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Spectral Analysis of the Euler-Bernoulli Beam Model with Fully Non-conservative Feedback Matrix

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Abstract

The Euler-Bernoulli beam model with fully non-conservative boundary conditions of feedback control type is investigated. The output vector (the shear and the moment at the right end) is connected to the observation vector (the velocity and its spatial derivative on the right end) by a 2×2 matrix (the boundary control matrix), all entries of which are non-zero real numbers. For any combination of the boundary parameters, the dynamics generator, $\mathcal{L}$, of the model is a non-selfadjoint matrix differential operator in the state Hilbert space. A set of 4 selfadjoint operators, defined by the same differential expression as $\mathcal{L}$ on different domains, is introduced. It is proven that each of these operators, as well as $\mathcal{L}$, is a finite-rank perturbation of the same selfadjoint dynamics generator of a cantilever beam model. It is also shown that the non-selfadjoint operator, $\mathcal{L}$, shares a number of spectral properties specific to its selfadjoint counterparts, such as (i) boundary inequalities for the eigenfunctions, (ii) the geometric multiplicities of the eigenvalues, (iii) the existence of real eigenvalues. These results are important for our next paper on the spectral asymptotics and stability for the multi-parameter beam model.

KEYWORDS:
non-selfadjoint operator, spectral equation, eigenvalues, eigenfunctions

1 | INTRODUCTION

The present paper is concerned with the spectral properties of the Euler-Bernoulli beam model subject to the general four-parameter family of non-conservative linear boundary conditions. The transverse displacement of the beam, $h(x,t)$, at position x and time t is governed by the hyperbolic partial differential equation

$$\rho A(x)\frac{\partial^2 h(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 h(x,t)}{\partial x^2}\right) = 0, \qquad 0 \le x \le L, \qquad t \ge 0. \tag{1.1}$$

This equation represents a commonly used model for the motion of a straight beam of length L, cross-sectional area $A(x)$, mass density $\rho(x)$, modulus of elasticity of the beam material $E(x)$, and cross-sectional moment of inertia $I(x)$ ($EI(x)$ is the bending stiffness). The model is obtained by using Hooke's law and the simplifying assumptions that the thickness and width of the beam are small compared to the length and the cross-sections of the beam remain plane during deformation (1, 2). We assume that the beam is clamped at the left end, i.e.

$$h(0,t) = h_x(0,t) = 0 \qquad \text{(left end clamped)}, \tag{1.2}$$

and subject to specific boundary conditions at the right end. To describe the right-end conditions, we use the moment $M(x,t)$, and the shear $Q(x,t)$ defined by:

$$M(x,t) = EI(x)h_{xx}(x,t), \qquad Q(x,t) = \left(EI(x)h_{xx}(x,t)\right)_x. \tag{1.3}$$

Let the input, $U(t)$, and output, $Y(t)$, be given as $\mathbb{R}^2$-vectors

$$U(t) = [-Q(L,t),\ M(L,t)]^T \qquad \text{and} \qquad Y(t) = \left[h_t(L,t),\ h_{xt}(L,t)\right]^T, \tag{1.4}$$

where the superscript "T" stands for transposition. The feedback control law is given by

$$U(t) = \mathbb{K}Y(t), \qquad \text{and} \qquad \mathbb{K} = \begin{bmatrix} -\alpha & -k_2 \\ -k_1 & -\beta \end{bmatrix}, \tag{1.5}$$

where α, β, k_1, k_2 are the control parameters. The feedback (1.5) can be written in the form

$$EI(1)h_{xx}(L,t) = -k_1 h_t(L,t) - \beta h_{xt}(L,t), \qquad \left(EI(x)h_{xx}(x,t)\right)_x\Big|_{x=L} = \alpha h_t(L,t) + k_2 h_{xt}(L,t). \tag{1.6}$$

If all parameters are equal to zero, then the right-end conditions become $h_{xx}(L,t) = h_{xxx}(L,t) = 0$, and the problem corresponds to the clamped-free beam. Consider the energy functional for the beam

$$\mathcal{E}(t) = \frac{1}{2}\int_0^L \left[\rho A(x)\, h_t^2(x,t) + EI(x)\, h_{xx}^2(x,t)\right] \mathrm{d}x\,. \tag{1.7}$$

Evaluating $\mathcal{E}_t(t)$ on the solutions of Eq.(1.1) satisfying the left-end conditions (1.2), we obtain that $\mathcal{E}_t(t) = Y(t)\cdot U(t)$, where "$\cdot$" denotes the dot-product in $\mathbb{R}^2$. If we use (1.5), then

$$\mathcal{E}_t(t) = Y(t)\cdot \mathbb{K}Y(t) = -\alpha h_t^2(L,t) - \beta h_{xt}^2(L,t) - (k_1 + k_2)h_t(L,t)h_{xt}(L,t), \tag{1.8}$$

which means that if $k_1 + k_2 = 0$ and $\alpha \geq 0$, $\beta \geq 0$, then $\mathcal{E}_t(t) \leq 0$ and the system is dissipative. Different combinations of the boundary parameters yield different energy dynamics for the structure. In our forthcoming paper, we generalize the model by incorporating additional energy-dissipating mechanisms. In particular, if the beam rests on an elastic foundation with γ being the modulus of elasticity, and the beam is subjected to an axial (tensile or compressive) force $S(x)$, then we get the equation

$$\rho A(x)\,\frac{\partial^2 h(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\,\frac{\partial^2 h(x,t)}{\partial x^2}\right) - S(x)\frac{\partial^2 h(x,t)}{\partial x^2} + \gamma h(x,t) = 0. \tag{1.9}$$

In addition, we can consider the so-called external or viscous damping of the form $a_0(x)h_t(x,t)$ and also the damping of the form $-a_1(x)h_{xxt}(x,t)$ (for physical interpretation, see (4, 5)). Therefore, the aforementioned generalization of the beam model (under the presence of damping terms, axial force, and a non-linear elastic foundation term $\mathfrak{S}(h)$) has the following form:

$$\rho A(x)\,\frac{\partial^2 h(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\,\frac{\partial^2 h(x,t)}{\partial x^2}\right) - S(x)\frac{\partial^2 h(x,t)}{\partial x^2} + \mathfrak{S}(h) + a_0(x)\frac{\partial h(x,t)}{\partial t} - a_1(x)\frac{\partial^3 h(x,t)}{\partial t \partial x^2} = 0. \tag{1.10}$$

In the present paper we consider the initial boundary-value problem defined by Eq.(1.1), conditions (1.2) and (1.6), and a standard set of the initial conditions,

$$h(x,0) = h_0(x), \qquad \frac{\partial h(x,0)}{\partial t} = h_1(x). \tag{1.11}$$

This initial-boundary value problem is reduced to the first order in time evolution equation in the Hilbert state space, $\mathcal{H}$, equipped with the energy metric. The dynamics generator, $\mathcal{L}$, is an unbounded non-selfadjoint matrix differential operator in $\mathcal{H}$. As is shown in the paper, the operator $\mathcal{L}$ is a finite-rank perturbation of a selfadjoint operator corresponding to the clamped-free beam model (i.e. when $k_1 = k_2 = \alpha = \beta = 0$). The main question in the present paper is the following: *which spectral properties of the selfadjoint operator will be preserved when some parameters (or all of them) are non-zero in the boundary matrix* $\mathbb{K}$*?*

Before we discuss the content and findings of the present paper, we would like to emphasize that there exists an extensive literature on the Euler-Bernoulli beam model. We mention below a few recent works where the model has been used in contemporary research directions. One of them is concerned with developing unmanned aerial vehicles (UAV) in aeronautics. For intelligence missions, surveillance, and environmental research, highly flexible unmanned airframes have been designed recently, that allow for high-altitude and long-endurance flights (6, 7). In particular, a light and flexible long-span object in flight (high aspect-ratio "flying wing" configuration) can be modeled as an elastic beam with both ends free. A boundary feedback stabilization of such beam-like structures could be of great interest both in control theory and in engineering practice. In works (6, 7) a computer theoretical methodology for a highly flexible wing has been presented. The authors use geometrically exact beam theory for elastic deformations coupled with aerodynamic theory of large motion airfoils. The analysis accounts for realistic design space requirements including concentrated payloads, multiple engines, and multiple control surfaces. Another important area of applications is the modeling of large space structures, e.g. a large communication satellite or a space platform. In such a structure, different types of damping devices are installed at the joints of the beam elements to suppress vibrations. Without these dampers, small vibrations would persist and even slowly build up. In (8) the model of serially connected non-collinear Euler-Bernoulli beams with dissipative joints has been considered and numerical simulation results presented. Obviously, rigorous analytical results on the problem would be desirable.

A class of Euler-Bernoulli beam models with boundary and structural damping has been discussed in (9, 10, 11, 12, 13). We also mention works (14, 15, 16) dealing with the models with viscous and Kelvin-Voigt damping. For a cantilever model, the lack of exponential stabilization under velocity feedback has been proven in (17); in paper (18) the energy multiplier method has been used to prove the exponential stabilization under the linear boundary feedback control $\alpha h_t(1,t) + h_{xxxt}(1,t) = 0$. In paper (19) the authors have shown the existence of different classes of non-homogeneous Euler-Bernoulli beam models with continuous density and flexural rigidity functions and different end conditions, that are analytically solvable and “isospectral” to a homogeneous beam model of clamped-clamped end conditions.

From the numerous works on the inverse problem for the Euler-Bernoulli model, we refer to the interesting paper (20). It has been long known that two scaling factors and three spectra, corresponding to three different end conditions, are required to determine the cross-sectional area $A(x)$ and the cross-sectional moment of inertia $I(x)$. However, the necessary and sufficient conditions on the spectral data that yield “positive” functions $A(x)$ and $I(x)$ have not been known. Such conditions have been derived in (20).

In paper (21) the authors study a slender beam with spatially non-homogeneous viscous damping and structural damping. For constant damping coefficients, it is well known that the structural damping induces a strong attenuation rate that is frequency proportional, while the viscous damping induces a constant attenuation rate for all frequencies. It is shown in (21) that for the case of variable damping coefficients, the asymptotic patterns of the spectra remain the same, i.e. the viscous damping causes an asymptotically constant shift in the attenuation rates; hence it is overwhelmed by the structural damping effect.

We also mention paper (22), where the asymptotic distribution of the eigenfrequencies of in-plane vibrations of an Euler-Bernoulli beam model with dissipative joints has been computed, which allows the beam to be curved as an arc of a circle. This result could be instrumental in the analysis of Euler-Bernoulli beam system describing UAVs (6). Finally, we mention papers (23, 24), where it is proven that the semigroup generated by the initial value problem, being non-analytic, belongs to the Gevrey class. It means that the differentiability of such semigroup is slightly weaker than that of an analytic semigroup.

The present paper is organized as follows. In Sec. 2, we present an operator reformulation of the initial boundary-value problem (1.1), (1.2), (1.6), (1.11) and define its dynamics generator $\mathcal{L}$. Then we introduce four selfadjoint operators closely related to the main non-selfadjoint operator $\mathcal{L}$ defined in (2.3) and (2.4) below. We derive explicit formulae for the inverse operators for both the selfadjoint and non-selfadjoint cases. It follows from those formulae that each operator is in fact a finite-dimensional perturbation of the same selfadjoint operator, describing the dynamics of the cantilever beam model.

In Sec. 3 we reduce the spectral problem ($\mathcal{L}\Psi = \lambda\Psi$, $\Psi \in \mathcal{D}(\mathcal{L})$, $\lambda \in \mathbb{C}$) for the operator $\mathcal{L}$ to the spectral problem for the non-selfadjoint polynomial operator pencil $\mathcal{P}(\lambda)$ generated by the operator $\mathcal{L}$. We derive the spectral equation for the pencil $\mathcal{P}(\lambda)$ (see Eq.(3.19) below) and discuss under what conditions on the control parameters one obtains the spectral equations for the selfadjoint operators introduced in Sec. 2. These spectral equations will be used in our forthcoming paper for the derivation of the asymptotic formulae for the eigenvalues of the corresponding operators.

In Sec. 4 we derive important inequalities that describe the boundary behavior of the eigenfunctions corresponding to the aforementioned selfadjoint operators. The results on the boundary behavior of the eigenfunctions, being important in their own right, could be generalized to the non-selfadjoint operator $\mathcal{L}$ (see Sec. 5). We also address the issue of the geometric multiplicities of the eigenvalues and prove that each of the above selfadjoint operators has a *simple* spectrum.

In Sec. 5 we present the generalization of the results of Sec. 4 to the non-selfadjoint operator $\mathcal{L}$. It turns out that for some combinations of the boundary parameters, the boundary inequalities for the eigenfunctions of the operator $\mathcal{L}$ are similar to the boundary inequalities for the selfadjoint operators, while for other combinations they are quite different.

In Sec. 6 we address the question of geometric multiplicity of each eigenvalue. We show that the result proven for a selfadjoint problem, i.e. that each eigenvalue is simple, is valid for the operator $\mathcal{L}$ as well, in the sense that the geometric multiplicity of each eigenvalue of $\mathcal{L}$ is 1. Since $\mathcal{L}$ is non-selfadjoint, the algebraic multiplicity of each eigenvalue is finite but not necessarily 1, i.e. for each eigenvalue of $\mathcal{L}$ there could be an eigenfunction and a finite chain of associate functions.

