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Journal of Functional Analysis

www.elsevier.com/locate/jfa



Polynomial control on stability, inversion and powers of matrices on simple graphs



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ARTICLE INFO

Article history: Received 24 May 2017 Accepted 27 September 2018 Available online 10 October 2018 Communicated by L. Saloff-Coste

Keywords:

Banach algebra of infinite matrices Norm-controlled inversion Lower stability bounds Polynomial growth for powers

ABSTRACT

Spatially distributed networks of large size arise in a variety of science and engineering problems, such as wireless sensor networks and smart power grids. Most of their features can be described by properties of their state-space matrices whose entries have indices in the vertex set of a graph. In this paper, we introduce novel algebras of Beurling type that contain matrices on a connected simple graph having polynomial off-diagonal decay, and we show that they are Banach subalgebras of $\mathcal{B}(\ell^p), 1 \leq p \leq \infty$, the space of all bounded operators on the space ℓ^p of all p-summable sequences. The ℓ^p -stability of state-space matrices is an essential hypothesis for the robustness of spatially distributed networks. In this paper, we establish the equivalence among ℓ^p -stabilities of matrices in Beurling algebras for different exponents 1 < $p \leq \infty$, with quantitative analysis for the lower stability bounds. Admission of norm-control inversion plays a crucial role in some engineering practice. In this paper, we prove that matrices in Beurling subalgebras of $\mathcal{B}(\ell^2)$ have normcontrolled inversion and we find a norm-controlled polynomial with close to optimal degree. Polynomial estimate to powers of matrices is important for numerical implementation of spatially distributed networks. In this paper, we apply our results on norm-controlled inversion to obtain a polynomial estimate to powers of matrices in Beurling algebras. The polynomial estimate is a noncommutative extension about convolution powers of a complex function and is applicable to

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estimate the probability of hopping from one agent to another agent in a stationary Markov chain on a spatially distributed network.

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1. Introduction

A spatially distributed network (SDN) contains a large amount of agents with limited sensing, data processing, and communication capabilities for information transmission. It arises in a variety of science and engineering problems ([1,12,27,63]). The topology of an SDN can be described by a graph

$$\mathcal{G} := (V, E) \tag{1.1}$$

of large size, where a vertex in V represents an agent and an edge $(\lambda, \lambda') \in E$ between two vertices λ and $\lambda' \in V$ means that a direct communication link exists [11]. In this paper, we always assume that \mathcal{G} is connected and simple. Here a simple graph means that it is an unweighted, undirected graph containing no graph loops or multiple edges. Our motivating examples are 1) circular graphs $\mathcal{Z}_N := (\mathbb{Z}/N\mathbb{Z}, W)$ of order $N \geq 3$, where $W \subset \mathbb{Z}/N\mathbb{Z}$ and $(m,n) \in E_N$ means $m-n \in N\mathbb{Z} \pm W$; and 2) lattice graphs $\mathcal{Z}^d := (\mathbb{Z}^d, E^d), d \geq 1$, where $(m,n) \in E^d$ implies that m and $n \in \mathbb{Z}^d$ have distance one.

SDNs could give extraordinary capabilities especially when creating a data exchange network requires significant efforts or when establishing a centralized facility to process and store all the information is formidable. A comprehensive mathematical analysis of SDNs does not appear to exist yet, and there is a huge research gap between mathematical theory and engineering practice [5,11,18,35,38,49]. This inspires us to consider various properties of state-space matrices

$$A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in V} \tag{1.2}$$

of SDNs with indices in the vertex set V of a graph. This work is also motivated by the emerging field of signal processing on graphs, where matrices of the form (1.2) are used for linear processing such as filtering, translation, modulation, dilation and downsampling [10.29,40,44,43,47].

An abundant family of SDNs is spatially decaying linear systems whose state-space matrices have off-diagonal decay. Examples of such systems include smart power grids with sparse interconnection topologies, multi-agent systems with nearest-neighbor coupling structures, and (wireless) sensor networks for environment monitoring ([1,11,12, 18,27,35,63]). To describe off-diagonal decay property of matrices of the form (1.2), we introduce Banach algebras $\mathcal{B}_{r,\alpha}(\mathcal{G})$ of Beurling type for $1 \leq r \leq \infty$ and $\alpha \geq 0$, see (3.1) and (3.2) in Section 3. Matrices $A = (a(\lambda, \lambda'))_{\lambda,\lambda' \in V}$ in $\mathcal{B}_{r,\alpha}(\mathcal{G})$ have their entries dominated by a positive decreasing function h_A with polynomial decay,

$$|a(\lambda, \lambda')| \le h_A(\rho(\lambda, \lambda'))$$
 for all $\lambda, \lambda' \in V$, (1.3)

where $\rho(\lambda, \lambda')$ is the geodesic distance between vertices $\lambda, \lambda' \in V$. For the lattice graph \mathbb{Z}^d , Banach algebras $\mathcal{B}_{r,\alpha}(\mathbb{Z}^d)$ are introduced by Beurling in [8] for r = d = 1 and $\alpha = 0$, Jaffard in [28] for $r = \infty$ and Sun in [54] for $1 \leq r \leq \infty$.

Let $\ell^p := \ell^p(V), 1 \le p \le \infty$, be Banach spaces of *p*-summable sequences on *V* with standard norm $\|\cdot\|_{\ell^p}$, and $\mathcal{B}(\ell^p)$ be the Banach algebra of all bounded operators on ℓ^p with norm $\|\cdot\|_{\mathcal{B}(\ell^p)}$. We say that a matrix $A \in \mathcal{B}(\ell^p)$ has ℓ^p -stability if there exists a positive constant A_p such that

$$||Ac||_p \ge A_p ||c||_p \quad \text{for all } c \in \ell^p. \tag{1.4}$$

The optimal lower ℓ^p -stability bound of a matrix A is the maximal number A_p for (1.4) to hold. The ℓ^p -stability is an essential hypothesis for some matrices arising in time–frequency analysis, sampling theory, wavelet analysis and many other applied mathematical fields [3,14,20,52,57]. For the robustness against bounded noises, the sensing matrix arising in the sampling and reconstruction procedure of signals on a SDN is required in [11] to have ℓ^{∞} -stability, however there are some difficulties to verify ℓ^p -stability of a matrix in a distributed manner for $p \neq 2$ [37].

For a finite graph $\mathcal{G} = (V, E)$ and a matrix A with indices in its vertex set V, its ℓ^p -stability and ℓ^q -stability are equivalent to each other for any $1 \leq p, q \leq \infty$, and its optimal lower stability bounds satisfy

$$M^{-|1/p-1/q|} \le \frac{A_q}{A_p} \le M^{|1/p-1/q|},$$
 (1.5)

where M = #V is the number of vertices of the graph \mathcal{G} . The above estimation on lower stability bounds is unfavorable for matrices of large size but it cannot be improved if there is no restriction on the matrix A. Let d be the Beurling dimension of the graph \mathcal{G} . Matrices $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ with $1 \leq r \leq \infty$ and $\alpha > d(1-1/r)$ are bounded operators on $\ell^p, 1 \leq p \leq \infty$, and there exists a positive constant C such that

$$||A||_{\mathcal{B}(\ell^p)} \leq C||A||_{\mathcal{B}_{r,\alpha}}$$
 for all $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ and $1 \leq p \leq \infty$.

For their lower stability bounds, it is proved that

$$A_q > 0$$
 if and only if $A_p > 0$

in [2,54,58] for the infinite lattice graph \mathbb{Z}^d , and that

$$A_q > 0$$
 if $A_2 > 0$

in [11] for any infinite graph \mathcal{G} with finite Beurling dimension and $r = \infty$, where $1 \leq p, q \leq \infty$. In Theorem 4.1 of this paper, we provide a quantitative version of ℓ^p -stability for different $1 \leq p \leq \infty$ and prove the following result,

$$D_1 \left(\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_p} \right)^{-D_0|1/p - 1/q|} \le \frac{A_q}{A_p} \le D_2 \left(\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_p} \right)^{D_0|1/p - 1/q|}, \tag{1.6}$$

for any matrix $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ with $1 \leq r \leq \infty$ and $\alpha > d(1-1/r)$, where D_0, D_1, D_2 are absolute constants independent of matrices $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$, exponents $1 \leq p, q \leq \infty$ and the size M of the graph \mathcal{G} , cf. (1.5). The proof of Theorem 4.1 depends on an important estimate to the commutator between a matrix in the Beurling algebra $\mathcal{B}_{r,\alpha}(\mathcal{G})$ and a truncation operator. Similar estimate has been used by Sjöstrand in [48] to establish invertibility of infinite matrices in the Baskakov–Gohberg–Sjöstrand class.

A Banach subalgebra \mathcal{A} of \mathcal{B} is said to be *inverse-closed* if an element in \mathcal{A} , that is invertible in \mathcal{B} , is also invertible in \mathcal{A} . The inverse-closed subalgebras have numerous applications in time–frequency analysis, sampling theory, numerical analysis and optimization, and it has been established for matrices, integral operators, pseudo-differential operators satisfying various off-diagonal decay conditions. The reader may refer to [4, 6,16,19,22,21,23,26,28,33,36,48,54,53] and the survey papers [22,32,46] for historical remarks and recent advances.

