proven in a forthcoming paper. So, the use of the term "dynamics generator" for $i\mathcal{L}$ is justified (Curtain & Zwart, 1995).*

Now we reproduce an important result proven in (Shubov & Kindrat, 2018).

THEOREM 2.1 1) $\mathcal{L}$ is an unbounded operator with a compact resolvent, whose spectrum consists of a countable set of normal eigenvalues (i.e. isolated eigenvalues, each of at most finite multiplicity (Locker, 2000; Marcus, 1988; Gohberg & Krein, 1996)). This set can accumulate only at infinity.

2) For any combination of the parameters, such that $|k_1|+|k_2|+|\alpha|+|\beta|>0$, the operator $\mathcal{L}$ is a finite-rank perturbation of the selfadjoint operator $\mathcal{L}_0$, corresponding to the case $k_1=k_2=\alpha=\beta=0$. The fact that $\mathcal{L}$ is a perturbation of $\mathcal{L}_0$ should be understood in the following sense. The operators $\mathcal{L}^{-1}$ and $\mathcal{L}_0^{-1}$ exist and are related by the rule

$$\mathcal{L}^{-1}=\mathcal{L}_0^{-1}+\mathcal{T}, \tag{2.14}$$

where $\mathcal{T}$ is a finite-rank operator. The following formulae are valid for any $G=(g_0,g_1)^T\in\mathcal{H}$:

$$(\mathcal{L}_0^{-1}G)(x)=\left[\mathbb{R}[g_1](x),\, ig_0(x)\right]^T, \tag{2.15}$$

with $\mathbb{R}[\cdot]$ being defined as the Volterra integral operator in $\mathcal{H}$:

$$\mathbb{R}[f](x)=-i\int_0^x \mathrm{d}\tau\int_0^\tau \frac{\mathrm{d}\eta}{EI(\eta)}\int_\eta^1 \mathrm{d}\xi\int_\xi^1 \rho(\nu)\,f(\nu)\,\mathrm{d}\nu, \tag{2.16}$$

and

$$(\mathcal{T}G)(x)=\frac{i}{2EI(1)}\left[p(x)g_0'(1)+q(x)g_0(1),\ 0\right]^T, \tag{2.17}$$

where

$$\begin{aligned} p(x)&=\frac{1}{3}\left[k_2+\frac{EI'(1)}{EI(1)}\beta\right]x^3-\left[k_2+\left(\frac{EI'(1)}{EI(1)}+1\right)\beta\right]x^2,\\ q(x)&=\frac{1}{3}\left[\frac{EI'(1)}{EI(1)}k_1+\alpha\right]x^3-\left[\left(\frac{EI'(1)}{EI(1)}+1\right)k_1+\alpha\right]x^2. \end{aligned} \tag{2.18}$$

3) The decomposition similar to (2.14) is valid for the adjoint operator, i.e. $(\mathcal{L}^*)^{-1}=\mathcal{L}_0^{-1}+\mathcal{T}^*$, where $\mathcal{T}^*$ is given by formulae (2.17) and (2.18) in which k_1 and k_2 have been replaced by $(-k_2)$ and $(-k_1)$, and α and β by $(-\alpha)$ and $(-\beta)$, respectively.

4) If $k_1+k_2=0$ and $\alpha\geqslant 0,\beta\geqslant 0$, then the operator $\mathcal{L}$ is dissipative (Gohberg & Krein, 1996; Dunford & Schwartz, 1963).

The spectral equation. From now on we assume that the structural parameters, ρ and EI, are positive constants. We mention here that dealing with a variable coefficient case would make the calculations extremely lengthy without providing essential changes to our main asymptotic results (Locker, 2000; Fedoryuk, 1993; Mennicken & Möller, 2003). The eigenvalue-eigenfunction equation for the operator $\mathcal{L}$, $\mathcal{L}\Phi=\lambda\Phi$, can be equivalently written in terms of the polynomial operator pencil $\mathcal{P}(\lambda)$ (Marcus, 1988; Shubov & Kindrat, 2018; Shubov & Shubov, 2016) defined by the expression

$$\mathcal{P}(\lambda)\varphi_0=\varphi_0''''-\lambda^2\frac{\rho}{EI}\varphi_0 \tag{2.19}$$

on the domain

$$\begin{aligned}\mathcal{D}(\mathcal{P}(\lambda))=\big\{\varphi_0\in H^4(0,1);\quad \varphi_0(0)=\varphi_0'(0)=0;\quad EI\,\varphi_0''(1)=-k_1 i\lambda\,\varphi_0(1)-\beta i\lambda\,\varphi_0'(1),\\ EI\,\varphi_0'''(1)=\alpha i\lambda\,\varphi_0(1)+k_2 i\lambda\,\varphi_0'(1)\big\}.\end{aligned} \tag{2.20}$$

To simplify the presentation below, let us define the scaled quantities $\widetilde{\lambda},\widetilde{\alpha},\widetilde{\beta},\widetilde{k_1}$, and $\widetilde{k_2}$ by the formulae

$$\lambda=\sqrt{\frac{EI}{\rho}}\,\widetilde{\lambda},\quad \alpha=\sqrt{EI\rho}\,\widetilde{\alpha},\quad \beta=\sqrt{EI\rho}\,\widetilde{\beta},\quad k_1=\sqrt{EI\rho}\,\widetilde{k}_1,\quad k_2=\sqrt{EI\rho}\,\widetilde{k}_2. \tag{2.21}$$

If we substitute these parameters into (2.19)–(2.20), then after omitting the tilde, we arrive at the following spectral problem for the pencil:

$$\varphi'''' = \lambda^2\varphi, \qquad \varphi(0) = \varphi'(0) = 0, \tag{2.22}$$

$$\varphi''(1) = -k_1 i\lambda\varphi(1) - \beta i\lambda\varphi'(1), \qquad \varphi'''(1) = \alpha i\lambda\varphi(1) + k_2 i\lambda\varphi'(1). \tag{2.23}$$

In what follows, it is convenient to introduce new functions

$$C_\pm(z) = \cosh z \pm \cos z, \qquad S_\pm(z) = \sinh z \pm \sin z. \tag{2.24}$$

Notice, the function

$$\varphi(\lambda, x) = \mathcal{A}(\lambda)C_-(\sqrt{\lambda}x) + \mathcal{B}(\lambda)S_-(\sqrt{\lambda}x), \tag{2.25}$$

with $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ being unknown functions of λ, satisfies the differential equation and left-end boundary conditions of (2.22). Substituting (2.25) into the α-boundary condition $\varphi'''(1) = i\alpha\lambda\varphi(1) + ik_2\lambda\varphi'(1)$, and denoting $\mu = \sqrt{\lambda}$, we obtain that the function

$$\begin{aligned}\varphi(\lambda, x) = {} & [-\mu C_+(\mu) + ik_2\mu C_-(\mu) + i\alpha S_-(\mu)]C_-(\mu x) + \\ & [\mu S_-(\mu) - ik_2\mu S_+(\mu) - i\alpha C_-(\mu)]S_-(\mu x)\end{aligned} \tag{2.26}$$

satisfies the clamped conditions at the left end and the α-boundary condition at the right end. The β-boundary condition, $\varphi''(1) = -ik_1\lambda\varphi(1) - i\beta\lambda\varphi'(1)$, generates the following *spectral equation* for the pencil:

$$\begin{aligned}&\mu[i(k_1+k_2)\sinh\mu\sin\mu + k_1k_2(1-\cosh\mu\cos\mu) + (1+\cosh\mu\cos\mu)] = \\ &-i\beta\mu^2(\cosh\mu\sin\mu + \sinh\mu\cos\mu) + \alpha\beta\mu(1-\cosh\mu\cos\mu) \\ &-i\alpha(\cosh\mu\sin\mu - \sinh\mu\cos\mu).\end{aligned} \tag{2.27}$$

3. Asymptotic analysis of the spectrum

The first result of this section (Theorem 3.1) is concerned with the location of the set of the eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}'}$ of the operator $\mathcal{L}$ on the complex plane. It is convenient to prove the result on the spectrum's location in terms of a new complex parameter $\mu = \sqrt{\lambda}$, with the branch being fixed by the requirement that $\sqrt{\lambda} > 0$ for $\lambda > 0$. It is clear that if $\lambda \in \mathbb{C}$, then $\mu \in \overline{\mathbb{C}^+}$ (the closed upper half-plane). The set of points $\{\mu_n = \sqrt{\lambda_n}\}_{n\in\mathbb{Z}'}$ is in fact the set of solutions of the spectral equation (2.27).

THEOREM 3.1 1) If the parameters of the model are such that $1 - k_1k_2 \neq \pm(k_1+k_2)$, then the set of roots of the spectral equation (2.27) splits into two subsets. The first subset is confined to a strip parallel to the real axis in the upper half-plane of the μ-plane. The second subset is symmetric to the first one with respect to the bisectors of the first and second coordinate angles of the μ-plane. That is, the second subset is confined to the strip parallel to the positive imaginary semi-axis on the μ-plane.

2) If $\beta \neq 0$, then Statement (1) of the theorem can be strengthened. Namely, in this case the roots of the spectral equation are located in the strip parallel to the real axis on the complex λ-plane.

Proof. It is convenient to represent the set of eigenvalues of the operator $\mathcal{L}$ in the form $\{\lambda_n^+\}_{n\in\mathbb{Z}'} \cup \{\lambda_n^-\}_{n\in\mathbb{Z}'}$, where λ_n^+ is the notation for an eigenvalue located in the closed upper half plane, $\lambda_n^+ \in$

$\overline{\mathbb{C}^+}$, and λ_n^- is the notation for an eigenvalue located in the open lower half plane, $\lambda_n^- \in \mathbb{C}^-$. As is known (Russell, 1986; Locker, 2000; Mennicken & Möller, 2003), the set of the eigenvalues of $\mathcal{L}$ is symmetric with respect to the imaginary axis on the λ-plane, i.e. if $\lambda_n^\pm$ is an eigenvalue then $\left(-\overline{\lambda_n^\pm}\right)$ is an eigenvalue as well. It can be easily seen that the set of the eigenvalues $\{\lambda_n^+\}_{n\in\mathbb{Z}'}$ generates the set of complex points $\left\{\mu_n^+ = \sqrt{\lambda_n^+}\right\}_{n\in\mathbb{Z}'}$ in the first coordinate angle of the μ-plane ($\mu = \sqrt{\lambda}$), which is symmetric with respect to its bisector. The set of the eigenvalues $\{\lambda_n^-\}_{n\in\mathbb{Z}'}$ generates the set $\left\{\mu_n^- = \sqrt{\lambda_n^-}\right\}_{n\in\mathbb{Z}'}$ in the second coordinate angle of the μ-plane, which is symmetric with respect to its bisector. Statement (1) will be proven if we show that when $|\Re\,\mu_n^\pm| \to \infty$ as $|n| \to \infty$, then $\Im\,\mu_n^\pm \leqslant C < \infty$ as $|n| \to \infty$ (or when $\Im\,\mu_n^\pm \to \infty$ as $|n| \to \infty$, then $|\Re\,\mu_n^\pm| \leqslant C < \infty$ as $|n| \to \infty$).

1) Let us use contradiction argument and assume that there exists a sequence of roots of the spectral equation (2.27), $\{\mu_n\}_{n=1}^\infty$, such that $\Re\,\mu_n \to \infty$ and $\Im\,\mu_n \to \infty$ as $n \to \infty$. In what follows, it is convenient to use the notation

$$\left\{\begin{matrix}\psi_n\\ \chi_n\end{matrix}\right\} = g_n(1+O(1)), \tag{3.1}$$

which means that two different sequences, $\{\psi_n\}_{n\in\mathbb{Z}'}$ and $\{\chi_n\}_{n\in\mathbb{Z}'}$, can be asymptotically approximated by one and the same sequence $\{g_n\}_{n\in\mathbb{Z}'}$, i.e. $\sup_{n\in\mathbb{Z}'}|\psi_n/g_n - 1| \leqslant C_1 < \infty$ and $\sup_{n\in\mathbb{Z}'}|\chi_n/g_n - 1| \leqslant C_2 < \infty$ (see (Fedoryuk, 1993; Murray, 1994)).