Finally, Sec. 7 is concerned with the existence of the real eigenvalues of the operator $\mathcal{L}$. As shown in Theorem 7.1 below, if only one control parameter is not equal to zero, then there are no real eigenvalues. However, when there are at least two non-zero boundary parameters, there could be real eigenvalues depending on which of the parameters are non-zero.

2 | PROBLEM STATEMENT

Let us rewrite problem (1.1), (1.2), and (1.6), as the first order in time evolution equation in the state space of the system (the energy space). Without loss of generality, we assume that the cross-sectional area $A = 1$, and the spatial extent of the beam is $L = 1$. We also assume that EI and ρ are strictly positive functions and such that $EI,\ \rho \in C^2[0,1]$.

Let $\mathcal{H}$ be the Hilbert space of two-component complex vector-valued functions obtained as the closure of smooth functions $\Phi(x) = \left[\varphi_0(x),\ \varphi_1(x)\right]^T$, such that $\varphi_0(0) = \varphi_0'(0) = 0$, in the following norm:

$$\|\Phi\|^2_{\mathcal{H}} = \frac{1}{2}\int_0^1 \left[EI(x)\left|\varphi_0''(x)\right|^2 + \rho(x)\left|\varphi_1(x)\right|^2\right] \mathrm{d}x\,. \tag{2.1}$$

The energy space $\mathcal{H}$ is topologically equivalent to the space $\widetilde{H}^2_0(0,1) \times L^2(0,1)$, where

$$\widetilde{H}^2_0(0,1) = \left\{\varphi \in H^2(0,1) \,:\, \varphi(0) = \varphi'(0) = 0\right\}.$$

The problem (1.1), (1.2), (1.6), and (1.11) can be represented as an evolution problem

$$\Phi_t(x,t) = i(\mathcal{L}\Phi)(x,t) \qquad \text{and} \qquad \Phi(x,0) = \left[\varphi_0(x), \varphi_1(x)\right]^T, \quad 0 \le x \le 1, \quad t \ge 0, \tag{2.2}$$

where the dynamics generator, $\mathcal{L}$, is given by the following matrix differential expression:

$$\mathcal{L} = -i\begin{bmatrix} 0 & 1 \\ -\dfrac{1}{\rho(x)}\dfrac{\partial^2}{\partial x^2}\left(EI(x)\dfrac{\partial^2}{\partial x^2}\,\cdot\right) & 0 \end{bmatrix}, \tag{2.3}$$

defined on the domain

$$\begin{aligned} \mathcal{D}(\mathcal{L}) = \Big\{\Phi = \left(\varphi_0, \varphi_1\right)^T \in \mathcal{H} \,:\, &\varphi_0 \in H^4(0,1),\ \varphi_1 \in H^2(0,1);\ \ \varphi_1(0) = \varphi_1'(0) = 0; \\ &EI(1)\varphi_0''(1) = -k_1\varphi_1(1) - \beta\varphi_1'(1),\ \ \left(EI(x)\varphi_0''(x)\right)'_{x=1} = k_2\varphi_1'(1) + \alpha\varphi_1(1)\Big\}. \end{aligned} \tag{2.4}$$

Remark 2.1. *1) We introduce the factor "i" in the definition* (2.3) *of the dynamics generator and into Eq.*(2.2) *for convenience. As is shown below, the operator $\mathcal{L}$ is a finite-rank perturbation of the selfadjoint operator corresponding to the cantilever beam (the model with clamped-free end conditions). So owing to this factor we deal with a selfadjoint operator rather than with a skew-selfadjoint operator.*

2) The problem (1.1), (1.2), *and* (1.6) *defines a C_0-semigroup in $\mathcal{H}$ and the operator $i\mathcal{L}$ is an infinitesimal generator of this semigroup. This fact follows, e.g. from the Riesz-basis property of the generalized eigenfunctions of $\mathcal{L}$, which will be proven in a forthcoming paper. So, the use of the term "dynamics generator" for $i\mathcal{L}$ is justified and we use the same term for $\mathcal{L}$ as well.*

Now we introduce four selfadjoint operators denoted by $\mathcal{L}_{CF}$, $\mathcal{L}_{CS}$, $\mathcal{L}_{CH}$, and $\mathcal{L}_{CC}$, whose spectral properties (as shown below) are similar to the spectral properties of the non-selfadjoint operator $\mathcal{L}$ considered for different combinations of control parameters. Each selfadjoint operator is defined by the same matrix differential expression (2.3) on functions $G = (g_0, g_1)^T \in \mathcal{H} \,:\ g_0 \in H^4(0,1),\ g_1 \in H^2(0,1)$, and such that $g_1(0) = g_1'(0) = 0$. These operators differ from one another only by the right-end conditions. More precisely, the right-end conditions together with the above left-end conditions define the domains of these operators and the subspaces on which the domains are dense. Namely, let us introduce the following definitions.

(i) $\mathcal{L}_{CF}$ is a selfadjoint operator in $\mathcal{H}$ corresponding to the *clamped-free beam model*, when $g_0''(1) = \left(EI(x)g_0''(x)\right)'_{x=1} = 0$.

(ii) $\mathcal{L}_{CS}$ is a selfadjoint operator in the space $\widetilde{\mathcal{H}} = \left\{G = (g_0, g_1)^T \,:\, G \in \mathcal{H} \text{ and } g_0'(1) = 0\right\}$, corresponding to the *clamped-sliding beam model*, when $\left(EI(x)g_0''(x)\right)'_{x=1} = 0$.

(iii) $\mathcal{L}_{CH}$ is a selfadjoint operator in the space $\widehat{\mathcal{H}} = \left\{G = (g_0, g_1)^T \,:\, G \in \mathcal{H} \text{ and } g_0(1) = 0\right\}$, corresponding to the *clamped-hinged beam model*, when $g_0''(1) = 0$.

(iv) $\mathcal{L}_{CC}$ corresponds to the *clamped-clamped beam model*, and it is a selfadjoint operator in the space

$$\mathcal{H}^0 = \left\{G = (g_0, g_1)^T \,:\, G \in \mathcal{H} \text{ and } g_0(1) = g_0'(1) = 0\right\}.$$

The fact that the above operators are indeed selfadjoint follows from straightforward calculations of the corresponding quadratic forms and demonstration that these forms are real. The following result holds for the differential operators introduced in (i)–(iv) above.

Lemma 2.2. *Each Hilbert space, $\widetilde{\mathcal{H}}$, $\widehat{\mathcal{H}}$, and $\mathcal{H}^0$, is a proper subspace of $\mathcal{H}$ and such that*

$$(i)\ \dim\left(\mathcal{H}/\widetilde{\mathcal{H}}\right) = 1, \qquad (ii)\ \dim\left(\mathcal{H}/\widehat{\mathcal{H}}\right) = 1, \qquad (iii)\ \dim\left(\mathcal{H}/\mathcal{H}^0\right) = 2. \tag{2.5}$$

Proof. To prove (i), it suffices to show that the orthogonal complement of the subspace $\widetilde{\mathcal{H}}$ of $\mathcal{H}$ (in the norm (2.1)) is one-dimensional. Namely, we show that any vector $F = (f_0, f_1)^T \in \mathcal{H}$ can be represented as an orthogonal sum of the form

$$F(x) = \widetilde{F}(x) + \alpha\widetilde{F}_0(x), \qquad \widetilde{F} \in \widetilde{\mathcal{H}}, \qquad \alpha \in \mathbb{C}, \qquad \widetilde{F}_0 \perp \widetilde{F}. \tag{2.6}$$

Let us take $\widetilde{F}_0(x) = \left(x^2,\, 0\right)^T \in \mathcal{H}$ and $\alpha = \frac{1}{2}f_0'(1)$ and denote the vector $\left[f_0(x) - \frac{1}{2}f_0'(1)x^2,\, f_1(x)\right]^T$ by $\widetilde{F}(x)$. The decomposition $F = \widetilde{F} + \alpha\widetilde{F}_0$ is obvious. Let us check that $\widetilde{F}$ and $\widetilde{F}_0$ are orthogonal in $\mathcal{H}$. We have

$$\left(\widetilde{F}, \widetilde{F}_0\right)_{\mathcal{H}} = EI\int_0^1 \left[f_0''(x) - f_0'(1)\right] \mathrm{d}x = EI\left[f_0'(x)\Big|_0^1 - f_0'(1)\right] = 0,$$

which justifies (i).

To prove relation (ii), it suffices to show that any vector $F = (f_0, f_1)^T \in \mathcal{H}$ can be represented as an orthogonal sum

$$F(x) = \widehat{F}(x) + \beta\widehat{F}_0(x), \qquad \widehat{F} \in \widehat{\mathcal{H}}, \qquad \beta \in \mathbb{C}, \qquad \widehat{F}_0 \perp \widehat{F}. \tag{2.7}$$

Let us take $\widehat{F}_0(x) = \left(x^3 - 3x^2,\, 0\right)^T \in \mathcal{H}$ and $\beta = -\frac{1}{2}f_0(1)$ and denote the vector $\left[f_0(x) + \frac{1}{2}f_0(1)(x^3 - 3x^2),\, f_1(x)\right]^T$ by $\widehat{F}(x)$. We check that $\widehat{F}(1) = \left(0, f_1(1)\right)^T$, then show that $\widehat{F}$ and $\widehat{F}_0$ are orthogonal in the norm of $\mathcal{H}$. We have

$$\left(\widehat{F}, \widehat{F}_0\right)_{\mathcal{H}} = \frac{EI}{2}\int_0^1 \left[f_0''(x) + 3(x-1)f_0(1)\right] 6(x-1)\,\mathrm{d}x = 0.$$

Thus relation (ii) is shown. To prove relation (iii), it suffices to show that $\dim\left(\widetilde{\mathcal{H}}/\mathcal{H}^0\right) = 1.$ and take into account (i). ☐

Our next result is concerned with the explicit formulae for the inverse operators.

Theorem 2.3. *Let $\mathbb{R}[\cdot]$ be the following Volterra integral operator defined on $\mathcal{H}$:*

$$\mathbb{R}[f](x) = -i\int_0^x \mathrm{d}\tau \int_0^\tau \frac{\mathrm{d}\eta}{EI(\eta)} \int_\eta^1 \mathrm{d}\xi \int_\xi^1 \rho(\nu)\, f(\nu)\,\mathrm{d}\nu\,. \tag{2.8}$$

The following statements are valid.

(i) The inverse operator $\mathcal{L}_{CF}^{-1}$ exists as a compact operator in $\mathcal{H}$ and is given by:

$$\left(\mathcal{L}_{CF}^{-1}G\right)(x) = \left[\mathbb{R}[g_1](x),\, ig_0(x)\right]^T, \qquad G \in \mathcal{H}. \tag{2.9}$$

(ii) The inverse operator $\mathcal{L}_{CS}^{-1}$ exists as a compact operator in $\widetilde{\mathcal{H}}$ and is given by:

$$\left(\mathcal{L}_{CS}^{-1}G\right)(x) = \left[\mathbb{R}[g_1](x) - \frac{1}{2}x^2\left(\mathbb{R}[g_1]\right)'(1),\, ig_0(x)\right]^T, \qquad G \in \widetilde{\mathcal{H}}, \tag{2.10}$$

where

$$\left(\mathbb{R}[g_1]\right)'(x) = -i\int_0^x \frac{\mathrm{d}\eta}{EI(\eta)} \int_\eta^1 \mathrm{d}\xi \int_\xi^1 \rho(\nu)\, g_1(\nu)\,\mathrm{d}\nu\,. \tag{2.11}$$

(iii) The inverse operator $\mathcal{L}_{CH}^{-1}$ exists as a compact operator in $\widehat{\mathcal{H}}$ and is given by:

$$\left(\mathcal{L}_{CH}^{-1}G\right)(x) = \left[\mathbb{R}[g_1](x) + \frac{1}{2}(x^3 - 3x^2)\mathbb{R}[g_1](1),\ ig_0(x)\right]^T, \qquad G \in \widehat{\mathcal{H}}. \tag{2.12}$$

(iv) The inverse $\mathcal{L}_{CC}^{-1}$ exists as a compact operator in $\mathcal{H}^0$ and is given by:

$$\left(\mathcal{L}_{CC}^{-1}G\right)(x) = \left[\mathbb{R}[g_1](x) + \left[2\mathbb{R}[g_1](1) - \left(\mathbb{R}[g_1]\right)'(1)\right]x^3 - \left[3\mathbb{R}[g_1](1) - \left(\mathbb{R}[g_1]\right)'(1)\right]x^2,\ ig_0(x)\right]^T, \qquad G \in \mathcal{H}^0. \tag{2.13}$$

It is convenient to prove Theorem 2.3 after we formulate and prove our next result – Theorem 2.4 below. This result is concerned with the properties of the main non-selfadjoint operator, $\mathcal{L}$, defined in (2.3) and (2.4).