A quantitative version of inverse-closedness is the norm-controlled inversion [24,25, 39,42,50]. Here an inverse-closed Banach subalgebra \mathcal{A} of \mathcal{B} is said to admit norm control in \mathcal{B} if there exists a continuous function h from $[0,\infty)\times[0,\infty)$ to $[0,\infty)$ such that

$$||A^{-1}||_{\mathcal{A}} \le h(||A||_{\mathcal{A}}, ||A^{-1}||_{\mathcal{B}}) \tag{1.7}$$

for all $A \in \mathcal{A}$ with $A^{-1} \in \mathcal{B}$. Admission of norm-control inversion plays a crucial role in [55] to solve nonlinear sampling problems termed with instantaneous compansion and local identification of signals with finite rate of innovation. Norm-controlled inversion was first studied by Stafney in [50], where it is shown that $\mathcal{B}_{1,0}(\mathcal{Z})$ does not admit a norm-controlled inversion in $\mathcal{B}(\ell^2)$. The polynomial norm-control inversion is established in [24] for matrices in differential algebras and [25] for matrices in Besov algebras, Bessel algebras, Dales-Davie algebras and Jaffard algebra. In Theorem 5.1 of this paper, we show that Banach algebras $\mathcal{B}_{r,\alpha}(\mathcal{G})$ with $1 \leq r \leq \infty$ and $\alpha > d(1-1/r)+1$ admit norm-controlled inversion in $\mathcal{B}(\ell^2)$, and there exists an absolute constant C such that

$$||A^{-1}||_{\mathcal{B}_{r,\alpha}} \le C||A^{-1}||_{\mathcal{B}(\ell^2)} (||A^{-1}||_{\mathcal{B}(\ell^2)} ||A||_{\mathcal{B}_{r,\alpha}})^{\alpha+d/r} \tag{1.8}$$

hold for all $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ with $A^{-1} \in \mathcal{B}(\ell^2)$. Moreover, the above polynomial norm-control inversion is close to optimal, as shown in Example 5.2 that the exponent $\alpha + d/r$ in (1.8) cannot be replaced by $\alpha + d/r - 1 - \epsilon$ for any $\epsilon > 0$. We remark that a weak version of the norm-controlled estimate (1.8), with the exponent $\alpha + d/r$ in (5.2) replaced by a larger exponent $2\alpha + 2 + 2/(\alpha - 2)$, is established in [25] for the Jaffard algebra $\mathcal{J}_{\alpha}(\mathcal{Z}) = \mathcal{B}_{\infty,\alpha}(\mathcal{Z})$.

Let \mathcal{A} be a Banach subalgebra of \mathcal{B} . We say that \mathcal{A} is its differential subalgebra of order $\theta \in (0,1]$ ([9,13,31,55,51]) if there exists a positive constant C such that

$$||AB||_{\mathcal{A}} \le C||A||_{\mathcal{A}}||B||_{\mathcal{A}} \left(\left(\frac{||A||_{\mathcal{B}}}{||A||_{\mathcal{A}}} \right)^{\theta} + \left(\frac{||B||_{\mathcal{B}}}{||B||_{\mathcal{A}}} \right)^{\theta} \right) \quad \text{for all } A, B \in \mathcal{A}. \tag{1.9}$$

The differential subalgebras was introduced in [9,31] for $\theta = 1$ and [13,55,51] for $\theta \in (0,1)$. In [13,23], it is shown that a C^* -subalgebra \mathcal{A} of \mathcal{B} with a common unit admits norm controlled inversion if \mathcal{A} is also a differential subalgebra. The reader may refer to [9,13,31,36,46,55,56] and references therein for historical remarks and recent advances in operator theory, harmonic analysis, non-commutative geometry, numerical analysis and optimization. The differential norm inequality (1.9) is satisfied by many Banach subalgebras of $\mathcal{B}(\ell^2)$ [23,54,53,51]. For $1 \leq r \leq \infty$ and $\alpha > d(1-1/r)$, it is shown in Proposition 3.6 that Banach algebras $\mathcal{B}_{r,\alpha}(\mathcal{G})$ are differential subalgebra of $\mathcal{B}(\ell^2)$ with order $\theta_{r,\alpha} = 2(\alpha - d + d/r)/(1 + 2\alpha - 2d + 2d/r)$. Applying the differential property (1.9) repeatedly and using the argument in [55, Proposition 2.4], we have the following subexponential estimate for the norms of powers $A^n, n \geq 1$, in $\mathcal{B}_{r,\alpha}(\mathcal{G})$,

$$||A^n||_{\mathcal{B}_{r,\alpha}} \le ||A||_{\mathcal{B}(\ell^2)}^n \left(C \frac{||A||_{\mathcal{B}_{r,\alpha}}}{||A||_{\mathcal{B}(\ell^2)}} \right)^{\frac{\theta_{r,\alpha}}{1+\theta_{r,\alpha}}} n^{\log_2(1+\theta_{r,\alpha})}, \quad n \ge 1,$$

where C is an absolute constant independent of integers $n \geq 1$ and matrices $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$. In Theorem 6.1 of this paper, we refine the above subexponential estimate to show that powers of a matrix in $\mathcal{B}_{r,\alpha}(\mathcal{G})$ with $1 \leq r \leq \infty$ and $\alpha > d(1-1/r)+1$ have polynomial growth,

$$||A^n||_{\mathcal{B}_{r,\alpha}} \le Cn \left(\frac{n||A||_{\mathcal{B}_{r,\alpha}}}{||A||_{\mathcal{B}(\ell^2)}}\right)^{\alpha+d/r} ||A||_{\mathcal{B}(\ell^2)}^n, \ n \ge 1.$$
(1.10)

Moreover, the above estimate is close to optimal, as shown in (6.3) that the exponent $\alpha + d/r$ in (1.10) cannot be replaced by $\alpha + d/r - 1 - \epsilon$ for any $\epsilon > 0$. The polynomial norm estimate in (1.10) is a noncommutative extension about convolution powers of a complex function on \mathbb{Z}^d , cf. [17,41,60,59] and (6.4). The power estimate in (1.10) is also applicable to estimate the probability $\Pr(X_n = \lambda | X_1 = \lambda')$ of hopping from one agent λ to another agent λ' in a stationary Markov chain $X_n, n \geq 1$, on a spatially distributed network, see Corollary 6.2.

The paper is organized as follows. In Section 2, we recall some concepts on connected simple graphs \mathcal{G} and we provide two basic estimates on their geometry. In Section 3, we introduce novel Banach algebra of matrices $\mathcal{B}_{r,\alpha}(\mathcal{G})$ with $1 \leq r \leq \infty$ and $\alpha \geq 0$, and we prove that they are differential and inverse-closed subalgebra of $\mathcal{B}(\ell^2)$. In Section 4, we establish the equivalence among ℓ^p -stabilities of matrices in the Beurling algebra $\mathcal{B}_{r,\alpha}(\mathcal{G})$ for different exponents $1 \leq p \leq \infty$, and we further show that their lower stability bounds are controlled by some polynomials. In Section 5, we prove that the Beurling algebra $\mathcal{B}_{r,\alpha}(\mathcal{G})$ admit norm-controlled inversion and a polynomial can be selected to be the norm-controlled function h in (1.7). In Section 6, we consider noncommutative extension

of convolution powers and show that norms of powers $A^n, n \geq 1$, of a matrix $A \in \mathcal{B}_{r,\alpha}$ with $1 \leq r \leq \infty$ and $\alpha > d(1 - 1/r)$ are dominated by a polynomial.

Notation: \mathbb{Z}_+ contains all nonnegative integers, for a real number t, denote by $\lfloor t \rfloor$, $\lceil t \rceil$, $t_+ = \max(t,0)$ and $t_- = \min(t,0)$ its greatest preceding integer, least succeeding integers, positive part and negative part respectively, and for a set F denote its cardinality and characteristic function by #F and χ_F respectively. In this paper, the capital letter C is an absolute constant which is not necessarily the same at each occurrence.

2. Preliminaries on connected simple graphs

In this section, we recall some concepts on connected simple graphs, and we establish two estimates about geometry of a connected simple graph with finite Beurling dimension.

In this paper, we assume that the underlying graph $\mathcal{G} := (V, E)$ is connected and simple. For the graph to describe an SDN distributed over a spatial domain with direct communication between agents via signal broadcasting within a finite range, where each vertex represents an agent and every edge shows that agents at the two ends of that edge have their distance in the spatial domain not greater than their communication range. Therefore the assumptions on its connectivity and simpleness are satisfied if agents in the SDN can communicate across the entire network either directly or through a chain of intermediate agents, agents of those SDNs use identical communication modules with identical communication range, and communication components are not used for internal data transmission [11].

Denote by ρ the geodesic distance on \mathcal{G} , which is the nonnegative function on $V \times V$ such that $\rho(\lambda, \lambda) = 0$ for all vertices $\lambda \in V$, and $\rho(\lambda, \lambda')$ is the number of edges in a shortest path connecting distinct vertices $\lambda, \lambda' \in V$ ([15]). For an SDN where two distinct agents exchange information through a chain of intermediate agents connecting them via a shortest path, the geodesic distance on the underlying graph is widely used to measure the communication cost/delay to data transmission.

The geodesic distance ρ on \mathcal{G} is a metric on V. For a circular graph $\mathcal{Z}_N := (\mathbb{Z}/N\mathbb{Z}, W)$ with $W = \{w_l, 1 \leq l \leq L\} \subset \mathbb{Z}/N\mathbb{Z}$, one may verify that it is simple and connected if and only if $N\mathbb{Z} \cap W = \emptyset$ and any integer in $\mathbb{Z}/N\mathbb{Z}$ is a linear combination of elements in W with integer coefficients, and also that the geodesic distance $\rho(m, n)$ between distinct integers $m, n \in \mathbb{Z}/N\mathbb{Z}$ is

$$\rho(m,n) = \min_{m-n-\sum_{k=1}^{L} k_l w_l \in N\mathbb{Z}, (k_1,\dots,k_L) \in \mathbb{Z}^L} \sum_{l=1}^{L} |k_l|.$$

The graph \mathcal{Z}^d is simple and connected, and the geodesic distance between $\mathbf{m} = (m_1, \dots, m_d)$ and $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ is given by $\rho(\mathbf{m}, \mathbf{n}) = \sum_{i=1}^d |m_i - n_i|$.