For the aforementioned sequence $\{\mu_n = x_n + iy_n\}$, the following asymptotic approximations hold:

$$\left\{\begin{matrix}\cosh\mu_n\\ \sinh\mu_n\end{matrix}\right\} = \frac{1}{2}\mathrm{e}^{x_n+iy_n}\left(1+O\left(\mathrm{e}^{-2x_n}\right)\right), \qquad \left\{\begin{matrix}\cos\mu_n\\ -i\sin\mu_n\end{matrix}\right\} = \frac{1}{2}\mathrm{e}^{-ix_n+y_n}\left(1+O\left(\mathrm{e}^{-2y_n}\right)\right). \tag{3.2}$$

If we denote

$$E_n = \tfrac{1}{4}\mathrm{e}^{x_n+y_n+i(-x_n+y_n)}, \tag{3.3}$$

then the following approximations are valid for the products appearing in Eq.(2.27):

$$\begin{aligned}\left\{\begin{matrix}\cosh\mu_n\cos\mu_n\\ \sinh\mu_n\cos\mu_n\end{matrix}\right\} &= E_n\left(1+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right),\\ \left\{\begin{matrix}\cosh\mu_n\sin\mu_n\\ \sinh\mu_n\sin\mu_n\end{matrix}\right\} &= iE_n\left(1+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right).\end{aligned} \tag{3.4}$$

Substituting approximations (3.4) into Eq.(2.27), we get

$$\begin{aligned}&\mu_n\Big\{\left[1+E_n\left(1+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right)\right]-(k_1+k_2)E_n\left(1+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right)\\ &+k_1k_2\left[1-E_n\left(1+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right)\right]\Big\} = -\mu_n^2 i\beta E_n\left[(1+i)+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right]\\ &+\mu_n\alpha\beta\left[1-E_n\left(1+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right)\right]+i\alpha E_n\left[(1-i)+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right].\end{aligned} \tag{3.5}$$

If $\beta \neq 0$, dividing both sides of Eq.(3.5) by $\mu_n^2 E_n$, we obtain a new equation

$$-i\beta\left[1+i+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right]+O\left(\mu_n^{-1}\right) = 0.$$

This equation fails when $|\mu_n| \to \infty$ as $n \to \infty$. If $\beta = 0$, dividing Eq.(3.5) by E_n, we obtain

$$-\mu_n\left[1-k_1-k_2-k_1k_2+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right]+\alpha\left[1+i+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right] = 0. \tag{3.6}$$

Since $1-k_1-k_2-k_1k_2\neq 0$, Eq.(3.6) becomes $1-k_1-k_2-k_1k_2=O\left(\mu_n^{-1}\right)$, which does not hold when n is large enough. So, we have shown that a sequence $\{\mu_n\}_{n\in\mathbb{Z}'}$, such that $\Re\,\mu_n\to\infty$ and $\Im\,\mu_n\to\infty$ as $n\to\infty$ does not exist.

2) Now we assume that there exists a sequence of roots $\{\widetilde{\mu}_n\}_{n\in\mathbb{Z}'}$ of the spectral equation, such that $\Re\,\widetilde{\mu}_n\to-\infty$ and $\Im\,\widetilde{\mu}_n\to\infty$ as $n\to\infty$, and show that this assumption leads to a contradiction as well. Without misunderstanding we omit the tilde "$\sim$" in the proof below. Denoting $\mu_n=-x_n+iy_n$, we get the following asymptotic approximations:

$$\begin{Bmatrix}\cosh\mu_n\\ \sinh\mu_n\end{Bmatrix}=\frac{1}{2}\mathrm{e}^{x_n-iy_n}\left(1+O\left(\mathrm{e}^{-2x_n}\right)\right),\qquad \begin{Bmatrix}\cos\mu_n\\ -i\sin\mu_n\end{Bmatrix}=\frac{1}{2}\mathrm{e}^{ix_n+y_n}\left(1+O\left(\mathrm{e}^{-2y_n}\right)\right). \tag{3.7}$$

Using (3.3), we obtain the following approximations for the products appearing in Eq.(2.27):

$$\begin{aligned}\begin{Bmatrix}\cosh\mu_n\cos\mu_n\\ -\sinh\mu_n\cos\mu_n\end{Bmatrix}&=\overline{E_n}\left(1+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right),\\ \begin{Bmatrix}\cosh\mu_n\sin\mu_n\\ -\sinh\mu_n\sin\mu_n\end{Bmatrix}&=i\overline{E_n}\left(1+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right).\end{aligned} \tag{3.8}$$

Substituting these approximations into Eq.(2.27), we get

$$\begin{aligned}&\mu_n\Big\{\left[1+\overline{E_n}\left(1+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right)\right]+(k_1+k_2)\overline{E_n}\left(1+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right)\\ &+k_1k_2\left[1-\overline{E_n}\left(1+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right)\right]\Big\}=-i\beta\mu_n^2\overline{E_n}\left[(-1+i)+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right]\\ &+\alpha\beta\mu_n\left[1-\overline{E_n}\left(1+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right)\right]-i\alpha\overline{E_n}\left[(1+i)+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right].\end{aligned} \tag{3.9}$$

If $\beta\neq 0$, we divide this equation by $\mu_n^2\overline{E_n}$ and have a new equation

$$\beta\left[-1+i+O\left(\mathrm{e}^{-2x_n}\right)+O\left(\mathrm{e}^{-2y_n}\right)\right]=O\left(\mu_n^{-1}\right),\qquad n\to\infty.$$

This equation fails when $|\mu|_n\to\infty$ as $n\to\infty$. If $\beta=0$, dividing Eq.(3.9) by $\overline{E_n}$, we obtain

$$1-k_1k_2+k_1+k_2=O\left(\mu_n^{-1}\right),\qquad n\to\infty.$$

Since $1-k_1k_2+k_1+k_2\neq 0$, the above equation is not valid.

Statement (1) of the theorem is proven.

Now we prove Statement (2), i.e. we show that for $\beta\neq 0$, the entire spectrum is located in a strip parallel to the real axis on the λ-*plane*. More precisely, we show that for the set $\left\{\mu_n^{\pm}=\sqrt{\lambda_n^{\pm}}\equiv x_n^{\pm}+iy_n^{\pm}\right\}_{n\in\mathbb{Z}'}$, the following results hold:

$$\sup_{n\in\mathbb{Z}'}x_n^+y_n^+\leqslant C<\infty,\qquad \sup_{n\in\mathbb{Z}'}\left|x_n^-y_n^-\right|\leqslant C<\infty. \tag{3.10}$$

We will prove the first estimate from (3.10); the second estimate can be shown in a similar fashion. Based on Statement (1) we know that the subset $\{\mu_n^+\}_{n\in\mathbb{Z}'}$ is located in two semi-infinite strips symmetric with respect to the bisector of the first coordinate angle. The first strip is parallel to the positive real semi-axis and the second one is parallel to the positive imaginary semi-axis. Let us denote the horizontal strip by S and its width by d. We can choose the numeration in such a way that $\{\mu_n^+\}_{n=1}^{\infty}\subset S$. In the proof below we will omit the superscript "+."

It can be easily checked that the following estimates hold:

$$2\{\cosh\mu_n, \sinh\mu_n\}^T = \mathrm{e}^{\mu_n}\left(1+O\left(\mathrm{e}^{-2x_n}\right)\right), \qquad x_n = \Re\mu_n, \tag{3.11}$$

where the superscript "T" means transposition. Substituting estimates (3.11) into Eq.(2.27) we obtain

$$\begin{aligned}
&\mu\Big\{i(k_1+k_2)\mathrm{e}^{\mu_n}\sin\mu_n\Big[1+O\Big(\mathrm{e}^{-2x_n}\Big)\Big]+k_1k_2\Big(2-\mathrm{e}^{\mu_n}\cos\mu_n\Big[1+O\Big(\mathrm{e}^{-2x_n}\Big)\Big]\Big)\\
&+\Big(2+\mathrm{e}^{\mu_n}\cos\mu_n\Big[1+O\Big(\mathrm{e}^{-2x_n}\Big)\Big]\Big)\Big\} = -i\beta\mu_n^2\mathrm{e}^{\mu_n}\Big\{\sin\mu_n\Big[1+O\Big(\mathrm{e}^{-2x_n}\Big)\Big]+\cos\mu_n\Big[1+O\Big(\mathrm{e}^{-2x_n}\Big)\Big]\Big\}\\
&+\alpha\beta\mu_n\Big\{2-\mathrm{e}^{\mu_n}\cos\mu_n\Big[1+O\Big(\mathrm{e}^{-2x_n}\Big)\Big]\Big\}-i\alpha\mathrm{e}^{\mu_n}\Big\{\sin\mu_n\Big[1+O\Big(\mathrm{e}^{-2x_n}\Big)\Big]-\cos\mu_n\Big[1+O\Big(\mathrm{e}^{-2x_n}\Big)\Big]\Big\}.
\end{aligned} \tag{3.12}$$

Dividing Eq.(3.12) by e^{μ_n} and taking into account that $|\cos\mu_n|+|\sin\mu_n| \leqslant C_0 < \infty$ for $\mu_n \in S$, we obtain the following asymptotic form of the spectral equation:

$$\begin{aligned}
&\mu_n\left[i(k_1+k_2)\sin\mu_n - k_1k_2\cos\mu_n+\cos\mu_n+O\left(\mathrm{e}^{-x_n}\right)\right] = -i\beta\mu_n^2\left[\sin\mu_n+\cos\mu_n+O\left(\mathrm{e}^{-2x_n}\right)\right]\\
&-\alpha\beta\mu_n\left[\cos\mu_n+O\left(\mathrm{e}^{-x_n}\right)\right]+i\alpha\left[\cos\mu_n-\sin\mu_n+O\left(\mathrm{e}^{-2x_n}\right)\right].
\end{aligned} \tag{3.13}$$

Since $\beta \neq 0$, dividing this equation by $\beta\mu_n^2$ yields $\sin\mu_n+\cos\mu_n = O\left(\mu_n^{-1}\right) = O\left(x_n^{-1}\right)$. Separating the real and imaginary parts of this equation and using

$$\sin\mu_n+\cos\mu_n = \sqrt{2}\left[\sin(x_n+\pi/4)\cosh y_n + i\cos(x_n+\pi/4)\sinh y_n\right],$$

we obtain the following system:

$$\sin(x_n+\pi/4)\cosh y_n = O\left(x_n^{-1}\right), \quad \cos(x_n+\pi/4)\sinh y_n = O\left(x_n^{-1}\right), \quad n\to\infty. \tag{3.14}$$

Taking into account that $1 \leqslant \cosh y_n \leqslant C_2 < \infty$, we rewrite the first equation of (3.14) as

$$\sin(x_n+\pi/4) = O\left(x_n^{-1}\right), \qquad x_n \to \infty, \tag{3.15}$$

which yields $x_n = \pi n - \pi/4 + O(1)$, and thus $x_n^{-1} = O\left(n^{-1}\right)$. For $\cos(x_n+\pi/4)$, we immediately obtain the estimate $\cos(x_n+\pi/4) = 1+O\left(n^{-2}\right)$, which transforms the second equation of (3.14) to the form $\sinh y_n = O\left(n^{-1}\right)$, and therefore $y_n = O\left(n^{-1}\right)$. The obtained estimates imply (3.10). □

As will be shown below, the spectral asymptotics are different for the cases $\beta \neq 0$ and $\beta = 0$. Our first result, Theorem 3.2 below, is concerned with the case $\beta = 0$. In the rest of the paper we consider only non-negative boundary control parameters, in order too keep the paper of a moderate size. If needed, the case of arbitrary complex parameters can be treated in a similar fashion.

THEOREM 3.2 Assume that $\beta = 0$ and $k_1, k_2 > 0$. Let A, B, and K be defined by

$$A = k_1+k_2, \qquad B = 1-k_1k_2, \qquad K = B/A. \tag{3.16}$$

1) If $0 \leqslant K < 1$, then $0 < (1-K)(1+K)^{-1} \leqslant 1$ and there could be a finite number of eigenvalues located in the lower half-plane, while there is an infinite sequence of eigenvalues in the upper half-plane, with the following asymptotic approximation as $n \to \infty$:

$$\lambda_n = (\pi n)^2 - i\pi n \ln\frac{1-K}{1+K} - \ln^2\sqrt{\frac{1-K}{1+K}} + \frac{\alpha(A-iB)}{A^2-B^2} + O\left(\frac{1}{n}\right), \quad \lambda_{-n} = -\overline{\lambda_n}. \tag{3.17}$$

2) If $-1 < K < 0$, then $1 < (1-K)(1+K)^{-1} < \infty$ and there could be a finite number of eigenvalues located in the upper half-plane, while there is an infinite sequence of eigenvalues in the lower half-plane, with the asymptotic approximation given by (3.17).

3) If $1 < K < \infty$, then $-1 < (1-K)(1+K)^{-1} < 0$ and there could be a finite number of eigenvalues located in the lower half-plane, while there is an infinite sequence of eigenvalues in the upper half-plane, with the following asymptotic approximation as $n \to \infty$:

$$\lambda_n = \pi^2\left(n+\frac{1}{2}\right)^2 - i\pi\left(n+\frac{1}{2}\right)\ln\left|\frac{1-K}{1+K}\right| \\ - \ln^2\sqrt{\left|\frac{1-K}{1+K}\right|} + \frac{\alpha(A-iB)}{A^2-B^2} + O\left(\frac{1}{n}\right), \quad \lambda_{-n} = -\overline{\lambda_n}. \tag{3.18}$$

4) If $-\infty < K < -1$, then $-\infty < (1-K)(1+K)^{-1} < -1$ and there could be a finite number of eigenvalues located in the upper half-plane, while there is an infinite sequence of eigenvalues in the lower half-plane, with the asymptotic approximation given by (3.18).

