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Location of eigenmodes of Euler-Bernoulli beam model under fully non-dissipative boundary conditions

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The Euler-Bernoulli beam model with non-conservative feedback type boundary conditions is investigated. Components of the two-dimensional input vector are shear and moment at the right end, and components of the observation vector are time derivative of displacement and slope at the right end. The boundary matrix containing four control parameters relates input and observation. The following results are presented: (i) if one and only one of the control parameters is positive and the rest of them are equal to zero, then the set of the eigenmodes is located in the open left half plane of the complex plane, which means that all eigenmodes are stable; (ii) if the diagonal elements of the boundary matrix are positive and off-diagonal elements are zeros, then the set of the eigenmodes is located in the open left half plane, which imply stability of all eigenmodes; (iii) specific combinations of the diagonal and off-diagonal elements have been found to ensure the stability results. To prove the results, two special relations between the eigenmodes and mode shapes of the non-selfadjoint problem and clamped-free selfadjoint problem have been established.

1. Introduction

In the present paper, we consider the stability problem for the Euler-Bernoulli beam model subject to a special type of non-dissipative boundary conditions. The left end the beam is clamped, while the right end is subject to linear conditions that are represented as a feedback control law. The input and the observation of this control law are the 2-dimensional vectors: U and Y, respectively. The components of the input, U, are the shear and the moment at the right end. The components of the

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observation, Y, are the time derivatives of the displacement (tip velocity) and of the slope (angular velocity) at the right end. U and Y are related by a 2×2 matrix $\mathbb{K}: \quad U = \mathbb{K}Y$. The entries of the feedback matrix, $\mathbb{K}$, are the control parameters. The corresponding closed loop system is non-dissipative in the sense that for the general $\mathbb{K}$, the energy functional is not a strictly decreasing function of time.

Our main result consists of the following: even though the system is non-dissipative all the eigenmodes are stable (i.e., they belong to the left half-plane of the spectral parameter) for *certain ranges* of the control parameters (see the precise statement in Theorem 2.4 below).

The main tools in proving this result are two relations between the eigenmodes and the mode shapes of the non-selfadjoint operator, which is the dynamics generator of our closed loop control system, and the selfadjoint operator, which is the dynamics generator of the beam with clamped-free boundary conditions. To the best of our knowledge, such relations (the main identities) did not occur in the literature on spectral analysis of differential operators before. We suggest that they might be of purely theoretical interest in their own right as a tool in spectral analysis.

The present work is a generalization of the results obtained in [Shubov & Shubov, 2016 (1)], where the boundary feedback matrix contained only two non-trivial parameters: the codiagonal entries of the feedback matrix, $\mathbb{K}$. It is shown in [Shubov & Shubov, 2016 (1)], that if only one control parameter is not equal to zero, then the model (still non-dissipative) is stable, i.e., all vibrational modes are located in the closed left half-plane of the complex plane.

Now we present a motivation for choosing the above described specific input U and observation Y. The reason for this choice is twofold.

(i) The engineering origin of the above control law resides in the problem of control of a flexible robot arm. We refer to [Guiver & Opmeer, 2010] where the same control law (with the diagonal entries of the feedback matrix, $\mathbb{K}$, are equal to zero) was used to investigate the stability of the Rayleigh and Timoshenko beams. The authors of this paper in turn referred to [Luo & Guo, 1997] where a shear force feedback control was used for an Euler-Bernoulli beam model of a robot arm with a revolute joint. We also mention [Cannon & Schmitz, 1984] and [Sakawa, et. al., 1985] for the experimental origin of the feedback controls for a robot arm.

(ii) The second reason for the choice of the above control law is purely theoretical. This reason was pointed out in [Guiver & Opmeer, 2010] (in the case of the Rayleigh and Timoshenko beams). Namely, if one calculates the time derivative of the energy functional, $\mathcal{E}$, of the beam, taking into account only the equation of motion and the clamped boundary conditions at the left end, then the result turns out to be $\mathcal{E}_t = Y \cdot U$. We find that the dot product of the above input and observation appears in a natural way. If we set $U = \mathbb{K}Y$ then $\mathcal{E}_t = Y \cdot \mathbb{K}Y$ and therefore, $\mathcal{E}_t < 0$ if $\mathbb{K}$ is a negative definite matrix. In other words, we get a feedback that produces a dissipative system. (This feedback is actually stabilizing, though this fact requires a separate proof. The proof based on the Riesz basis property of the mode shapes will be given in our forthcoming work.) Our feedback is a generalization of the aforementioned approach: we replace $\mathbb{K}$ by the general 2×2 matrix, i.e., $\mathbb{K}$ is not necessarily negative definite.

Feedback control of beams is an extensively studied area. Euler-Bernoulli, Rayleigh, Timoshenko, and other beam models have been studied with different control laws. We mention, e.g., [Chen, et. al., 1987], [Conrad & Morgul, 1998], [Ozer & Hansen, 2011, 2014]. We also mention [Guiver & Opmeer, 2010; Curtan & Zwart, 1995; Zwart, 2010; Lions, 1988; Shubov, 2006, 2008; Russell, 1986].

The area of beam control has numerous applications to the control of robotic manipulators, aircraft wings, propeller blades, large space structures, (see [Dowell, 2004; Joshi, 1988; Balakrishnan & Shubov, 2004; Balakrishnan, 2012] and references herein). Additional applications include dynamics of carbon nanotubes [Shubov & Rojas-Arenaza, 2008, 2011] and rapidly developing area of energy harvesting mathematical modeling [Erturk & Inman, 2011; Erturk, 2012; Shubov, 2018; Shubov & Shubov, 2016 (2)].

We also point out that non-dissipative feedbacks are important for the energy harvesting purposes since destabilization is desirable for energy harvesters. Our results describing the spectrum for the off-diagonal case may be useful for engineers designing control laws for energy harvesters. In this connection we mention [Ozer, 2018], which studies a voltage-controlled piezoelectric laminate model based on the Euler-Bernoulli beam. The model studied in this work can be viewed as an application of non-dissipative feedback in consideration of energy harvesting.

The non-dissipative boundary conditions similar to the conditions considered in the present paper for the Euler-Bernoulli beam can be also considered for the Rayleigh beam model (that adds rotary inertia effects of the Euler-Bernoulli beam) and for the Timosheko beam model (that takes into account the effects of shear distortion and rotary inertia). Those boundary conditions for the Rayleigh and Timoshenko beams with constant structural parameters have been considered in [Guiver & Opmeer, 2010]. The authors have demonstrated that each of these models has an infinite sequence of unstable eigenmodes. This is a sharp contrast to the Euler-Bernoulli beam which, as was mentioned above, may have only stable eigenmodes.

The present paper can be considered as a counterpart of the works [Shubov, 2014, 2018]. The author in [Shubov, 2014] examines the asymptotic and spectral analysis of the Rayleigh beam with similar non-dissipative boundary conditions. The main result of that paper is a set of asymptotic formulas for the eigenmodes of the Rayleigh beam pertaining to all possible ranges of control parameters. The instability result proven in [Guiver & Opmeer, 2010] follows from the asymptotic formulas of [Shubov, 2014] as an immediate corollary. The leading asymptotical term shows that the eigenmodes approach the imaginary axis. The second order asymptotical term contains an alternating sign factor. This fact together with the estimate of the remainder implies that starting from a certain number, all *even numbered modes* (in the asymptotic numeration) are unstable.

The main findings of the paper. We mention at this point that we use a nonstandard notation for the dynamics generator of our system. Namely, we denote the dynamics generator of the closed loop system by $i\mathcal{L}$ and the dynamics generator of the clamped-free beam by $i\mathcal{L}_0$. Then we apply the term "dynamics generator" for the operators $\mathcal{L}$ and $\mathcal{L}_0$ as well. The reason for such notations consists of the following. $\mathcal{L}_0$ is a selfadjoint operator in the state space while $i\mathcal{L}_0$ is skew-selfadjoint. $\mathcal{L}$ can be viewed as a finite rank perturbation of $\mathcal{L}_0$. The present paper is one in a series of our works devoted to the above control system. In this series we extensively use the classical results (Krein theorem, Keldysh theorem, etc.) on completeness and Riesz basis property of the eigenvectors of non-selfadjoint operators that are perturbations of selfadjoint (not skew-selfadjoint) operators [Gohberg & Krein, 1996]. Thus, our notations make it convenient to refer directly to the results presented in the literature.

Below we denote the diagonal entries of the feedback matrix, $\mathbb{K}$, by $(-\alpha)$ and $(-\beta)$ and the codiagonal entries by $(-\kappa_1)$ and $(-\kappa_2)$. In what follows, we discuss the cases when all control parameters are non-negative; the cases of non-positive boundary control parameters can be treated in a similar fashion. The present paper contains four main results formulated as Theorem 2.4 (see Section 2 of the paper). 1) The first group of results is concerned with the case when one control parameter is positive, and the rest are equal to zero. We show that the entire spectrum of the operator, $\mathcal{L}$, is located in the open upper half-plane of the complex plane (and, therefore, the spectrum of $i\mathcal{L}$ is located in the open left half-plane.) 2) The second group of results is concerned with the cases when two boundary control parameters are positive. It turns out that the location of the spectrum depends on a specific choice of two non-zero parameters. Namely, for the case when $\alpha > 0$, $\beta > 0$ and $\kappa_1 = \kappa_2 = 0$, the main non-selfadjoint operator, $\mathcal{L}$, is in fact a dissipative operator, i.e. $\Im(\mathcal{L}F, F)_{\mathcal{H}} \geq 0$ for $F \in \mathcal{D}(\mathcal{L})$. For such $\mathcal{L}$, the spectrum is located in the closed upper half-plane. 3) The situation when the combination of two positive control parameters involves one diagonal element of the control matrix (α or β) and one co-diagonal element (κ_1 or κ_2) is more complicated. It is proven in Statements (2) and (3) of Theorem 2.4 that for two cases when either

$\alpha > 0$, $\kappa_1 > 0$ and $\beta = \kappa_2 = 0$ or $\beta > 0$, $\kappa_2 > 0$ and $\alpha = \kappa_1 = 0$, the stability result holds, i.e. the entire spectra of the corresponding operators are located in the open upper half-plane.

As mentioned above to prove such results we derive “two main identities". Each identity relates the eigenmodes and mode shapes corresponding to the self-adjoint operator, $\mathcal{L}_0$, and the eigenmodes and mode shapes of the non-selfadjoint operator, $\mathcal{L}$. The first main identity is used to prove the result for the case $\alpha > 0$, $\kappa_1 > 0$; $\beta = \kappa_2 = 0$ and the second is used for the case $\beta > 0$, $\kappa_2 > 0$; $\alpha = \kappa_1 = 0$. The choice of non-trivial control parameters is important, meaning that the pair of positive parameters must be α and κ_1 or β and κ_2. Preliminary numerical simulations show that the combination $\alpha > 0$, $\kappa_2 > 0$, and $\beta = \kappa_1 = 0$ can produce several eigenvalues located in the closed lower half-plane.

The organization of the paper. In Section 2, we describe the Euler-Bernoulli beam equation and the boundary conditions. Then, we reformulate the original initial boundary value problem as an operator evolution equation in the state space and give an explicit description of the dynamics generator. Finally, we represent the spectral problem for the dynamics generator in the form of the spectral problem for a polynomial operator pencil. In the same section, we provide formulation of the results on the eigenvalue distribution in the model from [Shubov & Kindrat, 2018, 2019]. From these results (see Theorems 2.2 and 2.3 below) it follows that using the spectral asymptotics, one can claim that for specific combinations of control parameters, all distant eigenvalues are located in the upper half-plane. However, based on asymptotic results it is impossible to prove the absence of a possible finite number of eigenvalues that might be located in the lower half-plane (generating unstable vibrational modes) or on the real axis (generating marginally stable modes).