Theorem 2.4. *1) $\mathcal{L}$ is an unbounded operator with a compact resolvent, whose spectrum consists of a countable set of normal eigenvalues (i.e. isolated eigenvalues, each of at most finite multiplicity (25, 26, 27, 28, 29)), the set which can accumulate only at infinity.*

2) For any combination of the parameters, such that $|k_1| + |k_2| + |\alpha| + |\beta| > 0$, the operator $\mathcal{L}$ is a finite-rank perturbation of the selfadjoint operator $\mathcal{L}_{CF}$, corresponding to the case when $k_1 = k_2 = \alpha = \beta = 0$. The fact that $\mathcal{L}$ is a perturbation of $\mathcal{L}_{CF}$ should be understood in the following sense. The operators $\mathcal{L}^{-1}$ and $\mathcal{L}_{CF}^{-1}$ exist and are related by the rule

$$\mathcal{L}^{-1} = \mathcal{L}_{CF}^{-1} + \mathcal{T}, \tag{2.14}$$

where $\mathcal{T}$ is a finite-rank operator. The following formulae are valid for $\boldsymbol{G} = (g_0, g_1)^T \in \mathcal{H}$:

$$\left(\mathcal{L}_{CF}^{-1}G\right)(x) = \left[\mathbb{R}[g_1](x),\ ig_0(x)\right]^T, \tag{2.15}$$

with $\mathbb{R}[\cdot]$ being defined in (2.8) *and*

$$(\mathcal{T}G)(x) = \frac{i}{2EI(1)}\left[p(x)g_0'(1) + q(x)g_0(1),\ 0\right]^T, \tag{2.16}$$

where

$$\begin{aligned} p(x) &= \frac{1}{3}\left[k_2 + \frac{EI'(1)}{EI(1)}\beta\right]x^3 - \left[k_2 + \left(\frac{EI'(1)}{EI(1)} + 1\right)\beta\right]x^2, \\ q(x) &= \frac{1}{3}\left[\frac{EI'(1)}{EI(1)}k_1 + \alpha\right]x^3 - \left[\left(\frac{EI'(1)}{EI(1)} + 1\right)k_1 + \alpha\right]x^2. \end{aligned} \tag{2.17}$$

3) A similar decomposition is valid for the adjoint operator, i.e. $(\mathcal{L}^*)^{-1} = \mathcal{L}_{CF}^{-1} + \mathcal{T}^*$, where $\mathcal{T}^*$ is given by formulae (2.16) *and* (2.17) *in which k_1 and k_2 have been replaced by $(-k_2)$ and $(-k_1)$, and α and β by $(-\alpha)$ and $(-\beta)$, respectively.*

4) If $k_1 + k_2 = 0$ and $\alpha \geq 0, \beta \geq 0$, then the operator $\mathcal{L}$ is dissipative (30, 31).

Proof. To show the existence of the inverse operator, it suffices to show that the equation $\mathcal{L}\Psi = F$ has a unique solution $\Psi \in \mathcal{D}(\mathcal{L})$ for any $F \in \mathcal{H}$. If we denote $\Psi = (\psi_0, \psi_1)^T$ and $F = (f_0, f_1)^T$, then the above equation can be written component-wise as follows:

$$\left(EI(x)\psi_0''(x)\right)'' = -i\rho(x)f_1(x), \qquad \psi_1(x) = if_0(x). \tag{2.18}$$

Integrating the first equation of the system, we obtain its general solution in the form

$$\psi_0(x) = -i\int_0^x \mathrm{d}\tau \int_0^\tau \frac{\mathrm{d}\eta}{EI(\eta)} \int_\eta^1 \mathrm{d}\xi \int_\xi^1 \rho(\nu)\, f_1(\nu)\,\mathrm{d}\nu + Ax^3 + Bx^2 + Cx + D. \tag{2.19}$$

Now we choose the constants A, B, C, and D in such a way that the function $\psi_0(x)$ satisfies four boundary conditions. To satisfy the left-end conditions, $\psi_0(0) = \psi_0'(0) = 0$, we take $C = D = 0$. From the requirement $EI(1)\psi_0''(1) = -k_1\psi_1(1) - \beta\psi_1'(1)$, we obtain

$$EI(1)\,(6A + 2B) = -ik_1 f_0(1) - i\beta f_0'(1). \tag{2.20}$$

From the requirement that $\left(EI(x)\psi_0''(x)\right)'_{x=1} = k_2\psi_1'(1) + \alpha\psi_1(1)$, we obtain

$$6EI(1)\,A + EI'(1)\,(6A + 2B) = ik_2 f_0'(1) + i\alpha f_0(1). \tag{2.21}$$

Solving system (2.20)–(2.21), we get

$$\begin{aligned} A &= \frac{i}{6EI(1)}\left[\frac{EI'(1)}{EI(1)}k_1+\alpha\right]f_0(1)+\frac{i}{6EI(1)}\left[k_2+\frac{EI'(1)}{EI(1)}\beta\right]f_0'(1), \\ B &= -\frac{i}{2EI(1)}\left[\left(\frac{EI'(1)}{EI(1)}+1\right)k_1+\alpha\right]f_0(1)-\frac{i}{2EI(1)}\left[k_2+\left(\frac{EI'(1)}{EI(1)}+1\right)\beta\right]f_0'(1), \end{aligned} \tag{2.22}$$

which yields the following representation for ψ_0:

$$\begin{aligned} \psi_0(x) = &-i\int_0^x \mathrm{d}\tau\int_0^\tau \frac{\mathrm{d}\eta}{EI(\eta)}\int_\eta^1 \mathrm{d}\xi\int_\xi^1 \rho(\nu)\,f_1(\nu)\,\mathrm{d}\nu + \\ &\frac{i}{2EI(1)}\left\{\frac{1}{3}\left[k_2+\frac{EI'(1)}{EI(1)}\beta\right]x^3-\left[k_2+\left(\frac{EI'(1)}{EI(1)}+1\right)\beta\right]x^2\right\}f_0'(1)+ \\ &\frac{i}{2EI(1)}\left\{\frac{1}{3}\left[\frac{EI'(1)}{EI(1)}k_1+\alpha\right]x^3-\left[\left(\frac{EI'(1)}{EI(1)}+1\right)k_1+\alpha\right]x^2\right\}f_0(1). \end{aligned} \tag{2.23}$$

It can be easily seen that when $k_1 = k_2 = \alpha = \beta = 0$, the solution Ψ can be given in the form $\Psi(x) = \left[\mathbb{R}\left[f_1\right](x),\, i f_0(x)\right]^T$, which means that the first component of Ψ satisfies the conditions $\psi_0''(1) = \psi_0'''(1) = 0$ and $\psi_1(1) = \psi_1'(1) = 0$, i.e. for such Ψ one has $\Psi \in \mathcal{D}(\mathcal{L}_{CF})$ and $(\mathcal{L}_{CF}^{-1}F)(x) = \left[\mathbb{R}\left[f_1\right](x),\, i f_0(x)\right]^T$. Hence, formulae (2.14)–(2.17) are shown.

Finally, to show that the operator $\mathcal{L}^{-1}$ is compact, we note that the domain $\mathcal{D}(\mathcal{L}_{CF})$ is a closed subspace of $\mathcal{H}_1 \equiv H^4(0,1)\times H^2(0,1)$. From the above proof it follows that $\mathcal{L}^{-1}$ is a bounded operator from $\mathcal{H}$ into $\mathcal{D}(\mathcal{L}_{CF})$, if $\mathcal{D}(\mathcal{L}_{CF})$ is equipped with the norm of $\mathcal{H}_1$. Since the embedding $\mathcal{H}_1 \hookrightarrow \mathcal{H}$ is compact, $\mathcal{L}_{CF}^{-1}$ is a compact operator defined on $\mathcal{H}$. Addition of a finite-rank operator $\mathcal{T}$ cannot change the result.

(3) It can be verified directly that the adjoint operator $\mathcal{L}^*$ is defined by the same differential expression (2.3) on the domain

$$\begin{aligned} \mathcal{D}(\mathcal{L}^*) = \big\{\Phi = (\varphi_0,\varphi_1)^T \in \mathcal{H} \,:\, \varphi_0 \in H^4(0,1),\ \varphi_1\in H^2(0,1);\ \ \varphi_1(0)=\varphi_1'(0)=0; \\ EI(1)\,\varphi_0''(1) = k_2\varphi_1(1)+\beta\varphi_1'(1),\ \ \big(EI(x)\varphi_0''(x)\big)'_{x=1} = -k_1\varphi_1'(1)-\alpha\varphi_1(1)\big\}. \end{aligned} \tag{2.24}$$

So (2.24) is obtained from (2.4) by the replacement of the parameters: $(k_1,k_2)\to(-k_2,-k_1)$ and $(\alpha,\beta)\to(-\alpha,-\beta)$.

(4) We say that a linear operator is *dissipative* (see (30, 31)), if its quadratic form has a non-negative imaginary part. Let us evaluate $(\mathcal{L}F,F)_{\mathcal{H}}$ on any $F = (f_0,f_1)^T \in \mathcal{D}(\mathcal{L})$:

$$(\mathcal{L}F,F)_{\mathcal{H}} = -\operatorname{Im}\int_0^1 EI(x)f_0''(x)\overline{f_1''(x)}\,\mathrm{d}x - \frac{i}{2}EI(1)f_0''(1)\overline{f_1'(1)}+\frac{i}{2}\big(EI(x)f_0''(x)\big)'_{x=1}\overline{f_1(1)}. \tag{2.25}$$

Taking into account the right-end boundary conditions from (2.4) we obtain

$$\operatorname{Im}(\mathcal{L}F,F)_{\mathcal{H}} = \frac{1}{2}\left[\alpha\left|f_1(1)\right|^2+\beta\left|f_1'(1)\right|^2\right]+\frac{1}{2}(k_1+k_2)\operatorname{Re}\left[f_1(1)\overline{f_1'(1)}\right].$$

This relation shows that for $\alpha > 0$, $\beta > 0$ and $k_1+k_2 = 0$, the operator $\mathcal{L}$ is dissipative. If $k_1+k_2\neq 0$, the dissipativity of $\mathcal{L}$ does not hold. □

Now we are in a position to prove Theorem 2.3.

Proof. To prove formulae (2.9)–(2.13) for the inverse operators, we have to start with the equation, $\mathcal{M}F = G$, where $\mathcal{M}$ is either $\mathcal{L}_{CF}$, $\mathcal{L}_{CS}$, $\mathcal{L}_{CH}$, or $\mathcal{L}_{CC}$, and show that this equation is solvable for any G taken from the appropriate Hilbert space. Assuming $F = (f_0,\ f_1)^T$ and $G = (g_0,\ g_1)^T$, we reduce the above equation to a system similar to (2.18). Upon integration, we obtain that the first component, f_0, satisfying the left-end conditions $f(0) = f_0'(0) = 0$, can be given in the form

$$f_0(x) = \mathbb{R}[g_1](x) + A'x^3 + B'x^2. \tag{2.26}$$

To specify the constants A' and B' from (2.26), we have to apply the corresponoding right-end boundary conditions. Hence, the remaining step in the proof is to check that with our choice of the constants A' and B', the necessary conditions are satisfied.

(i) Formula (2.9) follows immediately from the proof of Theorem 2.4.

(ii) Let us take $A' = 0$ and $B' = 1/2\mathbb{R}'\left[g_1\right](1)$, and show that $(f_0, f_1)^T \in \mathcal{D}(\mathcal{L}_{CS})$. The only fact to check is that the function f_0 satisfies the conditions $f_0'(1) = \left(EI(x)f_0''(x)\right)'_{x=1} = 0$. Since $f_0(x) = \mathbb{R}[g_1](x) - \frac{1}{2}x^2\mathbb{R}'[g_1](1)$, we get $f_0'''(1) = 0$ and

$$f_0'(1) = -i\int_0^x \frac{\mathrm{d}\eta}{EI(\eta)}\int_\eta^1 \mathrm{d}\xi \int_\xi^1 \rho(\nu)g_1(\nu)\,\mathrm{d}\nu\Bigg|_{x=1} - i\int_0^1 \frac{\mathrm{d}\eta}{EI(\eta)}\int_\eta^1 \mathrm{d}\xi \int_\xi^1 \rho(\nu)g_1(\nu)\,\mathrm{d}\nu = 0.$$

Formula (2.10) is shown.

(iii) Taking A' and B' in such a way that $f_0(x) = \mathbb{R}[g_1](x) + \frac{1}{2}(x^3 - 3x^2)\mathbb{R}[g_1](1)$, we get $f_0(1) = 0$ and

$$f_0''(1) = \left[-\frac{i}{EI(x)}\int_x^1 \mathrm{d}\xi \int_\xi^1 \rho(\nu)g_1(\nu)\,\mathrm{d}\nu + \frac{1}{2}(6x-6)\mathbb{R}[g_1](1)\right]_{x=1} = 0,$$

which justifies formula (2.12).