Proof. The spectral equation (2.27) for $\beta = 0$ becomes

$$\mu\left[i(k_1+k_2)\sinh\mu\sin\mu + k_1k_2(1-\cosh\mu\cos\mu) + (1+\cosh\mu\cos\mu)\right] \\ = i\alpha(\sinh\mu\cos\mu - \cosh\mu\sin\mu). \tag{3.19}$$

Let us consider this equation in the horizontal strip S of a finite width d, which is parallel to the positive real semi-axis on the μ-plane. Using approximations (3.11), we rewrite Eq.(3.19) in the asymptotic form

$$(1-k_1k_2)\cos\mu + i(k_1+k_2)\sin\mu - \frac{i\alpha}{\mu}(\cos\mu - \sin\mu) = O\left(e^{-\mu}\right), \quad \mu \in S, \quad |\mu| \to \infty. \tag{3.20}$$

Let

$$\mathcal{K}(\mu) = \frac{1-k_1k_2-\frac{i\alpha}{\mu}}{k_1+k_2+\frac{\alpha}{\mu}}, \tag{3.21}$$

then Eq.(3.20) can be written in the form $i\sin\mu = -\mathcal{K}(\mu)\cos\mu + O(e^{-\mu})$, or equivalently,

$$e^{2i\mu} = \frac{1-\mathcal{K}(\mu)}{1+\mathcal{K}(\mu)} + O\left(e^{-\mu}\right). \tag{3.22}$$

Using the notations (3.16), we have the following representations:

$$\frac{1-\mathcal{K}(\mu)}{1+\mathcal{K}(\mu)} = \mathcal{A} + \frac{1}{\mu}\mathcal{B} + O\left(\frac{1}{\mu^2}\right), \quad \text{where} \quad \mathcal{A} = \frac{A-B}{A+B}, \quad \text{and} \quad \mathcal{B} = \frac{2\alpha(B+iA)}{(A+B)^2}, \tag{3.23}$$

which reduce Eq.(3.22) to

$$e^{2i\mu} = \mathcal{A} + \frac{\mathcal{B}}{\mu} + O\left(\frac{1}{\mu^2}\right), \qquad \mu \in S, \qquad |\mu| \to \infty. \tag{3.24}$$

To find the approximations for the roots of this equation, we have to distinguish the following cases: (i) $0 < \mathcal{A} \leqslant 1$; (ii) $-1 < \mathcal{A} < 0$; (iii) $1 < \mathcal{A} < \infty$; (iv) $-\infty < \mathcal{A} < -1$, It is technically more convenient to prove Statements (1) and (3) of the theorem corresponding to cases (i) and (ii), then to prove Statements (2) and (4) corresponding to cases (iii) and (iv).

Proof of Statement (1). If $0 \leqslant K < 1$, then $0 < \mathcal{A} \leqslant 1$ and the following formula holds for the roots of Eq.(3.24) located in the first coordinate angle of the μ-plane:

$$\mu_n = \pi n - i \ln \sqrt{\frac{1-K}{1+K}} + \frac{\alpha(A-iB)}{(A^2-B^2)\pi n} + O\left(\frac{1}{n^2}\right), \qquad n \to \infty. \tag{3.25}$$

To prove this formula, let us represent Eq.(3.24) in the form convenient for application of Rouché's Theorem:

$$f(\mu) = g(\mu), \quad \text{where} \quad f(\mu) = \mathrm{e}^{2i\mu} - \mathcal{A} - \frac{\mathcal{B}}{\mu} \quad \text{and} \quad g(\mu) = O\left(\mu^{-2}\right). \tag{3.26}$$

Let $\{\mu_n^0\}_{n=1}^{\infty}$ be the set of points defined explicitly by the formula

$$\mu_n^0 = \pi n - i \ln \sqrt{\mathcal{A}} - \frac{\mathcal{B} i}{2\mathcal{A}\pi n}, \qquad n \in \mathbb{N}^+. \tag{3.27}$$

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive numbers converging to zero; let $\mathcal{D}_{\varepsilon_n}(\mu_n^0)$ and $\mathcal{C}_{\varepsilon_n}(\mu_n^0)$ be the disk and the circle centered at μ_n^0 of radius ε_n. Let $\mu \in \mathcal{C}_{\varepsilon_n}(\mu_n^0)$, i.e. $\mu = \mu_n^0 + \varepsilon_n \mathrm{e}^{i\varphi}$, $0 \leqslant \varphi < 2\pi$. Let us evaluate both functions, f and g, on $\mathcal{C}_{\varepsilon_n}(\mu_n^0)$. We have

$$\begin{aligned} f(\mu) &= f(\mu_n^0 + \varepsilon_n \mathrm{e}^{i\varphi}) = \exp\left(2\pi n i + \ln \mathcal{A} + \frac{\mathcal{B}}{\mathcal{A}\pi n} + 2i\varepsilon_n \mathrm{e}^{i\varphi}\right) - \mathcal{A} - \frac{\mathcal{B}}{\mu_n^0 + \varepsilon_n \mathrm{e}^{i\varphi}} \\ &= \mathcal{A} \exp\left(\frac{\mathcal{B}}{\mathcal{A}\pi n}\right)\left[1 + 2i\varepsilon_n \mathrm{e}^{i\varphi} + O\left(\varepsilon_n^2\right)\right] - \mathcal{A} - \frac{\mathcal{B}}{\pi n}\left[1 + \frac{i \ln \sqrt{\mathcal{A}}}{\pi n} + O\left(\frac{1}{n^2}\right)\right] \\ &= 2i\mathcal{A}\varepsilon_n \mathrm{e}^{i\varphi} + O\left(\frac{1}{n^2}\right) + O\left(\frac{\varepsilon_n}{n}\right) + O\left(\varepsilon_n^2\right). \end{aligned}$$

Since $\varepsilon_n \to 0$ as $n \to \infty$, for large enough n we can write

$$|f(\mu)|_{\mu \in \mathcal{C}_{\varepsilon_n}(\mu_n^0)} \geqslant \mathcal{A}\varepsilon_n. \tag{3.28}$$

At the same time, the following bound holds for $g(\mu)$:

$$|g(\mu)|_{\mu \in \mathcal{C}_{\varepsilon_n}(\mu_n^0)} \leqslant \frac{C}{n^2}, \tag{3.29}$$

with some absolute constant C. If we choose $\varepsilon_n = 2C/(\mathcal{A}n^2)$, then on the circle $\mathcal{C}_{\varepsilon_n}(\mu_n^0)$ we obtain that $|f(\mu)| > |g(\mu)|$. Therefore, by Rouché's Theorem $f(\mu)$ and $f(\mu) + g(\mu)$ have the same number of roots (i.e. exactly one) in the disk $\mathcal{D}_{\varepsilon_n}(\mu_N^0)$. This yields the justification of formula (3.25). Squaring (3.25) we obtain (3.17).

The possible existence of a finite number of roots of the spectral equation in the second quadrant will be shown at the end of the proof of this theorem (see the last paragraph of the proof).

Proof of Statement (3). If $1 < K < \infty$, then $-1 < \mathcal{A} < 0$ and the following asymptotic formula holds for the roots of Eq.(3.24) located in the first coordinate angle of the $\mu-plane$:

$$\mu_n = \pi\left(n+\frac{1}{2}\right) - i\ln\sqrt{\left|\frac{1-K}{1+K}\right|} + \frac{\alpha(A-iB)}{(A^2-B^2)\pi n} + O\left(\frac{1}{n^2}\right), \qquad n\to\infty. \tag{3.30}$$

To prove this formula, we reduce Eq.(3.20) to the form (3.24) and define the functions f and g by formulae (3.26). Now we introduce a set of points $\{\widetilde{\mu}_n\}_{n=1}^{\infty}$ by

$$\widetilde{\mu}_n = \pi\left(n+\frac{1}{2}\right) - i\ln\sqrt{|\mathcal{A}|} - \frac{\mathcal{B}i}{2\mathcal{A}\pi n}, \qquad n\in\mathbb{N}^+. \tag{3.31}$$

Let $\mathcal{C}_{\varepsilon_n}(\widetilde{\mu}_n)$ be the circle centered at $\widetilde{\mu}_n$ of radius ε_n, i.e. if $\mu\in\mathcal{C}_{\varepsilon_n}(\widetilde{\mu}_n)$, then $\mu = \widetilde{\mu}_n + \varepsilon_n e^{i\varphi}$, $0\leqslant\varphi\leqslant 2\pi$. Let us evaluate $f(\mu)$ on this circle and have

$$\begin{aligned} f(\mu) &= \exp\left(2\pi n i + \pi i + \ln|\mathcal{A}| + \frac{\mathcal{B}}{\mathcal{A}\pi n} + 2i\varepsilon_n e^{i\varphi}\right) - \mathcal{A} - \frac{\mathcal{B}}{\widetilde{\mu}_n + \varepsilon_n e^{i\varphi}} \\ &= -|\mathcal{A}|\exp\left(\frac{\mathcal{B}}{\mathcal{A}\pi n}\right)\left[1 + 2i\varepsilon_n e^{i\varphi} + O\left(\varepsilon_n^2\right)\right] - \mathcal{A} - \frac{\mathcal{B}}{\pi n} + O\left(\frac{1}{n^2}\right) \\ &= 2i\mathcal{A}\varepsilon_n e^{i\varphi} + O\left(\frac{1}{n^2}\right) + O\left(\frac{\varepsilon_n}{n}\right) + O\left(\varepsilon_n^2\right), \end{aligned}$$

which implies for large enough n the following estimates:

$$|f(\mu)|_{\mu\in\mathcal{C}_{\varepsilon_n}(\widetilde{\mu}_n)} \geqslant \mathcal{A}\varepsilon_n, \qquad |g(\mu)|_{\mu\in\mathcal{C}_{\varepsilon_n}(\widetilde{\mu}_n)} \leqslant \frac{C}{n^2}, \tag{3.32}$$

with some absolute constant C. Choosing $\varepsilon_n = 2C/(2n^2)$, we combine these estimates and use Rocuhé's Theorem to obtain (3.30), which implies (3.18).

REMARK 3.1 If K is such that $-1 < K < 0$, then $|\mathcal{A}| > 1$. In this case, formula (3.25) generates a sequence of roots located in the lower half-plane. Even though these roots formally satisfy Eq.(3.24), they cannot be treated as approximations for the spectrum of our problem, since Eq.(3.24) is an approximation for the spectral equation (2.27) for $\mu\in S\subset\overline{\mathbb{C}^+}$. For similar reasons, (3.30) are not valid approximations for the case $-\infty < K < -1$.

So far, we have formulae (3.17) and (3.18) proven when $K\in[0,1)\cup(1,\infty)$. To prove these formulas for the complementary interval when $K\in(-\infty,-1)\cup(-1,0)$, we return to Eq.(3.19) and consider this equation in a strip, $\widehat{S}$, parallel to the negative real semi-axis in the second quadrant of the μ-plane, i.e. let $\mu\in\widehat{S}\subset\overline{\mathbb{C}^+}$ and such that $\Re\mu\to-\infty$. For $\mu\in\widehat{S}$, one has

$$2\{\cosh\mu, -\sinh\mu\}^T = e^{-\mu}\left(1+O\left(e^{2\mu}\right)\right). \tag{3.33}$$

Applying these approximations, we rewrite Eq.(3.19) in the asymptotic form as

$$(1-k_1k_2)\cos\mu - i(k_1+k_2)\sin\mu + \frac{i\alpha}{\mu}(\cos\mu+\sin\mu) = O\left(e^{\mu}\right), \quad \mu\in S, \quad |\mu|\to\infty. \tag{3.34}$$

Using (3.21), we obtain the following representation for Eq.(3.34):

$$e^{2i\mu} = \frac{1+\mathcal{K}(-\mu)}{1-\mathcal{K}(-\mu)} + O\left(e^{\mu}\right), \qquad \mu\in\widehat{S}. \tag{3.35}$$

Introducing $\nu = -\mu$, we rewrite this equation in the form

$$e^{2i\nu} = \frac{1-\mathcal{K}(\nu)}{1+\mathcal{K}(\nu)} + O\left(e^{-\nu}\right), \qquad \Re\,\nu \to \infty. \tag{3.36}$$

Due to the relation between μ and ν, one can see that we look for the asymptotic approximation of the roots of Eq.(3.36), when ν is located in a strip in the fourth quadrant, centrally symmetric to the strip $\widehat{S}$. Let S^* be a notation for this strip. Using (3.23), Eq.(3.36) can be represented as

$$e^{2i\nu} = \mathcal{A} + \frac{\mathcal{B}}{\nu} + O\left(\frac{1}{\nu^2}\right), \qquad |\nu| \to \infty, \qquad \nu \in S^*. \tag{3.37}$$

Proof of Statement (2). If $-1 < K < 0$, then $1 < \mathcal{A} < \infty$. In this case, the asymptotic approximations for the roots of Eq.(3.37) for $\nu \in S^* \subset \mathbb{C}^-$ can be given in the form

$$\nu_n = \pi n - i \ln \sqrt{\frac{1-K}{1+K}} + \frac{\alpha A}{(A^2-B^2)\pi n} - \frac{\alpha B i}{(A^2-B^2)\pi n} + O\left(\frac{1}{n^2}\right), \qquad n \to \infty. \tag{3.38}$$

If we return to the original parameter, μ, we obtain the following asymptotic approximation:

$$\mu_n = -\pi n + i \ln \sqrt{\frac{1-K}{1+K}} - \frac{\alpha A}{(A^2-B^2)\pi n} + \frac{\alpha B i}{(A^2-B^2)\pi n} + O\left(\frac{1}{n^2}\right), \qquad n \to \infty. \tag{3.39}$$

Proof of Statement (4). If $-\infty < K < -1$, which corresponds to $-\infty < \mathcal{A} < -1$. The asymptotic approximation for the roots of (3.37) for $\nu \in S^* \subset \mathbb{C}^-$ can be given as follows:

$$\nu_n = \pi\left(n + \frac{1}{2}\right) - i \ln \sqrt{\left|\frac{1-K}{1+K}\right|} + \frac{\alpha A}{(A^2-B^2)\pi n} - \frac{\alpha B i}{(A^2-B^2)\pi n} + O\left(\frac{1}{n^2}\right), \tag{3.40}$$

where $n \to \infty$. In terms of the original spectral parameter, we obtain

$$\mu_n = -\pi\left(n + \frac{1}{2}\right) + i \ln \sqrt{\left|\frac{1-K}{1+K}\right|} - \frac{\alpha A}{(A^2-B^2)\pi n} + \frac{\alpha B i}{(A^2-B^2)\pi n} + O\left(\frac{1}{n^2}\right), \tag{3.41}$$

It can be readily checked that formula (3.18) follows from (3.30) and (3.41).