In Section 3, we derive the specific relation between the eigenfunctions of the non-selfadjoint operator, $\mathcal{L}$, and its selfadjoint counterpart, $\mathcal{L}_0$, corresponding to the clamped-free model. We call this relation “the first main identity" and use it in Section 5 for the proof of Statement 3 of Theorem 2.4.

In Section 4, we derive the second relation connecting the eigenfunctions of the operators $\mathcal{L}$ and $\mathcal{L}_0$. We call this relation “the second main identity" and make use of it in Section 5 for the proof of Statement 4 of Theorem 2.4.

2. Problem statement, spectral asymptotics, and main results

The transverse displacement of the Euler-Bernoulli beam model, $h(x,t)$, at position x and time t is governed by the hyperbolic partial differential equation

$$\rho(x)\ \frac{\partial^2 h(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\ \frac{\partial^2 h(x,t)}{\partial x^2}\right) = 0, \qquad 0 \le x \le L, \qquad t \ge 0. \tag{2.1}$$

This equation represents a commonly used model for the motion of a straight beam of length L, linear density $\rho(x)$, modulus of elasticity of the beam material $E(x)$, and cross-sectional moment of inertia, $I(x)$ ($EI(x)$ is the bending stiffness). The model is obtained by using Hooke's law and the simplifying assumptions that the thickness and width of the beam are small compared to the length, and the cross-sections of the beam remain plane during deformation [Shubov & Shubov, 2016(1); Benaroya, 1998; Gladwell, 2005]. We assume that the beam is clamped at the left end

$$h(0,t) = h_x(0,t) = 0. \tag{2.2}$$

To describe the right-end conditions, we use the moment, $M(x,t)$, and the shear, $Q(x,t)$ defined by [Benaroya, 1998; Gladwell, 2005]:

$$M(x,t) = EI(x)h_{xx}(x,t), \qquad Q(x,t) = \big(EI(x)h_{xx}(x,t)\big)_x. \tag{2.3}$$

Let the input, $U(t)$, and output, $Y(t)$, be given as $\mathbb{R}^2$-vectors

$$U(t) = \big[-Q(L,t),\ \ M(L,t)\big]^T \qquad \text{and} \qquad Y(t) = \big[h_t(L,t),\ \ h_{xt}(L,t)\big]^T, \tag{2.4}$$

where the superscript "T" stands for transposition. The feedback control law is given by

$$U(t)=\mathbb{K}Y(t), \qquad \text{and} \qquad \mathbb{K}=\begin{bmatrix} -\alpha & -\kappa_2 \\ -\kappa_1 & -\beta \end{bmatrix}, \tag{2.5}$$

where α, β, κ_1, κ_2 are the control parameters. The feedback (2.5) can be written in the form

$$EI(L)h_{xx}(L,t)=-\kappa_1 h_t(L,t)-\beta h_{xt}(L,t), \qquad \big(EI(x)h_{xx}(x,t)\big)_x\big|_{x=L}=\alpha h_t(L,t)+\kappa_2 h_{xt}(L,t). \tag{2.6}$$

If $\alpha=\beta=\kappa_1=\kappa_2=0$, then the right-end conditions become $h_{xx}(L,t)=\big(EI(x)h_{xx}(x,t)\big)_x\big|_{x=L}=0$, and the problem corresponds to the clamped-free model.

Consider the energy functional for the beam [Benaroya, 1998; Gladwell, 2005]:

$$\mathcal{E}(t)=\frac{1}{2}\int_0^L \big[\rho(x)h_t^2(x,t)+EI(x)h_{xx}^2(x,t)\big]dx. \tag{2.7}$$

Evaluating $\mathcal{E}_t(t)$ on the solutions of Eq.(2.1) satisfying the left-end conditions (2.2), we obtain that $\mathcal{E}_t(t)=Y(t)\cdot U(t)$, where "$\cdot$" denotes the dot-product in $\mathbb{R}^2$. If we use (2.5), then

$$\mathcal{E}_t(t)=Y(t)\cdot\mathbb{K}Y(t)=-\alpha h_t^2(L,t)-\beta h_{xt}^2(L,t)-\big(\kappa_1+\kappa_2\big)h_t(L,t)h_{xt}(L,t), \tag{2.8}$$

which means that if $\kappa_1+\kappa_2=0$ and $\alpha\geq 0$, $\beta\geq 0$, then $\mathcal{E}_t(t)\leq 0$ and the system is dissipative. Different combinations of the boundary parameters yield different energy dynamics for the structure. If the above conditions on the control parameters are not satisfied, the system is not dissipative. In the present paper we consider the initial boundary-value problem defined by Eq.(2.1), conditions (2.2) and (2.6), and a standard set of the initial conditions

$$h(x,0)=h_0(x), \qquad \frac{\partial h(x,0)}{\partial t}=h_1(x). \tag{2.9}$$

Let us rewrite problem (2.1), (2.2), and (2.6), as the first order in time evolution equation in the state space of the system (the energy space). Without loss of generality we assume that the spatial extent of the beam is $L=1$. We also assume that EI and ρ are smooth, strictly positive functions i.e.

$$EI(\cdot),\ \rho(\cdot)\in C^2[0,1]; \qquad EI(x)>0, \qquad \rho(x)>0, \qquad x\in[0,1]. \tag{2.10}$$

Let $\mathcal{H}$ be the Hilbert space of two-component complex vector-valued functions obtained as the closure of smooth functions $\Phi(x)=\big[\varphi_0(x),\varphi_1(x)\big]^T$, such that $\varphi_0(0)=\varphi_0'(0)=0$, in the following norm:

$$\|\Phi\|_{\mathcal{H}}^2=\frac{1}{2}\int_0^1 \Big[EI(x)\big|\varphi_0''(x)\big|^2+\rho(x)\big|\varphi_1(x)\big|^2\Big]dx. \tag{2.11}$$

The energy space $\mathcal{H}$ is topologically equivalent to the space $\widetilde{H}_0^2(0,1)\times L^2(0,1)$, where

$$\widetilde{H}_0^2(0,1)=\Big\{\varphi\in H^2(0,1):\varphi(0)=\varphi'(0)=0\Big\}.$$

The problem (2.1), (2.2), (2.6), and (2.9) can be represented as an evolution problem

$$\Phi_t(x,t)=i\big(\mathcal{L}\Phi\big)(x,t) \qquad \text{and} \qquad \Phi(x,0)=\big[h_0(x),\ h_1(x)\big]^T,\ \ 0\leq x\leq 1,\ \ t\geq 0. \tag{2.12}$$

The dynamics generator is $i\mathcal{L}$, where $\mathcal{L}$ is given by the following matrix differential expression:

$$\mathcal{L}=-i\begin{bmatrix} 0 & 1 \\ -\frac{1}{\rho(x)}\frac{d^2}{dx^2}\Big(EI(x)\frac{d^2}{dx^2}\,\cdot\Big) & 0 \end{bmatrix} \tag{2.13}$$

defined on the domain

$$\begin{aligned}\mathcal{D}(\mathcal{L})=\Big\{&\Phi=\big(\varphi_0,\varphi_1\big)^T\in\mathcal{H}:\ \varphi_0\in H^4(0,1),\ \varphi_1\in H^2(0,1);\ \varphi_1(0)=\varphi_1'(0)=0;\\ &EI(1)\varphi_0''(1)=-\kappa_1\varphi_1(1)-\beta\varphi_1'(1),\ \big(EI(x)\varphi_0''(x)\big)'_{x=1}=\kappa_2\varphi_1'(1)+\alpha\varphi_1(1)\Big\}.\end{aligned} \tag{2.14}$$

Remark 2.1. 1) We introduce the factor "i" in the definition (2.13) and into Eq.(2.12) for convenience. As shown in Theorem 3.1 below, the operator $\mathcal{L}$ is a finite-rank perturbation of the selfadjoint operator corresponding to the cantilever beam (the model with clamped-free end conditions). So owing to this factor we deal with a selfadjoint (or symmetric) operator rather than with a skew-selfadjoint (or skew-symmetric) operator.

2) The problem (2.1), (2.2), and (2.6) defines a C_0-semigroup in $\mathcal{H}$ and the operator $i\mathcal{L}$ is an infinitesimal generator of this semigroup. This fact follows, e.g. from the Riesz basis property of the generalized eigenfunctions of $\mathcal{L}$, which will be proven in a forthcoming paper. So, the use of the term "dynamics generator" for $i\mathcal{L}$ is justified [Curtain & Zwart, 1995].

It can be verified directly that the eigenvalue/eigenfunction equation for the operator $\mathcal{L}$, $\left(\mathcal{L}\Psi = \lambda\Psi\right)$ can be equivalently written in terms of the polynomial operator pencil $\mathcal{P}(\lambda)$ [Marcus, 1988; Shubov & Kindrat, 2018; Shubov & Shubov, 2016(1)] which is given by the expression

$$\left[\mathcal{P}(\lambda)\psi\right](x) = \left(EI(x)\psi''(x)\right)'' - \lambda^2\rho(x)\psi(x), \tag{2.15}$$

defined on the domain

$$\begin{aligned} &\mathcal{D}\left(\mathcal{P}(\lambda)\psi\right) = \Big\{\psi \in H^4(0,1);\ \ \psi(0) = \psi'(0) = 0; \\ &EI(1)\psi''(1) = -\kappa_1 i\lambda\psi(1) - \beta i\lambda\psi'(1);\ \ \left(EI(x)\psi''(x)\right)'\Big|_{x=1} = \kappa_2 i\lambda\psi'(1) + \alpha i\lambda\psi(1).\Big\} \end{aligned} \tag{2.16}$$

We say that λ_n is *an eigenvalue of the pencil* $\mathcal{P}(\lambda)$ if there exists a function ψ_n from the domain $\mathcal{D}\left(\mathcal{P}(\lambda)\right)$ such that the following equation is satisfied:

$$\left(EI(x)\psi_n''(x)\right)'' = \lambda_n^2\rho(x)\psi_n(x). \tag{2.17}$$

The corresponding solution, ψ_n, is called *an eigenfunction of the pencil*.

We mentioned that $\mathcal{P}(\lambda)$ is a non-standard pencil since the spectral parameter λ enters the boundary conditions explicitly. If ψ_n is an eigenfunction of the pencil in the space $L^2_\rho(0,1)$, then the two-component vector-function $\left(\frac{1}{i\lambda_n}\psi_n(x), \psi_n(x)\right)^T$ is an eigenfunction of the operator $\mathcal{L}$ in the space $\mathcal{H}$.

Now we address the problem of the eigenfunctions and associate functions numeration. As is known [Gohberg & Krein, 1996], for each eigenvalue λ_n of the operator $\mathcal{L}$, there could be a non-trivial set of the associate functions. Due to the fact that the operator $\mathcal{L}$ (generating the pencil $\mathcal{P}(\lambda)$) has a compact inverse, each root space (eigenvectors and associate vectors together corresponding to λ_n) is finite-dimensional. However, in general, it is quite difficult task to obtain an upper bound on the dimensions of the root spaces corresponding to different eigenvalues. As follows from [Shubov & Kindrat, 2018], the distant eigenvalues are simple, i.e. the set of all associate functions of the operator $\mathcal{L}$ is finite. It means that one can use *through* numeration for the root vectors of the operator $\mathcal{L}$ making no difference between the numeration of the eigenfunction and associate functions. Concerning the self-adjoint counterpart of the operator $\mathcal{L}$, the operator $\mathcal{L}_0$ does not have associate functions. As shown in [Shubov & Kindrat, 2017], the spectrum of $\mathcal{L}_0$ is simple, i.e., each root subspace (corresponding to each λ_n^0) of $\mathcal{L}_0$ is one-dimensional, which means that the numeration of the eigenvalues of $\mathcal{L}_0$ can be done in a straightforward fashion.

Now, we provide formulation of the results from [Shubov & Kindrat, 2018] derived for the case of $EI = \rho = 1$ that will be needed for the present work.

As shown in [Shubov & Kindrat, 2018], the spectral asymptotics are different for the cases $\beta \neq 0$ and $\beta = 0$.