With the geodesic distance ρ on $\mathcal{G} := (V, E)$, we denote the closed ball with center $\lambda \in V$ and radius R by

$$B(\lambda, R) := \{ \lambda' \in V : \rho(\lambda, \lambda') \le R \},$$

and the counting measure on V by μ , where $\mu(F)$ is the number of vertices in F for any $F \subset V$. The counting measure μ is said to be a doubling measure ([34,62]) if there exists a positive constant $D_0(\mathcal{G})$ such that

$$\mu(B(\lambda, 2R)) \le D_0(\mathcal{G})\mu(B(\lambda, R))$$
 for all $\lambda \in V$ and $R \ge 0$. (2.1)

The minimal constant $D_0(\mathcal{G})$ in (2.1) is known as the doubling constant of the measure μ . Clearly, the doubling constant $D_0(\mathcal{G})$ dominates maximal node degree of the graph \mathcal{G} , i.e., $d_{\max}(\mathcal{G}) \leq D_0(\mathcal{G})$. Under the doubling assumption to the measure μ , the triple (V, ρ, μ) is a space of homogeneous type [34,62]. For a communication graph to describe an SDN, the doubling property for its counting measure means that number of agents in the R-neighborhood and 2R-neighborhood of each agent are comparable for all $R \geq 0$, and that every agent communicates directly with at most $D_0(\mathcal{G})$ agents in the network.

We say that the counting measure μ on the graph \mathcal{G} has polynomial growth if there exist positive constants $D_1(\mathcal{G})$ and $d := d(\mathcal{G})$ such that

$$\mu(B(\lambda, R)) \le D_1(\mathcal{G})(R+1)^d \text{ for all } \lambda \in V \text{ and } R \ge 0.$$
 (2.2)

The minimal constants d and $D_1(\mathcal{G})$ in (2.2) are called as the Beurling dimension and density of the graph \mathcal{G} respectively ([11]). The circular graph $\mathcal{Z}_N := (\mathbb{Z}/N\mathbb{Z}, W)$ and the lattice graph \mathcal{Z}^d have their Beurling dimension being one and the Euclidean dimension d respectively. We remark that a simple graph \mathcal{G} with its counting measure μ satisfying the doubling condition (2.1) has finite Beurling dimension,

$$\mu(B(\lambda, R)) \le D_0(\mathcal{G})(R+1)^{\log_2 D_0(\mathcal{G})}$$
 for all $\lambda \in V$ and $R \ge 0$,

where $D_0(\mathcal{G})$ is the doubling constant of the measure μ . However, the Beurling dimension d of the graph \mathcal{G} is usually much smaller than $\log_2 D_0(\mathcal{G})$ in the above estimate.

We say that the counting measure μ on the graph \mathcal{G} is normal if there exist $D_1(\mathcal{G})$ and $D_2(\mathcal{G})$ such that

$$D_2(\mathcal{G})(R+1)^d \le \mu(B(\lambda, R)) \le D_1(\mathcal{G})(R+1)^d$$
 (2.3)

for all $\lambda \in V$ and $0 \le R \le \text{diam}(\mathcal{G})$, the diameter of the graph \mathcal{G} . A normal measure has the doubling property (2.1) and the polynomial growth property (2.2), see [30,34,61,62] and references therein. We remark that the circular graph $\mathcal{Z}_N := (\mathbb{Z}/N\mathbb{Z}, W)$ and the lattice graph \mathcal{Z}^d have normal counting measures. For SDNs with agents distributed at $\mathbb{Z}^d + [-1/2, 1/2)^d$ and signal broadcasting range not less than $2\sqrt{d}$, one may verify that graphs to describe those SDNs have normal counting measures.

We conclude this section with a proposition on geometry of a connected simple graph with finite Beurling dimension.

Proposition 2.1. Let $\mathcal{G} := (V, E)$ be a connected simple graph with Beurling dimension d, and $h := \{h(n)\}_{n=0}^{\infty}$ be a positive decreasing sequence. Then the following statements hold.

(i) For any vertex $\lambda \in V$ and integer $s \geq 0$,

$$\sum_{\rho(\lambda,\lambda')\leq s} \rho(\lambda,\lambda')h(\rho(\lambda,\lambda')) \leq (d+1)D_1(\mathcal{G})\sum_{n=0}^s h(n)(n+1)^d.$$
 (2.4)

(ii) If $\sum_{n=0}^{\infty} h(n)(n+1)^{d-1} < \infty$, then

$$\sum_{\rho(\lambda,\lambda')>s} h(\rho(\lambda,\lambda')) \le D_1(\mathcal{G}) \left((s+1)^d h(s) + d \sum_{n=s+1}^{\infty} h(n)(n+1)^{d-1} \right)$$
 (2.5)

for any vertex $\lambda \in V$ and integer $s \geq 0$.

Proof. (i). Given a vertex $\lambda \in V$ and an integer $s \geq 0$, we obtain

$$\sum_{\rho(\lambda,\lambda') \leq s} \rho(\lambda,\lambda') h(\rho(\lambda,\lambda'))$$

$$= \sum_{n=0}^{s} nh(n) (\mu(B(\lambda,n)) - \mu(B(\lambda,n-1)))$$

$$= sh(s)\mu(B(\lambda,s)) + \sum_{n=0}^{s-1} \mu(B(\lambda,n)) (nh(n) - (n+1)h(n+1))$$

$$\leq D_1(\mathcal{G}) \Big(sh(s)(s+1)^d + \sum_{n=0}^{s-1} n(h(n) - h(n+1))(n+1)^d \Big)$$

$$= D_1(\mathcal{G}) \sum_{n=1}^{s} h(n) (n(n+1)^d - (n-1)n^d),$$

where the inequality follows from the polynomial growth property (2.2) and the monotonic assumption on the nonnegative sequence $\{h(n)\}_{n=0}^{\infty}$. Hence (2.4) follows.

(ii). Take a vertex $\lambda \in V$ and an integer $s \geq 0$. Similar to the first argument, we have

$$\sum_{\rho(\lambda,\lambda')\geq s} h(\rho(\lambda,\lambda'))$$

$$\leq \lim_{N\to\infty} h(N)\mu(B(\lambda,N)) + \sum_{n=s}^{N-1} \mu(B(\lambda,n))(h(n) - h(n+1))$$

$$\leq D_1(\mathcal{G}) \lim_{N\to\infty} \left(h(N)(N+1)^d + \sum_{n=s}^{N-1} (n+1)^d (h(n) - h(n+1))\right)$$

$$= D_1(\mathcal{G})\Big((s+1)^d h(s) + \sum_{n=s+1}^{\infty} h(n)((n+1)^d - n^d)\Big).$$

This proves (2.5). \square

3. Matrices with polynomial off-diagonal decay

Let \mathcal{G} be a connected simple graph with Beurling dimension d. For $1 \leq r \leq \infty$ and $\alpha \geq 0$, define

$$\mathcal{B}_{r,\alpha}(\mathcal{G}) := \left\{ A = \left(a(\lambda, \lambda') \right)_{\lambda, \lambda' \in V} : \|A\|_{\mathcal{B}_{r,\alpha}} < \infty \right\}, \tag{3.1}$$

where $h_A(n) = \sup_{\rho(\lambda, \lambda') \ge n} |a(\lambda, \lambda')|, n \ge 0$, and

$$||A||_{\mathcal{B}_{r,\alpha}} := \begin{cases} \left(\sum_{n=0}^{\infty} |h_A(n)|^r (n+1)^{\alpha r + d - 1} \right)^{1/r} & \text{if } 1 \le r < \infty \\ \sup_{n > 0} h_A(n) (n+1)^{\alpha} & \text{if } r = \infty, \end{cases}$$
(3.2)

[7,8,11,28,54]. We will use the abbreviated notation $\mathcal{B}_{r,\alpha}$ instead of $\mathcal{B}_{r,\alpha}(\mathcal{G})$ if there is no confusion. The commutative subalgebra

$$A^* := \left\{ \left(a(k - k') \right)_{k, k' \in \mathbb{Z}} : \sum_{n=0}^{\infty} \sup_{|k| \ge n} |a(k)| < \infty \right\}$$
 (3.3)

of the class $\mathcal{B}_{r,\alpha}(\mathcal{G})$ with $r=1, \alpha=0$ and $\mathcal{G}=\mathcal{Z}$ was introduced by Beurling to study contraction of periodic functions [8]. The set $\mathcal{B}_{r,\alpha}(\mathcal{G})$ with $r=+\infty, \alpha\geq 0$ and $\mathcal{G}=\mathcal{Z}$ is the Jaffard class $\mathcal{J}_{\alpha}(\mathcal{Z})$ of matrices with polynomial off-diagonal decay ([11,28]), since

$$||A||_{\mathcal{J}_{\alpha}} := \sup_{i,j \in \mathbb{Z}} |a(i,j)| (1+|i-j|)^{\alpha} = ||A||_{\mathcal{B}_{\infty,\alpha}} \text{ for } A := (a(i,j))_{i,j \in \mathbb{Z}}.$$

The set $\mathcal{B}_{r,\alpha}(\mathcal{G})$ with $1 \leq r < \infty, \alpha \geq 0$ and $\mathcal{G} = \mathcal{Z}$ is defined in [54] to contain all matrices $A = (a(i,j))_{i,j \in \mathbb{Z}}$ with the following norm

$$||A||_{\mathcal{B}_{r,\alpha}}^* = \left(\sum_{n=0}^{\infty} \left(\sup_{|i-j| \ge n} |a(i,j)| (1+|i-j|)^{\alpha}\right)^r\right)^{1/r} < \infty.$$
 (3.4)