To complete the proof of the theorem, we have to show that in Cases (1) and (3), there is at most a finite number of eigenvalues of the operator $\mathcal{L}$ in the lower half-plane (and in Cases (2) and (4), in the upper half-plane). This can be seen immediately: since in Case (1) we have $0 < \mathcal{A} < 1$, the equation (3.37) can have only a finite number of solutions for $\Im\,\nu < 0$. Similar arguments can be used for the cases (2)–(4). □

REMARK 3.2 As follows from formulae (3.17) and (3.18), when $|\mathcal{A}| < 1$, there exist at most a finite number of eigenvalues located in the lower half-plane ($\Im\,\lambda_n < 0$) and/or a finite number of real eigenvalues ($\Im\,\lambda_n = 0$), and an infinite set of eigenvalues located in the upper half-plane ($\Im\,\lambda_n > 0$). It means that when $|\mathcal{A}| < 1$ (which corresponds to the case $k_1 k_2 < 1$), there could be a finite number of unstable and/or neutrally stable eigenmodes and an infinite set of stable eigenmodes. When $|\mathcal{A}| > 1$, there could be a finite number of stable and/or neutrally stable eigenmodes, and an infinite set of unstable eigenmodes.

In the next statement we consider the cases excluded in Theorem 3.2, i.e. the cases $|K| = 1$. In these cases, the condition from Theorem 3.1, that $1 - k_1k_2 \neq \pm(k_1 + k_2)$, fails, i.e. the spectrum is not confined to the semi-infinite strips parallel to the coordinate axes on the μ-plane. As is shown below, when $|K| = 1$, the asymptotic distribution of the eigenvalues strongly depends on the parameter α. In the next theorem we consider the case when $\alpha = 0$. In Section 4 we carry out the analysis of the spectrum when $\alpha > 0$, which is significantly more complicated than the case $\alpha = 0$.

THEOREM 3.3 Assume that $\alpha = \beta = 0$.

1) When $K = 1$ (or $1 - k_1k_2 = k_1 + k_2$), the set of the eigenvalues splits into two branches denoted by $\{\lambda_n^+\}_{n=n_0^+}^{\infty}$ and $\{\lambda_n^-\}_{n=n_0^-}^{\infty}$. If

$$L = \frac{1 + k_1k_2}{1 - k_1k_2}, \tag{3.42}$$

then the lower limits, n_0^+ and n_0^- for the above sequences, are given as

$$n_0^+ = \left\lceil -\frac{1}{2} + \frac{\ln\sqrt{L + \sqrt{L^2 - 1}}}{\pi} \right\rceil, \qquad n_0^- = \left\lceil -\frac{1}{2} - \frac{\ln\sqrt{L + \sqrt{L^2 - 1}}}{\pi} \right\rceil, \tag{3.43}$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. The following formulae are valid:

$$\begin{aligned} \lambda_n^- &= -4\pi\left(n + \frac{1}{2}\right)\ln\sqrt{L + \sqrt{L^2 - 1}} + 2i\left[\pi^2\left(n + \frac{1}{2}\right)^2 - \ln^2\sqrt{L + \sqrt{L^2 - 1}}\right], \\ \lambda_n^+ &= -\overline{\lambda_n^-}, \qquad n \geqslant n_0^+. \end{aligned} \tag{3.44}$$

2) When $K = -1$ (or $1 - k_1k_2 = -(k_1 + k_2)$), the set of eigenvalues splits into two branches denoted by $\{\widetilde{\lambda}_m^+\}_{m=-\infty}^{m_0^+}$ and $\{\widetilde{\lambda}_m^-\}_{m=-\infty}^{m_0^-}$. If

$$M = \frac{k_1k_2 + 1}{k_1k_2 - 1}, \tag{3.45}$$

then the lower limits, m_0^+ and m_0^-, are given as

$$m_0^+ = \left\lfloor \frac{\ln\sqrt{M + \sqrt{M^2 - 1}}}{\pi} \right\rfloor, \qquad m_0^- = \left\lfloor -\frac{\ln\sqrt{M + \sqrt{M^2 - 1}}}{\pi} \right\rfloor, \tag{3.46}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. The following formulae are valid:

$$\begin{aligned} \widetilde{\lambda}_m^+ &= 4\pi m \ln\sqrt{M + \sqrt{M^2 - 1}} - 2i\left[\pi^2 m^2 - \ln^2\sqrt{M + \sqrt{M^2 - 1}}\right], \\ \widetilde{\lambda}_m^- &= -\overline{\widetilde{\lambda}_m^+}, \qquad m \geqslant m_0^+. \end{aligned} \tag{3.47}$$

Proof. Case (1). Let us represent Eq.(3.19) when $K = 1$, i.e. when $k_1 + k_2 = 1 - k_1k_2$, in the form

$$\cosh\mu\cos\mu + i\sinh\mu\sin\mu = -\frac{1 + k_1k_2}{1 - k_1k_2}. \tag{3.48}$$

Verifying that $\cosh\mu\cos\mu + i\sinh\mu\sin\mu = \cosh\mu(1 + i)$, we rewrite Eq.(3.48) in a new form

$$\cosh\mu(1 + i) = -L, \tag{3.49}$$

with L being defined in (3.42). Let us show that the set of roots of this equation is symmetric with respect to both the bisector of the first and the third coordinate angles, and the bisector of the second and fourth coordinate angles (see Fig. 1(a)). It suffices to show that if μ_0 is a solution of Eq.(3.49), then $(i\overline{\mu_0})$ and $(-i\overline{\mu_0})$ are also solutions. We have

$$\cosh(i\overline{\mu_0}(1+i)) = \cosh(\overline{\mu_0}(-1+i)) = \overline{\cosh(\mu_0(-1-i))} = \overline{\cosh(\mu_0(1+i))} = -L,$$

and also

$$\cosh(-i\overline{\mu_0}(1+i)) = \cosh(i\overline{\mu_0}(1+i)) = -L,$$

which yields the aforementioned symmetry.

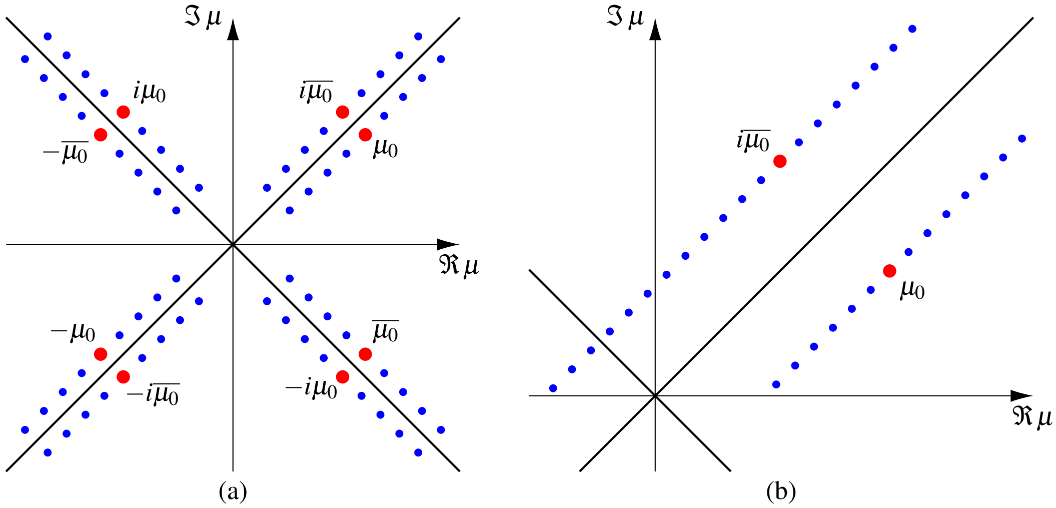

FIG. 1. Distribution of the eigenvalues on the μ-plane when $K = 1$ and $\alpha = 0$.

Eq.(3.49) generates a quadratic equation, $Y^2 + 2YL + 1 = 0$, with respect to $Y(\mu) = e^{\mu(1+i)}$. The roots of this equation are

$$Y_{1,2} = -L \pm \sqrt{L^2-1}. \tag{3.50}$$

Since $K = 1$ and $k_1, k_2 > 0$, we obtain that $1 - k_1k_2 > 0$ and thus $L > 1$, which means that both roots (3.50) are negative. If $Y_1 = -L + \sqrt{L^2-1}$, then we have the equation $e^{\mu(1+i)} = -\left(L - \sqrt{L^2-1}\right)$ with the following set of roots:

$$\mu_n^- = \pi\left(n+\frac{1}{2}\right)(1+i) + (1-i)\ln\sqrt{L-\sqrt{L^2-1}}, \qquad n \in \mathbb{Z}. \tag{3.51}$$

Taking into account that $L - \sqrt{L^2-1} = \left(L+\sqrt{L^2-1}\right)^{-1}$, we rewrite (3.51) in the form

$$\mu_n^- = \left[\pi\left(n+\frac{1}{2}\right) - \ln\sqrt{L+\sqrt{L^2-1}}\right] + i\left[\pi\left(n+\frac{1}{2}\right) + \ln\sqrt{L+\sqrt{L^2-1}}\right], \qquad n \in \mathbb{Z}. \tag{3.52}$$

We recall that Eq.(3.19) is valid only for $\mu \in \overline{\mathbb{C}^+}$. Therefore, one can take only those μ_n^- from set (3.52), for which the following inequality holds:

$$\Im\,\mu_n^- = \pi\left(n+\frac{1}{2}\right) + \ln\sqrt{L+\sqrt{L^2-1}} \geqslant 0, \qquad \text{or} \qquad n \geqslant -\frac{1}{2} - \frac{\ln\sqrt{L+\sqrt{L^2-1}}}{\pi}, \tag{3.53}$$

which means that for the lower bound for n one can take n_0^- given by formula (3.43). The geometry of the spectrum in this case shows that the model has an infinite set of stable vibrational modes and at most a finite number of unstable modes.

The second root of Eq.(3.49) generates the equation $\mathrm{e}^{\mu(1+i)} = -\left(L+\sqrt{L^2-1}\right)$, whose set of roots can be given as

$$\mu_n^+ = i\overline{\mu_n^-} = \left[\pi\left(n+\frac{1}{2}\right) + \ln\sqrt{L+\sqrt{L^2-1}}\right] + i\left[\pi\left(n+\frac{1}{2}\right) - \ln\sqrt{L+\sqrt{L^2-1}}\right], \quad n \in \mathbb{Z}. \tag{3.54}$$

From the set (3.54) one has to take only those values of μ_n^+ for which

$$\Im\,\mu_n^+ = \pi\left(n+\frac{1}{2}\right) - \ln\sqrt{L+\sqrt{L^2-1}} \geqslant 0, \qquad \text{or} \qquad n \geqslant -\frac{1}{2} + \frac{\ln\sqrt{L+\sqrt{L^2-1}}}{\pi}, \tag{3.55}$$

which yields the formula for n_0^+ from (3.43) (see Fig. 1(b)).

Statement (1) of the theorem is shown.

Now we briefly discuss *Case (2)*, i.e. when $K=-1$. Taking into account that $k_1k_2-1 = k_1+k_2 > 0$, we represent the spectral equation (3.48) in the form

$$\cosh\mu(1-i) = M, \tag{3.56}$$

with M being given in (3.45). This equation generates a quadratic equation, $Y^2-2YM+1=0$, with respect to $Y(\mu) = \mathrm{e}^{\mu(1-i)}$. This equation has two positive roots

$$Y_{1,2} = M \pm \sqrt{M^2-1}. \tag{3.57}$$

If $Y_1 = M+\sqrt{M^2-1}$, then the roots of the equation $\mathrm{e}^{\mu(1-i)} = Y_1$ are

$$\widehat{\mu}_m^+ = \left[\pi m + \ln\sqrt{M+\sqrt{M^2-1}}\right] - i\left[\pi m - \ln\sqrt{M+\sqrt{M^2-1}}\right], \qquad m \in \mathbb{Z}. \tag{3.58}$$

Taking into account that one has to consider only such m for which

$$\Im\,\widehat{\mu}_m^+ = -\pi m + \ln\sqrt{M+\sqrt{M^2-1}} \geqslant 0, \qquad \text{or} \qquad m \leqslant \frac{\ln\sqrt{M+\sqrt{M^2-1}}}{\pi}, \tag{3.59}$$

we obtain the lower limit m_0^+ given in formula (3.46). The second root of Eq.(3.57) generates $\mathrm{e}^{\mu(1-i)} = M-\sqrt{M^2-1}$, whose roots can be given as

$$\widehat{\mu}_m^- = i\overline{\widehat{\mu}_m^+} = \left[\pi m - \ln\sqrt{M+\sqrt{M^2-1}}\right] - i\left[\pi m + \ln\sqrt{M+\sqrt{M^2-1}}\right], \qquad m \in \mathbb{Z}. \tag{3.60}$$

Taking into account that one has to consider only such m for which

$$\Im\,\widehat{\mu}_m^- = -\pi m - \ln\sqrt{M+\sqrt{M^2-1}} \geqslant 0, \qquad \text{or} \qquad m \leqslant -\frac{\ln\sqrt{M+\sqrt{M^2-1}}}{\pi}, \tag{3.61}$$

we obtain the lower limit m_0^- given in formula (3.46). Using the symmetry of the spectrum, we arrive at formulae (3.47). □

4. Spectral asymptotics for the case $|K| = 1$ and $\alpha > 0$

We begin this section with a technical result (Lemma 4.1) that we need for the proof of the main result, Theorem 4.2.