Theorem 2.2. *Assume that* $\beta = \kappa_2 = 0$*, and* $\kappa_1 > 0$, $\alpha > 0$*. Let*

$$\mathcal{K} = \frac{\kappa_1 - 1}{\kappa_1 + 1}. \tag{2.18}$$

1) *If* $0<\mathcal{K}<1$, *then there is an infinite sequence of eigenvalues in the upper half-plane with the following asymptotic approximation as* $n\to\infty$*:*

$$\lambda_n=(\pi n)^2-i\pi n\ln\mathcal{K}-\ln^2\sqrt{\mathcal{K}}+\frac{\alpha(\kappa_1-i)}{\kappa_1^2-1}+O\left(\frac{1}{n}\right),\qquad \lambda_{-n}=-\overline{\lambda}_n. \tag{2.19}$$

There could be a finite number of eigenvalues located in the closed lower half plane.

2) *If* $-1<\mathcal{K}<0$, *then there is an infinite sequence of eigenvalues in the upper half-plane with the following asymptotic approximation as* $n\to\infty$:

$$\lambda_n=\pi^2\left(n+\frac{1}{2}\right)^2-i\pi\left(n+\frac{1}{2}\right)\ln|\mathcal{K}|+\frac{\alpha(\kappa_1-i)}{\kappa_1^2-1}+O\left(\frac{1}{n}\right),\qquad \lambda_{-n}=-\overline{\lambda}_n. \tag{2.20}$$

There could be a finite number of eigenvalues located in the closed lower half-plane.

3) *If* $\mathcal{K}=0$, *i.e.* $\kappa_1=1$, *then there is an infinite sequence of eigenvalues in the upper half-plane with the following asymptotic approximation as* $n\to\infty$*:*

$$\begin{aligned}&\lambda_n=(\pi n)^2-\frac{1}{4}\ln^2(2\pi nD)-\frac{1}{2}\ln(2\pi nD)+\frac{1}{2}D+i\pi n\ln(2\pi nD)+O\left(\frac{\ln^2 n}{n}\right),\\ &\lambda_{-n}=-\overline{\lambda}_n,\qquad \text{where}\qquad D=\frac{1-i}{2\alpha}.\end{aligned} \tag{2.21}$$

There could be a finite number of eigenvalues in the closed lower half-plane.

Theorem 2.3. *Assume that* $\alpha=\kappa_1=0$ *and* $\beta>0$, $\kappa_2>0$. *Then there is an infinite sequence of eigenvalues in the upper half-plane with the following asymptotic approximation as* $n\to\infty$*:*

$$\lambda_n=(\pi n)^2+\frac{\pi^2 n}{2}+\left(\frac{\pi}{4}\right)^2+\frac{\kappa_2+i}{\beta}+O\left(\frac{1}{n}\right),\qquad \lambda_{-n}=-\overline{\lambda}_n. \tag{2.22}$$

There could be a finite number of eigenvalues in the closed lower half-plane.

We point out that for the case of smooth variable structural parameters, the leading asymptotical terms in formulas (2.19) – (2.22) remain essentially the same; in particular, one has $\lim_{n\to\infty}\lambda_n/(\pi n)^2\to Const$. Finally, we present the main result of the paper.

Theorem 2.4. *The spectrum* $\mathfrak{S}(\mathcal{L})$ *of the operator* $\mathcal{L}$ *is located in the open upper half-plane and, moreover,* $\inf\{\Im\lambda:\ \lambda\in\mathfrak{S}(\mathcal{L})\}>0$ *if one of the following four sets of conditions is satisfied:*

1) one control parameter (*any from the set* $\{\kappa_1,\kappa_2,\alpha,\beta\}$) *is positive, and the rest of them are equal to zero;*

2) $\alpha>0$, $\beta>0$, *and* $\kappa_1=\kappa_2=0$;

3) $\kappa_1>0$, $\alpha>0$, *and* $\kappa_2=\beta=0$;

4) $\kappa_2>0$, $\beta>0$, *and* $\kappa_1=\alpha=0$.

Remark 2.5. 1) To prove Theorem 2.4 it suffices to show that in each of the aforementioned four cases, the spectrum $\mathfrak{S}(\mathcal{L})$ is located in the open upper half-plane. If the latter fact is shown, then the fact that $\inf\{\Im\lambda:\ \lambda\in\mathfrak{S}(\mathcal{L})\}>0$ becomes an immediate corollary of Theorems 2.2 and 2.3. Indeed, owing to the results of these theorems, in all considered cases the spectrum of $\mathcal{L}$ is asymptotically close to certain horizontal lines in the upper half-plane, and, therefore, no subsequence can approach the real axis.

2) As mentioned in Remark 2.1, the operator $i\mathcal{L}$ is the generator of a C_0-semigroup. By Theorem 2.4 the spectral abscissa of this semigroup is such that $\sup\{\Re(i\lambda):\ \lambda\in\mathfrak{S}(\mathcal{L})\}<0$. However, the fact that the spectral abscissa of the generator is negative does not guarantee the exponential decay of the semigroup (see, e.g., Pazy, 1983, Sec. 4.4, Example 4.2). Based on the location of the spectrum, one can claim that the system modeled by (2.1) – (2.6) is stable.

3) As is well known (Pazy, 1983; Engel & Nagel, 1999), a semigroup decays exponentially if the spectral abscissa is negative and the semigroup is analytic. However, the semigroup generated by our system (2.1) – (2.6) is not analytic. It has been shown in (Shubov, 2006) that this semigroup belongs to the Gevrey class.

3. Derivation of the first main identity

Let $\mathcal{L}_0$ be a selfadjoint operator in $\mathcal{H}$ obtained from $\mathcal{L}$ by placing $\alpha=\beta=\kappa_1=\kappa_2=0$, i.e., $\mathcal{L}_0$ corresponds to the clamped-free boundary conditions. As shown in [Shubov & Kindrat, 2018], the operator $\mathcal{L}$ has a compact inverse and thus it has purely discrete spectrum of normal eigenvalues. (We recall that a *normal eigenvalue* is an isolated point of a discrete spectrum whose multiplicity is finite [Gohberg & Krein, 1996]).

Lemma 3.1. *1) The operator $\mathcal{L}^*$ adjoint to operator $\mathcal{L}$, is given by the same differential expression (2.13) as $\mathcal{L}$ and defined on the domain*

$$\mathcal{D}(\mathcal{L}^*)=\Big\{\Phi=(\varphi_0,\varphi_1)^T\in\mathcal{H}:\ \ \varphi_0\in H^4(0,1),\ \ \varphi_1\in H^2(0,1);\ \ \varphi_1(0)=\varphi_1'(0)=0; \\ EI(1)\varphi_0''(1)=\beta\varphi_1'(1),\ \ (EI(x)\varphi_0''(x))'_{x=1}=-\alpha\varphi_1(1)\Big\}. \tag{3.1}$$

2) The operator $\mathcal{L}$ is a finite-rank perturbation of the self-adjoint operator $\mathcal{L}_0$ in the sense that the operators $\mathcal{L}^{-1}$ and $\mathcal{L}_0^{-1}$ are related by the rule:

$$\mathcal{L}^{-1}=\mathcal{L}_0^{-1}+\mathcal{T}, \tag{3.2}$$

where $\mathcal{T}$ is a rank-two operator. The following formulas hold for any $G=(g_0,g_1)^T\in\mathcal{H}$:

$$\left(\mathcal{L}_0^{-1}G\right)(x)=-\begin{pmatrix}-i\displaystyle\int_0^x d\eta\int_0^\eta\frac{d\tau}{EI(\tau)}\int_\tau^1 d\xi\int_\xi^1\rho(\nu)g_1(\nu)d\nu\\ ig_0(x)\end{pmatrix} \tag{3.3}$$

and

$$\left(\mathcal{T}G\right)(x)=\begin{pmatrix}-i\chi_1(x)\Big[\kappa_1 g_0(1)+\beta g_0'(1)\Big]-i\chi_2(x)\Big[\kappa_2 g_0'(1)+\alpha g_0(1)\Big]\\ 0\end{pmatrix}, \tag{3.4}$$

where

$$\chi_1(x)=\int_0^x d\tau\int_0^\tau\frac{d\eta}{EI(\eta)},\qquad \chi_2(x)=\int_0^x d\tau\int_0^\tau\frac{1-\eta}{EI(\eta)}d\eta. \tag{3.5}$$

Proof. To prove Statement (1) it suffices to check that for the operators $\mathcal{L}$ and $\mathcal{L}^*$ defined by (2.13), (2.14) and (2.13), (3.1) respectively, one has

$$(\mathcal{L}U,V)_{\mathcal{H}}=(U,\mathcal{L}^*V),\qquad U\in\mathcal{D}(\mathcal{L}),\qquad V\in\mathcal{D}(\mathcal{L}^*). \tag{3.6}$$

Indeed, by direct calculation, one can establish that the left hand side of (3.6) (for $U\in\mathcal{D}(\mathcal{L})$) and the right hand side of (3.6) (for $V\in\mathcal{D}(\mathcal{L}^*)$) are both equal to the same expression given by

$$-\frac{i}{2}\left[-\alpha u_1(1)\overline{v_1(1)}-\beta u_1'(1)\overline{v_1'(1)}+\int_0^1\Big[EI(x)u_1''(x)\overline{v_0''(x)}-EI(x)u_0''(x)\overline{v_1''(x)}\Big]dx\right].$$

Statement (1) is proven.

To prove Statement (2), we consider the equation $\mathcal{L}\Psi=F$ and show it has the unique solution $\Psi\in\mathcal{D}(\mathcal{L})$ for any $F\in\mathcal{H}$. Let $\Psi=(\psi_0,\psi_1)^T$ and $F=(f_0,f_1)^T$, then the afore equation can be written component-wise as follows:

$$\left(EI(x)\psi_0''(x)\right)''=-i\rho(x)f_1(x),\qquad \psi_1(x)=if_0(x). \tag{3.7}$$

The boundary conditions are:

$$EI(1)\psi_0''(1)=-i\kappa_1 f_0(1)-i\beta f_0'(1), \tag{3.8}$$

$$\left(EI(x)\psi_0''(x)\right)'\Big|_{x=1}=i\kappa_2 f_0'(1)+i\alpha f_0(1). \tag{3.9}$$

Integrating (3.7) and using condition (3.9), we obtain

$$\Big(EI(x)\psi_0''(x)\Big)' = i\kappa_2 f_0'(1) + i\alpha f_0(1) + i\int_x^1 \rho(\tau) f_1(\tau)d\tau. \tag{3.10}$$

Integrating (3.10) and using condition (3.8), we obtain

$$EI(1)\psi_0''(1) - EI(x)\psi_0''(x) = i\kappa_2 f_0'(1)(1-x) + i\alpha f_0(1)(1-x) + i\int_x^1 d\tau \int_\tau^1 \rho(\eta) f_1(\eta)d\eta,$$

which yields

$$\begin{aligned} EI(x)\psi_0''(x) = &- i\kappa_1 f_0(1) - i\beta f_0'(1) - i\kappa_2 f_0'(1)(1-x) \\ &- i\alpha f_0(1)(1-x) - i\int_x^1 d\tau \int_\tau^1 \rho(\eta) f_1(\eta)d\eta. \end{aligned} \tag{3.11}$$

Integrating this equation we have

$$\begin{aligned} \psi_0'(x) = &- i\kappa_1 f_0(1)\int_0^x \frac{d\tau}{EI(\tau)} - i\beta f_0'(1)\int_0^x \frac{d\tau}{EI(\tau)} - i\kappa_2 f_0'(1)\int_0^x \frac{1-\tau}{EI(\tau)}d\tau \\ &- i\alpha f_0(1)\int_0^x \frac{1-\tau}{EI(\tau)}d\tau - i\int_0^x \frac{d\tau}{EI(\tau)}\int_\tau^1 d\eta \int_\eta^1 f_1(\xi)d\xi. \end{aligned} \tag{3.12}$$