We remark that norms in (3.2) and (3.4) are equivalent to each other,

$$||A||_{\mathcal{B}_{r,\alpha}} \le ||A||_{\mathcal{B}_{r,\alpha}}^* \le 2^{2(\alpha+1/r)} ||A||_{\mathcal{B}_{r,\alpha}} \text{ for all } A \in \mathcal{B}_{r,\alpha}(\mathcal{Z}^d).$$
 (3.5)

The first inequality in (3.5) follows immediately from (3.2) and (3.4), while the second estimate holds because for any $A := (a(i,j))_{i,j\in\mathbb{Z}} \in \mathcal{B}_{r,\alpha}(\mathcal{Z})$ we have

$$(\|A\|_{\mathcal{B}_{r,\alpha}}^*)^r \leq \sup_{i,j \in \mathbb{Z}} |a(i,j)|^r (1+|i-j|)^{\alpha r}$$

$$+ \sum_{l=0}^{\infty} 2^l \Big(\sup_{|i-j| \geq 2^l} |a(i,j)| (1+|i-j|)^{\alpha} \Big)^r$$

$$\leq (h_A(0))^r + \sum_{m=0}^{\infty} (h_A(2^m))^r 2^{(m+1)\alpha r}$$

$$+ \sum_{l=0}^{\infty} 2^l \sum_{m=l}^{\infty} (h_A(2^m))^r 2^{(m+1)\alpha r}$$

$$\leq (h_A(0))^r + 2^{2\alpha r + 2} \sum_{m=0}^{\infty} (h_A(2^m))^r 2^{(m-1)(\alpha r + 1)}$$

$$\leq 2^{2\alpha r + 2} (\|A\|_{\mathcal{B}_{r,\alpha}})^r.$$

Due to the above equivalence (3.5), we follow the terminology in [54] to call $\mathcal{B}_{r,\alpha}$ as a Beurling class of matrices with polynomial off-diagonal decay.

Given any $1 \leq r \leq \infty$ and $\alpha \geq 0$, one may verify from the definition (3.1) that the Beurling class $\mathcal{B}_{r,\alpha}$ with norm $\|\cdot\|_{\mathcal{B}_{r,\alpha}}$ is a *solid Banach space*. Here solidness of the Banach space $\mathcal{B}_{r,\alpha}$ means that

$$||A||_{\mathcal{B}_{r,\alpha}} \le ||B||_{\mathcal{B}_{r,\alpha}} \tag{3.6}$$

hold for all matrices $A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in V}$ and $B = (b(\lambda, \lambda'))_{\lambda, \lambda' \in V}$ satisfying $|a(\lambda, \lambda')| \leq |b(\lambda, \lambda')|$ for all $\lambda, \lambda' \in V$.

Banach spaces $\mathcal{B}_{r,\alpha}$ and $\mathcal{B}_{r',\alpha'}$ with different exponents $1 \leq r, r' \leq \infty$ and $\alpha, \alpha' \geq 0$ are closely related.

Proposition 3.1. Let $1 \le r \le r' \le \infty$ and $\alpha, \alpha' \ge 0$, and $\mathcal{G} := (V, E)$ be a connected simple graph with Beurling dimension d. Then the following statements hold.

(i) If $\alpha' > \alpha + d(1/r - 1/r')$, then $\mathcal{B}_{r',\alpha'}(\mathcal{G})$ is a Banach subspace of $\mathcal{B}_{r,\alpha}(\mathcal{G})$, and

$$||A||_{\mathcal{B}_{r,\alpha}} \le \left(\frac{\alpha' - \alpha - (d-1)(1/r - 1/r')}{\alpha' - \alpha - d(1/r - 1/r')}\right)^{1/r - 1/r'} ||A||_{\mathcal{B}_{r',\alpha'}}$$
(3.7)

hold for all $A \in \mathcal{B}_{r',\alpha'}(\mathcal{G})$.

(ii) If $\alpha' \leq \alpha + d(1/r - 1/r')$, then $\mathcal{B}_{r',\alpha'}(\mathcal{G})$ is a Banach supspace of $\mathcal{B}_{r,\alpha}(\mathcal{G})$. Moreover for all $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ we have

$$||A||_{\mathcal{B}_{r',\alpha'}} \le ||A||_{\mathcal{B}_{r,\alpha}} \tag{3.8}$$

if r' = r, and

$$||A||_{\mathcal{B}_{r',\alpha'}} \le (\alpha r + d)^{1/r - 1/r'} ||A||_{\mathcal{B}_{r,\alpha}}$$
(3.9)

if r' > r.

Proof. (i). Take $A := (a(\lambda, \lambda'))_{\lambda, \lambda' \in V} \in \mathcal{B}_{r', \alpha'}$ with $r' = \infty$. The conclusion (3.7) is obvious for $r = r' = \infty$. For $1 \le r < \infty$, we have

$$||A||_{\mathcal{B}_{r,\alpha}}^r \le ||A||_{\mathcal{B}_{\infty,\alpha'}}^r \sum_{n=0}^{\infty} (n+1)^{(\alpha-\alpha')r+d-1} \le \frac{\alpha'-\alpha'-d/r+1/r}{\alpha'-\alpha-d/r} ||A||_{\mathcal{B}_{\infty,\alpha'}}^r.$$

This proves (3.7) with $1 \le r < r' = \infty$. We can use similar argument to prove (3.7) with $r \le r' < \infty$.

(ii). The conclusion (3.8) is obvious. Now we prove (3.9). Take $A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in V} \in \mathcal{B}_{r,\alpha}$ and set $h_A(n) = \sup_{\rho(\lambda, \lambda') \geq n} |a(\lambda, \lambda')|, n \geq 0$. Then for $r' = \infty$, we obtain from monotonicity of the nonnegative sequence $\{h_A(n)\}_{n=0}^{\infty}$ that

$$||A||_{\mathcal{B}_{r',\alpha'}}^{r} \leq ||A||_{\mathcal{B}_{\infty,\alpha+d/r}}^{r} = \sup_{N\geq 0} (h_{A}(N))^{r} (N+1)^{\alpha r+d}$$

$$\leq (\alpha r + d)(h_{A}(N))^{r} \int_{0}^{N+1} t^{\alpha r+d-1} dt$$

$$\leq (\alpha r + d) \sum_{n=0}^{N} (h_{A}(n))^{r} (n+1)^{\alpha r+d-1} \leq (\alpha r + d) ||A||_{\mathcal{B}_{r,\alpha}}^{r}.$$
(3.10)

Similarly for $r < r' < \infty$,

$$||A||_{\mathcal{B}_{r',\alpha'}}^{r'} \leq ||A||_{\mathcal{B}_{\infty,\alpha+d/r}}^{r'-r} \sum_{n=0}^{\infty} (h_A(n))^r (n+1)^{(\alpha'-\alpha-d(1/r-1/r'))r'+\alpha r+d-1}$$

$$\leq (\alpha r + d)^{r'/r-1} ||A||_{\mathcal{B}_{r,\alpha}}^{r'-r} \sum_{n=0}^{\infty} (h_A(n))^r (n+1)^{\alpha r+d-1}$$

$$= (\alpha r + d)^{r'/r-1} ||A||_{\mathcal{B}_{r,\alpha}}^{r'}, \tag{3.11}$$

where the second inequality follows from (3.10). Combining (3.10) and (3.11) proves (3.9). \Box

A matrix $A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in V}$ is said to have bandwidth N if

$$a(\lambda, \lambda') = 0 \text{ if } \rho(\lambda, \lambda') > N.$$
 (3.12)

Then a matrix A with finite bandwidth belongs to Beurling families $\mathcal{B}_{r,\alpha}$, $1 \leq r \leq \infty$, $\alpha \geq 0$ if and only if it has bounded entries (i.e., $||A||_{\mathcal{B}_{\infty,0}} < \infty$), since for a matrix A with bandwidth N,

$$(N+1)^{(\alpha-\alpha')_{-}} \|A\|_{\mathcal{B}_{r',\alpha'}} \le \|A\|_{\mathcal{B}_{r,\alpha}} \le (N+1)^{(\alpha-\alpha')_{+} + d(1/r - 1/r')} \|A\|_{\mathcal{B}_{r',\alpha'}}$$

where $1 \leq r \leq r' \leq \infty$ and $\alpha, \alpha' \geq 0$. On the other hand, any matrix $A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in V} \in \mathcal{B}_{r,\alpha}$ with $1 \leq r < \infty$ and $\alpha \geq 0$ is well approximated by their truncations,

$$A_N = \left(a(\lambda, \lambda') \chi_{[0,1]}(\rho(\lambda, \lambda')/N) \right)_{\lambda, \lambda' \in G}, \ N \ge 0, \tag{3.13}$$

(and hence by bounded matrices with finite bandwidth), since

$$||A - A_N||_{\mathcal{B}_{r,\alpha}}^r = (h_A(N+1))^r \sum_{n=0}^N (n+1)^{\alpha r + d - 1}$$

$$+ \sum_{n=N+1}^\infty (h_A(n))^r (n+1)^{d-1}$$

$$\leq (2^{\alpha r + d} - 1) \sum_{n \geq N/2} (h_A(n))^r (n+1)^{\alpha r + d - 1} \to 0$$
(3.14)

as $N \to \infty$, where $h_A(n) = \sup_{\rho(\lambda, \lambda') \ge n} |a(\lambda, \lambda')|, n \ge 0$, the limit holds by the assumption $||A||_{\mathcal{B}_{r,\alpha}} < \infty$, and the inequality follows from the monotonicity of $\{h_A(n)\}_{n=0}^{\infty}$ and the estimate

$$\sum_{n=0}^{N} (n+1)^{\alpha r + d - 1} \le \left(2^{\alpha r + d} - 1\right) \sum_{N/2 \le n \le N} (n+1)^{\alpha r + d - 1}.$$

This proves the following proposition about well approximation of matrices with finite bandwidth in Beurling classes of matrices.