DEFINITION 4.1 We say that two positive functions f_1 and f_2 are related by $f_1(x) \asymp f_2(x)$, $x \in [0,L]$, if there exist two positive constants C_1 and C_2, such that

$$C_1 f_1(x) \leqslant f_2(x) \leqslant C_2 f_1(x), \qquad 0 \leqslant x \leqslant L. \tag{4.1}$$

We say that two positive functions f_1 and f_2 are related by $f_1(x) \prec f_2(x)$, if there exists a positive constant C, such that

$$f_1(x) \leqslant C f_2(x), \qquad 0 \leqslant x \leqslant L. \tag{4.2}$$

LEMMA 4.1 A complex-valued function

$$f(z) = \mathrm{e}^{-iz} - Az, \qquad A \in \mathbb{C}, \qquad A \neq \overline{A}, \tag{4.3}$$

has a countable set of roots in the upper half-plane. There could be a finite number of roots in the close lower half-plane and there are no purely imaginary roots. The set of roots in the upper half-plane consists of two branches denoted by $\{z_n^+\}_{n=1}^\infty$ and $\{z_m^-\}_{m=1}^\infty$. The following asymptotic approximation is valid for the branch $\{z_n^+\}_{n=1}^\infty$ located in the first coordinate angle:

$$z_n^+ = 2\pi n + i\ln(2\pi nA) - \frac{\ln(2\pi nA)}{2\pi n} + O\left(\frac{\ln^2 n}{n^2}\right), \qquad n \to \infty. \tag{4.4}$$

The following asymptotic approximation is valid for the branch $\{z_m^-\}_{m=1}^\infty$ located in the second coordinate angle:

$$z_m^- = -2\pi m + i\ln(-2\pi mA) + \frac{\ln(-2\pi mA)}{2\pi m} + O\left(\frac{\ln^2 m}{m^2}\right), \qquad m \to \infty. \tag{4.5}$$

Proof. 1) To solve the equation $f(z) = 0$, we use Kantorovich's Theorem on Newton's method (see Theorem 4.4.2 of (Atkinson & Han, 2001) or Ch. XVIII of (Kantorovich & Akilov, 1982)). According to this theorem, the iterative method of Newton, applied to a quite general nonlinear equation, converges to a solution z^* near some point z^0, provided that $f(z)$ satisfies certain boundedness conditions near z^0 (see (4.7) and (4.8) below).

Let z_n^0 be the initial approximation to the root of the function f given by the formula

$$z_n^0 = 2\pi n + i\ln(2\pi nA) - \frac{\ln(2\pi nA)}{2\pi n}. \qquad n \in \mathbb{N}^+. \tag{4.6}$$

Let $\Omega \subset \mathbb{C}$ be the closed unit disk centered at z_n^0, i.e. if $z \in \Omega$, then $z = z_n^0 + \varepsilon \mathrm{e}^{i\varphi}$, $0 \leqslant \varepsilon \leqslant 1$, $0 \leqslant \varphi < 2\pi$. We introduce two constantes, η and K, by the rule:

$$\eta \geqslant \left|\frac{f(z_n^0)}{f'(z_n^0)}\right|, \qquad K \geqslant \max_{z\in\Omega}\left|\frac{f''(z)}{f'(z_n^0)}\right|. \tag{4.7}$$

By Kantorovich's Theorem, if the function, f, is such that the condition

$$\frac{\left|f(z_n^0)f''(z)\right|}{\left|f'(z_n^0)\right|^2} \leqslant \frac{1}{2} \tag{4.8}$$

is satisfied, then the root of the function f is located in a closed disk

$$\left|z - z_n^0\right| \leqslant r_0 = \frac{1-\sqrt{1-2h}}{h}\,\eta, \qquad h = \eta K, \tag{4.9}$$

and is unique on the open disk

$$\left|z - z_n^0\right| < r_1 = \frac{1+\sqrt{1-2h}}{h}\,\eta. \tag{4.10}$$

To apply the theorem, we evaluate η and K corresponding to our case and have

$$(i)\ f(z_n^0) = \exp -2\pi n i + \ln(2\pi nA) + i\frac{\ln(2\pi nA)}{2\pi n} - A\left\{2\pi n + i\ln(2\pi nA) - \frac{\ln(2\pi nA)}{2\pi n}\right\}$$
$$= 2\pi nA\left\{i\frac{\ln(2\pi nA)}{2\pi n} + O\left(\frac{\ln^2 n}{n^2}\right)\right\} - A\left\{i\ln(2\pi nA) - \frac{\ln(2\pi nA)}{2\pi n}\right\} = O\left(\frac{\ln^2 n}{n}\right),$$
$$(ii)\ \eta \geqslant \left|\frac{f(z_n^0)}{f'(z_n^0)}\right| \asymp \frac{\ln^2 n}{n^2}, \qquad (iii)\ K \geqslant \max_{z\in\Omega}\left|\frac{f''(z)}{f'(z_n^0)}\right| \asymp \max_{z\in\Omega}\left|\mathrm{e}^{-i(z-z_n^0)}\right| \asymp 1.$$

From these estimates we obtain that

$$h = \eta K \prec \frac{\ln^2 n}{n^2},$$

which means that the function $f(z)$ has a unique root z_n^* located in a circle of radius r_0 around the point z_n^0, where

$$r_0 = \frac{1-\sqrt{1-2h}}{h}\eta \asymp \frac{1-(1-h)}{h}\eta \asymp \frac{\ln^2 n}{n^2}. \tag{4.11}$$

The result (4.11) means that the function f has exactly one root in a small vicinity of each point z_n^0, $n \geqslant n_0$, given in (4.6), i.e. if z_n^* is the root, then $\left|z_n^0 - z_n^*\right| \leqslant C\ln^2 n/n^2$, with C being an absolute constant.

To complete the proof, we have to show that there are no other roots except those obtained by Kantorovich's Theorem in the subdomain of the first coordinate angle bounded by the real positive semi-axis ($z = x$, $z \geqslant R$), the arc of radius R ($z = R\mathrm{e}^{i\varphi}$, $0 \leqslant \varphi \leqslant \pi/2$), and the positive imaginary semi-axis ($z = x + iy$, $y \geqslant R$). It is convenient to consider two subdomains located above and below the bisector of the first coordinate angle ($z = x + ix$, $x \geqslant R$). In what follows, we provide a detailed proof for the more complicated case of the subdomain located below the bisector. The fact that there could be at most a finite number of roots in the upper triangular subdomain can be shown without difficulties.

Let $\widetilde{\Omega}$ be the subdomain of the first coordinate angle bounded by the diagonal ($\Re z = \Im z$), the arc of a circle of a large enough radius R ($z = R\mathrm{e}^{i\varphi}$, $0 \leqslant \varphi \leqslant \pi/4$), and the real axis ($\Re z \geqslant R$, $\Im z = 0$). In $\widetilde{\Omega}$ we distinguish three subdomains: Ω_0, Ω^-, and Ω^+ (see Fig. 2). Namely, let Ω_0 be a curved strip with the center line given by the equation

$$z = 2\pi x + i\ln(2\pi xA) - \frac{\ln(2\pi xA)}{2\pi x},$$

so that if $z \in \Omega_0$, then

$$z = 2\pi x + i\ln(2\pi xA) - \frac{\ln(2\pi xA)}{2\pi x} + \delta i, \qquad \text{with} \qquad -1 \leqslant \delta \leqslant 1. \tag{4.12}$$

As is shown, the roots of the function $f(z)$ are asymptotically located on the center line near the points with $x = n \in \mathbb{N}^+$. The fact that there are no other roots in this curved strip follows from the second part

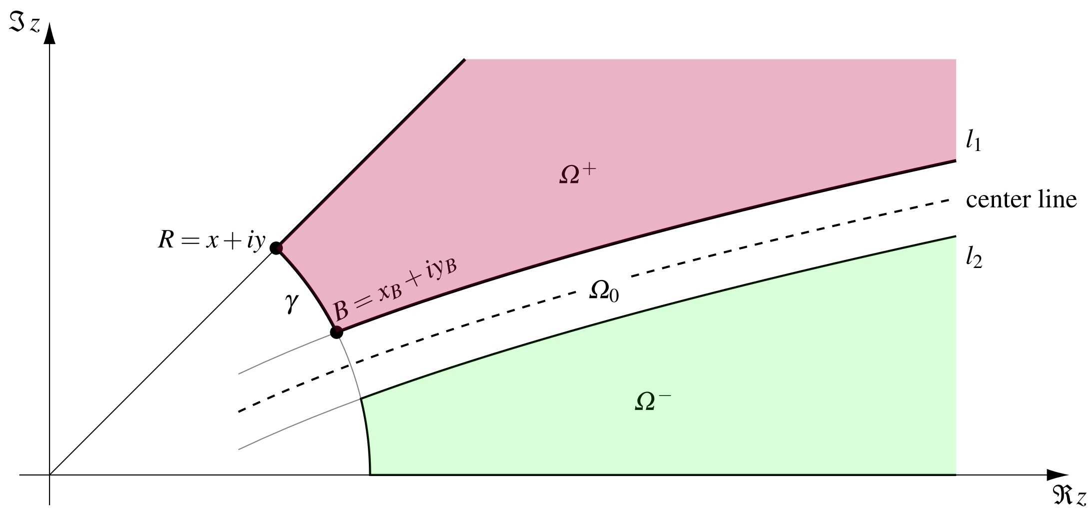

FIG. 2. Subdomains of the complex plane in the first coordinate angle.

of Kantorovich's Theorem (Kantorovich & Akilov, 1982; Atkinson & Han, 2001). Let Ω^+ be a domain in $\widetilde{\Omega}$ located above Ω_0 and Ω^- be a domain in $\widetilde{\Omega}$ located below Ω_0 (see Fig. 2).

Let us show that the function $f(z)$ does not have any roots in the subdomain Ω^+ (the proof for Ω^- is similar). It is technically convenient to deal with the function $\varphi(z) = 1 - Ae^{iz}z$ whose roots coincide with the roots of $f(z)$. The result will be obtained if we show that the function $\psi(z) \equiv 1 - \varphi(z) = Ae^{iz}z$ satisfies the estimate $|\psi(z)| \leqslant 1/2$ when $z \in \Omega^+$. Evaluating $|\psi(z)|$ on the upper boundary of Ω_0 given by (4.12) with $\delta = 1$ yields

$$\begin{aligned} |\psi(z)| &= \left| A \exp i2\pi x - \ln(2\pi xA) - i\frac{\ln(2\pi xA)}{2\pi x} - 1 \left\{ 2\pi x + i\ln(2\pi xA) - \frac{\ln(2\pi xA)}{2\pi x} + i \right\} \right| \\ &= |A| \left| \frac{e^{-1}}{2\pi xA} \left[1 - i\frac{\ln(2\pi xA)}{2\pi x} + O\left(\frac{\ln^2 x}{x^2}\right) \right] 2\pi x \left[1 + O\left(\frac{\ln x}{x}\right) \right] \right| = e^{-1} + O\left(\frac{\ln x}{x}\right). \end{aligned} \tag{4.13}$$

It is clear that $|\psi(x)| \leqslant 1/2$ for large x. Now we evaluate $|\psi(z)|$ on the bisector for large x:

$$|\psi(x+ix)| = \left| A(x+ix)e^{ix-x} \right| = \sqrt{2}\,|A|\,xe^{-x} \leqslant 1/2. \tag{4.14}$$

Finally we evaluate $|\psi(z)|$ on the arc γ (see Fig. 2). For $z \in \gamma$ we have

$$|\psi(z)| = |A| \cdot R \left| \exp(iRe^{i\mathrm{Arg}\{z\}}) \right| = |A|\,R\,e^{-R\sin(\mathrm{Arg}\{z\})}. \tag{4.15}$$

The expression $\exp(-R\sin(\mathrm{Arg}\{z\}))$ attains its maximal value when $\mathrm{Arg}\{z\}$ corresponds to the point B on Fig. 2. Since $B = x_B + iy_B$ belongs to the upper boundary of Ω_0, the following representation holds:

$$z = 2\pi x + i\ln|2\pi xA| - \mathrm{Arg}\{A\} - \frac{\ln|2\pi xA|}{2\pi x} - i\frac{\mathrm{Arg}\{A\}}{2\pi x} + i. \tag{4.16}$$

Thus, $\operatorname{Arg}\{B\}$ can be approximated in the following way:

$$\begin{aligned}\operatorname{Arg}\{B\} &= \tan^{-1}\left(\frac{y_B}{x_B}\right) = \tan^{-1}\left\{\frac{\ln|2\pi x_B A| + 1 - \frac{\operatorname{Arg}\{A\}}{2\pi x_B}}{2\pi x_B}\left[1 + O\left(\frac{1}{x_B}\right)\right]\right\}\\ &= \tan^{-1}\left\{\frac{\ln|2\pi x_B A|+1}{2\pi x_B} + O\left(\frac{\ln x_B}{x_B^2}\right)\right\} = \frac{\ln|2\pi x_B A|+1}{2\pi x_B} + O\left(\frac{\ln^3 x_B}{x_B^3}\right),\end{aligned}$$

and therefore for $\sin(\operatorname{Arg}\{B\})$ we obtain

$$\sin(\operatorname{Arg}\{B\}) = \frac{\ln|2\pi x_B A|+1}{2\pi x_B} + O\left(\frac{\ln^3 x_B}{x_B^3}\right). \tag{4.17}$$

$|B|$ can be approximated by

$$|B| = x_B\sqrt{1+\frac{y_B^2}{x_B^2}} = x_B\sqrt{1+O\left(\frac{\ln^2 x_B}{x_B^2}\right)} = 2\pi x_B + O(1). \tag{4.18}$$

Substituting (4.17) and (4.18) into (4.15), we obtain

$$\begin{aligned}|\varphi(x)| &\asymp |A|\left[2\pi x_B + O(1)\right]\exp -\left[2\pi x_B + O(1)\right]\left[\frac{\ln|2\pi x_B A|+1}{2\pi x_B} + O\left(\frac{\ln^3 x_B}{x_B^3}\right)\right]\\ &= |A|\left[2\pi x_B + O(1)\right]\exp -(\ln|2\pi x_B A|+1) + O\left(\frac{\ln x_B}{x_B}\right) = \mathrm{e}^{-1} + O\left(\frac{\ln x_B}{x_B}\right),\end{aligned} \tag{4.19}$$

which implies that $|\varphi(z)| \leqslant 1/2$.