Finally, integrating (3.12) we obtain

$$\begin{aligned} \psi_0(x) = &- i\kappa_1 f_0(1)\int_0^x d\tau \int_0^\tau \frac{d\eta}{EI(\eta)} - i\beta f_0'(1)\int_0^x d\tau \int_0^\tau \frac{d\eta}{EI(\eta)} \\ &- i\kappa_2 f_0'(1)\int_0^x d\tau \int_0^\tau \frac{1-\eta}{EI(\eta)}d\eta - i\alpha f_0(1)\int_0^x d\tau \int_0^\tau \frac{1-\eta}{EI(\eta)}d\eta \\ &- i\int_0^x d\tau \int_0^\tau \frac{d\eta}{EI(\eta)}\int_\eta^1 d\xi \int_\xi^1 \rho(\omega) f_1(\omega)d\omega. \end{aligned} \tag{3.13}$$

In terms of the functions χ_1 and χ_2 introduced in (3.4), formula (3.13) can be written as

$$\begin{aligned} \psi_0(x) = &- i\chi_1(x)\Big[\kappa_1 f_0(1) + \beta f_0'(1)\Big] - i\chi_2(x)\Big[\kappa_2 f_0'(1) + \alpha f_0(1)\Big] \\ &- i\int_0^x d\tau \int_0^\tau \frac{d\eta}{EI(\eta)}\int_\eta^1 d\xi \int_\xi^1 \rho(\omega) f_1(\omega)d\omega, \end{aligned}$$

which implies (3.3) and (3.4). ■

At this point we make the following comments on the relation (3.2). To the best of our knowledge, a finite-dimensional perturbation (in our case $\mathcal{T}$) of a compact self-adjoint operator $(\mathcal{L}_0^{-1})$ can significantly affect the geometry of the spectrum of the operator $\mathcal{L}$ which is inverse to $(\mathcal{L}_0^{-1} + \mathcal{T})$. The reason for such a behavior is the fact that in our case the perturbation $\mathcal{T}$ of $\mathcal{L}_0^{-1}$ cannot be reformulated in terms of the perturbation of the unbounded operator $\mathcal{L}_0$. In particular, one cannot claim that $\mathcal{L}$ is a bounded (or relatively bounded) perturbation of $\mathcal{L}_0$, which means that the behavior of the spectra of $\mathcal{L}$ and $\mathcal{L}_0$ can be very different (which is observed in our case).

Based on the spectral asymptotics, we claim that there are only a finite number of real eigenvalues and/or a finite number of the eigenvalues with negative imaginary parts (that generate marginally stable or unstable vibrational mode shapes). Let $\mathbb{Z}' = \mathbb{Z}\setminus\{0\}$ and let

$$\left\{\Psi_m(x) = \left(\frac{1}{i\lambda_m}\psi_m(x), \psi_m(x)\right)^T\right\}_{m\in\mathbb{Z}'} \quad \text{and} \quad \left\{\Phi_n(x) = \left(\frac{1}{i\lambda_n^0}\varphi_n(x), \varphi_n(x)\right)^T\right\}_{n\in\mathbb{Z}'} \tag{3.14}$$

be the sets of the normalized to unity in $\mathcal{H}$ eigenfunctions corresponding to the eigenvalues λ_m and λ_n^0 of the operators $\mathcal{L}$ and $\mathcal{L}_0$ respectively.

It can be verified directly that $\psi_m(x)$ and $\varphi_n(x)$ are the eigenfunctions (corresponding to the eigenvalues λ_m and λ_n^0) of the pencils $\mathcal{P}(\lambda)$ and $\mathcal{P}_0(\lambda)$ generated by the operators $\mathcal{L}$ and $\mathcal{L}_0$.

Now we are in a position to prove the first main result to be used in the proof of Theorem 2.4.

Theorem 3.2. *Let λ_m and $\psi_m(x)$ be an eigenvalue and the corresponding eigenfunction of the pencil $\mathcal{P}(\lambda)$. Let $\lambda_m \neq \lambda_n^0$, $n \in \mathbb{Z}'$. Then the following expansion of $\psi_m(x)$ in terms of the eigenfunctions $\{\varphi_n(x)\}_{n=1}^{\infty}$ of the pencil $\mathcal{P}_0(\lambda)$ is valid:*

$$\begin{aligned}\psi_m(x) = \frac{\lambda_m}{i}\Bigg\{ &\sum_{n=1}^{\infty}\Big[\kappa_1\varphi_n'(1)+\alpha\varphi_n(1)\Big]\frac{\psi_m(1)}{\left(\lambda_n^0\right)^2-\lambda_m^2}\ \varphi_n(x)+ \\ &\sum_{n=1}^{\infty}\Big[\kappa_2\varphi_n(1)+\beta\varphi_n'(1)\Big]\frac{\psi_m'(1)}{\left(\lambda_n^0\right)^2-\lambda_m^2}\ \varphi_n(x)\Bigg\}.\end{aligned} \tag{3.15}$$

The series (3.15) converges in the space $L_0^2(0,1)$.

Proof. Consider the eigenvalue/eigenfunction equation corresponding to the eigenvalue λ_m of the operator $\mathcal{L}$, i.e., $\mathcal{L}\Psi_m = \lambda_m\Psi_m$. Using decomposition (3.2), we rewrite this equation in the form:

$$\Psi_m(x) = \lambda_m\mathcal{L}^{-1}\Psi_m(x) = \lambda_m\left(\mathcal{L}_0^{-1}+\mathcal{T}\right)\Psi_m(x). \tag{3.16}$$

Since λ_m does not coincide with any λ_n^0, Eq. (3.16) can be transformed to

$$\Psi_m = \lambda_m\left(I-\lambda_m\mathcal{L}_0^{-1}\right)^{-1}\mathcal{T}\Psi_m = \lambda_m\mathcal{L}_0\left(\mathcal{L}_0-\lambda_m I\right)^{-1}\mathcal{T}\Psi_m. \tag{3.17}$$

Using the spectral decomposition for the selfadjoint operator $\mathcal{L}_0$ [Gohberg & Krein, 1996; Birman & Solomyak, 1987]:

$$\mathcal{L}_0 = \sum_{n\in\mathbb{Z}'}\lambda_n^0(\cdot,\Phi_n)_{\mathcal{H}}\Phi_n, \tag{3.18}$$

we obtain that the following representation holds for the operator $\left(I-\lambda_m\mathcal{L}_0^{-1}\right)$:

$$I-\lambda_m\mathcal{L}_0^{-1} = \sum_{n\in\mathbb{Z}'}\left[(\cdot,\Phi_n)_{\mathcal{H}}\Phi_n - \lambda_m\frac{(\cdot,\Phi_n)}{\lambda_n^0}\Phi_n\right] = \sum_{n\in\mathbb{Z}'}\frac{\lambda_n^0-\lambda_m}{\lambda_n^0}(\cdot,\Phi_n)_{\mathcal{H}}\Phi_n,$$

and therefore

$$\left(I-\lambda_m\mathcal{L}_0^{-1}\right)^{-1} = \sum_{n\in\mathbb{Z}'}\frac{\lambda_n^0}{\lambda_n^0-\lambda_m}(\cdot,\Phi_n)_{\mathcal{H}}\Phi_n. \tag{3.19}$$

Substituting decomposition (3.19) into Eq.(3.17), we get

$$\Psi_m(x) = \lambda_m\sum_{n\in\mathbb{Z}'}\frac{\lambda_n^0}{\lambda_n^0-\lambda_m}\ \left(\mathcal{T}\Psi_m,\Phi_n\right)_{\mathcal{H}}\ \Phi_n(x), \tag{3.20}$$

where the series converges in the space $\mathcal{H}$. Using formulas (3.4), (3.5), and (3.14) we evaluate the scalar product $\mathcal{A}_{mn} \equiv \left(\mathcal{T}\Psi_m,\Phi_n\right)_{\mathcal{H}}$ and have

$$\mathcal{A}_{mn} = -i\left(\begin{pmatrix}\dfrac{\chi_1(x)}{i\lambda_m}\left(\kappa_1\psi_m(1)+\beta\psi_m'(1)\right)+\dfrac{\chi_2(x)}{i\lambda_m}\left(\kappa_2\psi_m'(1)+\alpha\psi_m(1)\right)\\ 0\end{pmatrix},\begin{pmatrix}\dfrac{\varphi_n(x)}{i\lambda_n^0}\\ \varphi_n(x)\end{pmatrix}\right)_{\mathcal{H}}. \tag{3.21}$$

Using formula (2.11) for the norm in $\mathcal{H}$, we obtain that

$$\begin{aligned}\mathcal{A}_{mn} &= \frac{1}{2}\int_0^1\left\{EI(x)\Big(-i\chi_1(x)\Big)''\left[\frac{\kappa_1}{i\lambda_m}\psi_m(1)+\frac{\beta}{i\lambda_m}\psi_m'(1)\right]\overline{\frac{1}{i\lambda_n^0}\ \psi_n''(x)}\right\}dx \\ &+\frac{1}{2}\int_0^1\left\{EI(x)\Big(-i\chi_2(x)\Big)''\left[\frac{\kappa_2}{i\lambda_m}\psi_m'(1)+\frac{\alpha}{i\lambda_m}\psi_m(1)\right]\overline{\frac{1}{i\lambda_n^0}\ \psi_n''(x)}\right\}dx.\end{aligned} \tag{3.22}$$

Taking into account that $\chi_1''(x) = 1/EI(x)$ and $\chi_2''(x) = (1-x)/EI(x)$ we get from (3.22):

$$\begin{aligned}\mathcal{A}_{mn} &= \frac{1}{2}\int_0^1 (-i)\left[\frac{\kappa_1}{i\lambda_m}\psi_m(1) + \frac{\beta}{i\lambda_m}\psi_m'(1)\right]\overline{\frac{1}{i\lambda_n^0}\varphi_n''(x)dx} \\ &+ \frac{1}{2}\int_0^1 (-i)(1-x)\left[\frac{\kappa_2}{i\lambda_m}\psi_m'(1) + \frac{\alpha}{i\lambda_m}\psi_m(1)\right]\overline{\frac{1}{i\lambda_n^0}\varphi_n''(x)dx} \\ &= \frac{-i}{2\lambda_m\lambda_n^0}\left\{\left[\kappa_1\psi_m(1) + \beta\psi_m'(1)\right]\int_0^1 \varphi_n''(x)dx \right. \\ &\left. + \left[\kappa_2\psi_m'(1) + \alpha\psi_m(1)\right]\int_0^1 (1-x)\varphi_n''(x)dx\right\}.\end{aligned} \tag{3.23}$$

Evaluating both integrals, one has

$$\int_0^1 \varphi_n''(x)dx = \varphi_n'(1), \qquad \int_0^1 (1-x)\varphi_n''(x)dx = \varphi_n'(x) - x\varphi_n'(x)\Big|_0^1 + \int_0^1 \varphi_n'(x)dx = \varphi_n(1). \tag{3.24}$$

Substituting (3.24) into (3.23) yields the following formula for $\mathcal{A}_{mn}$:

$$\mathcal{A}_{mn} = -\frac{i}{2\lambda_m\lambda_n^0}\left\{\left[\kappa_1\psi_m(1) + \beta\psi_m'(1)\right]\varphi_n'(1) + \left[\kappa_2\psi_m'(1) + \alpha\psi_m(1)\right]\varphi_n(1)\right\}. \tag{3.25}$$

Using this formula, we obtain the following representation for Ψ_m of (3.20):

$$\begin{aligned}\Psi_m(x) &= \lambda_m \sum_{n\in\mathbb{Z}'} \frac{\lambda_n^0}{\lambda_n^0 - \lambda_m}\left\{ -\frac{i}{2\lambda_m\lambda_n^0}\left(\left[\kappa_1\psi_m(1) + \beta\psi_m'(1)\right]\varphi_n'(1)\right.\right. \\ &\qquad \left.\left. + \left[\kappa_2\psi_m'(1) + \alpha\psi_m(1)\right]\varphi_n(1)\right)\right\}\Phi_n(x) \\ &= -\frac{i}{2}\sum_{n\in\mathbb{Z}'}\frac{1}{\lambda_n^0 - \lambda_m}\left(\left[\kappa_1\psi_m(1) + \beta\psi_m'(1)\right]\varphi_n'(1) + \left[\kappa_2\psi_m'(1) + \alpha\psi_m(1)\right]\varphi_n(1)\right)\Phi_n(x).\end{aligned} \tag{3.26}$$

Rewriting this equation component-wise we obtain the following two equations:

$$\frac{1}{i\lambda_m}\psi_m(x) = -\frac{i}{2}\sum_{n\in\mathbb{Z}'}\frac{\varphi_n(x)}{\lambda_n^0(\lambda_n^0 - \lambda_m)}\left(\left[\kappa_1\psi_m(1) + \beta\psi_m'(1)\right]\varphi_n'(1) + \left[\kappa_2\psi_m'(1) + \alpha\psi_m(1)\right]\varphi_n(1)\right). \tag{3.27}$$

and

$$\psi_m(x) = -\frac{i}{2}\sum_{n\in\mathbb{Z}'}\frac{\varphi_n(x)}{\lambda_n^0 - \lambda_m}\left(\left[\kappa_1\psi_m(1) + \beta\psi_m'(1)\right]\varphi_n'(1) + \left[\kappa_2\psi_m'(1) + \alpha\psi_m(1)\right]\varphi_n(1)\right). \tag{3.28}$$

The series in (3.27) converges in $\widetilde{H}_0(0,1)$ and the series in (3.28) converges in $L^2(0,1)$.