Proposition 3.2. Let $1 \leq r < \infty, \alpha \geq 0$, and $\mathcal{G} := (V, E)$ be a connected simple graph with Beurling dimension d. Then the set of all matrices in $\mathcal{B}_{r,\alpha}(\mathcal{G})$ with finite bandwidth is dense in $\mathcal{B}_{r,\alpha}(\mathcal{G})$.

We remark that the approximation property in the above proposition does not hold in general for Beurling classes $\mathcal{B}_{r,\alpha}$ with $r = \infty$ and $\alpha \geq 0$. For instance, for the lattice graph \mathcal{Z}^d , the matrix $A = ((1+|i-j|)^{-\alpha})_{i,j\in\mathbb{Z}^d}$ belongs to $\mathcal{B}_{\infty,\alpha}(\mathcal{Z}^d)$, while it cannot be approximated by matrices with finite bandwidth in the norm $\|\cdot\|_{\mathcal{B}_{\infty,\alpha}}$.

The Beurling class $\mathcal{B}_{1,0}$ plays a special role in Beurling family $\mathcal{B}_{r,\alpha}$, $1 \leq r \leq \infty$, $\alpha \geq 0$, of matrices. It is a Banach subalgebra of the Schur algebra \mathcal{S} under matrix multiplication.

Here the Schur algebra $\mathcal{S} := \mathcal{S}(\mathcal{G})$ contain all matrices $A := (a(\lambda, \lambda'))_{\lambda, \lambda' \in V}$ with finite Schur norm

$$\|A\|_{\mathcal{S}} = \max \left(\sup_{\lambda \in V} \sum_{\lambda' \in V} |a(\lambda, \lambda')|, \sup_{\lambda' \in V} \sum_{\lambda \in V} |a(\lambda, \lambda')| \right) < \infty, \tag{3.15}$$

which is a Banach subalgebra of $\mathcal{B}(\ell^q)$, $1 \leq q \leq \infty$.

Proposition 3.3. Let $1 \leq q \leq \infty$ and $\mathcal{G} := (V, E)$ be a connected simple graph with Beurling dimension d. Then $\mathcal{B}_{1,0}(\mathcal{G})$ is a Banach subalgebra of the Schur algebra $\mathcal{S}(\mathcal{G})$ and hence also a Banach subalgebra of $\mathcal{B}(\ell^q), 1 \leq q \leq \infty$. Moreover,

$$||A||_{\mathcal{B}(\ell^q)} \le ||A||_{\mathcal{S}} \le dD_1(\mathcal{G})||A||_{\mathcal{B}_{1,0}},$$
 (3.16)

and

$$||AB||_{\mathcal{B}_{1,0}(\mathcal{G})} \le 2^{d+1} dD_1(\mathcal{G}) ||A||_{\mathcal{B}_{1,0}(\mathcal{G})} ||B||_{\mathcal{B}_{1,0}(\mathcal{G})}$$
(3.17)

for all $A, B \in \mathcal{B}_{1,0}(\mathcal{G})$.

Proof. The first inequality in (3.16) is well known. Take $A := (a(\lambda, \lambda'))_{\lambda, \lambda' \in V} \in \mathcal{B}_{1,0}$. Then it follows from Proposition 2.1 that

$$\begin{split} & \sum_{\lambda' \in V} |a(\lambda, \lambda')| \leq \sum_{\lambda' \in V} h_A(\rho(\lambda, \lambda')) \\ & \leq D_1(\mathcal{G}) \left(h_A(0) + d \sum_{n=1}^{\infty} h_A(n) (n+1)^{d-1} \right) \leq dD_1(\mathcal{G}) \|A\|_{\mathcal{B}_{1,0}}, \end{split}$$

where $h_A(n) = \sup_{\rho(\lambda,\lambda') \geq n} |a(\lambda,\lambda')|, n \geq 0$. This proves the second estimate in (3.16). We follow the argument in [54] to prove (3.17), where the conclusion with $\mathcal{G} = \mathcal{Z}^d$ is proved. Take $A := (a(\lambda,\lambda'))_{\lambda,\lambda' \in V}$ and $B := (b(\lambda,\lambda'))_{\lambda,\lambda' \in V} \in \mathcal{B}_{r',\beta}$, and write $AB := (c(\lambda,\lambda'))_{\lambda,\lambda' \in V}$. Then for all $\lambda,\lambda' \in V$ we have

$$|c(\lambda, \lambda')| \leq \sum_{\lambda'' \in V} |a(\lambda, \lambda'')| |b(\lambda'', \lambda')|$$

$$\leq h_A(\lfloor \rho(\lambda, \lambda')/2 \rfloor) \sum_{\lambda'' \in V} |b(\lambda'', \lambda')|$$

$$+h_B(\lfloor \rho(\lambda, \lambda')/2 \rfloor) \sum_{\lambda'' \in V} |a(\lambda, \lambda'')|$$

$$\leq ||B||_{\mathcal{S}} h_A(\lfloor \rho(\lambda', \lambda)/2 \rfloor) + ||A||_{\mathcal{S}} h_B(\lfloor \rho(\lambda, \lambda')/2 \rfloor), \tag{3.18}$$

where $h_A(n) = \sup_{\rho(\lambda, \lambda') \geq n} |a(\lambda, \lambda')|$ and $h_B(n) = \sup_{\rho(\lambda, \lambda') \geq n} |b(\lambda, \lambda')|, n \in \mathbb{Z}_+$. Therefore

$$||AB||_{\mathcal{B}_{1,0}} \le ||B||_{\mathcal{S}} \sum_{n=0}^{\infty} h_A(\lfloor n/2 \rfloor) (n+1)^{\alpha+d-1} + ||A||_{\mathcal{S}} \sum_{n=0}^{\infty} h_B(\lfloor n/2 \rfloor) (n+1)^{\alpha+d-1} \le 2^{\alpha+d+1} dD_1(\mathcal{G}) ||A||_{\mathcal{B}_{1,0}} ||B||_{\mathcal{B}_{1,0}},$$
(3.19)

where the first and second inequality follows from (3.18) and (3.16) respectively. \Box

In this paper, we mainly consider the Beurling class of matrices $\mathcal{B}_{r,\alpha}$ with $1 \leq r \leq \infty$ and $\alpha > d(1-/r)$. By Propositions 3.1 and 3.3, all of them are Banach subspaces of $\mathcal{B}_{1,0}$. In the following proposition, we show that they are Banach subalgebras of $\mathcal{B}_{1,0}$ (and hence the Schur algebra \mathcal{S} and the operator algebra $\mathcal{B}(\ell^q), 1 \leq q \leq \infty$) under matrix multiplication.

Proposition 3.4. Let $\mathcal{G} := (V, E)$ be a connected simple graph with Beurling dimension d. Then for any $1 \le r \le \infty$ and $\alpha > d(1 - /r)$, $\mathcal{B}_{r,\alpha}(\mathcal{G})$ is a Banach algebra. Moreover,

$$||AB||_{\mathcal{B}_{r,\alpha}} \le 2^{\alpha+1+d/r} dD_1(\mathcal{G}) \left(\frac{\alpha - (d-1)(1-1/r)}{\alpha - d(1-1/r)}\right)^{1-1/r} ||A||_{\mathcal{B}_{r,\alpha}} ||B||_{\mathcal{B}_{r,\alpha}}$$
(3.20)

hold for all $A, B \in \mathcal{B}_{r,\alpha}(\mathcal{G})$.

Proof. Take $A, B \in \mathcal{B}_{r,\alpha}$. Following the argument used in the proof of Proposition 3.3, we have

$$||AB||_{\mathcal{B}_{r,\alpha}} \le 2^{\alpha + d/r} (||B||_{\mathcal{S}} ||A||_{\mathcal{B}_{r,\alpha}} + ||A||_{\mathcal{S}} ||B||_{\mathcal{B}_{r,\alpha}}). \tag{3.21}$$

This together with (3.7) and (3.16) proves (3.20). \square

By Propositions 3.1 and 3.2, matrices in $\mathcal{B}_{1,0}$ and $\mathcal{B}_{r,\alpha}$, $1 \leq r < \infty$ are well approximated by bounded matrices with finite bandwidth in the Schur norm. In the following proposition, we show that given any matrix in a Banach algebra $\mathcal{B}_{r,\alpha}$ with $1 \leq r \leq \infty$ and $\alpha > d(1-1/r)$, the approximation order by matrices with finite bandwidth in the Schur norm (and hence by the operator norm in $\mathcal{B}(\ell^q)$, $1 \leq q \leq \infty$ by (3.16)) is $\alpha - d(1-/r) > 0$.