(iv) Let $f_0(x) = \mathbb{R}[g_1](x) + \left[2\mathbb{R}[g_1](1) - \left(\mathbb{R}[g_1]\right)'(1)\right]x^3 + \left[-3\mathbb{R}[g_1](1) + \left(\mathbb{R}[g_1]\right)'(1)\right]x^2$. Then $f_0(1) = 0$ and $f_0'(1) = \left(\mathbb{R}[g_1]\right)'(1) + 3\left[2\mathbb{R}[g_1](1) - \left(\mathbb{R}[g_1]\right)'(1)\right] + 2\left[-3\mathbb{R}[g_1](1) + \left(\mathbb{R}[g_1]\right)'(1)\right] = 0$, which justifies formula (2.13). □

3 | SPECTRAL EQUATION

In this section we derive the *spectral equation*, whose solutions coincide with the eigenvalues of the operator $\mathcal{L}$. It can be verified directly that the eigenvalue problem for the operator $\mathcal{L}$ in the space $\mathcal{H}$ is equivalent to the eigenvalue problem for the non-selfadjoint polynomial operator pencil (27) $\mathcal{P}(\lambda)$ in the space $L^2_\rho(0,1)$:

$$[\mathcal{P}(\lambda)\varphi](x) = \left(EI(x)\varphi''(x)\right)'' - \lambda^2\rho(x)\varphi(x), \tag{3.1}$$

defined on the domain

$$\begin{aligned}\mathcal{D}(\mathcal{P}(\lambda)) = \big\{\varphi_0 \in H^4(0,1) :\ & \varphi_0(0) = \varphi_0'(0) = 0;\\ & EI(1)\,\varphi_0''(1) = -k_1 i\lambda\varphi_0(1) - \beta i\lambda\varphi_0'(1),\ \left(EI(x)\varphi_0''(x)\right)'_{x=1} = \alpha i\lambda\varphi_0(1) + k_2 i\lambda\varphi_0'(1)\big\}.\end{aligned} \tag{3.2}$$

By Theorem 2.4, the spectral problem for the pencil, $\mathcal{P}(\lambda)\varphi = 0$, has a countable set of solutions $\left\{\lambda_n\right\}_{n\in\mathbb{Z}'}$. Let $\left\{\psi_n(x)\right\}_{n\in\mathbb{Z}'}$ be a set of normalized in $L^2_\rho(0,1)$ eigenfunctions of the eigenvalue pencil problem. As is known (see (29) and references therein), the set $\left\{\lambda_n\right\}_{n\in\mathbb{Z}'}$ is symmetric with respect to the imaginary axis, i.e. if $\operatorname{Re}\lambda_n \neq 0$, then $\lambda_{-|n|} = -\overline{\lambda_{|n|}}$, and in addition $\psi_{-|n|}(x) = -\overline{\psi_{|n|}}(x)$. If $\left\{\Phi_n(x)\right\}_{n\in\mathbb{Z}'}$ is the set of the eigenfunctions of the operator $\mathcal{L}$, then the following relation holds:

$$\Phi_n(x) = \left[\frac{1}{i\lambda_n}\psi_n(x),\ \psi_n(x)\right]^T, \qquad n \in \mathbb{Z}'. \tag{3.3}$$

Since the eigenfunctions of $\mathcal{P}(\lambda)$ are normalized to unity in $L^2_\rho(0,1)$, then the eigenfunctions of $\mathcal{L}$ are almost normalized in $\mathcal{H}$, i.e. there exists two absolute constants, C_1 and C_2, such that

$$0 < C_1 = \inf_{n\in\mathbb{Z}'}\left\|\Phi_n\right\|_{\mathcal{H}} \le \sup_{n\in\mathbb{Z}'}\left\|\Phi_n\right\|_{\mathcal{H}} = C_2 < \infty. \tag{3.4}$$

Now we are in a position to derive the spectral equation for the pencil $\mathcal{P}(\lambda)$ (which is the same for the operator $\mathcal{L}$). From this moment on we assume that the structural parameters of the model are constant. The case of variable parameters can be treated in a similar fashion, but the derivation of the asymptotic approximations for the solution of the spectral equation is extremely lengthy (see, e.g. (5, 21, 32) and references therein).

Let us define the scaled quantities $\widetilde{\lambda}, \widetilde{\alpha}, \widetilde{\beta}, \widetilde{k}_1$, and $\widetilde{k}_2$ by the formulae

$$\lambda = \sqrt{\frac{EI}{\rho}}\,\widetilde{\lambda}, \qquad \widetilde{\alpha} = \sqrt{EI\rho}\,\alpha, \qquad \widetilde{\beta} = \sqrt{EI\rho}\,\beta, \qquad \widetilde{k}_1 = \sqrt{EI\rho}\,k_1, \qquad \widetilde{k}_2 = \sqrt{EI\rho}\,k_2. \tag{3.5}$$

If we substitute these parameters into (3.1)–(3.2), then after omitting the tilde, we arrive at the following spectral problem:

$$\varphi'''' = \lambda^2\varphi, \qquad \varphi(0) = \varphi'(0) = 0, \tag{3.6}$$

$$\varphi''(1) = -k_1 i\lambda\varphi(1) - \beta i\lambda\varphi'(1), \qquad \varphi'''(1) = \alpha i\lambda\varphi(1) + k_2 i\lambda\varphi'(1). \tag{3.7}$$

Notice that the function

$$\varphi(x,\lambda) = \mathcal{A}(\lambda)\Big[\cosh\sqrt{\lambda}x - \cos\sqrt{\lambda}x\Big] + \mathcal{B}(\lambda)\Big[\sinh\sqrt{\lambda}x - \sin\sqrt{\lambda}x\Big], \tag{3.8}$$

with $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ being arbitrary functions of λ, satisfies the differential equation and the left-end boundary conditions from (3.6).

Derivation of the spectral equation. In what follows, it is convenient to introduce new functions

$$C_\pm(z) = \cosh z \pm \cos z, \qquad S_\pm(z) = \sinh z \pm \sin z, \tag{3.9}$$

for which the following properties can be easily checked:

$$\begin{aligned} C'_\pm(z) &= S_\mp(z), & C''_\pm(z) &= C_\mp(z), & C'''_\pm(z) &= S_\pm(z),\\ S'_\pm(z) &= C_\pm(z), & S''_\pm(z) &= S_\mp(z), & S'''_\pm(z) &= C_\mp(z). \end{aligned} \tag{3.10}$$

Using formulae (3.9), we can express $\varphi(x,\lambda)$ as

$$\varphi(x,\lambda) = \mathcal{A}(\lambda)C_-(\sqrt{\lambda}x) + \mathcal{B}(\lambda)S_-(\sqrt{\lambda}x). \tag{3.11}$$

Denoting $\mu = \sqrt{\lambda}$ and using (3.10), we get

$$\begin{aligned} \varphi(1,\lambda) &= \mathcal{A}(\lambda)C_-(\mu) + \mathcal{B}(\lambda)S_-(\mu), & \varphi'(1,\lambda) &= \mu\big[\mathcal{A}(\lambda)S_+(\mu) + \mathcal{B}(\lambda)C_-(\mu)\big],\\ \varphi''(1,\lambda) &= \mu^2\big[\mathcal{A}(\lambda)C_+(\mu) + \mathcal{B}(\lambda)S_+(\mu)\big], & \varphi'''(1,\lambda) &= \mu^3\big[\mathcal{A}(\lambda)S_-(\mu) + \mathcal{B}(\lambda)C_+(\mu)\big]. \end{aligned} \tag{3.12}$$

Substituting (3.12) into the α-boundary condition, $\varphi'''(1) = i\alpha\lambda\varphi(1) + ik_2\lambda\varphi'(1)$, we get

$$\mu\big[\mathcal{A}(\lambda)S_-(\mu) + \mathcal{B}(\lambda)C_+(\mu)\big] = i\alpha\big[\mathcal{A}(\lambda)C_-(\mu) + \mathcal{B}(\lambda)S_-(\mu)\big] + ik_2\mu\big[\mathcal{A}(\lambda)S_+(\mu) + \mathcal{B}(\lambda)C_-(\mu)\big]. \tag{3.13}$$

Collecting all terms containing $\mathcal{A}(\lambda)$ on the left-hand side and the terms containing $\mathcal{B}(\lambda)$ on the right-hand side, we obtain a new equation

$$\mathcal{A}(\lambda)\big[\mu S_-(\mu) - ik_2\mu S_+(\mu) - i\alpha C_-(\mu)\big] = \mathcal{B}(\lambda)\big[-\mu C_+(\mu) + ik_2\mu C_-(\mu) + i\alpha S_-(\mu)\big]. \tag{3.14}$$

It is clear that if one takes $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ as follows:

$$\mathcal{A}(\lambda) = -\mu C_+(\mu) + ik_2\mu C_-(\mu) + i\alpha S_-(\mu), \qquad \mathcal{B}(\lambda) = \mu S_-(\mu) - ik_2\mu S_+(\mu) - i\alpha C_-(\mu), \tag{3.15}$$

then the function $\varphi(x,\lambda)$ from (3.8) satisfies the clamped conditions at the left end and the α-boundary condition at the right end. Thus the pencil eigenfunction becomes

$$\varphi(x,\lambda) = \big[-\mu C_+(\mu) + ik_2\mu C_-(\mu) + i\alpha S_-(\mu)\big]C_-(\mu x) + \big[\mu S_-(\mu) - ik_2\mu S_+(\mu) - i\alpha C_-(\mu)\big]S_-(\mu x). \tag{3.16}$$

The β-boundary condition, $\varphi''(1) = -ik_1\lambda\varphi(1) - i\beta\lambda\varphi'(1)$, generates the spectral equation for the pencil. Using (3.12) we obtain that

$$\mathcal{A}(\lambda)C_+(\mu) + \mathcal{B}(\lambda)S_+(\mu) = -ik_1\big[\mathcal{A}(\lambda)C_-(\mu) + \mathcal{B}(\lambda)S_-(\mu)\big] - i\beta\mu\big[\mathcal{A}(\lambda)S_+(\mu) + \mathcal{B}(\lambda)C_-(\mu)\big]. \tag{3.17}$$

Using formulae (3.15) for $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$, we evaluate the terms in (3.17) and have

$$\begin{aligned} \mathcal{A}(\lambda)C_+(\mu) + \mathcal{B}(\lambda)S_+(\mu) &= -2ik_2\mu\sinh\mu\sin\mu - 2\mu(1+\cosh\mu\cos\mu) - 2i\alpha(\cosh\mu\sin\mu - \sinh\mu\cos\mu),\\ \mathcal{A}(\lambda)C_-(\mu) + \mathcal{B}(\lambda)S_-(\mu) &= 2ik_2\mu(1-\cosh\mu\cos\mu) - 2\mu\sinh\mu\sin\mu,\\ \mathcal{A}(\lambda)S_+(\mu) + \mathcal{B}(\lambda)C_-(\mu) &= -2\mu(\cosh\mu\sin\mu + \sinh\mu\cos\mu) - 2i\alpha(1-\cosh\mu\cos\mu). \end{aligned} \tag{3.18}$$

Substituting (3.18) into Eq.(3.17) we arrive at the following *spectral equation:*

$$\begin{aligned} &\mu\big[i(k_1+k_2)\sinh\mu\sin\mu + k_1k_2(1-\cosh\mu\cos\mu) + (1+\cosh\mu\cos\mu)\big] =\\ &-i\beta\mu^2(\cosh\mu\sin\mu + \sinh\mu\cos\mu) + \alpha\beta\mu(1-\cosh\mu\cos\mu) - i\alpha(\cosh\mu\sin\mu - \sinh\mu\cos\mu). \end{aligned} \tag{3.19}$$

Remark 3.1. *For different combinations of the boundary parameters, Eq.*(3.19) *generates the spectral equations corresponding to the selfadjoint operators defined in Sec. 2. Namely, we have the following cases:*

(1) *If* $\alpha = \beta = k_1 = k_2 = 0$, *then we have* $\varphi''(1) = \varphi'''(1) = 0$. *Eq.*(3.19) *generates the spectral equation for the clamped-free pencil, or equivalently for the operator* $\mathfrak{L}_{CF}$*:*

$$1 + \cosh\sqrt{\lambda}\cos\sqrt{\lambda} = 0. \tag{3.20}$$

(2) *If* $\alpha = k_2 = 0$, $|k_1| < \infty$, *and* $\beta \to \infty$, *then we have* $\varphi'(1) = \varphi'''(1) = 0$. *Eq.*(3.19) *generates the spectral equation for the clamped-sliding pencil, or equivalently for the operator* $\mathcal{L}_{CS}$*:*

$$\cosh\sqrt{\lambda}\sin\sqrt{\lambda} + \sinh\sqrt{\lambda}\cos\sqrt{\lambda} = 0. \tag{3.21}$$

(3) *If* $k_1 = \beta = 0$, $|k_2| < \infty$, *and* $\alpha \to \infty$, *then we have* $\varphi(1) = \varphi''(1) = 0$. *Eq.*(3.19) *generates the spectral equation for the clamped-hinged pencil, or equivalently for the operator* $\mathcal{L}_{CH}$*:*

$$\cosh\sqrt{\lambda}\sin\sqrt{\lambda} - \sinh\sqrt{\lambda}\cos\sqrt{\lambda} = 0. \tag{3.22}$$

(4) *If* $k_1, k_2 \to \infty$ *with* $|\alpha|, |\beta| < \infty$, *or* $\alpha, \beta \to \infty$ *with* $k_1, k_2 < \infty$, *then we have* $\varphi(1) = \varphi'(1) = 0$. *Eq.*(3.19) *generates the spectral equation for the clamped-clamped pencil, or equivalently for the operator* $\mathcal{L}_{CC}$*:*

$$1 - \cosh\sqrt{\lambda}\cos\sqrt{\lambda} = 0. \tag{3.23}$$

4 | SELFADJOINT PROBLEMS

In this section we discuss several properties of the eigenfunctions of the polynomial pencils corresponding to different types of selfadjoint boundary conditions. In particular we establish the boundary behavior of the eigenfunctions and prove that the selfadjoint problems cannot share eigenvalues; we also prove that the geometric multiplicity of an eigenvalue of each selfadjoint problem is one. These properties, being important in their own right, remain valid for the multi-parameter non-selfadjoint problems. The extension of the results to the non-selfadjoint problems is discussed in Sec. 5.