It can be readily seen that the set of the eigenvalues of the operator $\mathcal{L}_0$ is symmetric with respect to the origin, i.e., if λ_n^0 is an eigenvalue of $\mathcal{L}_0$, then $(-\lambda_n^0)$ is an eigenvalue as well. It follows from the boundary-value problem for the eigenfunction $\varphi_n(x)$ corresponding to λ_n^0, i.e. $\varphi_n(x)$ is the solution of the problem

$$\left(EI(x)\varphi''(x)\right)'' = \left(\lambda_n^0\right)^2\rho(x)\varphi(x), \qquad \varphi(0) = \varphi'(0) = \varphi''(L) = \left(EI(x)\varphi''(x)\right)\big|_{x=1} = 0.$$

Thus, taking into account that $\lambda^0_{-|n|} = -\lambda^0_{|n|}$ and $\varphi_{-|n|} = \varphi_{|n|}$, we modify Eq. (3.28) to the form:

$$\begin{aligned} 2i\psi_m(x) &= \sum_{n=1}^{\infty} \varphi_n(x)\psi_m(1)\Big[\kappa_1\varphi_n'(1) + \alpha\varphi_n(1)\Big]\left\{\frac{1}{\lambda_n^0 - \lambda_m} + \frac{1}{\lambda_{-n}^0 - \lambda_m}\right\} \\ &+ \sum_{n=1}^{\infty} \varphi_n(x)\psi_m'(1)\Big[\kappa_2\varphi_n(1) + \beta\varphi_n'(1)\Big]\left\{\frac{1}{\lambda_n^0 - \lambda_m} + \frac{1}{\lambda_{-n}^0 - \lambda_m}\right\} \\ &= \sum_{n=1}^{\infty} \varphi_n(x)\psi_m(1)\Big[\kappa_1\varphi_n'(1) + \alpha\varphi_n(1)\Big]\left\{\frac{1}{\lambda_n^0 - \lambda_m} - \frac{1}{\lambda_n^0 + \lambda_m}\right\} \\ &+ \sum_{n=1}^{\infty} \varphi_n(x)\psi_m'(1)\Big[\kappa_2\varphi_n(1) + \beta\varphi_n'(1)\Big]\left\{\frac{1}{\lambda_n^0 - \lambda_m} + \frac{1}{\lambda_{-n}^0 - \lambda_m}\right\}, \end{aligned} \tag{3.29}$$

from which representation (3.15) follows immediately. ■

Corollary 3.3. Let us justify that series (3.15) converges uniformly with respect to $x \in [0,1]$. Using the results of [Shubov & Kindrat, 2018], it is not difficult to show that there exists an absolute constant C_0 such that for the normalized $\varphi_n(x)$ the following estimates hold:

$$\left|\varphi_n'(x)\right| \le C_0\sqrt{\lambda_n^0} \quad \text{and} \quad \left|\varphi_n(x)\right| \le C_0, \quad x \in [0,1], \quad n \ge 1. \tag{3.30}$$

Indeed, using the symmetry of the spectrum of $\mathcal{L}_0$, we have modified Eq. (3.28) to the form of (3.29) (or, equivalently to the form of (3.15)). Accounting for (3.30) we obtain that for each m, there exists a constant $C_1(m)$ such that the following estimates hold:

$$\left|\frac{\left[\kappa_1\varphi_n'(x) + \alpha\varphi_n(x)\right]\varphi_n(x)}{\left(\lambda_n^0\right)^2 - \lambda_m^2}\right| \le \frac{C_1(m)}{n^3}, \qquad \left|\frac{\left[\kappa_2\varphi_n(x) + \beta\varphi_n'(x)\right]\varphi_n(x)}{\left(\lambda_n^0\right)^2 - \lambda_m^2}\right| \le \frac{C_1(m)}{n^3}.$$

Even though $C_1(m)$ may grow with m, since the summation in (3.29) (as well as in (3.28) and (3.15)) takes place with respect to n, the uniform convergence of (3.29) (and (3.28), (3.15)) with respect to x, $x \in [0,1]$ is not affected. Setting $x = 1$, in (3.15) we obtain the following identity for each $m \in \mathbb{N}^+$:

$$\frac{i}{\lambda_m}\psi_m(1) = \psi_m(1)\sum_{n=1}^{\infty}\frac{\Big[\kappa_1\varphi_n'(1) + \alpha\varphi_n(1)\Big]\varphi_n(1)}{\left(\lambda_n^0\right)^2 - \lambda_m^2} + \psi_m'(1)\sum_{n=1}^{\infty}\frac{\Big[\kappa_2\varphi_n(1) + \beta\varphi_n'(1)\Big]\varphi_n(1)}{\left(\lambda_n^0\right)^2 - \lambda_m^2}. \tag{3.31}$$

In what follows, we call (3.31) *the first main identity*.

4. Derivation of the second main identity

In this section we deal with the first component (3.27) of the eigenvector Ψ_m. It is convenient to introduce a new function $\mathcal{B}(x)$

$$\mathcal{B}(x) = \int_0^x d\tau \int_0^\tau \frac{d\eta}{EI(\eta)}. \tag{4.1}$$

Let $\mathbb{L}$ be a linear functional defined by

$$\mathbb{L}[g] = \lambda_m^2 \int_0^1 \mathcal{B}(x)g(x)\rho(x)dx. \tag{4.2}$$

Theorem 4.1. *Let λ_m and $\psi_m(x)$ be an eigenvalue and the corresponding eigenfunction of the pencil $\mathcal{P}(\lambda)$. Let $\lambda_m \neq \lambda_n^0$, $n \in \mathbb{Z}'$. Then the following relation is valid:*

$$\begin{aligned}
&\Big[\kappa_2\psi_m'(1)+\alpha\psi_m(1)\Big]\left\{\mathcal{B}(1)-\sum_{n=1}^{\infty}\frac{\varphi_n(1)\varphi_n'(1)}{\big(\lambda_n^0\big)^2}\right\}\\
&+\Big[\kappa_1\psi_m(1)+\beta\psi_m'(1)\Big]\left\{\mathcal{B}'(1)-\sum_{n=1}^{\infty}\frac{\big(\varphi_n'(1)\big)^2}{\big(\lambda_n^0\big)^2}\right\}+\frac{1}{i\lambda_m}\psi_m'(1)\\
&=-\Big[\kappa_1\psi_m(1)+\beta\psi_m'(1)\Big]\sum_{n=1}^{\infty}\frac{\big(\varphi_n'(1)\big)^2}{\big(\lambda_n^0\big)^2-\lambda_m^2}-\Big[\kappa_2\psi_m'(1)+\alpha\psi_m(1)\Big]\sum_{n=1}^{\infty}\frac{\varphi_n(1)\varphi_n'(1)}{\big(\lambda_n^0\big)^2-\lambda_m^2}.
\end{aligned}\tag{4.3}$$

Proof. As we know, series (3.27) converges in the space $\widetilde{H}_0^2(0,1)$. Due to the embedding theorem, we have $\widetilde{H}_0^2(0,1)\hookrightarrow C^1(0,1)$, which means that series (3.27) converges uniformly with respect to $x\in[0,1]$. Therefore, series (3.27) can be integrated term-wise with the weight $\rho(x)$. Applying $\mathbb{L}$ to both sides of Eq. (3.27) we get

$$\begin{aligned}
\frac{1}{i\lambda_m}\mathbb{L}[\psi_m]=&-\frac{1}{2}\sum_{n\in\mathbb{Z}'}\frac{1}{\lambda_n^0\left(\lambda_n^0-\lambda_m\right)}\mathbb{L}[\varphi_n]\\
&\times\left[\Big(\kappa_1\psi_m(1)+\beta\psi_m'(1)\Big)\varphi_n'(1)+\Big(\kappa_2\psi_m'(1)+\alpha\psi_m(1)\Big)\varphi_n(1)\right].
\end{aligned}\tag{4.4}$$

Now we evaluate $\mathbb{L}[\psi_m]$ and $\mathbb{L}[\varphi_m]$. We have

$$\mathbb{L}[\psi_m]=\lambda_m^2\int_0^1\mathcal{B}(x)\psi_m(x)\rho(x)dx.\tag{4.5}$$

Using Eq.(2.17) we modify (4.5) and have

$$\begin{aligned}
\mathbb{L}[\psi_m]=&\int_0^1\mathcal{B}(x)\Big(EI(x)\psi_m''(x)\Big)''dx\\
=&\mathcal{B}(1)\Big(EI(x)\psi_m''(x)\Big)'_{x=1}-\int_0^x\frac{d\eta}{EI(\eta)}EI(x)\psi_m''(x)\Big|_0^1+\int_0^1\psi_m''(x)dx\\
=&\mathcal{B}(1)\Big(EI(x)\psi_m''(x)\Big)'_{x=1}-\mathcal{B}'(1)EI(1)\psi_m''(1)+\psi_m'(1).
\end{aligned}\tag{4.6}$$

Applying the boundary conditions from (2.16), we proceed with (4.6) as

$$\mathbb{L}[\psi_m]=i\lambda_m\left\{\mathcal{B}(1)\Big[\kappa_2\psi_m'(1)+\alpha\psi_m(1)\Big]+\mathcal{B}'(1)\Big[\kappa_1\psi_m(1)+\beta\psi_m'(1)\Big]\right\}+\psi_m'(1).\tag{4.7}$$

Evaluating $\mathbb{L}[\varphi_n]$ we obtain

$$\mathbb{L}[\varphi_n]=\frac{\lambda_m^2}{\big(\lambda_n^0\big)^2}\ \varphi_n'(1).\tag{4.8}$$

Substituting (4.7) and (4.8) into relation (4.4) we obtain

$$\begin{aligned}
&\mathcal{B}(1)\Big[\kappa_2\psi_m'(1)+\alpha\psi_m(1)\Big]+\mathcal{B}'(1)\Big[\kappa_1\psi_m(1)+\beta\psi_m'(1)\Big]+\frac{1}{i\lambda_m}\psi_m'(1)\\
&=-\frac{\lambda_m^2}{2}\sum_{n\in\mathbb{Z}'}\frac{\Big(\kappa_1\psi_m(1)+\beta\psi_m'(1)\Big)\varphi_n'(1)+\Big(\kappa_2\psi_m'(1)+\alpha\psi_m(1)\Big)\varphi_n(1)}{\big(\lambda_n^0\big)^3\left(\lambda_n^0-\lambda_m\right)}\ \varphi_n'(1).
\end{aligned}\tag{4.9}$$