Proposition 3.5. Let $\mathcal{G} := (V, E)$ be a connected simple graph with Beurling dimension d. Take $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ and define $A_N, N \geq 0$, as in (3.13), where $1 \leq r \leq \infty$ and $\alpha > d(1 - /r)$. Then

$$||A - A_N||_{\mathcal{S}} \le \frac{dD_1(\mathcal{G})}{(\alpha - d(1 - 1/r))^{1 - 1/r}} ||A||_{\mathcal{B}_{r,\alpha}} (N + 1)^{-\alpha + d(1 - 1/r)}, N \ge 0.$$
 (3.22)

Proof. Take $A := (a(\lambda, \lambda'))_{\lambda, \lambda' \in V} \in \mathcal{B}_{r,\alpha}$. By Proposition 2.1, we have

$$||A - A_N||_{\mathcal{S}} \le D_1(\mathcal{G}) \Big((N+2)^d h_A(N+1) + d \sum_{n=N+2}^{\infty} h_A(n)(n+1)^{d-1} \Big),$$

where $h_A(n) = \sup_{\rho(\lambda,\lambda') > n} |a(\lambda,\lambda')|, n \geq 0$. Therefore

$$||A - A_N||_{\mathcal{S}} \le dD_1(\mathcal{G}) ||A||_{\mathcal{B}_{r,\alpha}} (N+1)^{-\alpha}$$

if r = 1. Similarly for $1 < r \le \infty$, we have

$$||A - A_N||_{\mathcal{S}} \le dD_1(\mathcal{G}) ||A||_{\mathcal{B}_{r,\alpha}} \Big(\sum_{n=N+1}^{\infty} (n+1)^{-\alpha r' + d - 1} \Big)^{1/r'}$$

$$\le d(\alpha r' - d)^{-1/r'} D_1(\mathcal{G}) ||A||_{\mathcal{B}_{r,\alpha}} (N+1)^{-\alpha + d/r'}$$
(3.23)

where 1/r' = 1 - 1/r. \Box

By (3.7) and (3.16), Banach algebras $\mathcal{B}_{r,\alpha}$ with $1 \leq r \leq \infty$ and $\alpha > d(1-1/r)$ are subalgebras of $\mathcal{B}(\ell^p)$, $1 \leq p \leq \infty$,

$$||A||_{\mathcal{B}(\ell^p)} \le \frac{\alpha - (d-1)(1-1/r)}{\alpha - d(1-1/r)} dD_1(\mathcal{G}) ||A||_{\mathcal{B}_{r,\alpha}} \text{ for all } A \in \mathcal{B}_{r,\alpha}(\mathcal{G}).$$
 (3.24)

Moreover, following the argument in [54,51] and applying (2.5), we obtain that $\mathcal{B}_{r,\alpha}$ are differential subalgebras of $\mathcal{B}(\ell^2)$.

Proposition 3.6. Let \mathcal{G} be a connected simple graph with Beurling dimension d, $1 \le r \le \infty$, $\alpha > d(1-1/r)$, and set

$$\theta_{r,\alpha} = \frac{2(\alpha - d + d/r)}{1 + 2\alpha - 2d + 2d/r}.$$

Then there exists an absolute constant C such that

$$||AB||_{\mathcal{B}_{r,\alpha}} \le C||A||_{\mathcal{B}_{r,\alpha}} ||B||_{\mathcal{B}_{r,\alpha}} \left(\left(\frac{||A||_{\mathcal{B}(\ell^2)}}{||A||_{\mathcal{B}_{r,\alpha}}} \right)^{\theta_{r,\alpha}} + \left(\frac{||B||_{\mathcal{B}(\ell^2)}}{||B||_{\mathcal{B}_{r,\alpha}}} \right)^{\theta_{r,\alpha}} \right)$$
(3.25)

hold for all $A, B \in \mathcal{B}_{r,\alpha}(\mathcal{G})$.

Applying (3.25) repeatedly and using the argument in [55, Proposition 2.4], we can find an absolute constant C such that

$$||A_1 \dots A_n||_{\mathcal{B}_{r,\alpha}} \le \left(\max_{1 \le k \le n} ||A_k||_{\mathcal{B}(\ell^2)}\right)^n \times \left(C \frac{\max_{1 \le k \le n} ||A_k||_{\mathcal{B}_{r,\alpha}}}{\max_{1 \le k \le n} ||A_k||_{\mathcal{B}(\ell^2)}}\right)^{\frac{\theta_{r,\alpha}}{1+\theta_{r,\alpha}} n^{\log_2(1+\theta_{r,\alpha})}}$$

$$(3.26)$$

for all $A_1, \ldots, A_n \in \mathcal{B}_{r,\alpha}, n \geq 1$. This implies that Banach algebras $\mathcal{B}_{r,\alpha}$ admit norm-controlled inversions in $\mathcal{B}(\ell^2)$.

Corollary 3.7. Let \mathcal{G} be a connected simple graph with Beurling dimension d, and let $1 \leq r \leq \infty$ and $\alpha > d(1-1/r)$. Then matrices in the Banach algebra $\mathcal{B}_{r,\alpha}(\mathcal{G})$ admit norm-controlled inversions in $\mathcal{B}(\ell^2)$.

Proof. Take $A \in \mathcal{B}_{r,\alpha}$ with $A^{-1} \in \mathcal{B}(\ell^2)$. Set $B = I - A^*A/\|A\|_{\mathcal{B}(\ell^2)}^2$. One may verify that

$$||B||_{\mathcal{B}(\ell^2)} \le 1 - (\kappa(A))^{-2} < 1 \text{ and } ||B||_{\mathcal{B}_{r,\alpha}} \le 1 + ||A||_{\mathcal{B}(\ell^2)}^{-2} ||A^*A||_{\mathcal{B}_{r,\alpha}},$$
 (3.27)

where $\kappa(A) = ||A||_{\mathcal{B}(\ell^2)} ||A^{-1}||_{\mathcal{B}(\ell^2)}$. Therefore by (3.20), (3.26) and (3.27), we obtain

$$\begin{split} \|A^{-1}\|_{\mathcal{B}_{r,\alpha}} &= \|(A^*A)^{-1}A^*\|_{\mathcal{B}_{r,\alpha}} \le C\|A^*\|_{\mathcal{B}_{r,\alpha}} \|A\|_{\mathcal{B}(\ell^2)}^{-2} \sum_{n=0}^{\infty} \|B^n\|_{\mathcal{B}_{r,\alpha}} \\ &\le C\|A\|_{\mathcal{B}_{r,\alpha}} \|A\|_{\mathcal{B}(\ell^2)}^{-2} \sum_{n=0}^{\infty} \left(1 - (\kappa(A))^{-2}\right)^n \\ &\times \left(C\frac{1 + \|A\|_{\mathcal{B}(\ell^2)}^{-2} \|A\|_{\mathcal{B}_{r,\alpha}}^2}{1 - (\kappa(A))^{-2}}\right)^{\frac{\theta_{r,\alpha}}{1 + \theta_{r,\alpha}} n^{\log_2(1 + \theta_{r,\alpha})}} < \infty, \end{split}$$

where C is an absolute constant independent of the matrix A. \square

4. ℓ^p -stability bound control

In this section, we prove the following result on lower ℓ^p -stability bounds of matrices in Beurling algebras for different exponent $1 \le p \le \infty$.

Theorem 4.1. Let $1 \leq p, q, r \leq \infty$, r' = r/(r-1), \mathcal{G} be a connected simple graph with Beurling dimension d, the counting measure μ on \mathcal{G} have the doubling property (2.1), and let $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ for some $\alpha > d/r'$. If there exists a positive constant A_p such that

$$||Ac||_p \ge A_p ||c||_p \quad \text{for all } c \in \ell^p, \tag{4.1}$$

then there exists a positive constant A_q such that

$$||Ad||_q \ge A_q ||d||_q \quad \text{for all } d \in \ell^q. \tag{4.2}$$

Moreover, there exists an absolute constant C, independent of matrices $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ and exponents $1 \leq p, q \leq \infty$, such that the lower ℓ^q -stability bound A_q in (4.2) satisfies

$$\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_q} \le C \begin{cases} \left(\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_p}\right)^{(1+\theta(p,q))^{K_0}} & \text{if } \alpha \ne 1 + d/r' \\ \left(\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_p} \ln\left(1 + \frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_p}\right)\right)^{(1+\theta(p,q))^{K_0}} & \text{if } \alpha = 1 + d/r', \end{cases}$$
(4.3)

where

$$\theta(p,q) = \frac{d|1/p - 1/q|}{K_0 \min(\alpha - d/r', 1) - d|1/p - 1/q|}$$

and K_0 is a positive integer with

$$K_0 > \frac{d}{\min(\alpha - d/r', 1)}.$$

To prove Theorem 4.1, we introduce a truncation operator χ_{λ}^{N} and its smooth version Ψ_{λ}^{N} by

$$\chi_{\lambda}^{N}: (c(\lambda'))_{\lambda' \in V} \longmapsto (\chi_{[0,1]}(\rho(\lambda, \lambda')/N)c(\lambda'))_{\lambda' \in V}$$
 (4.4)

and

$$\Psi_{\lambda}^{N}: (c(\lambda'))_{\lambda' \in V} \longmapsto (\psi_{0}(\rho(\lambda, \lambda')/N)c(\lambda'))_{\lambda' \in V}, \tag{4.5}$$

where ψ_0 is the trapezoid function given by

$$\psi_0(t) = \begin{cases} 1 & \text{if } |t| \le 1/2\\ 2 - 2|t| & \text{if } 1/2 < |t| \le 1\\ 0 & \text{if } |t| > 1. \end{cases}$$

The truncation operator χ_{λ}^{N} and its smooth version Ψ_{λ}^{N} localize a vector to the N-neighborhood of the vertex λ , and it can also be considered as diagonal matrices with diagonal entries $\chi_{[0,1]}(\rho(\lambda,\lambda')/N) = \chi_{B(\lambda,N)}(\lambda')$ and $\psi_{0}(\rho(\lambda,\lambda')/N), \lambda' \in V$, respectively. Our proof of Theorem 4.1 depends on the estimate (4.19) for the commutator between a matrix in the Beurling algebra and the truncation operator Ψ_{λ}^{N} . Similar estimate has been used by Sjöstrand in [48] to establish inverse-closedness of the Baskakov–Gohberg–Sjöstrand subalgebra in $\mathcal{B}(\ell^{2})$.