Collecting (4.13), (4.14), and (4.19), we obtain that $|\psi(z)|$ is bounded above by $1/2$ along the boundaries of the curved triangle. Using the Phragmén-Lindelöf Theorem (Evgrafov (1978), p.247), we claim that $|\psi(z)| \leqslant 1/2$ in Ω^+, which means that $\varphi(z) \geqslant 1/2$.

2) To prove formula (4.5), we notice that the two equations, $f(z) = 0$ and $\overline{f(z)} = 0$ have the same set of roots. Let us rewrite the second equation in terms of a new variable $\xi = -\bar{z}$ and we have

$$\mathrm{e}^{i\bar{z}} - \overline{A}\bar{z} = \mathrm{e}^{-i\xi} - (-\overline{A})\xi = 0.$$

As is already shown, this equation has the set of roots in the first quadrant whose asymptotics are given by formula (4.4), in which A has been replaced by $(-\overline{A})$. Therefore, we have

$$\xi_m = 2\pi m + i\ln(2\pi m(-\overline{A})) - \frac{\ln(2\pi m(-\overline{A}))}{2\pi m} + O\left(\frac{\ln^2 m}{m^2}\right), \qquad m \to \infty,$$

or in terms of z the following approximation holds:

$$\overline{z_{\bar{m}}} = -2\pi m - i\ln(-2\pi m\overline{A}) + \frac{\ln(-2\pi m\overline{A})}{2\pi m} + O\left(\frac{\ln^2 m}{m^2}\right),$$

which yields formula (4.5).

3) To show that Eq.(4.3) does not have purely imaginary roots, we use contradiction argument, and assume that iy_0, $y_0 \in \mathbb{R}$ is a root of Eq.(4.3). This means that $\exp(y_0) = iAy_0$. Since $A \neq \overline{A}$, the left-hand

side of this equation is real, while the right-hand side has a non-trivial imaginary part, which means that such equation cannot hold.

Finally, we assume that there exists a sequence of roots of Eq.(4.3) in the lower half-plane, $\{z_n = x_n - iy_n, x_n > 0, y_n \geqslant 0\}_{n=1}^{\infty}$, and show that this assumption leads to a contradiction as well. Indeed, we have $\exp(-ix_n - y_n) = A(x_n - iy_n)$. When $n \to \infty$, the left-hand side stays bounded while the right-hand side tends to infinity. □

Now we are in a position to present the main result of this section.

THEOREM 4.2 Assume that $\beta = 0$ and $\alpha > 0$.

(1) When $K = 1$ (or $1 - k_1k_2 = k_1 + k_2$), the following asymptotic approximation holds as $n \to \infty$:

$$\lambda_n = (\pi n)^2 - \frac{\ln^2(2\pi nD)}{4} - \frac{\ln(2\pi nD)}{2} + \frac{1}{2D} + i\pi n\ln(2\pi nD) + O\left(\frac{\ln^2 n}{n}\right), \qquad \lambda_{-n} = -\overline{\lambda_n}, \quad (4.20)$$

where

$$D = \frac{1 - k_1k_2}{(1+i)\alpha}. \quad (4.21)$$

There exists at most a finite number of eigenvalues in the closed lower half-plane.

(2) When $K = -1$ (or $k_1k_2 - 1 = k_1 + k_2$), the following asymptotic approximation holds as $m \to \infty$:

$$\lambda_m = (\pi m)^2 - \frac{\ln^2(2\pi mD)}{4} - \frac{\ln(2\pi mD)}{2} - \frac{1}{2D} - i\pi m\ln(2\pi mD) + O\left(\frac{\ln^2 m}{m}\right), \quad \lambda_{-m} = -\overline{\lambda_m}. \quad (4.22)$$

There exists at most a finite number of eigenvalues in the closed upper half-plane.

Proof. The spectral equation (3.19) written for the case $\alpha > 0$ and $K = 1$ has the form:

$$\cosh\mu(1+i) = -L + \frac{i\alpha}{\mu(1-k_1k_2)}(\sinh\mu\cos\mu - \cosh\mu\sin\mu), \quad (4.23)$$

with L being given in (3.42). Let us show that the set of roots of this equation cannot be confined to a strip of a finite width parallel to the real axis and, due to the symmetry of the spectrum, cannot be confined to a strip parallel to the imaginary axis. Arguing by contradiction, we assume that there exists a sequence of roots of Eq.(4.23), such that $\{\mu_n = x_n + iy_n\}_{n=1}^{\infty}$ and $x_n \to \infty$ as $n \to \infty$ and $0 \leqslant y_n \leqslant C < \infty$. The following estimates can be readily checked:

$$2\cosh\mu_n(1+i) = \mathrm{e}^{(x_n-y_n)+i(x_n+y_n)} + O\left(\mathrm{e}^{-x_n}\right), \quad 2\{\cosh\mu_n, \sinh\mu_n\}^T = \mathrm{e}^{x_n+iy_n} + O\left(\mathrm{e}^{-x_n}\right). \quad (4.24)$$

Substituting (4.24) into Eq.(4.23), and multiplying both sides by $2\mathrm{e}^{-x_n}$, one gets

$$\mathrm{e}^{-y_n+i(x_n+y_n)} = -2L\mathrm{e}^{-x_n} + \frac{i\alpha}{\mu_n(1-k_1k_2)}\mathrm{e}^{iy_n}(\cos\mu_n - \sin\mu_n) + O\left(\mathrm{e}^{-2x_n}\right). \quad (4.25)$$

Since $0 \leqslant y_n \leqslant C < \infty$, the right-hand side of Eq.(4.25) tends to zero as $n \to \infty$, while the left-hand side is bounded below. The obtained contradiction means that if $x_n \to \infty$ as $n \to \infty$, then $y_n \to \infty$. Using a similar contradiction argument, it can be shown that if $y_n \to \infty$ as $n \to \infty$, then $x_n \to \infty$.

To prove the formula for $\{\mu_n^+\}$ we assume $\mu = x + iy$, with $x \geqslant y \to \infty$, and rewrite Eq.(4.23) in the form

$$\cos\mu + i\tanh\mu\sin\mu = -\frac{L}{\cosh\mu} + \frac{i\alpha}{\mu(1-k_1k_2)}(\tanh\mu\cos\mu - \sin\mu). \quad (4.26)$$

Taking into account that $\tanh\mu = 1-2e^{-2\mu}+O\left(e^{-4\mu}\right)$, we get the following approximations for the entries of Eq.(4.26) as $|\mu|\to\infty$:

$$\begin{aligned}\cos\mu + i\tanh\mu\sin\mu &= \cos\mu + i\sin\mu - 2ie^{-2\mu}\sin\mu + O\left(e^{-4\mu}\sin\mu\right),\\ \tanh\mu\cos\mu - \sin\mu &= (\cos\mu-\sin\mu) - 2e^{-2\mu}\cos\mu + O\left(e^{-4\mu}\cos\mu\right).\end{aligned} \tag{4.27}$$

Since $x \geqslant y \to \infty$, the following estimates hold for the remainder terms in (4.27):

$$\left|e^{-4\mu}\sin\mu\right| \asymp e^{-4x}e^{y} \prec e^{-3x} = \left|e^{-3\mu}\right|, \qquad \left|e^{-4\mu}\cos\mu\right| \asymp e^{-4x}e^{y} \prec e^{-3x} = \left|e^{-3\mu}\right|. \tag{4.28}$$

Using these estimates we represent Eq.(4.26) in the asymptotic form

$$e^{i\mu} - 2ie^{-2\mu}\sin\mu = -\frac{L}{\cosh\mu} + \frac{i\alpha}{\mu(1-k_1k_2)}\left[(\cos\mu-\sin\mu) - 2e^{-2\mu}\cos\mu\right] + O\left(e^{-3\mu}\right). \tag{4.29}$$

Since $\left|e^{-2\mu}\sin\mu\right| \prec e^{-x} = \left|e^{-\mu}\right|$, $\left|e^{-2\mu}\cos\mu\right| \prec e^{-x} = \left|e^{-\mu}\right|$, we have

$$\left|\frac{L}{\cosh\mu} + \frac{2i\alpha e^{-2\mu}\cos\mu}{\mu(1-k_1k_2)} - 2ie^{-2\mu}\sin\mu\right| \prec e^{-x} = \left|e^{-\mu}\right|, \tag{4.30}$$

which yields the following modification of Eq.(4.29):

$$e^{i\mu} = \frac{i\alpha}{2\mu(1-k_1k_2)}\left[(1+i)e^{i\mu} + (1-i)e^{-i\mu}\right] + O\left(e^{-\mu}\right). \tag{4.31}$$

Let us rewrite Eq.(4.31) in the form

$$e^{-iz} - Dz - i = O\left(e^{-(1+i)\mu}\right), \tag{4.32}$$

where $z = 2\mu$ and D is given in (4.21). We modify this equation in such a way that for the corresponding model equation (i.e. the equation obtained from (4.32) by replacing the right-hand side with zero) one can apply Lemma 4.1. If we multiply both sides of Eq.(4.32) by $e^{1/D}$ and set $\widetilde{z} = z + i/D$, then we get a new equation

$$e^{-i\widetilde{z}} - \widetilde{D}\widetilde{z} = O\left(e^{-(1+i)\mu}\right), \qquad \text{where} \qquad \widetilde{D} = De^{1/D}. \tag{4.33}$$

Applying formula (4.4) to the model equation induced by (4.33) (i.e. $\exp(-i\widetilde{z}) - \widetilde{D}\widetilde{z} = 0$), we obtain the following asymptotic approximation for its roots:

$$\begin{aligned}\widetilde{z}_n &= 2\pi n + i\ln(2\pi n\widetilde{D}) - \frac{\ln(2\pi n\widetilde{D})}{2\pi n} + O\left(\frac{\ln^2 n}{n^2}\right)\\ &= 2\pi n + i\left[\ln(2\pi nD) + \frac{1}{D}\right] - \left[\frac{\ln(2\pi nD)}{2\pi n} + \frac{1}{2\pi nD}\right] + O\left(\frac{\ln^2 n}{n^2}\right), \qquad n\to\infty.\end{aligned} \tag{4.34}$$

Now we show that, due to the exponential decay rate of the right-hand side of Eq.(4.33), the asymptotic approximation for $\widetilde{z}_n$ that holds for the roots of the model equation is valid for the roots of Eq.(4.33) as well. We represent this equation in the form $f(\widetilde{z}) = g(\widetilde{z})$, where $f(\widetilde{z}) = e^{-i\widetilde{z}} - \widetilde{D}\widetilde{z}$, and for the analytic function g, the following estimate holds:

$$|g(\widetilde{z})| \asymp \left|e^{-(1+i)\widetilde{z}/2}\right|. \tag{4.35}$$

Let us evaluate both functions, f and g, on the circle of a small radius ε centered at the point

$$\widetilde{z}_n^0 = 2\pi n + i\ln(2\pi n\widetilde{D}) - \frac{\ln(2\pi n\widetilde{D})}{2\pi n}, \tag{4.36}$$

with n being large enough, i.e. let $\widetilde{z} = \widetilde{z}_n^0 + \varepsilon e^{i\varphi}$, $0 \leqslant \varphi < 2\pi$. For such $\widetilde{z}$ we have

$$\begin{aligned} f(\widetilde{z}) &= f(\widetilde{z}_n^0 + \varepsilon e^{i\varphi}) = \exp -i\widetilde{z}_n^0 - i\varepsilon e^{i\varphi} - \widetilde{D}\left(\widetilde{z}_n^0 + \varepsilon e^{i\varphi}\right) \\ &= 2\pi n\widetilde{D}\left\{1 + i\frac{\ln(2\pi n\widetilde{D})}{2\pi n} - \frac{\ln^2(2\pi n\widetilde{D})}{4\pi^2 n^2} + O\left(\frac{\ln^3 n}{n^3}\right)\right\}\left\{1 - i\varepsilon e^{i\varphi} + O\left(\varepsilon^2\right)\right\} \\ &\quad - \widetilde{D}\left\{2\pi n + i\ln(2\pi n\widetilde{D}) - \frac{\ln(2\pi n\widetilde{D})}{2\pi n} + \varepsilon e^{i\varphi}\right\}. \end{aligned} \tag{4.37}$$

Let us show that from (4.35) and (4.37) we obtain the estimates of f and g that allow application of Rouché's Theorem. Namely, we have

$$|f(\widetilde{z})| \succ \left[\frac{\ln^2 n}{n} + \varepsilon n\right] \succ \frac{\ln^2 n}{n^2} \quad \text{if} \quad \varepsilon \geqslant \frac{\ln^2 n}{n^2}, \qquad |g(\widetilde{z})| \prec \left|e^{-\widetilde{z}_n^0/2} e^{-i\widetilde{z}_n^0/2}\right| \prec n e^{-n},$$

which means that representation (4.34) holds for the roots of (4.33).