Let us denote by $\mathcal{S}$ the series in the right hand side of Eq.(4.9). It is convenient to modify $\mathcal{S}$ as follows:

$$\mathcal{S}\equiv\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)\frac{\Big(\kappa_1\psi_m(1)+\beta\psi_m'(1)\Big)\varphi_n'(1)+\Big(\kappa_2\psi_m'(1)+\alpha\psi_m(1)\Big)\varphi_n(1)}{\big(\lambda_n^0\big)^3\left(\lambda_n^0-\lambda_m\right)}\ \varphi_n'(1).$$

Taking into account that $\varphi_{-n}(x)=\varphi_n(x)$ and $\lambda_n^0=-\lambda_{-n}^0$, we obtain that $\mathcal{S}$ can be represented in the form

$$\begin{aligned}\mathcal{S}=&\sum_{n=1}^{\infty}\frac{\Big(\kappa_1\psi_m(1)+\beta\psi_m'(1)\Big)\varphi_n'(1)+\Big(\kappa_2\psi_m'(1)+\alpha\psi_m(1)\Big)\varphi_n(1)}{\left(\lambda_n^0\right)^3}\\ &\times\left[\frac{1}{\lambda_n^0-\lambda_m}-\frac{1}{-\lambda_n^0-\lambda_m}\right]\varphi_n'(1)\\ =&\,2\sum_{n=1}^{\infty}\frac{\Big(\kappa_1\psi_m(1)+\beta\psi_m'(1)\Big)\varphi_n'(1)+\Big(\kappa_2\psi_m'(1)+\alpha\psi_m(1)\Big)\varphi_n(1)}{\left(\lambda_n^0\right)^2\Big(\left(\lambda_n^0\right)^2-\lambda_m^2\Big)}\quad\varphi_n'(1).\end{aligned}\tag{4.10}$$

Substituting $\mathcal{S}$ of (4.10) into relation (4.9) yields:

$$\begin{aligned}&\mathcal{B}(1)\Big[\kappa_2\psi_m'(1)+\alpha\psi_m(1)\Big]+\mathcal{B}'(1)\Big[\kappa_1\psi_m(1)+\beta\psi_m'(1)\Big]+\frac{1}{i\lambda_m}\psi_m'(1)\\ =&-\lambda_m^2\Big[\kappa_1\psi_m(1)+\beta\psi_m'(1)\Big]\sum_{n=1}^{\infty}\frac{\left(\varphi_n'(1)\right)^2}{\left(\lambda_n^0\right)^2\left(\left(\lambda_n^0\right)^2-\lambda_m^2\right)}\\ &-\lambda_m^2\Big[\kappa_2\psi_m'(1)+\alpha\psi_m(1)\Big]\sum_{n=1}^{\infty}\frac{\varphi_n(1)\varphi_n'(1)}{\left(\lambda_n^0\right)^2\left(\left(\lambda_n^0\right)^2-\lambda_m^2\right)}.\end{aligned}\tag{4.11}$$

Using the fact that

$$\frac{\lambda_m^2}{\left(\lambda_n^0\right)^2\left(\left(\lambda_n^0\right)^2-\lambda_m^2\right)}=\frac{1}{\left(\lambda_n^0\right)^2-\lambda_m^2}-\frac{1}{\left(\lambda_n^0\right)^2},$$

we rewrite (4.11) as

$$\begin{aligned}&\mathcal{B}(1)\Big[\kappa_2\psi_m'(1)+\alpha\psi_m(1)\Big]+\mathcal{B}'(1)\Big[\kappa_1\psi_m(1)+\beta\psi_m'(1)\Big]+\frac{1}{i\lambda_m}\psi_m'(1)\\ =&-\Big[\kappa_1\psi_m(1)+\beta\psi_m'(1)\Big]\left\{\sum_{n=1}^{\infty}\frac{\left(\varphi_n'(1)\right)^2}{\left(\lambda_n^0\right)^2-\lambda_m^2}-\sum_{n=1}^{\infty}\frac{\left(\varphi_n'(1)\right)^2}{\left(\lambda_n^0\right)^2}\right\}\\ &-\Big[\kappa_2\psi_m'(1)+\alpha\psi_m(1)\Big]\left\{\sum_{n=1}^{\infty}\frac{\varphi_n(1)\varphi_n'(1)}{\left(\lambda_n^0\right)^2-\lambda_m^2}-\sum_{n=1}^{\infty}\frac{\varphi_n(1)\varphi_n'(1)}{\left(\lambda_n^0\right)^2}\right\}.\end{aligned}\tag{4.12}$$

Rearranging terms in this equation, we finally obtain (4.3). ■

To reduce (4.3) to the desired form we need the results proven in Lemma 4.1 and 4.2 below.

Lemma 4.2. 1) *The set of eigenfunctions $\{\varphi_n(x)\}_{n=1}^{\infty}$ of the pencil $\mathcal{P}_0(\lambda)$ normalized to unity*

$$\int_0^1|\varphi_n(x)|^2\rho(x)dx=1,\qquad n\in\mathbb{N}^+,\tag{4.13}$$

where $\mathbb{N}^+$ denotes the set of all positive integers, forms an orthonormal basis in the space $L^2_\rho(0,1)$.

2) *Let $\widetilde{H}^2_0(0,1)$ be equipped with the norm*

$$\|f\|^2_{H_0^2(0,1)}=\frac{1}{2}\int_0^1 EI(x)|f''(x)|^2dx.$$

The set $\left\{\frac{1}{i\lambda_n}\varphi_n(x)\right\}_{n=1}^{\infty}$ forms an orthonormal basis in the space $\widetilde{H}^2_0(0,1)$.

Proof. To prove Statement (1) it suffices to show that $\{\varphi_n(x)\}_{n=1}^{\infty}$ is an orthonormal set in $L^2_\rho(0,1)$ and that any function $g\in L^2_\rho(0,1)$ can be represented in the form of the expansion with respect to this set.

Since $\{\Phi_n(x)\}_{n\in\mathbb{Z}'}$ is the set of eigenfunctions of the self-adjoint operator $\mathcal{L}_0$, this set is complete and orthogonal in $\mathcal{H}$. Let us show that with normalization (4.13), this set is also normalized to unity in $\mathcal{H}$. Evaluating the norm of $\frac{1}{i\lambda_n^0}\varphi_n(x)$ in the space $\widetilde{H}^2_0(0,1)$, we take into account that φ_n satisfies the equation $\left(EI(x)\varphi_n''(x)\right)'' = \left(\lambda_n^0\right)^2 \rho(x)\varphi_n(x)$ and have

$$\int_0^1 EI(x)\frac{1}{\left(\lambda_n^0\right)^2}\left|\varphi_n''(x)\right|^2 dx = EI(x)\frac{1}{\left(\lambda_n^0\right)^2}\varphi_n''(x)\overline{\varphi_n'(x)}\Big|_0^1 - \frac{1}{\left(\lambda_n^0\right)^2}\left(EI(x)\varphi_n''(x)\right)' \ \overline{\varphi_n(x)}\Big|_0^1$$
$$+\int_0^1 \frac{1}{\left(\lambda_n^0\right)^2}\left(EI(x)\varphi_n''(x)\right)''\overline{\varphi_n(x)}dx + \int_0^1 \rho(x)\left|\varphi_n(x)\right|^2 dx. \tag{4.14}$$

Out of integral terms are equal to zero due to the boundary conditions. Using normalization conditions (4.13), we obtain from (4.14) that

$$\|\Phi_n\|_{\mathcal{H}}^2 = \frac{1}{2}\int_0^1 \left[\frac{E(x)\left|\varphi_n''(x)\right|^2}{\lambda_n^2} + \rho_n(x)\left|\varphi_n(x)\right|^2\right]dx = 1,$$

which means that the set of eigenfunctions of $\mathcal{L}_0$ is normalized to unity is $\mathcal{H}$.

From the fact that $\{\Phi_n(x)\}_{n\in\mathbb{Z}'}$ forms an orthonormal basis in $\mathcal{H}$ it follows that any function $G = (g_0, g_1)^T$ from $\mathcal{H}$, can be expanded with respect to this basis, $G(x) = \sum_{n\in\mathbb{Z}'} a_n \Phi_n(x)$. In particular, if $G = (0, g_1), \ g_1 \in L^2_\rho(0,1)$, then

$$G(x) = \sum_{n\in\mathbb{Z}'} \left(G, \Phi_n\right)_{\mathcal{H}} \ \Phi_n(x) = \sum_{n\in\mathbb{Z}'} \left(\frac{1}{2}\int_0^1 \rho(x) g_1(x)\varphi_n(x)dx\right)\Phi_n(x). \tag{4.15}$$

Since $\varphi_{-n}(x) = \varphi_n(x)$, we obtain from (4.15) that

$$\begin{aligned} g_1(x) &= \left(\sum_{-\infty}^{n=-1} + \sum_{n=1}^{\infty}\right)\left(\frac{1}{2}\int_0^1 g_1(\xi)\varphi_n(\xi)\rho(\xi)d\xi\right)\varphi_n(x) \\ &= \sum_{n=1}^{\infty}\left(\int_0^1 g_1(\xi)\varphi_n(\xi)\rho(\xi)d\xi\right)\varphi_n(x). \end{aligned} \tag{4.16}$$

Now let us check the orthogonality of the set $\left\{\varphi_n(x)\right\}_{n=1}^{\infty}$ in $L^2_\rho(0,1)$. The orthogonality condition for the set $\left\{\Phi_n(x)\right\}_{n\in\mathbb{Z}'}$ yields:

$$\begin{aligned} &\frac{1}{2i\lambda_n^0 \ \overline{i\lambda_m^0}}\int_0^1 EI(\xi)\varphi_n''(\xi)\overline{\varphi_m''(\xi)}d\xi + \frac{1}{2}\int_0^1 \varphi_n(\xi)\overline{\varphi_m(\xi)}\rho(\xi)d\xi \\ =&\frac{\lambda_n^0}{2\lambda_m^0}\int_0^1 \rho(\xi)\varphi_n(\xi)\overline{\varphi_m(\xi)}d\xi + \frac{1}{2}\int_0^1 \rho(\xi)\varphi_n(\xi)\overline{\varphi_m(\xi)}d\xi \\ =&\frac{1}{2}\left(\frac{\lambda_n^0}{\lambda_m^0}+1\right)\left(\varphi_n, \varphi_m\right)_{L^2_\rho(0,1)} = \delta_{mn}. \end{aligned} \tag{4.17}$$

Since $\lambda_n^0 \neq \lambda_m^0$ [Shubov & Kindrat, 2017; Gladwell, 2005], we obtain that

$$\left(\varphi_n, \varphi_m\right)_{L^2_\rho(0,1)} = \delta_{nm}. \tag{4.18}$$

Taking into account decomposition (4.16) and condition (4.18), we claim that $\left\{\varphi_n(x)\right\}_{n=1}^{\infty}$ forms an orthonormal basis in the space $L^2_\rho(0,1)$.

Statement (1) of the lemma is proven.