To prove Theorem 4.1, we recall maximal N-disjoint subsets $V_N \subset V, N \geq 1$, which satisfy

$$B(\lambda, N) \cap (\bigcup_{\lambda_m \in V_N} B(\lambda_m, N)) \neq \emptyset$$
 for all $\lambda \in V$ (4.6)

and

$$B(\lambda_m, N) \cap B(\lambda_n, N) = \emptyset$$
 for all distinct vertices $\lambda_m, \lambda_n \in V_N$. (4.7)

There are many different ways to construct a maximal N-disjoint set. For instance, we can construct $V_N = \{\lambda_m\}_{m\geq 1}$ as follows: take a $\lambda_1 \in V$ and define λ_m , $m\geq 2$, recursively such that

$$\rho(\lambda_m, \lambda_1) = \min_{\lambda \in A_m} \rho(\lambda, \lambda_1),$$

$$\rho_{\mathcal{G}}(\lambda_m, \lambda_1) = \min\{\rho_{\mathcal{G}}(\lambda, \lambda_1), \lambda \in A_m\},$$

where

$$A_m = \{ \lambda \in V, \ B(\lambda, N) \cap B(\lambda_{m'}, N) = \emptyset, 1 \le m' \le m - 1 \}.$$

Vertices in a maximal N-disjoint set V_N are known as fusion vertices of the graph \mathcal{G} [11]. For a maximal N-disjoint set V_N , the N-neighborhoods $B(\lambda_m, N), \lambda_m \in V_N$, centered at fusion vertices have no common vertices by (4.7). It is shown in [11] that the (2N)-neighborhood $B(\lambda_m, 2N), \lambda_m \in V_N$, is a covering of the set V.

Proposition 4.2. Let $\mathcal{G} := (V, E)$ be a connected simple graph and μ have the doubling property (2.1). If V_N is a maximal N-disjoint subset of V, then

$$1 \leq \inf_{\lambda \in V} \sum_{\lambda_m \in V_N} \chi_{B(\lambda_m, N')}(\lambda)$$

$$\leq \sup_{\lambda \in V} \sum_{\lambda_m \in V_N} \chi_{B(\lambda_m, N')}(\lambda) \leq (D_0(\mathcal{G}))^{\lceil \log_2(2N'/N+1) \rceil}$$
(4.8)

for all $N' \geq 2N$.

To prove Theorem 4.1, we first establish its weak version, the equivalence between ℓ^p and ℓ^q -stabilities of a matrix with small |1/p - 1/q|.

Lemma 4.3. Let $p, r, r', d, \alpha, \mathcal{G}, A$ be as in Theorem 4.1. If $1 \leq q \leq \infty$ satisfies

$$d|1/p - 1/q| < \min(\alpha - d/r', 1), \tag{4.9}$$

then A has ℓ^q -stability, i.e., there exist positive constant A_q such that

$$||Ac||_q \le A_q ||c||_q$$
 for all $c \in \ell^q$.

Furthermore there exists an absolute constant C, independent of matrices $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ and exponents $1 \leq p, q \leq \infty$, such that the optimal lower ℓ^q -stability bound A_q of the matrix A satisfies

$$A_{q} \geq CA_{p} \times \left\{ \begin{pmatrix} \frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_{p}} \end{pmatrix}^{-\theta_{1}(p,q)} & \text{if } \alpha \neq d/r' + 1\\ \left(\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_{p}} \ln\left(1 + \frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_{p}}\right)\right)^{-\theta_{1}(p,q)} & \text{if } \alpha = d/r' + 1, \end{cases}$$

$$(4.10)$$

where

$$\theta_1(p,q) = \frac{d|1/p - 1/q|}{\min(\alpha - d/r', 1) - d|1/p - 1/q|}.$$

Proof. Let $N \geq 2$ be a positive integer chosen later, V_N be a maximal N-disjoint set of fusion vertices satisfying (4.6) and (4.7), and let Ψ^N_{λ} , $\lambda \in V$, be the localization operators in (4.5). Take $c = (c(\lambda))_{\lambda \in V} \in \ell^q$. Applying the covering property (4.8) of $\{B(\lambda_m, 2N), \lambda_m \in V_N\}$, we have

$$||c||_q \le ||(||\Psi_{\lambda_m}^{4N}c||_q)_{\lambda_m \in V_N}||_q$$

Combining it with the polynomial growth property (2.2) and the norm equivalence between $\|\Psi_{\lambda_m}^{4N}c\|_p$ and $\|\Psi_{\lambda_m}^{4N}c\|_q$, we obtain

$$||c||_{q} \leq ||((\mu(B(\lambda_{m}, 4N)))^{(1/q-1/p)_{+}} ||\Psi_{\lambda_{m}}^{4N} c||_{p})_{\lambda_{m} \in V_{N}}||_{q}$$

$$\leq CN^{d(1/q-1/p)_{+}} ||(||\Psi_{\lambda_{m}}^{4N} c||_{p})_{\lambda_{m} \in V_{N}}||_{q}. \tag{4.11}$$

Here in the proof, the capital letter C denotes an absolute constant independent of matrices A, sequences c, integers N, and exponents p and q, which is not necessarily the same at each occurrence.

For $\lambda \in V$, it follows from the ℓ^p -stability (4.1) for the matrix A that

$$A_p \|\Psi_{\lambda}^{4N} c\|_p \le \|A\Psi_{\lambda}^{4N} c\|_p. \tag{4.12}$$

Let $A_N, N \ge 2$, be matrices with finite bandwidth in (3.13). Combining (4.11) and (4.12), we get

$$\begin{aligned} A_{p} \| c \|_{q} &\leq C N^{d(1/q - 1/p) +} \| (\|A\Psi_{\lambda_{m}}^{4N} c\|_{p})_{\lambda_{m} \in V_{N}} \|_{q} \\ &\leq C N^{d(1/q - 1/p) +} (\| (\|(A - A_{N})\Psi_{\lambda_{m}}^{4N} c\|_{p})_{\lambda_{m} \in V_{N}} \|_{q} \\ &+ \| (\|[A_{N}, \Psi_{\lambda_{m}}^{4N}] c\|_{p})_{\lambda_{m} \in V_{N}} \|_{q} + \| (\|\Psi_{\lambda_{m}}^{4N} (A_{N} - A) c\|_{p})_{\lambda_{m} \in V_{N}} \|_{q} \\ &+ \| (\|\Psi_{\lambda_{m}}^{4N} A c\|_{p})_{\lambda_{m} \in V_{N}} \|_{q}), \end{aligned}$$

$$(4.13)$$

where $[A_N, \Psi_{\lambda_m}^{4N}] = A_N \Psi_{\lambda_m}^{4N} - \Psi_{\lambda_m}^{4N} A_N$ is the commutator between A_N and $\Psi_{\lambda_m}^{4N}$ ([48, 54]).

For any $d \in \ell^q$, we obtain from the support property for $\Psi_{\lambda_m}^{2N}$, the equivalence between two norms $\|\chi_{\lambda_m}^{4N}d\|_p$ and $\|\chi_{\lambda_m}^{4N}d\|_q$, the polynomial growth property (2.2) and the covering property in Proposition 4.2 that

$$\begin{split} & \left\| \left(\| \Psi_{\lambda_m}^{4N} d \|_p \right)_{\lambda_m \in V_N} \right\|_q \le \left\| \left(\| \chi_{\lambda_m}^{4N} d \|_p \right)_{\lambda_m \in V_N} \right\|_q \\ & \le \left\| \left(\| \chi_{\lambda_m}^{4N} d \|_q (\mu(B(\lambda_m, 4N))^{(1/p - 1/q)_+})_{\lambda_m \in V_N} \right\|_q \\ & \le C N^{d(1/p - 1/q)_+} \left\| \left(\| \chi_{\lambda_m}^{4N} d \|_q \right)_{\lambda_m \in V_N} \right\|_q \le C N^{d(1/p - 1/q)_+} \| d \|_q. \end{split}$$

This together with (3.16) yields the following three estimates:

$$\| \left(\| \Psi_{\lambda_m}^{4N} A c \|_p \right)_{\lambda_m \in V_N} \|_q \le C N^{d(1/p - 1/q)_+} \| A c \|_q, \tag{4.14}$$

$$\begin{split} & \| \left(\| (A - A_N) \Psi_{\lambda_m}^{4N} c \|_p \right)_{\lambda_m \in V_N} \|_q \le \| A - A_N \|_{\mathcal{S}} \| \left(\| \Psi_{\lambda_m}^{4N} c \|_p \right)_{\lambda_m \in V_N} \|_q \\ & \le C N^{d(1/p - 1/q)_+} \| A - A_N \|_{\mathcal{S}} \| c \|_q, \end{split}$$

$$(4.15)$$

and

$$\| \left(\| \Psi_{\lambda_m}^{4N}(A_N - A)c \|_p \right)_{\lambda_m \in V_N} \|_q \le C N^{d(1/p - 1/q)_+} \| (A_N - A)c \|_q$$

$$\le C N^{d(1/p - 1/q)_+} \| A - A_N \|_{\mathcal{S}} \| c \|_q.$$
(4.16)