Taking into account that $\mu_n^+ = 1/2(\widetilde{z}_n - i/D)$, we obtain an asymptotic approximation for the sequence $\{\mu_n^+\}_{n=1}^{\infty}$ as $n \to \infty$,

$$\mu_n^+ = \frac{1}{2}\left(\widetilde{z}_n - \frac{i}{D}\right) = \pi n + i\frac{\ln(2\pi nD)}{2} - \frac{\ln(2\pi nD)}{4\pi n} - \frac{1}{4\pi nD} + O\left(\frac{\ln^2 n}{n^2}\right), \tag{4.38}$$

which yields (4.20).

To complete the proof of Statement (1), one has to show that there might be only a finite number of roots of Eq.(4.23) in the second coordinate angle of the μ-plane. To this end, we represent (4.23) in the form (4.26) and assume that $\mu = x + iy$ and $|x| \geqslant y$, $x < 0$. Taking into account that $\tanh\mu = -1 + 2e^{2\mu} + O\left(e^{4\mu}\right)$, we obtain that the following approximations are valid as $|\mu| \to \infty$:

$$\begin{aligned} \cos\mu + i\tanh\mu\sin\mu &= \cos\mu - i\sin\mu + 2ie^{2\mu}\sin\mu + O\left(e^{4\mu}\sin\mu\right), \\ \tanh\mu\cos\mu - \sin\mu &= -(\cos\mu + \sin\mu) + 2e^{2\mu}\cos\mu + O\left(e^{4\mu}\cos\mu\right). \end{aligned} \tag{4.39}$$

Since $-x \geqslant y \to \infty$, the following estimates hold for the remainder terms in (4.39):

$$\left|e^{4\mu}\sin\mu\right| \asymp e^{4x}e^{y} \prec e^{3x} = \left|e^{3\mu}\right|, \qquad \left|e^{4\mu}\cos\mu\right| \asymp e^{4x}e^{y} \prec e^{3x} = \left|e^{3\mu}\right|. \tag{4.40}$$

Using these estimates, we represent Eq.(4.26) in the asymptotic form

$$e^{-i\mu} + 2ie^{2\mu}\sin\mu = -\frac{L}{\cosh\mu} - \frac{i\alpha}{\mu(1 - k_1 k_2)}\left[\cos\mu + \sin\mu - 2e^{2\mu}\cos\mu\right] + O\left(e^{3\mu}\right). \tag{4.41}$$

Since $\left|e^{2\mu}\sin\mu\right| \prec |e^{\mu}|$ and $\left|e^{2\mu}\cos\mu\right| \prec |e^{\mu}|$, we have

$$\left|\frac{L}{\cosh\mu} - \frac{2i\alpha e^{2\mu}\cos\mu}{\mu(1 - k_1 k_2)} + 2ie^{2\mu}\sin\mu\right| \prec e^{x} = |e^{\mu}|. \tag{4.42}$$

This yields the following modification of Eq.(4.41):

$$\mathrm{e}^{-i\mu} = -i\frac{i\alpha}{2\mu(1-k_1k_2)}\left[(1-i)\mathrm{e}^{i\mu}+(1+i)\mathrm{e}^{-i\mu}\right]+O\left(\mathrm{e}^{\mu}\right), \tag{4.43}$$

which can be represented in the form

$$\mathrm{e}^{-i\mu}\left[1+O\left(\frac{1}{\mu}\right)\right] = \frac{\alpha(1+i)\mathrm{e}^{i\mu}}{2\mu(1-k_1k_2)}.$$

As $|\mu|\to\infty$, the left-hand side of this equation tends to infinity, while the right-hand side tends to zero. This means that Eq.(4.26) might have a finite number of solutions in the second coordinate angle on the μ-plane.

Statement (1) is shown.

To prove Statement (2), we rewrite the spectral equation (4.23) for the case $K=-1$ and have

$$i\sinh\mu\sin\mu-\cosh\mu\cos\mu = -M+\frac{i\alpha}{\mu(k_1k_2-1)}(\sinh\mu\cos\mu-\cosh\mu\sin\mu), \tag{4.44}$$

with M given in (3.45). Let us introduce a new independent variable $\xi=-\overline{\mu}$ and rewrite Eq.(4.44) in terms of ξ. We have

$$i\sinh\xi\sin\xi+\cosh\xi\cos\xi = M-\frac{i\alpha}{\xi(1-k_1k_2)}(\sinh\xi\cos\xi-\cosh\xi\sin\xi). \tag{4.45}$$

Direct comparison of this equation and Eq.(4.26) shows that the only difference is the fact that α in Eq.(4.26) has been replaced by $(-\alpha)$ in Eq.(4.45). Since the value of α does not affect the asymptotic estimates, if we carry out steps similar to (4.27)–(4.31), we arrive at the equation similar to (4.32), where in place of D we have $(-D)$, i.e.

$$\mathrm{e}^{-i\xi}-(-D)\xi-i = O\left(\mathrm{e}^{-(1+i)\xi/2}\right). \tag{4.46}$$

Denoting $D'=-D$, we immediately obtain the asymptotic approximation for roots of Eq.(4.46):

$$\xi_m = \pi m+i\frac{\ln(2\pi mD')}{2}-\frac{\ln(2\pi mD')}{4}-\frac{1}{4\pi mD'}+O\left(\frac{\ln^2 m}{m^2}\right). \tag{4.47}$$

Therefore, using the relation between ξ and μ. we obtain the following approximation for $\{\mu_m^-\}_{m=1}^{\infty}$:

$$\mu_m^- = -\pi m+i\frac{\ln(2\pi m\overline{D'})}{2}+\frac{\ln(2\pi m\overline{D'})}{4}+\frac{1}{4\pi m\overline{D'}}+O\left(\frac{\ln^2 m}{m^2}\right). \tag{4.48}$$

Taking into account that $\overline{D'}=D$, we get

$$\mu_m^- = -\pi m+i\frac{\ln(2\pi mD)}{2}+\frac{\ln(2\pi mD)}{4}+\frac{1}{4\pi mD}+O\left(\frac{\ln^2 m}{m^2}\right), \tag{4.49}$$

which yields (4.22).

Finally, to justify that there could be a finite set of eigenvalues in the closed upper half-plane, one can use an argument similar to the one used in proving Statement (1). □

5. Spectral asymptotics for the case $\beta \neq 0$

In this section we derive asymptotic approximations for the solutions of Eq.(2.27) in a strip S on the complex λ-plane, which is parallel to the positive real semi-axis.

THEOREM 5.1 The following asymptotic approximation holds for the eigenvalue, when $n \to \infty$:

$$\lambda_n = (\pi n)^2 - \frac{\pi^2}{2} n + \left(\frac{\pi}{4}\right)^2 + \frac{k_1 + k_2}{\beta} + i\,\frac{1 - k_1 k_2 + \alpha\beta}{\beta} + O\left(\frac{1}{n}\right), \qquad \lambda_{-n} = -\overline{\lambda_n}. \tag{5.1}$$

Proof. Let us rewrite Eq.(2.27) in the form

$$\begin{aligned} &\beta\mu(\cosh\mu\sin\mu + \sinh\mu\cos\mu) + (k_1 + k_2)\sinh\mu\sin\mu - i(1 - k_1k_2 + \alpha\beta)\cosh\mu\cos\mu = \\ &-\frac{\alpha}{\mu}(\cosh\mu\sin\mu - \sinh\mu\cos\mu) + i(1 + k_1k_2 - \alpha\beta). \end{aligned} \tag{5.2}$$

Let $\widetilde{S}$ be the strip on the complex μ-plane, which is an image of the strip S on the λ-plane obtained under the transformation $\mu = \sqrt{\lambda}$. If $\lambda = x + iy \in S$, then the following relation holds for $\mu \in \widetilde{S}$:

$$\mu = \sqrt{\lambda} = \sqrt{x} + i(1/2)yx^{-1/2} + O\left(x^{-3/2}\right). \tag{5.3}$$

Indeed, we have

$$\mu = \sqrt{|\lambda|}\exp\tfrac{i}{2}\tan^{-1}(y/x) = \sqrt[4]{x^2+y^2}\left[\cos\left\{\tfrac{1}{2}\tan^{-1}(y/x)\right\} + i\sin\left\{\tfrac{1}{2}\tan^{-1}(y/x)\right\}\right].$$

Since y is bounded and $x \to \infty$, the following approximations are valid:

$$\cos\left\{\tfrac{1}{2}\tan^{-1}(y/x)\right\} = 1 + O\left(x^{-2}\right), \qquad \sin\left\{\tfrac{1}{2}\tan^{-1}(y/x)\right\} = y/(2x) + O\left(x^{-3}\right). \tag{5.4}$$

With these formulae, we get

$$\mu = \sqrt{x}\left[1 + (y/x)^2 + O\left(x^{-4}\right)\right]\left[1 + iy/(2x) + O\left(x^{-2}\right)\right],$$

which yields (5.3). The following approximations can be easily checked for $\mu \in \widetilde{S}$ and $x \to \infty$:

$$\begin{aligned} (i)\ \mathrm{e}^{\pm\mu} &= \mathrm{e}^{\pm\sqrt{x}}\mathrm{e}^{\pm iy/(2\sqrt{x})}\left[1 + O\left(x^{-3/2}\right)\right] = \mathrm{e}^{\pm\sqrt{x}}\left(1 \pm \frac{iy}{2\sqrt{x}} - \frac{y^2}{4x}\right)\left[1 + O\left(x^{-3/2}\right)\right], \\ (ii)\ \mathrm{e}^{\pm i\mu} &= \mathrm{e}^{\pm i\sqrt{x}}\mathrm{e}^{\mp y/(2\sqrt{x})}\left[1 + O\left(x^{-3/2}\right)\right] = \mathrm{e}^{\pm i\sqrt{x}}\left(1 \mp \frac{y}{2\sqrt{x}} + \frac{y^2}{4x}\right)\left[1 + O\left(x^{-3/2}\right)\right]. \end{aligned} \tag{5.5}$$

Let new functions $E(x,y)$, $F(x,y)$, and $G(x,y)$ be defined by

$$\begin{aligned} E(x,y) &= \frac{1}{2}\mathrm{e}^{\sqrt{x}}\left(1 + \frac{iy}{2\sqrt{x}} - \frac{y^2}{4x}\right), \\ F(x,y) &= \sin\sqrt{x} + \frac{iy}{2\sqrt{x}}\cos\sqrt{x} + \frac{y^2}{4x}\sin\sqrt{x}, \\ G(x,y) &= \cos\sqrt{x} - \frac{iy}{2\sqrt{x}}\sin\sqrt{x} + \frac{y^2}{4x}\cos\sqrt{x}. \end{aligned} \tag{5.6}$$

Making use of (5.5) we obtain the following approximations:

$$\begin{aligned}
\begin{Bmatrix}\cosh\mu\\ \sinh\mu\end{Bmatrix} &= \frac{1}{2}\mathrm{e}^{\sqrt{x}}\left(1+\frac{iy}{2\sqrt{x}}-\frac{y^2}{4x}\right)\left[1+O\left(\frac{1}{x^{3/2}}\right)\right] = E(x,y)\left[1+O\left(\frac{1}{x^{3/2}}\right)\right],\\
\cos\mu &= \cos\sqrt{x}-i\frac{y}{2\sqrt{x}}\sin\sqrt{x}+\frac{y^2}{4x}\cos\sqrt{x}+O\left(\frac{1}{x^{3/2}}\right) = G(x,y)+O\left(\frac{1}{x^{3/2}}\right),\\
\sin\mu &= \sin\sqrt{x}+i\frac{y}{2\sqrt{x}}\cos\sqrt{x}+\frac{y^2}{4x}\sin\sqrt{x}+O\left(\frac{1}{x^{3/2}}\right) = F(x,y)+O\left(\frac{1}{x^{3/2}}\right).
\end{aligned} \tag{5.7}$$

Substituting (5.7) into Eq.(5.2), and dividing by $E(x,y)$, we obtain a new equation

$$\begin{aligned}
&\beta\mu\left[F(x,y)+G(x,y)+O\left(\frac{1}{x^{3/2}}\right)\right]+(k_1+k_2)F(x,y)-i(1-k_1k_2+\alpha\beta)G(x,y)=\\
&-\frac{\alpha}{\mu}\left[F(x,y)-G(x,y)+O\left(\frac{1}{x^{3/2}}\right)\right]+O\left(\frac{1}{x^{3/2}}\right).
\end{aligned} \tag{5.8}$$