Now we prove that the set $\left\{\frac{1}{i\lambda_n^0}\varphi_n(x)\right\}_{n=1}^{\infty}$ forms an orthonormal basis in the space $\widetilde{H}^2_0(0,1)$. To this end, we prove that the above set is complete in $\widetilde{H}^2_0(0,1)$ and orthogonal. To prove completeness, we use the contradiction argument and assume that there exists $\psi \in H^2_0(0,1)$ such that $\psi \perp \varphi_n,\ n \in \mathbb{N}^+$. We have

$$\begin{aligned}(\psi, \varphi_n)_{H_0^2(0,1)} =& \frac{1}{2}\int_0^1 EI(\xi)\psi''(\xi)\overline{\varphi_n''(\xi)}d\xi = \frac{1}{2}EI(\xi)\psi'(\xi)\overline{\varphi_n''(\xi)}\Big|_0^1 \\ & -\frac{1}{2}\left(EI(\xi)\overline{\varphi_n''(\xi)}\right)'\psi(\xi)\Big|_0^1 + \frac{1}{2}\int_0^1 \left(EI(\xi)\overline{\varphi_n''(\xi)}\right)''\psi(\xi)d\xi \\ =& \frac{\lambda_n^2}{2}\int_0^1 \overline{\varphi_n(\xi)}\psi(\xi)\rho(\xi)d\xi = 0.\end{aligned}$$

Since $\{\varphi_n(x)\}_{n=1}^\infty$ is an orthonormal basis in $L^2_\rho(0,1)$, we get $\psi = 0$. Completeness is proven.
Orthogonality of the set can be shown by sequence of steps similar to (4.17). ■

Lemma 4.3. *For the function $\mathcal{B}$ defined in (4.1), the following relations are valid:*

$$\mathcal{B}(1) = \sum_{n=1}^\infty \frac{\varphi_n(1)\varphi_n'(1)}{\left(\lambda_n^0\right)^2} \qquad \textit{and} \qquad \mathcal{B}'(1) = \sum_{n=1}^\infty \frac{\left(\varphi_n'(1)\right)^2}{\left(\lambda_n^0\right)^2}. \tag{4.19}$$

Proof. Since $\mathcal{B}(0) = \mathcal{B}'(0) = 0$ and $\mathcal{B}''(x) = \frac{1}{EI(x)}$, we obtain that $\mathcal{B} \in \mathcal{H}$. The following expansion of $\mathcal{B}(x)$ with respect to the basis $\left\{\frac{1}{i\lambda_n^0}\varphi_n(x)\right\}_{n=1}^\infty$ takes place:

$$\mathcal{B}(x) = \sum_{n=1}^\infty a_n \ \frac{1}{i\lambda_n^0}\varphi_n(x), \tag{4.20}$$

where

$$a_n = \int_0^1 \frac{i}{\lambda_n^0}\varphi_n''(\xi)\mathcal{B}''(\xi)EI(\xi)d\xi = \frac{i}{\lambda_n^0}\int_0^1 \varphi_n''(\xi)d\xi = \frac{i}{\lambda_n^0}\varphi_n'(1). \tag{4.21}$$

Hence, substituting (4.21) into (4.20) we obtain

$$\mathcal{B}(x) = \sum_{n=1}^\infty \frac{1}{\left(\lambda_n^0\right)^2} \ \varphi_n'(1)\varphi_n(x). \tag{4.22}$$

Since the series converges in $\widetilde{H}_0^2(0,1)$, we use the embedding $\widetilde{H}_0^2(0,1) \hookrightarrow C^1(0,1)$ and claim that

$$\mathcal{B}'(x) = \sum_{n=1}^\infty \frac{\varphi_n'(1)}{\left(\lambda_n^0\right)^2} \ \varphi_n'(x). \tag{4.23}$$

The lemma is proven. ■

Corollary 4.4. Taking into account representations (4.19), we obtain that (4.12) can be modified to the form

$$\frac{1}{i\lambda_m}\psi_m'(1) = -\left[\kappa_1\psi_m(1) + \beta\psi_m'(1)\right]\sum_{n=1}^\infty \frac{\left(\varphi_n'(1)\right)^2}{\left(\lambda_n^0\right)^2 - \lambda_m^2} - \left[\kappa_2\psi_m'(1) + \alpha\psi_m(1)\right]\sum_{n=1}^\infty \frac{\varphi_n(1)\varphi_n'(1)}{\left(\lambda_n^0\right)^2 - \lambda_m^2}. \tag{4.24}$$

In what follows, we call (4.24) *the second main identity.*

5. Proof of Theorem 2.4

Using the first and second main identities (3.31) and (4.24) we prove the stability results formulated in Theorem 2.4.

Proof. In what follows in this proof, under Statements (1), (2), (3), or (4) we understand the main statement of Theorem 2.4 under the conditions (1), or (2), or (3), or (4) respectively.

1) It is technically convenient to start with the proof of Statement (2). To this end, we show that for the case when $\kappa_1 = \kappa_2 = 0$, the operator $\mathcal{L}$ is *dissipative*. We use the following definition

of a dissipative operator (see [Gohberg & Krein, 1996, and Dunford & Schwartz, 1963]). A linear operator A acting in a Hilbert space H defined on the domain $\mathcal{D}(A)$ is called *dissipative* if

$$\Im\big(Af, f\big)_H \geq 0 \qquad \text{for} \qquad f \in \mathcal{D}(A). \tag{5.1}$$

For a dissipative operator with purely discrete spectrum, all its eigenvalues are located in the closed upper half-plane. Let us show that the operator $\mathcal{L}$ defined by (2.13) on the domain

$$\begin{aligned} \mathcal{D}(\mathcal{L}) = \Big\{ \Phi = (\varphi_0, \varphi_1)^T \in \mathcal{H}: \ \varphi_0 \in H^4(0,1),\ \ \varphi_1 \in H^2(0,1);\ \ \varphi_1(0) = \varphi_1'(0) = 0; \\ EI(1)\varphi_0''(1) = -\beta\varphi_1'(1),\ \ \big(EI(x)\varphi_0''(x)\big)'_{x=1} = \alpha\varphi_1(1) \Big\} \end{aligned} \tag{5.2}$$

is dissipative. We evaluate $\big(\mathcal{L}F, F\big)_{\mathcal{H}}$ on any $F = (f_0, f_1)^T \in \mathcal{D}(\mathcal{L})$ and have

$$\big(\mathcal{L}F, F\big)_{\mathcal{H}} = -\Im \int_0^1 EI(x) f_0''(x)\overline{f_1''(x)}dx - \frac{i}{2}EI(1)f_0''(1)\overline{f_1'(1)} + \frac{i}{2}\big(EI(x)f_0''(x)\big)'_{x=1}\overline{f_1(1)}.$$

Taking into account the right-end boundary conditions from (5.2) we obtain that

$$\Im\big(\mathcal{L}F, F\big)_{\mathcal{H}} = \frac{1}{2}\Big[\alpha\big|f_1(1)\big|^2 + \beta\big|f_1'(1)\big|^2\Big] \geq 0. \tag{5.3}$$

To prove statement (2) it remains to be seen that for $\alpha + \beta > 0$, there are no real eigenvalues. Using the contradiction argument, we assume that $\mathcal{L}$ has a real eigenvalue $\lambda_0 > 0$. From (5.3) we obtain that if $\alpha\beta \neq 0$, then the eigenfunction corresponding to λ_0 has to satisfy the boundary conditions $f_1(1) = f_1'(1) = 0$. This means that λ_0 must be an eigenvalue of the clamped-clamped beam model [Shubov & Kindrat, 2017]. However, the same eigenfunction must satisfy the boundary conditions from $\mathcal{D}(\mathcal{L})$, i.e. $\big(EI(1)f_0''(1)\big) = -\beta f_1'(1)$ and $\big(EI(1)f_0''(1)\big)'(1) = \alpha f_1(1)$, which yields that $f_0''(1) = \big(EI(1)f_0''(1)\big)'(1) = 0$. From these equations we obtain that the same λ_0 must be an eigenvalue of the clamped-free beam model. As follows from Corollary 4.3 of [Shubov & Kindrat, 2017], two selfadjoint beam problems do not share eigenvalues. Therefore, the fact that for $\alpha > 0$ and $\beta > 0$, all eigenvalues are located in the open upper half-plane is proven.

Now let us consider the case when one of the parameters is zero, e.g., $\alpha > 0$ and $\beta = 0$. Then assuming that there is a real eigenvalue, λ_0, we obtain from dissipativity of $\mathcal{L}$ that $f_1(1) = 0$. Since $\beta = 0$, the boundary conditions from (5.2) become $f_1''(1) = \big(EI(1)f_0''(1)\big)' = 0$. From the fact that $f_0(1) = f_0''(1) = 0$, it follows that λ_0 is an eigenvalue of the self-adjoint problem corresponding the clamped-hinged beam model. From the fact that $f_0''(1) = \big(EI(1)f_0''(1)\big) = 0$ it follows that λ_0 is an eigenvalue of the clamped-free beam model. By Corollary 4.3 of [Shubov & Kindrat, 2017] this cannot happen.

Statement (2) is shown.

2) Now we are in a position to prove Statement (1). The fact that the spectrum is located in the closed upper half-plane follows immediately from the combination of the results of [Shubov & Shubov, 2016] (see Theorem 5.1 of [Shubov & Shubov, 2016]), and of Statement (2) proven above. The fact that there are no real eigenvalues follows from Theorem 7.1 of [Shubov & Kindrat, 2017].

3) To prove Statement (3), let us consider the first main identity (3.31) when $\beta = \kappa_2 = 0$. Based on the result of Theorem 5.3 [Shubov & Kindrat, 2017], we have $\psi_m(1) \neq 0$ for any m, which yields the following identity:

$$\frac{i}{\lambda_m} = \sum_{n=1}^{\infty} \frac{\Big[\kappa_1\varphi_n'(1) + \alpha\varphi_n(1)\Big]\varphi_n(1)}{\big(\lambda_n^0\big)^2 - \lambda_m^2}. \tag{5.4}$$

Using Theorem 4.2 of [Shubov & Kindrat, 2017], we claim that $\varphi_n(1)\varphi_n'(1) > 0$. Thus, $\kappa_1\varphi_n(1)\varphi_n'(1) + \alpha\varphi_n^2(1) > 0$ for $\kappa_1 + \alpha > 0$.

Setting $\lambda_m \equiv a_m + ib_m$, we rewrite relation (5.4) in the form

$$\frac{i(a_m - ib_m)}{a_m^2 + b_m^2} = \sum_{n=1}^{\infty} \frac{\gamma_n\left\{\left[\left(\lambda_n^0\right)^2 - a_m^2 + b_m^2\right] + 2ia_m b_m\right\}}{\left(\left(\lambda_n^0\right)^2 - a_m^2 + b_m^2\right)^2 + 4a_m^2 b_m^2}, \quad \text{where} \quad \gamma_n = \kappa_1 \varphi_n(1)\varphi_n'(1) + \alpha\left(\varphi_n(1)\right)^2. \tag{5.5}$$

Taking the real and imaginary parts of (5.5), we obtain the following relations:

$$\frac{a_m}{a_m^2 + b_m^2} = 2\sum_{n=1}^{\infty} \gamma_n \frac{a_m b_m}{\left(\left(\lambda_n^0\right)^2 - a_m^2 + b_m^2\right)^2 + 4a_m^2 b_m^2} \tag{5.6}$$

and

$$\frac{b_m}{a_m^2 + b_m^2} = \sum_{n=1}^{\infty} \frac{\gamma_n\left[\left(\lambda_n^0\right)^2 - a_m^2 + b_m^2\right]}{\left(\left(\lambda_n^0\right)^2 - a_m^2 + b_m^2\right)^2 + 4a_m^2 b_m^2}. \tag{5.7}$$

Assume that λ_m is such that $a_m \neq 0$. The relation (5.6) becomes

$$\frac{1}{2\left(a_m^2 + b_m^2\right)} = b_m \sum_{n=1}^{\infty} \frac{\gamma_n}{\left(\left(\lambda_n^0\right)^2 - a_m^2 + b_m^2\right)^2 + 4a_m^2 b_m^2},$$

which yields $b_m > 0$.

Assume that $a_m = 0$ for some $m > 0$. Then we turn to Eq. (5.7) which becomes

$$\frac{1}{b_m} = \sum_{n=1}^{\infty} \frac{\gamma_n\left(\left(\lambda_n^0\right)^2 + b_m^2\right)}{\left(\left(\lambda_n^0\right)^2 + b_m^2\right)^2},$$

and again, $b_m > 0$.