We consider four selfadjoint boundary value problems for the beam equation with the clamped left-end conditions and different right-end conditions. Namely, if the spectral problem for a model is given by Eqs.(3.6) and

$$\begin{aligned}
&(\alpha)\quad \text{the free end conditions, } \varphi''(1) = \varphi'''(1) = 0, \text{ it is called the } \textit{CF-model};\\
&(\beta)\quad \text{the sliding end conditions, } \varphi'(1) = \varphi'''(1) = 0, \text{ it is called the } \textit{CS-model;}\\
&(\gamma)\quad \text{the hinged end conditions, } \varphi(1) = \varphi''(1) = 0, \text{ it is called the } \textit{CH-model;}\\
&(\delta)\quad \text{the clamped end conditions, } \varphi(1) = \varphi'(1) = 0, \text{ it is called the } \textit{CC-model.}
\end{aligned} \tag{4.1}$$

It is convenient to formulate without proof (the proof can be done by using directly the definition (3.9) and formulae (3.10)) the following technical statement.

Lemma 4.1. *The following representation is valid for a solution* $\varphi(x,\lambda)$ *of Eqs.* (3.6) *satisfying the different right-end conditions. Namely,*

$$\textit{if } \varphi(1,\lambda) = 0, \textit{ then } \varphi(x,\lambda) = S_-(\mu)C_-(\mu x) - C_-(\mu)S_-(\mu x); \tag{4.2}$$
$$\textit{if } \varphi'(1,\lambda) = 0, \textit{ then } \varphi(x,\lambda) = C_-(\mu)C_-(\mu x) - S_+(\mu)S_-(\mu x); \tag{4.3}$$
$$\textit{if } \varphi''(1,\lambda) = 0, \textit{ then } \varphi(x,\lambda) = S_+(\mu)C_-(\mu x) - C_+(\mu)S_-(\mu x); \tag{4.4}$$
$$\textit{if } \varphi'''(1,\lambda) = 0, \textit{ then } \varphi(x,\lambda) = C_+(\mu)C_-(\mu x) - S_-(\mu)S_-(\mu x). \tag{4.5}$$

In the next statement, Theorem 4.2, we discuss the boundary properties of the eigenfuntions of different selfadjoint problems. Without misunderstanding, we will use the same notation, $\varphi_n(x)$, for eigenfunctions corresponding to different problems.

Theorem 4.2. *The following inequalities are valid for the eigenfunctions corresponding to different selfadjoint models:*

$$\begin{aligned}
&(a)\ \ \varphi_n(1)\varphi_n'(1) > 0 \textit{ for the CF-model}, &&(b)\ \ \varphi_n(1)\varphi_n''(1) > 0 \textit{ for the CS-model},\\
&(c)\ \ \varphi_n'(1)\varphi_n'''(1) > 0 \textit{ for the CH-model}, &&(d)\ \ \varphi_n''(1)\varphi_n'''(1) > 0 \textit{ for the CC-model}.
\end{aligned} \tag{4.6}$$

Proof. Case (4.6 (a)). The problem is defined by Eqs.(3.6) and conditions (α) of (4.1). Using contradiction argument, we assume that there exists an eigenvalue, λ_n, such that the corresponding eigenfunction, $\varphi_n(x)$, satisfies an additional condition: $\varphi_n(1) = 0$. Using formula (4.2) we obtain the following representation for $\varphi'''(x,\lambda)$ (see (3.10)):

$$\varphi'''(x,\lambda) = \mu^3\left[S_-(\mu)S_-(\mu x) - C_-(\mu)C_+(\mu x)\right]. \tag{4.7}$$

Evaluating φ''' at $x = 1$, we have $\varphi'''(1,\lambda) = \mu^3\left[S_-^2(\mu) - C_-(\mu)C_+(\mu)\right] = -2\mu^3\sinh\mu\sin\mu$, which means that $\varphi'''(1) = 0$ only if $\mu = \pi n$, for some $n \in \mathbb{N}^+$. Thus, the solution of the problem defined by (3.6) and (4.2) that satisfies $\varphi'''(1) = 0$ has to be of the form:

$$\varphi_n(x) = \sinh(\pi n)C_-(\pi n x) - \left[\cosh(\pi n) + (-1)^{n+1}\right]S_-(\pi n x). \tag{4.8}$$

Let us calculate $\varphi_n''(1)$ and show that $\left|\varphi_n''(1)\right| > 0$. We have

$$\varphi_n''(1) = (\pi n)^2\left\{\sinh(\pi n)C_+(\pi n) - \left[\cosh(\pi n) + (-1)^{n+1}\right]S_+(\pi n)\right\} = 2(-1)^n(\pi n)^2\sinh(\pi n).$$

Since $\sinh(\pi n) > 0$, the function $\varphi_n(x)$ cannot satisfy the condition $\varphi_n''(1) = 0$, which mean that the initial assumption was not correct and $\varphi_n(1) \neq 0$.

Now let us show that the assumption $\varphi_n'(1) = 0$ leads to a contradiction as well. Using formula (4.3), we obtain the following representation for $\varphi''(x, \lambda)$:

$$\varphi''(x, \lambda) = \mu^2\left[C_-(\mu)C_+(\mu x) - S_+(\mu)S_+(\mu x)\right].$$

Evaluating $\varphi''(1, \lambda)$ we obtain that $\varphi''(1, \lambda) = \mu^2\left[C_-(\mu)C_+(\mu) - S_+^2(\mu)\right] = -2\mu^2\sinh\mu\sin\mu$, which means that $\varphi''(1, \lambda) = 0$ only if $\mu = \pi n,\ n \in \mathbb{N}^+$. For such μ_n, we have the following representation for the corresponding eigenfunction:

$$\varphi_n(x) = \left[\cosh(\pi n) + (-1)^{n+1}\right]C_-(\pi n x) - \sinh(\pi n)S_-(\pi n x). \tag{4.9}$$

Let us evaluate $\varphi_n'''(1)$ and show that $\varphi_n'''(1) \neq 0$. We have

$$\varphi_n'''(1) = (\pi n)^3\left\{\left[\cosh(\pi n) + (-1)^{n+1}\right]S_-(\pi n) - \sinh(\pi n)C_+(\pi n)\right\} = 2(-1)^{n+1}(\pi n)^3\sinh(\pi n) \neq 0,$$

which contradicts (4.1 (β)) and therefore $\varphi_n'(1) \neq 0$. Case (4.6 (a)) is proven.

Case (4.6 (b)). The problem is defined by (3.6) and conditions (β) of (4.1). Using contradiction argument, we assume that there exists an eigenvalue, λ_n, such that the corresponding eigenfunction, $\varphi_n(x)$, satisfies an additional condition $\varphi_n(1) = 0$. If we repeat all steps completed at the beginning of the proof of Case (4.6 (a)), we arrive at formula (4.8) for the eigenfunction $\varphi_n(x)$. Now we evaluate $\varphi_n'(1)$ and show that $\varphi_n'(1) \neq 0$. We have

$$\varphi_n'(1) = \pi n\left\{\sinh(\pi n)S_+(\pi n) - \left[\cosh(\pi n) + (-1)^{n+1}\right]C_+(\pi n)\right\} = -2\pi n\left[1 + (-1)^{n+1}\cosh(\pi n)\right].$$

Since $\cosh(\pi n) > 1$, we get that $\varphi_n'(1) \neq 0$ and hence the initial assumption is not valid, which means that $\varphi_n(1) \neq 0$.

Assume now that $\varphi_n''(1) = 0$. It means that the solution of problem (3.6) has to satisfy the conditions: $\varphi'(1) = \varphi''(1) = \varphi'''(1) = 0$. Thus, $\varphi_n(x)$ is in fact an eigenfunction of the CF-model considered in Case (4.6 (a)). By formula (4.6 (a)), $\varphi_n(1)\varphi_n'(1) \neq 0$, which is in contradiction with the boundary conditions of the CS-model. Case (4.6 (b)) is proven.

Case (4.6 (c)). Assuming that $\varphi_n'(1) = 0$, we show that this assumption leads to a contradiction. Let us take the solution of problem (3.6) given by (4.2). By assumption, $\varphi'(1, \lambda) = 0$, i.e. $\varphi'(1, \lambda) = \mu\left[S_-(\mu)S_+(\mu) - C_-^2(\mu)\right] = -2\mu[1 - \cosh\mu\cos\mu] = 0$, where $\mu = \sqrt{\lambda}$. Since $\mu \neq 0$, this equation means that there exists μ_n satisfying the equation

$$\cosh\mu\cos\mu = 1. \tag{4.10}$$

Now we evaluate $\varphi_n''(1, \lambda_n)$, $\lambda_n = \mu_n^2$, and have

$$\varphi_n''(1) = \mu_n^2\left[S_-(\mu_n)C_+(\mu_n) - C_-(\mu_n)S_+(\mu_n)\right] = 2\mu_n^2\left[\sinh\mu_n\cos\mu_n - \cosh\mu_n\sin\mu_n\right]. \tag{4.11}$$

If $\varphi_n''(1) = 0$ then the system obtained from (4.10) and (4.11), i.e.

$$\sinh\mu\cos\mu - \cosh\mu\sin\mu = 0, \qquad \cosh\mu\cos\mu = 1, \tag{4.12}$$

must be consistent. Let us show that this is not the case. Squaring the first equation of (4.12) and taking into account the second equation of the system, we get

$$\sinh^2\mu\left(1 - \sin^2\mu\right) - 2\sinh\mu\sin\mu + \cosh^2\mu\sin^2\mu = (\sinh\mu - \sin\mu)^2 = 0. \tag{4.13}$$

Combining (4.13) with the second equation of (4.12) we get a new system

$$\sinh\mu = \sin\mu, \qquad \cosh\mu\cos\mu = 1. \tag{4.14}$$

Since $\cosh\mu \neq 0$, the second equation of (4.14) yields $\cos^2\mu = \left(\cosh^2\mu\right)^{-1}$ and thus $\cosh^4\mu + 2\cosh^2\mu + 1 = 0$. This equation does not have any solutions. The obtained contradiction means that $\varphi_n'(1) \neq 0$.

Now we assume that $\varphi_n'''(1) = 0$. In this case, $\varphi_n(x)$ must be an eigenfunction of the CF-model considered in Case (4.6 (a)). However, according to that case, one has $\varphi_n(1)\varphi_n'(1) \neq 0$, which contradicts the statement of the CH-model. Case (4.6 (c)) is proven.

Case (4.6 (d)). Assuming that $\varphi_n''(1) = 0$, we get that $\varphi_n(x)$ is an eigenfunction of the CH-model and thus, according to Case (4.6 (c)), one has $\varphi_n'(1)\varphi_n'''(1) > 0$, or $\varphi_n'(1) \neq 0$, which contradicts the statement of the CC-model.

Now assuming that $\varphi_n'''(1) = 0$, we get that $\varphi_n(x)$ is an eigenfunction of the CS-model and, due to the result of Case (4.6 (b)), one has $\varphi_n(1)\varphi_n''(1) \neq 0$, which contradicts the statement of the CC-model. Case (4.6 (d)) is proven. □

Corollary 4.3. *The selfadjoint problems (4.1 (α)–(δ)) do not share eigenvalues.*

Proof. 1) We use a contradiction argument and assume that there exists an eigenvalue λ_0, such that the CF-model has an eigenfunction $u(x)$, and the CS-model has an eigenfunction $v(x)$, for the same $\lambda_0 = \mu_0^2$. By Theorem 4.2, Cases (4.6 (a) and (b)), we have $u(1)v(1) \neq 0$. Construct the following function:

$$f(x) = \frac{u(x)}{u(1)} - \frac{v(x)}{v(1)}. \tag{4.15}$$

This function satisfies (3.6), the right-end conditions $f(1) = 0$ (by construction), and $f'''(1) = 0$. Using formula (4.2) for $f(x)$, we can identify the roots of the equation $f'''(1) = 0$ as

$$f'''(1) = \mu_0^3\left[S_-^2(\mu_0) - C_-(\mu_0)C_+(\mu_0)\right] = -2\mu_0^3 \sinh\mu_0 \sin\mu_0 = 0.$$

Hence the problem for $f(x)$ has a solution only if $\mu_0 = \pi n$, $n \in \mathbb{N}^+$. However, neither the spectral equation (3.20) for the CF-model, nor the spectral equation (3.21) for the CS-model can have $\mu_0 = \pi n$ as their solutions. Thus, the CF-model and the CS-model cannot share an eigenvalue.

2) Consider the CF-model and the CH-model. Assuming that these problems share an eigenvalue, $\lambda_0 = \mu_0^2$, with the eigenfunctions $u(x)$ and $v(x)$ respectively. By Theorem 4.2, Cases (4.6 (a) and (c)), we have $u'(1)v'(1) \neq 0$. Construct the following function:

$$g(x) = \frac{u(x)}{u'(1)} - \frac{v(x)}{v'(1)}. \tag{4.16}$$

This function, being a solution of Eqs.(3.6), satisfies the following right-end conditions: $g'(1) = 0$ (by construction) and $g''(1) = 0$. Using formula (4.4) for $g(x)$, we obtain that

$$g'(x) = \mu_0\left[S_+(\mu_0)S_+(\mu_0 x) - C_-(\mu_0)C_+(\mu_0 x)\right],$$

which means that $g'(1) = 2\sinh\mu_0 \sin\mu_0 = 0$ only if $\mu_0 = \pi n$, $n \in \mathbb{N}^+$. However, such $\lambda_0 = \mu_0^2$ does not satisfy the CF-model spectral equation (3.20).