Using (5.3) and (5.6), and taking into account that $\mu^{-1}=x^{-1/2}+O\left(x^{-3/2}\right)$ and

$$F(x,y)\pm G(x,y)=\left(1+\frac{y^2}{4x}\right)\left(\sin\sqrt{x}\pm\cos\sqrt{x}\right)+\frac{iy}{2\sqrt{x}}\left(\cos\sqrt{x}\mp\sin\sqrt{x}\right),$$

we reduce Eq.(5.8) to the form

$$\begin{aligned}
&\beta\left[\left(\sqrt{x}+\frac{y^2}{4\sqrt{x}}\right)\left(\sin\sqrt{x}+\cos\sqrt{x}\right)+\frac{iy}{2}\left(\cos\sqrt{x}-\sin\sqrt{x}\right)\right]=\\
&i\left(1-k_1k_2+\alpha\beta-\frac{y(k_1+k_2)}{2\sqrt{x}}\right)\cos\sqrt{x}-\left(k_1+k_2-\frac{y(1-k_1k_2+\alpha\beta)}{2\sqrt{x}}\right)\sin\sqrt{x}\\
&-\frac{\alpha}{\sqrt{x}}\left(\sin\sqrt{x}-\cos\sqrt{x}\right)+O\left(\frac{1}{x}\right).
\end{aligned} \tag{5.9}$$

Separating the real and imaginary parts, we obtain the following system:

$$\begin{aligned}
&\beta\sqrt{x}\left(\sin\sqrt{x}+\cos\sqrt{x}\right)+(k_1+k_2)\sin\sqrt{x}=-\beta\frac{y^2}{4\sqrt{x}}\left(\sin\sqrt{x}+\cos\sqrt{x}\right)+\\
&\quad\frac{y}{2\sqrt{x}}(1-k_1k_2+\alpha\beta)\sin\sqrt{x}-\frac{\alpha}{\sqrt{x}}\left(\sin\sqrt{x}-\cos\sqrt{x}\right)+O\left(\frac{1}{x}\right),
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
&\beta\frac{y}{2}\left(\cos\sqrt{x}-\sin\sqrt{x}\right)-(1-k_1k_2+\alpha\beta)\cos\sqrt{x}=-\beta\frac{y}{2\sqrt{x}}\left(\sin\sqrt{x}+\cos\sqrt{x}\right)-\\
&\quad\frac{y}{2\sqrt{x}}(k_1+k_2)\cos\sqrt{x}+O\left(\frac{1}{x}\right).
\end{aligned} \tag{5.11}$$

To find the approximations for the solutions of Eq.(5.9) we proceed in the following way: firstly, we use Eq.(5.10) to derive the approximations for the real parts of the solutions, and secondly, based on the results obtained for Eq.(5.10) we use Eq.(5.11) to derive the approximations for the imaginary parts of the solutions.

1) Let $t=\sqrt{x}$ and f and g be two functions defined by the formulae

$$f(t) = \mathrm{e}^{it}\left[\beta t(\sin t+\cos t)+(k_1+k_2)\sin t\right],$$
$$g(t) = \mathrm{e}^{it}\left[-\beta\frac{y^2}{4t}(\sin t+\cos t)+\frac{y}{2t}(1-k_1k_2+\alpha\beta)\sin t-\frac{\alpha}{t}(\sin t-\cos t)\right]+O\left(\frac{1}{t^2}\right).$$

It can be easily seen that Eq.(5.10) being multiplied by e^{it}, can be written as $f(t)=g(t)$. Since $|g(t)|\asymp 1/|t|$, the equation $f(t)=0$ can be considered as *the model equation*. To find the asymptotic distribution of the roots of this model equation, we use Lemma 5.1, whose formulation and proof will be given after the proof of Theorem 5.1.

Let us represent the model equation in the form convenient for application of Lemma 5.1:

$$(z+C)\mathrm{e}^{iz}=Az+B, \qquad \text{where} \qquad z=2t, \quad A=-i, \quad -B=C=(1-i)(k_1+k_2)/\beta. \tag{5.12}$$

Applying Lemma 5.1 we obtain the following approximation for the roots:

$$z_n=2\pi n-\frac{\pi}{2}+\frac{k_1+k_2}{\pi n\beta}+O\left(\frac{1}{n^2}\right), \qquad n\to\infty. \tag{5.13}$$

In the next step, we prove that a similar approximation holds for the roots of the equation $f(t)=g(t)$, which can be written in the form

$$[2t+(1-i)(k_1+k_2)/\beta]\mathrm{e}^{i2t}+2it+(1-i)(k_1+k_2)/\beta=(1-i)/\beta\, g(t). \tag{5.14}$$

Let

$$z_n^0=2\pi n-i\ln A+i\frac{C-B/A}{2\pi n}. \tag{5.15}$$

Using Rouché's Theorem, we show that in a small vicinity of z_n^0, there exists a root of Eq.(5.14). Let z belong to a closed disk centered at z_n^0 of a small radius $\varepsilon>0$, i.e. $z\in\mathcal{D}_\varepsilon(z_n^0)\cup\mathcal{C}_\varepsilon(z_n^0)$. Consider both functions, f and g, on the circle $z=z_n^0+\varepsilon\mathrm{e}^{i\varphi}, 0\leqslant\varphi<2\pi$. Denoting $F(z)=f(z/2)$, we have

$$\begin{aligned} F(z) &= (z+C)\mathrm{e}^{iz}-Az-B=\left(z_n^0+\varepsilon\mathrm{e}^{i\varphi}+C\right)\mathrm{e}^{i\left(z_n^0+\varepsilon\mathrm{e}^{i\varphi}\right)}-A\left(z_n^0+\varepsilon\mathrm{e}^{i\varphi}\right)-B \\ &= F(z_n^0)+i\varepsilon z_n^0\mathrm{e}^{i\varphi}+O(\varepsilon)=F(z_n^0)+2\pi ni\varepsilon\mathrm{e}^{i\varphi}+O(\varepsilon). \end{aligned} \tag{5.16}$$

Evaluating $F(z_n^0)$, we obtain

$$F(z_n^0)=A(2\pi n-i\ln A+B/A+O(1/n))-A(2\pi n-i\ln A)-B=O(1/n). \tag{5.17}$$

Combining (5.16) and (5.17) we obtain the lower bound on $|F(z)|$ as

$$|F(z)|=\left|2\pi ni\,\varepsilon\mathrm{e}^{i\varphi}+O(\varepsilon)+O\left(n^{-1}\right)\right|\geqslant|\pi n\varepsilon-C_1/n|. \tag{5.18}$$

For the right-hand side of Eq.(5.14) we have $|(1-i)\beta g(t)|\leqslant C_2/n$. If we take $\varepsilon=(2C_2+C_1)/(\pi n)$, then from (5.18) we get that $|F(z)|\geqslant 2C_2/n$, which yields $|F(z)|>|G(z)|$, where $G(z)=(1-i)/\beta g(z/2)$. Using Rouché's Theorem, we obtain the desired result, i.e. that the real parts of the roots can be approximated by

$$x_n=(\pi n)^2-\frac{\pi^2}{2}n+\left(\frac{\pi}{4}\right)^2+\frac{k_1+k_2}{\beta}+O\left(\frac{1}{n}\right), \qquad n\to\infty. \tag{5.19}$$

2) Based on the results obtained for Eq.(5.10) we can derive asymptotic approximations for the imaginary parts of the roots. Since $|y_n| \asymp 1$ and $|x_n| \asymp n^2$, we obtain that when $x = x_n$ and $y = y_n$, Eq.(5.11) has the form

$$\beta y_n(\cos\sqrt{x_n} - \sin\sqrt{x_n}) - 2(1 - k_1k_2 + \alpha\beta)\cos\sqrt{x_n} = O(1/n), \qquad n \to \infty. \tag{5.20}$$

Based on (5.13) we have $\sqrt{x_n} = \pi n - \pi/4 + O\left(n^{-1}\right)$, so evaluating $\cos\sqrt{x_n}$ and $\sin\sqrt{x_n}$, we get

$$\cos\sqrt{x_n} = (-1)^n\frac{\sqrt{2}}{2}\left[1 + O\left(\frac{1}{n}\right)\right], \qquad \sin\sqrt{x_n} = (-1)^{(n+1)}\frac{\sqrt{2}}{2}\left[1 + O\left(\frac{1}{n}\right)\right].$$

Therefore Eq.(5.20) can be reduced to

$$\beta y_n\sqrt{2}\left[1 + O\left(\frac{1}{n}\right)\right] - \frac{\sqrt{2}}{2}(1 - k_1k_2 + \alpha\beta)\left[1 + O\left(\frac{1}{n}\right)\right] = O\left(\frac{1}{n}\right),$$

which yields

$$y_n = \frac{1 - k_1k_2 + \alpha\beta}{2\beta} + O\left(\frac{1}{n}\right). \tag{5.21}$$

Combinding (5.19) and (5.21), we obtain the result of Theorem 5.1. □

LEMMA 5.1 Consider the following equation on the complex plane:

$$(z + C)\mathrm{e}^{iz} = Az + B, \tag{5.22}$$

where $A, B, C \in \mathbb{C}$ and $A \neq 0$. This equation has a countable set of roots whose asymptotic approximation can be given by the formula

$$z_n = 2\pi n - i\ln A + i\frac{C - B/A}{2\pi n} + O\left(\frac{1}{n^2}\right). \tag{5.23}$$

Proof. To find an asymptotic approximation for the roots of the function $F(z) = z\left(\mathrm{e}^{iz} - A\right) + C\mathrm{e}^{iz} - B$, we use Kantorovich's Theorem on Newton's method (see the proof of Lemma 4.1 above). Let z_n^0 be the initial approximation to the root

$$z_n^0 = 2\pi n - i\ln A + i\frac{C - B/A}{2\pi n}. \tag{5.24}$$

In the sequel, we need estimates for $\left|F(z_n^0)\right|$ and $\left|F'(z_n^0)\right|$. We have $\left|F(z_n^0)\right| \asymp 1$ and

$$F'(z_n^0) = (1 + iz_n^0 + iC)\mathrm{e}^{iz_n^0} - A = 2\pi nAi + A\ln A + Bi + O(1/n). \tag{5.25}$$

Let Ω be a closed unit disk centered at z_n^0, i.e. $z = z_n^0 + \varepsilon\mathrm{e}^{i\varphi}$, $0 \leqslant \varepsilon \leqslant 1$, $0 \leqslant \varphi < 2\pi$. We have

$$\max_{z\in\Omega}\left|F''(z)\right| = \max_{z\in\Omega}\left|-z_n^0 - \varepsilon\mathrm{e}^{i\varphi} + 2i - C\right|\left|\mathrm{e}^{iz_n^0}\right|\left|\exp(i\varepsilon\mathrm{e}^{i\varphi})\right| \asymp n. \tag{5.26}$$

Using (5.18), (5.25), and (5.26), we evaluate η and K needed for the application of Kantorovich's Theorem:

$$\eta \geqslant \left|\frac{F(z_n^0)}{F'(z_n^0)}\right| \asymp \frac{1}{n^2}, \qquad K \geqslant \max_{z\in\Omega}\left|\frac{F''(z)}{F'(z_n^0)}\right| \asymp 1, \tag{5.27}$$

which means that there exists an absolute constant C_0, such that $h = \eta K \leqslant C_0 n^{-2}$. Thus, Eq.(5.22) has a unique root z_n^* located in a circle of radius r_0 around the point z_n^0, where

$$r_0 = \frac{1-\sqrt{1-2h}}{h}\eta \asymp \frac{1-(1-h)}{h}\eta \asymp \frac{1}{n^2}. \tag{5.28}$$

To show that there are no other roots in the strip containing $\{z_n^*\}$, we refer to the second part of Kantorovich's Theorem. To prove that there are no roots above the horizontal strip containing the set $\{z_n^*\}$, consider $F(z)$ when z belongs to any horizontal line given by

$$z = 2\pi x - i\ln A + i\frac{C-B/A}{2\pi x} + di, \tag{5.29}$$

where x is large enough and $d \geqslant 1$. For $|F(z)|$ we have

$$\begin{aligned}|F(z)| &= \left|\left(2\pi x - i\ln A + i\frac{C-B/A}{2\pi x} + di + C\right)\left(\mathrm{e}^{iz} - A\right) - B\right| \\ &= \left|\left[2\pi x + i(d-\ln A - iC) + O\left(x^{-1}\right)\right] A\left(\mathrm{e}^{id}-1\right) - B + O\left(x^{-1}\right)\right| \\ &= 2\pi|A|\left(1-\mathrm{e}^{-d}\right)x + O(1).\end{aligned}$$

This means that for $d \geqslant 1$ we have $|F(z)| \succ x$. □

COROLLARY 5.1 Assume that $k_1 k_2 = 1$. The following asymptotic approximations are valid for the eigenvalues as $n \to \infty$.

(1) If $\beta = 0$, then

$$\lambda_n = (\pi n)^2 + \frac{\alpha}{k_1+k_2} + O\left(\frac{1}{n}\right), \qquad \lambda_{-n} = -\overline{\lambda_n}.$$

In this case the spectrum is asymptotically real, and the operator $\mathcal{L}$ is "close" to a selfadjoint operator.

(2) If $\beta \neq 0$, then

$$\lambda_n = (\pi n)^2 - \frac{\pi^2}{2}n + \left(\frac{\pi}{4}\right)^2 + \frac{k_1+k_2}{\beta} + i\alpha + O\left(\frac{1}{n}\right), \qquad \lambda_{-n} = -\overline{\lambda_n}.$$

REMARK 5.1 We would like to note that the cases considered in Sections 4 and 5 generate spectral equations that are of the polynomial-exponential type. Lemmas 4.1 and 5.1 state asymptotic results regarding this type of equations, and therefore might prove useful while investigating other spectral problems. It is also interesting to point out that while the boundary control law is formally continuous with respect to the control parameters, for any combination of control parameters the asymptotic approximations are discontinuous with respect to changing the parameters.

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