As follows from relation (5.6) for any $a_m > 0$, one gets $b_m > 0$, which means that λ_0 can be a real eigenvalue of the operator $\mathcal{L}$ if and only if both operators $\mathcal{L}$ and $\mathcal{L}_0$ share λ_0 as their eigenvalues. Let us show that this cannot happen. Using the contradiction argument, we assume that some $\lambda_{n_0} > 0$ is an eigenvalue of both $\mathcal{L}_0$ and $\mathcal{L}$ with the eigenvectors $\Phi_{n_0}(x)$ and $\Psi_{n_0}(x)$ respectively. From relation (3.17), which is valid for any m, one gets for $m = n_0$:

$$\left(I - \lambda_{n_0}\mathcal{L}_0^{-1}\right)\Psi_{n_0}(x) = \lambda_{n_0}\mathcal{T}\Psi_{n_0}(x). \tag{5.8}$$

Using the spectral decomposition (3.18) for $\mathcal{L}_0$, we obtain that

$$\left(I - \lambda_{n_0}\mathcal{L}_0^{-1}\right)\Psi_{n_0}(x) = \sum_{\substack{n\in\mathbb{Z}' \\ n\neq n_0}} \frac{\lambda_n^0 - \lambda_{n_0}}{\lambda_n^0}\left(\Psi_{n_0}, \Phi_n\right)_{\mathcal{H}}\Phi_n(x) + \lambda_{n_0}\mathcal{T}\Psi_{n_0}(x). \tag{5.9}$$

Let us prove that $\left(I - \lambda_{n_0}\mathcal{L}_0^{-1}\right)\Psi_{n_0}$ and Φ_{n_0} are orthogonal in $\mathcal{H}$. We have

$$\left(\left(I - \lambda_{n_0}\mathcal{L}_0^{-1}\right)\Psi_{n_0}, \Phi_{n_0}\right)_{\mathcal{H}} = \left(\Psi_{n_0}, \left(I - \lambda_{n_0}\mathcal{L}_0^{-1}\right)\Phi_{n_0}\right)_{\mathcal{H}} = 0,$$

since λ_{n_0} and $\Phi_{n_0}(x)$ are eigenvalue and the corresponding eigenvector of $\mathcal{L}_0$. The sum in (5.9) is orthogonal to Φ_{n_0} by construction. Therefore we get that $\left(\mathcal{T}\Psi_{n_0}, \Phi_{n_0}\right)_{\mathcal{H}} = 0$. Now, we use the formula in (3.25) in which $m = n = n_0$, $\kappa_2 = \beta = 0$, and obtain that

$$\left(\mathcal{T}, \Psi_{n_0}\right)_{\mathcal{H}} = -\frac{i}{2\lambda_{n_0}^2}\left(\kappa_1 \varphi'_{n_0}(1) + \alpha\varphi_{n_0}(1)\right)\psi_{n_0}(1) = 0, \tag{5.10}$$

where φ_{n_0} and ψ_{n_0} are the second components of the eigenvectors Φ_{n_0} and Ψ_{n_0} respectively. However, by Theorem 5.3 of [Shubov & Kindrat], $\psi_{n_0}(1) \neq 0$ and by Theorem 4.2 of [Shubov & Kindrat], $\varphi_{n_0}(1)\varphi'_{n_0}(1) > 0$. Therefore, (5.10) is not valid.

Due to the obtained contradiction, Statement (3) is proven.

4) To prove Statement (4), let us consider the second main identity (4.24). Assuming that $\kappa_1 = \alpha = 0$ and κ_2 and β are such that $\beta\kappa_2 > 0$, we have

$$\frac{i}{\lambda_m}\psi'_m(1) = \left[\beta \sum_{n=1}^{\infty} \frac{\left(\varphi'_n(1)\right)^2}{\left(\lambda_n^0\right)^2 - \lambda_m^2} + \kappa_2 \sum_{n=1}^{\infty} \frac{\varphi_n(1)\varphi'_n(1)}{\left(\lambda_n^0\right)^2 - \lambda_m^2}\right]\psi'_m(1). \tag{5.11}$$

Since $\psi'_m(1) \neq 0$, we can cancel out $\psi'_m(1)$ and obtain that for each $m \in \mathbb{N}^+$ the following relation holds:

$$\frac{i}{\lambda_m} = \sum_{n=1}^{\infty} \frac{\left[\kappa_2\varphi_n(1) + \beta\varphi'_n(1)\right]\varphi'_n(1)}{\left(\lambda_n^0\right)^2 - \lambda_m^2}. \tag{5.12}$$

If $\lambda_m = a_m + ib_m$, then separating the real and imaginary parts of Eq.(5.12) we arrive at the equations similar to Eqs.(5.6) and (5.7) in which γ_n from (5.5) has been replaced with δ_n, where $\delta_n \equiv \kappa_2\varphi_1(1)\varphi'_n(1) + \beta\left(\varphi'_n(1)\right)^2$. Therefore, the remaining part of the proof is quite similar to the end of the proof of Statement (3). ■

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References

1. Shubov, M. and Shubov, V., 2016(1), Stability of a flexible structure with destabilizing boundary conditions, *Proceed. Royal Soc. A: Mathem.*, **472**, p. 1-22.
2. Guiver, Ch. and Opmeer, M.R., 2010, Non-dissipative boundary feedback for Rayleigh and Timoshenko beam, *Syst. Contr. Let.*, **59** (9), p. 578-586.
3. Luo, Z.-H. and Guo, B.-Z., 1997, Shear force feedback control with a single-link flexible robot with a revolute joint, *IEEE Trans. Automat. Control*, **42** (1), p. 53-65.
4. Cannon, R.H. and Schmitz, R., 1984, Initial experiments on the end-point control of a flexible one-link robot, *Int'l J. Robotics Res.* **3**(3), p. 62-75.
5. Sakawa, Y., Matsuno, F., and Fukushima, S., 1985, Modeling and feedback control of a flexible arm, *J. Robotic Syst.*, **2**(4), p. 453-472.
6. Chen, G., Krantz, S.G., Ma, D.W., and Wayne, N.E., 1988, The Euler-Bernoulli beam equation with boundary energy dissipation, in *Operator Methods for Optimal Control Problems* (S.J. Lee, Ed.), Marcel Decker, p. 67-96.
7. Conrad, F. and Morgul, O., 1988, On a stabilization of a flexible beam with a tip mass, *SIAM J. Control Optim.*, **36**, p. 1962-1986.
8. Ozer, A.O. and Hansen, S.W., 2011, Exact controllability of a Rayleigh beam with a single boundary control, *Math. Control Signals Syst.*, **23**(1), p. 199-222.
9. Ozer, A.O. and Hansen, S.W., 2014, Exact boundary controllability results for a multilayer Rao-Nakra sandwich beam, *SIAM J. Control Optim.*, **52**(2), p. 1314-1337.
10. Curtain, R.F. and Zwart, H.J., 1995, *An Introduction to Infinite Dimensional Linear Systems Theory*, Springer-Verlag, New York.
11. Zwart, H., 2010, Riesz basis for strongly continuous groups, *J. Diff. Eqs.*, **249**, p. 2397-2408.
12. Lions, J.L., 1988, Exact controllability, stabilization, and perturbations for distributed parameter systems, *SIAM Rev.*, **30**, p. 1-68.
13. Shubov, M.A., 2008, Exact controllability non-selfadjoint Euler-Bernoulli beam model via spectral decomposition method, *IMA J. Contr. Inform*, **25**, p. 185-203.

14. Shubov, M.A., 2006, Exact controllability of coupled Euler-Bernoulli and Timoshenko beam model, *IMA J. Contr. Inform.*, **23**, p. 279-300.
15. Russell, D. L., 1986, Mathematical models for the elastic beam and their control-theoretical implications. In: *Semigroups, Theory and Applications, Vol. II;* Pitman Research Notes, **152**, p. 177-215.
16. Dowell, E.H. (Ed.), 2004, *A Modern Course in Aeroelasticity*, 4^{th} revised ed., Klumer Academic Publ.
17. Joshi, S.M., 1988, *Control of Large Flexible Structures*, Springer, Berlin, Germany.
18. Balakrishnan, A.V. and Shubov, M.A., 2004, Asymptotic behavior of aeroelastic modes for aircraft wing model in subsonic air flow, *Proceed. Royal Soc. A*, London, **460**, p.1057-1091.
19. Balakrishnan, A.V., 2012, *Aerolasticity, The Continuum Theory,* Springer, New York, Heidelberg, Lodon.
20. Shubov, M.A. and Rojas-Arenaza, M., 2008, Double-walled carbon nanotube as vibrational system: mathematical approach, *Mathem. Methods Appl. Sci*, **29**, p. 2181-2199.
21. Shubov, M.A. and Rojas-Arenaza, M., 2011, Four-branch vibrational spectrum of double-walled carbon nanotube model, *Proceed. Royal Soc. A*, London, **467**, p. 99-126.
22. Erturk, A. and Inman, D.J., 2011, *Piezoelectric Energy Harvesting*, Wiley Chichester, UK.
23. Erturk, A., 2012, Assumed-mode modeling of piezoelectric energy harvesters: Euler-Bernoulli, Rayleigh, and Timoshenko models with axial deformations, *Comput. Structures*, **106**, p. 214-227.
24. Shubov, M. A., 2018, Asymptotic and spectral analysis of a model of the piezoelectric energy harvester with the Timoshenko beam as a substructure, *Applied Sci.*, **8(9)**, p. 1434-1480.
25. Shubov, M.A. and Shubov, V.I., 2016(2), Asymptotic and spectral analysis and control problems for mathematical model of piezoelectric energy harvester, *Math. Probl. in Engin., Sci., and Aerospace*, **7**, p. 249-268.
26. Ozer, A.O., 2018, Experimental stabilization of the smart piezoelectric composite beam with only one boundary controler, *Proceed. of IFAC Conf. on Lagrangian and Hamiltonian Methods,* **51-3**, p. 80-85.
27. Shubov, M.A., 2014, Spectral asymptotics, instability, and Riesz basis propery of root vectors for Rayleigh beam model with non-dissipative boundary conditions, *Assymptotic Analysis,* **87**, p. 147-190.
28. Gohberg, I. Ts. and Krein, M.G., 1996, *Introduction to the Theory of Nonselfadjoint Operators in Hilbert Space*, Transl. Math. Monographs, **18**, AMS, Providence, RI.
29. Shubov, M.A. and Kindrat, L.P., 2018, Asymptotic distribution of the eigenmodes and stability of an elastic structure with general feedback matrix, to appear: *IMA J. Applied Math.*
30. Shubov, M.A., and Kindart, L.P., 2019, Spectral analysis and numerical investigation of a flexible structure with non-conservative bounding data, Book chapter in: *Matrix Calculus,* ISBN 978-1-83880-008-6, p. 1-21.
31. Benaroya, H., 1998, *Mechanical Vibration: Analysis, Uncertainties, and Control,* Prentice Hall, NJ.
32. Gladwell, Graham M. L., 2005, *Inverse Problems in Vibration,* 2^{nd} Ed., Klumer Academic, Boston, London.
33. Marcus, A.S., 1988, *Introduction to the Spectral Theory of Polynomial Operator Pencils,* Transl. Math. Monographs, **71**, AMS, Providence, RI.
34. Pazy, A., 1983, *Semigroup of Linear Operators and Applications to Partial Differential Equations,* Springer-Verlag, New York.
35. Engel, K.J., Nagel, R., 2000, *One-Parameter Semigroups for Linear Evolution Equations,* Springer-Verlag, New York, Berlin, Heidelberg.
36. Shubov, M.A., 2006, Generation of Gevrey class semigroup by non-selfadjoint Euler-Bernoulli beam model, *Math. Meth. Applied Sci,* **29**, p. 2181-2199.
37. Birman, M.S. and Solomyak, M.Z., 1987, *Spectral Theory of Self-Adjoint Operators in Hilbert Space,* E-library of Math. Dept., Moscow State University, Moscow, Russia.
38. Dunford, N. and Scwartz, J.T., 1963, *Linear Operators, Part III: Spectral Operators*, Interscience Publ., New York, NY.