3) Assuming that the CF-model and the CC-model have a common eigenvalue, λ_0, we obtain that the two spectral equations (3.20) and (3.23) must have the same root λ_0, which is impossible.

4) Assume that the CS-model and the CH-model share an eigenvalue, λ_0. It means that the two spectral equations, (3.21) and (3.22), have a common root, which cannot happen.

5) Assume that the CS-model and the CC-model share an eigenvalue, λ_0, with $u(x)$ and $v(x)$ being the corresponding eigenfunctions. Construct a new function

$$\psi(x) = \frac{u(x)}{u''(1)} - \frac{v(x)}{v''(1)}, \tag{4.17}$$

which satisfies Eqs.(3.6) and the right-end conditions: $\psi'(1) = 0$ and $\psi''(1) = 0$ (by construction). Using formula (4.4) for $\psi(x)$, we obtain that $\psi'(1) = 0$ only if $\mu_0 = \pi n$, $n \in \mathbb{N}^+$. However, at such point $\lambda_0 = (\pi n)^2$, the spectral equations (3.21) and (3.23) do not hold.

6) Finally, assume that λ_0 is a common eigenvalue of the CH-model and the CC-model. The following well defined function

$$\chi(x) = \frac{u(x)}{u'''(1)} - \frac{v(x)}{v'''(1)} \tag{4.18}$$

satisfies the conditions $\chi(1) = \chi'''(1) = 0$. Using formula (4.2) we obtain that $\chi'''(1) = 0$ only if $\mu_0 = \pi n$, $n \in \mathbb{N}^+$. However, at such point the spectral equation (3.23) does not hold. □

Our next statement is concerned with the geometric multiplicities of the eigenvalues of each selfadjoint problem.

Theorem 4.4. *The spectrum of each selfadjoint problem is simple.*

Proof. In each case below, we use a contradiction argument and we assume that there exists an eigenvalue, λ_n, such that the corresponding problem has two linearly independent eigenfunctions, $u(x)$ and $v(x)$.

Let us consider the CF-model, *Case (4.1 (α)).* By formula (4.6 (a)), one has $u(1)v(1) \neq 0$. Let us construct a new function, $f(x)$, by formula (4.15). This function satisfies Eqs.(3.6) with $\lambda = \lambda_n$, the right-end conditions $f''(1) = f'''(1) = 0$, and

$f(1) = 0$ by construction. Being an eigenfunction of the CF-model, $f(x)$ satisfies (4.6 (a)), i.e. $f(1)f'(1) \neq 0$, which contradicts our construction since $f(1) = 0$.

Let us consider the CS-model, *Case (4.1 (β))*. By formula (4.6 (b)), one has $u''(1)v''(1) \neq 0$. Let us construct a new function, $\psi(x)$, by formula (4.17). This function satisfies Eqs.(3.6), the right-end conditions $\psi'(1) = \psi'''(1) = 0$, and $\psi''(1) = 0$ by construction. Being an eigenfunction of the CS-model, $\psi(x)$ satisfies (4.6 (b)), i.e. $\psi(1)\psi''(1) \neq 0$, which contradicts our construction since $\psi''(1) = 0$.

Let us consider the CH-model, *Case (4.1 (γ))*. By formula (4.6 (c)), one has $u'(1)v'(1) \neq 0$. Let us construct a new function, $g(x)$, by formula (4.16). This function satisfies Eqs.(3.6), the right-end conditions $g(1) = g''(1) = 0$, and $g'(1) = 0$ by construction. Being an eigenfunction of the CH-model, $g(x)$ satisfies (4.6 (c)), i.e. $g'(1)g'''(1) \neq 0$, which contradicts our construction since $g'(1) = 0$.

Let us consider the CC-model, *Case (4.1 (δ))*. By formula (4.6 (d)), one has $u'''(1)v'''(1) \neq 0$. Let us construct a new function, $\chi(x)$, by formula (4.18). This function satisfies Eqs.(3.6), the right-end conditions $\chi(1) = \chi'(1) = 0$, and $\chi'''(1) = 0$ by construction. Being an eigenfunction of the CC-model, $\chi(x)$ satisfies (4.6 (d)), i.e. $\chi''(1)\chi'''(1) \neq 0$, which contradicts our construction since $\chi'''(1) = 0$. □

5 | MAIN BOUNDARY INEQUALITIES

In this section, we extend the results of Theorem 4.2 to the case of the non-selfadjoint operator $\mathcal{L}$ defined in (2.3) and (2.4). It turns out that the boundary behavior of an eigenfunction, $\varphi_n(x)$, of the corresponding non-selfadjoint operator pencil depends on which boundary parameters from (3.7) are non-zero. In Theorem 5.1 below, we consider the cases with one non-zero boundary parameter, Theorems 5.2 and 5.3 deal with the cases of two non-zero boundary parameters, and Theorem 5.4 deals with the cases when only one parameter is zero. Now we recall that the boundary control parameters are real.

Theorem 5.1. *If one of the boundary parameters is not equal to zero, while the remaining three parameters are zeros, then the following result holds:*

$$\varphi_n(1)\varphi_n'(1) > 0, \tag{5.1}$$

where $\varphi_n(x) = \varphi(x, \lambda_n) = \varphi(x, \mu_n^2)$ is any eigenfunction of the spectral problem (3.6)–(3.7).

Proof. Case (1). Let $\alpha \neq 0$ and $k_1 = k_2 = \beta = 0$. In this case, the problem is given by Eqs.(3.6) and the following conditions at $x = 1$ (see (3.7)):

$$\varphi'''(1) = i\alpha\lambda\varphi(1), \qquad \varphi''(1) = 0. \tag{5.2}$$

Arguing by contradiction, we assume that there exists an eigenfunction, $\varphi_n(x)$, for which (5.1) fails, e.g. $\varphi_n(1) = 0$. Then the following conditions (for $\varphi(x) = \varphi_n(x)$) at $x = 1$ have to be satisfied:

$$\varphi'''(1) = \varphi''(1) = \varphi(1) = 0. \tag{5.3}$$

Since the function satisfying (3.6) and $\varphi'''(1) = \varphi''(1) = 0$ is an eigenfunction of the CF-model, then by Theorem 4.2, Case (4.6 (a)), we have $\varphi(1) \neq 0$, which contradicts our assumption.

Now assume that for some eigenfunction $\varphi_n'(1) = 0$, which yields the problem with the right-end boundary conditions $\varphi'(1) = \varphi''(1) = 0$. Using (4.4) for $\varphi(x)$ we obtain that the remaining condition, $\varphi'(1) = 0$, yields $S_+^2(\mu) - C_+(\mu)C_-(\mu) = 2\sinh\mu\sin\mu = 0$, and we get $\mu = \mu_n = \pi n,\ n \in \mathbb{N}^+$. Therefore, $\varphi_n(x) = \sinh(\pi n)C_-(\mu_n x) - [\cosh(\pi n) - (-1)^n]S_-(\mu_n x)$. However, this real-valued function cannot satisfy the first boundary condition from (5.2). Thus $\varphi_n'(1) \neq 0$, and Case (1) is shown.

Case (2). Let $\beta \neq 0$ and $\alpha = k_1 = k_2 = 0$. In this case, the problem is given by Eqs.(3.6) and the following conditions at $x = 1$:

$$\varphi'''(1) = 0, \qquad \varphi''(1) = -i\beta\lambda\varphi'(1). \tag{5.4}$$

Arguing by contradiction, assume that there exists an eigenfunction, $\varphi(x) = \varphi_n(x)$, such that $\varphi_n(1) = 0$. Using formula (4.5) for $\varphi(x)$ we apply the condition $\varphi(1) = 0$ and have

$$\varphi(1) = C_+(\mu)C_-(\mu) - S_-^2(\mu) = 2\sinh\mu\sin\mu = 0,$$

which can happen only if $\mu_n = \pi n,\ n \in \mathbb{N}^+$. For such μ_n, $\varphi_n(x)$ is a real-valued function for which the second condition from (5.4) does not hold, meaning that $\varphi_n(1) \neq 0$.

If $\varphi_n'(1) = 0$, then $\varphi_n(x)$ must be an eigenfunction corresponding to the CF-model. Therefore μ_n must be real, and with real $\lambda_n = \mu_n^2$, the second boundary condition from (5.4) fails, i.e. $\varphi_n'(1) \neq 0$. Thus, Case (2) is shown.

Case (3). Let $k_1 \neq 0$ and $\alpha = \beta = k_2 = 0$. The right-end boundary conditions are

$$\varphi''(1) = -ik_1\lambda\varphi(1), \qquad \varphi'''(1) = 0. \tag{5.5}$$

If for some n, $\varphi_n(1) = 0$, then we have the problem with the following right-end conditions:

$$\varphi(1) = \varphi''(1) = \varphi'''(1) = 0. \tag{5.6}$$

If a function, φ, satisfies the conditions $\varphi''(1) = \varphi'''(1) = 0$, then it is an eigenfunction of the CF-model, and if it satisfies $\varphi(1) = \varphi''(1) = 0$, then it is an eigenfunction of the CH-model. By Corollary 4.3 the two problems cannot share an eigenvalue, and thus $\varphi_n(1) \neq 0$.

Now assume that $\varphi_n'(1) = 0$. In this case, φ_n is an eigenfunction of the CS-model and, therefore, can be taken as a real-valued function corresponding to a positive eigenvalue λ_n. For a real-valued function the first condition from (5.5) fails. Thus Case (3) is shown.

Case (4). Let $k_2 \neq 0$ and $\alpha = \beta = k_1 = 0$. The right-end boundary conditions are

$$\varphi'''(1) = ik_2\lambda\varphi(1), \qquad \varphi''(1) = 0. \tag{5.7}$$

If $\varphi_n(1) = 0$, we have the problem with the right-end conditions $\varphi(1) = \varphi''(1) = 0$, which means that φ_n is an eigenfunction of the CH-model. Hence the corresponding eigenvalue $\lambda_n > 0$ and the eigenfunction can be taken real-valued. However, for a real-valued function the first condition in (5.7) cannot be satisfied.

If $\varphi_n'(1) = 0$, then we have $\varphi'''(1) = \varphi''(1) = 0$, which corresponds to the CF-model. At the same time, $\varphi(x)$ satisfies the conditions $\varphi'''(1) = \varphi'(1) = 0$, which corresponds to the CS-model. This contradicts the fact that the above selfadjoint problems do not share eigenvalues, and so $\varphi_n'(1) \neq 0$. □

Theorem 5.2. *Consider the following four sets of conditions on the boundary parameters:*

$$\begin{aligned} &(1)\ |\alpha\beta| > 0 \text{ and } k_1 = k_2 = 0, \qquad &&(2)\ |k_1k_2| > 0 \text{ and } \alpha = \beta = 0, \\ &(3)\ |\alpha k_2| > 0 \text{ and } k_1 = \beta = 0, \qquad &&(4)\ |k_1\beta| > 0 \text{ and } \alpha = k_2 = 0. \end{aligned} \tag{5.8}$$

If any of these sets of conditions is satisfied, then relation (5.1) *holds for any eigenfunction, φ_n, of the corresponding operator pencil.*

Proof. Case (1). Since $k_1 = k_2 = 0$, the right-end boundary conditions are

$$\varphi'''(1) = i\alpha\lambda\varphi(1), \qquad \varphi''(1) = -i\beta\lambda\varphi'(1). \tag{5.9}$$

Using contradiction argument, assume that for some eigenvalue $\lambda_n = \mu_n^2$, the corresponding eigenfunction, $\varphi_n(x)$, is such that $\varphi_n(1) = 0$. Due to (5.9), this eigenfunction satisfies the following right-end boundary conditions:

$$\varphi(1) = \varphi'''(1) = 0 \qquad \text{and} \qquad \varphi''(1) = -i\beta\lambda_n\varphi'(1). \tag{5.10}$$

Using formula (4.5) for the solution of Eqs.(3.6), we apply the remaining condition, $\varphi(1) = 0$, and get the spectral equation: $C_+(\mu)C_-(\mu) - S_-^2(\mu) = 2\sinh\mu\sin\mu = 0$. Substituting $\mu_n = \pi n$ into (4.5), we get the following expression for the eigenfunction:

$$\varphi_n(x) = \left[\cosh(\pi n) + (-1)^n\right]C_-(\mu_n x) - \sinh(\pi n)S_-(\mu_n x). \tag{5.11}$$

For a real-valued function φ_n, the second condition of (5.9) cannot be satisfied.

Now we assume that $\varphi_n'(1) = 0$. The corresponding right-end boundary conditions are

$$\varphi'(1) = \varphi''(1) = 0, \qquad \varphi'''(1) = i\alpha\lambda\varphi(1). \tag{5.12}$$

Using formula (4.4), we apply the remaining condition, $\varphi'(1) = 0$, and get the spectral equation: $S_+^2(\mu) - C_+(\mu)C_-(\mu) = 2\sinh\mu\sin\mu = 0$. Substituting $\mu = \pi n$, $n \in \mathbb{N}^+$, into (4.4) generates a real-valued eigenfunction. However, for a real-valued function, the first condition of (5.9) cannot be satisfied. Case (1) is proven.

Case (2). The right-end boundary conditions corresponding to the case $|k_1k_2| > 0$ are

$$\varphi'''(1) = ik_2\lambda\varphi'(1), \qquad \varphi''(1) = -ik_1\lambda\varphi(1). \tag{5.13}$$

Assuming that $\varphi_n(1) = 0$ for some eigenvalue $\lambda_n = \mu_n^2$, we obtain that the corresponding eigenfunction, $\varphi_n(x)$, satisfies the following conditions: $\varphi(1) = \varphi''(1) = 0$. This means that $\varphi_n(x)$ is an eigenfunction of the CH-model. Since the problem has real spectrum, its eigenfunctions are real-valued, which contradicts the first condition of (5.13).

Assuming that $\varphi_n'(1) = 0$ for some λ_n, we obtain that λ_n must be an eigenvalue of the CS-model, and thus λ_n is real. Therefore, for the corresponding eigenfunction, the second condition of (5.13) cannot hold. Case (2) is proven.

Case (3). The right-end conditions corresponding to the case $\beta = k_1 = 0$ are

$$\varphi'''(1) = i\alpha\lambda\varphi(1) + ik_2\lambda\varphi'(1), \qquad \varphi''(1) = 0. \tag{5.14}$$

Assume that $\varphi_n(1) = 0$ for some λ_n. Then $\varphi_n(x)$ is in fact an eigenfunction of the CH-model, which has real spectrum and real-valued eigenfunctions. However, for such an eigenfunction the condition $\varphi'''(1) = ik_2\lambda\varphi'(1)$ does not hold.

Assuming that $\varphi_n'(1) = 0$, we obtain the problem with the following right-end conditions: $\varphi'(1) = \varphi''(1) = 0$. Using (4.4) and applying condition $\varphi'(1) = 0$, we obtain the spectral equation $\sinh\mu\sin\mu = 0$. The corresponding eigenfunction obtained from (4.4) evaluated at $\mu = \pi n$ is real-valued. For such a function the condition $\varphi'''(1) = i\alpha\lambda\varphi(1)$ of (5.14) fails. Case (3) is completely shown.

The result for Case (4) ($|\beta k_1| > 0$ and $\alpha = k_2 = 0$) can be shown using a similar argument. □

Theorem 5.3. *(1) If $|\alpha k_1| > 0$ and $\beta = k_2 = 0$, then $|\varphi_n(1)\varphi_n''(1)\varphi_n'''(1)| > 0$.*

(2) If $|\beta k_2| > 0$ and $\alpha = k_1 = 0$, then $|\varphi_n'(1)\varphi_n''(1)\varphi_n'''(1)| > 0$.

Proof. The right-end boundary conditions for Case (1) are

$$\varphi'''(1) = i\alpha\lambda\varphi(1) \qquad \text{and} \qquad \varphi''(1) = -ik_1\lambda\varphi(1). \tag{5.15}$$

If $\varphi_n(1) = 0$, then φ_n must be an eigenfunction of the CF-model (for which $\varphi_n''(1) = \varphi_n'''(1) = 0$) and of the CH-model (for which $\varphi_n(1) = \varphi_n''(1) = 0$), which cannot happen by Corollary 4.3. Hence, $\varphi_n(1) \neq 0$ and as follows from Eqs.(5.15), $\varphi_n''(1)\varphi_n'''(1) \neq 0$. Case (1) is proven.

Consider Case (2). The right-end boundary conditions are

$$\varphi''(1) = -i\beta\lambda\varphi'(1), \qquad \varphi'''(1) = ik_2\lambda\varphi'(1). \tag{5.16}$$

If $\varphi_n'(1) = 0$, then φ_n must be an eigenfunction of the CF-model ($\varphi_n''(1) = \varphi_n'''(1) = 0$) and of the CS-model (for which $\varphi_n'(1) = \varphi_n'''(1) = 0$), which cannot happen. □

Theorem 5.4. *If one of the boundary parameters is zero while the remaining three parameters are non-zeros, then the following results hold:*

$$\begin{aligned} &1)\ \textit{If } \beta = 0,\ \textit{then } |\varphi_n(1)\varphi_n''(1)| > 0; && 2)\ \textit{If } k_2 = 0,\ \textit{then } |\varphi_n(1)\varphi_n'''(1)| > 0;\\ &3)\ \textit{If } k_1 = 0,\ \textit{then } |\varphi_n'(1)\varphi_n''(1)| > 0; && 4)\ \textit{If } \alpha = 0,\ \textit{then } |\varphi_n'(1)\varphi_n'''(1)| > 0. \end{aligned} \tag{5.17}$$

Proof. Case (1). Let $\beta = 0$. Then the right-end conditions become

$$\varphi''(1) = -ik_1\lambda\varphi(1) \qquad \text{and} \qquad \varphi'''(1) = i\alpha\lambda\varphi(1) + ik_2\lambda\varphi'(1). \tag{5.18}$$

Arguing by contradiction, assume that there exists an eigenvalue, λ_n, whose eigenfunction, φ_n, satisfies $\varphi_n(1) = 0$. Such an eigenfunction has to satisfy the conditions

$$\varphi(1) = \varphi''(1) = 0 \qquad \text{and} \qquad \varphi'''(1) = ik_2\lambda\varphi'(1). \tag{5.19}$$

Since $\varphi_n(x)$ satisfies the first two conditions of (5.19), it is an eigenfunction of the CH-model, which means that $\lambda_n > 0$ and $\varphi_n(x)$ is a real-valued function. However, a real-valued function cannot satisfy the third boundary condition. The obtained contradiction yields $\varphi_n(1) \neq 0$.

Now assume that for some eigenfunction, φ_n, we have $\varphi_n''(1) = 0$. From (5.18) it follows that $\varphi_n(1) = 0$, which immediately yields a situation already discussed in the previous paragraph. Case (1) is proven.

Case (2). Let $k_2 = 0$, then the right-end conditions become

$$\varphi''(1) = -ik_1\lambda\varphi(1) - i\beta\lambda\varphi'(1) \qquad \text{and} \qquad \varphi'''(1) = i\alpha\lambda\varphi(1). \tag{5.20}$$

Assume that for some eigenvalue, λ_n, the corresponding eigenfunction, φ_n, satisfies $\varphi_n(1) = 0$. Such a function has to satisfy the following boundary conditions:

$$\varphi(1) = \varphi'''(1) = 0 \qquad \text{and} \qquad \varphi''(1) = -i\beta\lambda\varphi'(1). \tag{5.21}$$

Using formula (4.2) we apply the condition, $\varphi'''(1) = 0$, and obtain the spectral equation $\varphi'''(1) = \mu^3\left[S_-^2(\mu) - C_-(\mu)C_+(\mu)\right] = -2\mu^3 \sinh\mu \sin\mu = 0$. This equation is valid if $\mu = \pi n,\ n \in \mathbb{N}^+$, and thus $\lambda_n = (\pi n)^2 > 0$. With such λ_n, the eigenfunction is

$$\varphi_n(x) = \sinh(\pi n)C_-(\pi n x) - \left[\cosh(\pi n) + (-1)^{n+1}\right]S_-(\pi n x). \tag{5.22}$$

It is clear that for a real-valued function, the third condition of (5.21) fails.

Now assuming that $\varphi_n'''(1) = 0$, we also have $\varphi_n(1) = 0$. Repeating the argument that followed formula (5.21), we immediately obtain a contradiction. Case (2) is shown.

Case (3). Let $k_1 = 0$. Then the right-end conditions become

$$\varphi''(1) = -i\beta\lambda\varphi'(1) \qquad \text{and} \qquad \varphi'''(1) = i\alpha\lambda\varphi(1) + ik_2\lambda\varphi'(1). \tag{5.23}$$

Assume that for some λ_n we have $\varphi_n'(1) = 0$. It means that the eigenfunction, φ_n, is a solution of the problem with the following right-end conditions:

$$\varphi'(1) = \varphi''(1) = 0 \qquad \text{and} \qquad \varphi'''(1) = i\alpha\lambda\varphi(1). \tag{5.24}$$

Using (4.4) and applying condition $\varphi'(1) = 0$, one gets the spectral equation $\varphi'(1) = S_-^2(\mu) - C_-(\mu)C_+(\mu) = -2\sinh\mu\sin\mu = 0$, which means that $\lambda_n = (\pi n)^2$ and the eigenfunction, φ_n, becomes

$$\varphi_n(x) = S_+(\pi n)C_-(\pi n x) - C_+(\pi n)S_-(\pi n x) = \sinh(\pi n)C_-(\pi n x) - \left[\cosh(\pi n) + (-1)^n\right]S_-(\pi n x).$$

For a real-valued function the third equation of (5.24) fails.

Now assume that for some φ_n, we have $\varphi_n''(1) = 0$. From the first equation of (5.23), it follows that $\varphi_n'(1) = 0$, which immediately yields a problem with the right-end conditions (5.24). Case (3) is shown.

Case (4). The remaining case when $\alpha = 0$ and $\beta k_1 k_2 \neq 0$ leads to the following conditions:

$$\varphi''(1) = -ik_1\lambda\varphi(1) - i\beta\lambda\varphi'(1), \qquad \varphi'''(1) = ik_2\lambda\varphi'(1). \tag{5.25}$$

Arguing by contradiction, assume that for some eigenvalue, λ_n, the eigenfunction, φ_n, satisfies $\varphi'(1) = 0$. Such an eigenfunction has to satisfy the boundary conditions

$$\varphi'(1) = \varphi'''(1) = 0 \qquad \text{and} \qquad \varphi''(1) = -i\, k_1\lambda\varphi(1), \tag{5.26}$$

which means that φ_n is an eigenfunction of the CS-model. For a real-valued function the last condition of (5.26) fails.

Now assume that for some φ_n we have $\varphi_n'''(1) = 0$. From the second equation of (5.25) it follows that $\varphi_n'(1) = 0$, which immediately yields the problem already resolved. □

Corollary 5.5. *When all four control parameters are non-zero and either* $\varphi_n(1) = 0$ *or* $\varphi_n'(1) = 0$, *then the following result holds:*

$$\Big(|\varphi_n(1)| + |\varphi_n'(1)|\Big)|\varphi_n''(1)\varphi_n'''(1)| > 0.$$

If the boundary control matrix, $\mathbb{K}$, *is invertible, and if either* $\varphi_n''(1) = 0$ *or* $\varphi_n'''(1) = 0$, *then the following result holds:*

$$\Big(|\varphi_n''(1)| + |\varphi_n'''(1)|\Big)|\varphi_n(1)\varphi_n'(1)| > 0.$$

6 | ON THE GEOMETRIC MULTIPLICITIES OF THE EIGENVALUES

Theorem 6.1. *Assume that in the spectral problem* (3.6) *and* (3.7) *the boundary control parameters satisfy one of the following conditions: either only one parameter is non-zero, or only two parameters are non-zero, or only three parameters are non-zero. Then, the geometric multiplicity of any eigenvalue is one.*

Proof. In each case below we use the contradiction argument and assume that there exists an eigenvalue, λ_n, whose geometric multiplicity is at least 2. It means that the spectral problem with $\lambda = \lambda_n$ has two linearly independent solutions, $u(x)$ and $v(x)$ (the eigenfunctions).

One non-zero parameter.

Case (1). Let $\alpha \neq 0$ and $\beta = k_1 = k_2 = 0$, then the right-end boundary conditions are given by (5.2). By Theorem 5.1, we have $u(1)v(1) \neq 0$. Consider a new function, f, defined by formula (4.15). This function, being an eigenfunction of the pencil spectral problem defined by Eqs.(3.6) and conditions (5.2) satisfies (5.1) of Theorem 5.1, i.e. $|f(1)f'(1)| > 0$. However, by construction we have $f(1) = 0$, which means that the assumption on the existence of two linearly independent solutions, $u(x)$ and $v(x)$ is not valid.

Case (2). Let $\beta \neq 0$ and $\alpha = k_1 = k_2 = 0$, then the right-end boundary conditions are given by (5.4). By Theorem 5.1, we have $u'(1)v'(1) \neq 0$. Introduce a new function, g, by formula (4.16). This function satisfies Eqs.(3.6), conditions (5.4), and $g'(1) = 0$, which is in contradiction with Theorem 5.1.

Case (3). Let $k_1 \neq 0$ and $\alpha = \beta = k_2 = 0$, then the right-end boundary conditions are given by (5.5). By Theorem 5.1, we have $u(1)v(1) \neq 0$. If we introduce the function f as in (4.15), we can argue as in Case (1), that u and v cannot be linearly independent.

Case (4). Let $k_2 \neq 0$ and $\alpha = \beta = k_1 = 0$, then the right-end boundary conditions are given by (5.7). If we construct g by the rule of (4.16), then following the argument as in Case (2), we immediately obtain the result.

Two non-zero parameters.

Case (1). Let $\alpha\beta \neq 0$ and $k_1 = k_2 = 0$. Then the right-end boundary conditions are given by (5.9). By Theorem 5.2 we have $u'(1)v'(1) \neq 0$. Let us introduce a new function $g(x)$ by formula (4.16). This function satisfies Eqs.(3.6) and conditions (5.9). According to Case (2) of Theorem 5.2, g has to satisfy (5.1). However $g'(1) = 0$ by construction.

Case (2). Let $k_1k_2 \neq 0$ and $\alpha = \beta = 0$. Then the right-end boundary conditions are given by (5.13). By Theorem 5.2 we have $u(1)v(1) \neq 0$. Let us construct a new function $f(x)$ by formula (4.15). By Theorem 5.2, Case (2), this function has to satisfy $|f(1)f'(1)| > 0$, which is not true since by construction, $f(1) = 0$.

Case (3). Let $\alpha k_2 \neq 0$ and $\beta = k_1 = 0$. Then the right-end boundary conditions are given by (5.14). By Theorem 5.2 we have $u(1)v(1) \neq 0$. Construct a new function $g(x)$ by formula (4.16). By Theorem 5.2, Case (3), an eigenfunction of a problem defined by (3.6) and (5.14) has to satisfy relation (5.1). It is not the case since by construction $g'(1) = 0$.

Case (4). The proof for the case when $\beta k_1 \neq 0$ and $\alpha = k_2 = 0$ can be done similarly.

Case (5). Let $\alpha k_1 \neq 0$ and $\beta = k_2 = 0$. Then the right-end boundary conditions are given by (5.15). Define $f(x)$ by formula (4.15). By Theorem 5.3, Case (1), an eigenfunction of a problem defined by (3.6) and (5.15) has to satisfy (5.1). However, it is not the case since by construction $f(1) = 0$.

Case (6). Finally, we consider the case when $\beta k_2 \neq 0$ and $\alpha = k_1 = 0$. The right-end boundary conditions are given by (5.16). Let us construct a new function $g(x)$ by formula (4.16). By Theorem 5.3, Case (2), an eigenfunction g has to satisfy relation (5.1), which is not the case since $g'(1) = 0$.

Three non-zero parameters.

Case (1). Let $\alpha = 0$. The right-end boundary conditions are given by (5.25). By Theorem 5.4, Case (4), we have $u'(1)v'(1) \neq 0$. Let $g(x)$ be defined by (4.16). The function g satisfies (3.6), (5.25), and $g'(1) = 0$ by construction. The last property contradicts relation (4) of (5.17).

Case (2). Let $\beta = 0$. The right-end boundary conditions are given by (5.18). Let $f(x)$ be defined by (4.15). The function f satisfies (3.6), (5.18), and $f(1) = 0$ by construction. The last property contradicts relation (1) of (5.17).

Case (3). Let $k_1 = 0$, then the right-end boundary conditions are given by (5.23). Construct a new function $g(x)$ by (4.16). This function satisfies (3.6), (5.23), and $g'(1) = 0$ by construction. The last property contradicts relation (3) of (5.17).

Case (4). Let $k_2 = 0$. Then the right-end boundary conditions are given by (5.20). Construct a new function $f(x)$ by formula (4.15). This function satisfies (3.6), (5.20), and $f(1) = 0$ by construction. The last property contradicts relation (2) of (5.17). □

Remark 6.2. *From Theorem 6.1 it follows that for each eigenvalue there is only one eigenfunction. However, the problem is non-selfadjoint and there could be a finite chain of associate functions, i.e. the algebraic multiplicity could be greater than 1. As follows from the asymptotic approximation for the eigenvalues* $\{\lambda_n\}_{n\in\mathbb{Z}'}$ *as* $|n| \to \infty$*, the distant eigenvalues are simple (34). So the problem can have only a finite number of associate functions.*

7 | ON THE EXISTENCE OF REAL EIGENVALUES

When all boundary parameters are zeros, the spectral problem (3.6) and (3.7) is selfadjoint and corresponds to the cantilever beam model. In this case all eigenvalues are real. In this section, we examine under what conditions the problem has real eigenvalues, when some (or all) boundary parameters are non-zeros.

Theorem 7.1. *If only one control parameter is not equal to zero, the problem does not have real eigenvalues.*

Proof. Case (1). Let $\alpha \neq 0$ and $k_1 = k_2 = \beta = 0$, then the spectral equation (3.19) becomes

$$\sqrt{\lambda}(1 + \cosh \mu \cos \mu) = -i\alpha(\cosh \mu \sin \mu - \sinh \mu \cos \mu), \qquad \mu = \sqrt{\lambda}. \tag{7.1}$$

For $\mu > 0$, Eq.(7.1) is equivalent to the following system:

$$1 + \cosh \mu \cos \mu = 0, \qquad \cosh \mu \sin \mu = \sinh \mu \cos \mu. \tag{7.2}$$

Let us show that the system does not have any solution for $\mu > 0$. First we note that $\mu = 0$ is not a solution of the system. Squaring the second equation of (7.2) and counting for the first equation, we obtain $\cosh^2 \mu \sin^2 \mu = \left(\cosh^2 \mu - 1\right) \cos^2 \mu = 1 - \cos^2 \mu = \sin^2 \mu$. This equation is satisfied only for $\mu = \pi n$, $n \in \mathbb{N}^+$. However, in this case the first equation of (7.2) is not satisfied.

Case (2). Let $\beta \neq 0$ and $k_1 = k_2 = \alpha = 0$, then the spectral equation becomes

$$\sqrt{\lambda}(1 + \cosh \mu \cos \mu) = -i\mu\beta(\cosh \mu \sin \mu + \sinh \mu \cos \mu). \tag{7.3}$$

For $\mu > 0$, this equation is equivalent to the system

$$1 + \cosh \mu \cos \mu = 0, \qquad \cosh \mu \sin \mu = -\sinh \mu \cos \mu. \tag{7.4}$$

Using an argument similar to that of Case (1), one gets the desired result.

Case (3). Let $k_1 \neq 0$ and $k_2 = \alpha = \beta = 0$. The spectral equation (3.19) becomes

$$1 + \cosh \mu \cos \mu = -ik_1 \sinh \mu \sin \mu. \tag{7.5}$$

Separating the real and imaginary parts for $\mu > 0$ yields

$$1 + \cosh \mu \cos \mu = 0, \qquad \sinh \mu \sin \mu = 0. \tag{7.6}$$

The second equation is satisfied for $\mu_n = \pi n$, $n \in \mathbb{N}^+$. At such points the first equation of (7.6) fails.

Case (4). Let $k_2 \neq 0$ and $k_1 = \alpha = \beta = 0$. The argument for this case is similar to Case (3). □

In the two theorems below we discuss the results for practically important cases when the control parameters are non-negative. The generalization to arbitrary real parameters can be done in a straightforward manner.

Theorem 7.2. *Let $\alpha = \beta = 0$. 1) If $0 < k_1 k_2 < 1$, then a real eigenvalue $\lambda_0 = \mu_0^2$ exists if and only if the following condition is satisfied:*

$$\left[\cosh^{-1}\left(\frac{1 + k_1 k_2}{1 - k_1 k_2}\right) - \pi\right] (\mathrm{mod}\, 2\pi) = 0. \tag{7.7}$$

2) If $k_1 k_2 > 1$, then a real eigenvalue $\lambda_0 = \mu_0^2$ exists if and only if the following holds:

$$\left[\cosh^{-1}\left(\frac{1 + k_1 k_2}{k_1 k_2 - 1}\right)\right] (\mathrm{mod}\, 2\pi) = 0. \tag{7.8}$$

Proof. 1) Assume that k_1 and k_2 are such that condition (7.7) is satisfied. This means that for some $n \in \mathbb{N}^+$ one has

$$\cosh^{-1}\left(\frac{1 + k_1 k_2}{1 - k_1 k_2}\right) = \pi(2n + 1). \tag{7.9}$$

Let us show that $\mu_0 = \pi(2n + 1)$ is the solution of the spectral equation (3.19). Separating the real and imaginary parts of the spectral equation corresponding to the case $\alpha = \beta = 0$, one gets the following system:

$$1 + \cosh \mu \cos \mu + k_1 k_2 (1 - \cosh \mu \cos \mu) = 0, \qquad \sinh \mu \sin \mu = 0. \tag{7.10}$$

Obviously, the second equation of system (7.10) is satisfied for $\mu = \pi(2n+1)$. Let us check that the first equation is also satisfied. Using (7.9) we have

$$1 - \cosh(\pi(2n + 1)) + k_1 k_2 [1 + \cosh(\pi(2n + 1))] = 1 - \frac{1 + k_1 k_2}{1 - k_1 k_2} + k_1 k_2 \left[\frac{1 + k_1 k_2}{1 - k_1 k_2}\right] = 0.$$

Thus the fact that Eq.(3.19) has a real root when k_1 and k_2 satisfy (7.7) is proven.

Now let us prove the converse statement: if Eq.(3.19) has a real root, then k_1 and k_2 have to satisfy condition (7.7). Assume that a real eigenvalue $\mu = \mu_0$ exists, i.e. it is a solution of system (7.10). Being a solution of the second equation of (7.10), μ_0 can be represented as $\mu_0 = \pi m$, $m \in \mathbb{N}^+$.

(i) If $m = 2n+1$, then the first equation of (7.10) becomes $(k_1k_2+1)+(k_1k_2-1)\cosh\mu_0 = 0$. Since $k_1k_2 < 1$, this equation can be rewritten as $\cosh\mu_0 = (1+k_1k_2)/(1-k_1k_2)$, which yields formula (7.7).

(ii) If $m = 2n$, then the first equation of system (7.10) becomes $(1-k_1k_2)\cosh\mu_0+(1+k_1k_2) = 0$. For $k_1k_2 < 1$, this equation does not have any solutions.

Statement (1) is proven.

2) Consider the case when $k_1k_2 > 1$, and (7.8) holds. It means that for some $n \in \mathbb{N}^+$ one has

$$\cosh^{-1}\left(\frac{k_1k_2+1}{k_1k_2-1}\right) = 2\pi n. \tag{7.11}$$

Let us check that the first equation of system (7.10) is satisfied for $\mu_0 = 2\pi n$. We have

$$1+\cosh(2\pi n)+k_1k_2[1-\cosh(2\pi n)] = 1+\frac{k_1k_2+1}{k_1k_2-1}+k_1k_2\left[1-\frac{k_1k_2+1}{k_1k_2-1}\right] = 0$$

Thus the fact that Eq.(3.19) has a real root when k_1 and k_2 satisfy (7.8) is proven.

Now let us prove the converse statement: if Eq.(3.19) has a real root, then k_1 and k_2 have to satisfy (7.8). Assuming that a real eigenvalue exists, we obtain that it must be a solution of system (7.10), and thus can be represented in the form $\mu_0 = \pi m,\ m \in \mathbb{N}^+$.

(i) If $m = 2n$, then the first equation of (7.10) becomes $1+\cosh\mu_0+k_1k_2\big(1-\cosh\mu_0\big) = 0$. Since $k_1k_2 > 1$, this equation can be rewritten as $\cosh\mu_0 = (k_1k_2+1)/(k_1k_2-1)$, which yields formula (7.8).

(ii) If $m = 2n+1$, then the first equation of (7.10) becomes $(k_1k_2-1)\cosh\mu_0+(k_1k_2+1) = 0$. However, this equation does not have any real solutions. □

Theorem 7.3. *If $k_1 = k_2 = 0$ and $\alpha\beta > 0$, then the problem does not have real eigenvalues.*

Proof. We use a contradiction argument, i.e. assume that some $\lambda_0 > 0$ is a eigenvalue of $\mathcal{L}$ (and λ_0^{-1} is a real eigenvalue of $\mathcal{L}^{-1}$). Since $\mathcal{L}$ is a dissipative operator (see Theorem 2.4), the operator $\mathcal{M} = -\mathcal{L}^{-1}$ is also dissipative. However, if a compact dissipative operator has a real eigenvalue, then the corresponding root space is invariant with respect to both $\mathcal{M}$ and $\mathcal{M}^*$ (30, 33), i.e. such a subspace is spanned by the eigenvectors only (there are no associate vectors) and for each element, $\boldsymbol{G}$, from the above subspace, one has $\mathcal{M}\boldsymbol{G} = \mathcal{M}^*\boldsymbol{G}$. As follows from formulae (2.14)–(2.16), if $\boldsymbol{G} = \big[g_0(x),\ g_1(x)\big]^T$, then

$$(\mathcal{M}\boldsymbol{G})(x) = \big(-\mathcal{L}_{CF}^{-1}\boldsymbol{G}\big)(x)+(-\mathcal{T}\boldsymbol{G})(x),$$

where

$$(-\mathcal{T}\boldsymbol{G})(x) = \frac{i}{2EI}\left[\beta x^2 g_0'(1)-\alpha\left(\frac{x^3}{3}-x^2\right)g_0(1),\ 0\right]^T.$$

As follows from formula (2.24), for the same vector $\boldsymbol{G}$, one has

$$(\mathcal{M}^*\boldsymbol{G})(x) = -\big(\mathcal{L}_{CF}^{-1}\boldsymbol{G}\big)(x)+(-\mathcal{T}^*\boldsymbol{G})(x),$$

where

$$(-\mathcal{T}^*\boldsymbol{G})(x) = \frac{i}{2EI}\left[-\beta x^2 g_0'(1)+\alpha\left(\frac{x^3}{3}-x^2\right)g_0(1),\ 0\right]^T.$$

Therefore, $\mathcal{M}\boldsymbol{G} = \mathcal{M}^*\boldsymbol{G}$ if and only if $\boldsymbol{G}$ is such that $g_0(1) = g_0'(1) = 0$. By Theorem 5.2, neither $g_0(1)$ nor $g_0'(1)$ is equal to 0. The obtained contradiction proves the result. □

Author contributions

Each author contributed approximately the same amount of time and effort to this work.

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