Applying similar argument, we obtain

$$\| (\| [A_{N}, \Psi_{\lambda_{m}}^{4N}] c \|_{p})_{\lambda_{m} \in V_{N}} \|_{q}$$

$$\leq (\sup_{\lambda \in V} \| [A_{N}, \Psi_{\lambda}^{4N}] \|_{\mathcal{S}}) \| (\| \chi_{\lambda_{m}}^{5N} c \|_{p})_{\lambda_{m} \in V_{N}} \|_{q}$$

$$\leq C N^{d(1/p - 1/q) +} (\sup_{\lambda \in V} \| [A_{N}, \Psi_{\lambda}^{4N}] \|_{\mathcal{S}}) \| c \|_{q}.$$

$$(4.17)$$

Combining (4.13)–(4.17), we get

$$A_{p}\|c\|_{q} \leq CN^{d|1/p-1/q|} (\|A - A_{N}\|_{\mathcal{S}} + \sup_{\lambda \in V} \|[A_{N}, \Psi_{\lambda}^{4N}]\|_{\mathcal{S}}) \|c\|_{q}$$

$$+ CN^{d|1/p-1/q|} \|Ac\|_{q}.$$

$$(4.18)$$

For any $\lambda \in V$, we have

$$\|[A_N, \Psi_{\lambda}^{4N}]\|_{\mathcal{S}} = \left\| \left(a(\lambda', \lambda'') \chi_{[0,1]} \left(\frac{\rho(\lambda', \lambda'')}{N} \right) \times \left(\psi_0 \left(\frac{\rho(\lambda', \lambda)}{4N} \right) - \psi_0 \left(\frac{\rho(\lambda'', \lambda)}{4N} \right) \right)_{\lambda', \lambda'' \in V} \right\|_{\mathcal{S}}$$

$$\leq \frac{1}{2N} \left\| \left(|a(\lambda', \lambda'')| \rho(\lambda', \lambda'') |\chi_{[0,1]} \left(\frac{\rho(\lambda', \lambda'')}{N} \right) \right)_{\lambda', \lambda'' \in V} \right\|_{\mathcal{S}}$$

$$\leq CN^{-1} \sum_{n=0}^{N} h_A(n)(n+1)^d, \tag{4.19}$$

where the last inequality follows from (2.4). Therefore for any $\lambda \in V$,

$$\begin{aligned} & \|[A_N, \Psi_{\lambda}^{4N}]\|_{\mathcal{S}} \le CN^{-1} \|A\|_{\mathcal{B}_{r,\alpha}} \Big(\sum_{n=0}^{N} (n+1)^{-(\alpha-1)r'+d-1} \Big)^{1/r'} \\ & \le C \|A\|_{\mathcal{B}_{r,\alpha}} \times \begin{cases} N^{-1} & \text{if } \alpha > 1 + d/r' \\ N^{-1} (\ln(N+1))^{1-1/r} & \text{if } \alpha = 1 + d/r' \\ N^{-\alpha+d/r'} & \text{if } \alpha < 1 + d/r'. \end{cases}$$

$$(4.20)$$

For the Schur norm of $A - A_N$, there exists an absolute constant C_0 , independent of $N \geq 1$ and $A \in \mathcal{B}_{r,\alpha}$, such that

$$||A - A_N||_{\mathcal{S}} \le C \Big((N+2)^d h_A(N+1) + d \sum_{n=N+2}^{\infty} h_A(n)(n+1)^{d-1} \Big)$$

$$\le C_0 ||A||_{\mathcal{B}_{r,\alpha}} N^{-\alpha + d/r'}, \tag{4.21}$$

where the first inequality follows from (3.14) and the second inequality is true because

$$\sum_{n=0}^{N+1} (h_A(n))^r (n+1)^{\alpha r+d-1} \ge (h_A(N+1))^r \int_0^{N+2} t^{\alpha r+d-1} dt$$
$$= (\alpha r+d)^{-1} (h_A(N+1))^r (N+2)^{\alpha r+d}$$

and

$$\sum_{n=N+2}^{\infty} (n+1)^{-\alpha r'+d-1} \le \int_{N+2}^{\infty} t^{-\alpha r'+d-1} dt \le \frac{1}{\alpha r'-d} (N+2)^{-\alpha r'+d}.$$

Combining (4.18), (4.20) and (4.21), we obtain

$$\begin{split} A_p \|c\|_q &\leq C_1 \|A\|_{\mathcal{B}_{r,\alpha}} N^{d|1/p-1/q|-\min(\alpha-d/r',1)} \|c\|_q \\ &+ C N^{d|1/p-1/q|} \|Ac\|_q \end{split} \tag{4.22}$$

if $\alpha \neq 1 + d/r'$, and

$$A_{p} \|c\|_{q} \leq C_{1} \|A\|_{\mathcal{B}_{r,\alpha}} N^{d|1/p-1/q|-1} (\ln N)^{1/r'} \|c\|_{q}$$

$$+ C N^{d|1/p-1/q|} \|Ac\|_{q}$$

$$(4.23)$$

if $\alpha = 1 + d/r'$, where C_1 is an absolute constant independent of matrices A, integers N and sequences c.

For $\alpha \neq 1 + d/r'$, replacing N in (4.22) by

$$N_0 = \left[\left(2C_1 \|A\|_{\mathcal{B}_{r,\alpha}} / A_p \right)^{(\min(\alpha - d/r', 1) - d|1/p - 1/q|)^{-1}} \right],$$

we get from (4.9) and (4.23) that

$$A_p \|c\|_q \le \frac{A_p}{2} \|c\|_q + C \left(\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_p}\right)^{\theta_1(p,q)} \|Ac\|_q.$$

This proves (4.10) for $\alpha \neq 1 + d/r'$.

For $\alpha = 1 + d/r'$, set

$$C_2 := \frac{8C_1 ||A||_{\mathcal{B}_{r,\alpha}}}{(1 - d|1/p - 1/q|)A_p} \ge 8$$

and

$$N_1 := \left\lfloor \left(C_2 \ln C_2 \right)^{(1-d|1/p-1/q|)^{-1}} \right\rfloor \ge \frac{1}{2} \left(C_2 \ln C_2 \right)^{(1-d|1/p-1/q|)^{-1}}.$$

Then

$$C_1 \|A\|_{\mathcal{B}_{r,\alpha}} N_1^{d|1/p-1/q|-1} \le \frac{(1-d|1/p-1/q|)A_p}{4\ln C_2}$$
 (4.24)

and

$$N_1 \le (C_2 \ln C_2)^{(1-d|1/p-1/q|)^{-1}} \le C_2^{2(1-d|1/p-1/q|)^{-1}}.$$
(4.25)

Replacing N in (4.23) by N_1 and applying (4.24) and (4.25), we obtain

$$\begin{split} A_p \|c\|_q &\leq \frac{(1-d|1/p-1/q|)A_p}{4\ln C_2} \Big(2(1-d|1/p-1/q|)^{-1}\ln C_2\Big)^{1-1/r} \|c\|_q \\ &+ C(C_2\ln C_2)^{\theta_1(p,q)} \|Ac\|_q \\ &\leq \frac{A_p}{2} \|c\|_q + C\big(C_2\ln C_2\big)^{\theta_1(p,q)} \|Ac\|_q. \end{split}$$

This proves (4.10) for $\alpha = 1 + d/r'$. \square

Having the above technical lemma, we use a bootstrap approach to prove Theorem 4.1, cf. [28,45,54].

Proof of Theorem 4.1. Let K be a positive integer with

$$d|1/p - 1/q|/K < \min(\alpha - d/r', 1).$$

Then $K \leq K_0$. Let $\{p_k\}_{k=0}^K$ be a monotone sequence such that

$$p_0 = p, p_K = q \text{ and } |1/p_k - 1/p_{k+1}| = |1/p - 1/q|/K, 0 \le k \le K - 1.$$

Applying Lemma 4.3 repeatedly, we conclude that A has ℓ^{p_k} -stability for all $1 \le k \le K$. Moreover the lower ℓ^{p_k} -stability bound A_{p_k} satisfies

$$\frac{A_{p_k}}{A_{p_{k+1}}} \le C \begin{cases} \left(\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_{p_k}}\right)^{\theta_K(p,q)} & \text{if } \alpha \ne d/r' + 1\\ \left(\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_{p_k}} \ln\left(1 + \frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_{p_k}}\right)\right)^{\theta_K(p,q)} & \text{if } \alpha = d/r' + 1 \end{cases}$$
(4.26)

for all $0 \le k \le K - 1$, where

$$\theta_K(p,q) = \frac{d|1/p-1/q|}{K\min(\alpha-d/r',1)-d|1/p-1/q|}$$

and C is an absolute constant independent of $A \in \mathcal{B}_{r,\alpha}$.

For $\alpha \neq 1 + d/r'$, we obtain from (4.26) that

$$\frac{A_{p_{k+1}}}{\|A\|_{\mathcal{B}_{r,q}}} \ge C \left(\frac{A_p}{\|A\|_{\mathcal{B}_{r,q}}}\right)^{(1+\theta_K(p,q))^{k+1}}, \ 0 \le k \le K-1.$$
(4.27)

This proves (4.3) for $\alpha \neq 1 + d(1 - 1/r)$.

For $\alpha = 1 + d/r'$, it follows from (4.26) that

$$\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_{p_{k+1}}} \le C \left(\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_{p_k}}\right)^{1+\theta_K(p,q)} \left(\ln\left(1+\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_{p_k}}\right)\right)^{\theta_K(p,q)}, \ 0 \le k \le K-1.$$

Applying the above estimate repeatedly, we obtain

$$\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_{p_k}} \le C \left(\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_p}\right)^{(1+\theta_K(p,q))^k} \left(\ln\left(1 + \frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_p}\right)\right)^{(1+\theta_K(p,q))^k - 1}$$

by induction on $1 \le k \le K$. This proves (4.3) for $\alpha = 1 + d/r'$. \square

5. Norm-controlled inversion

By Corollary 3.7, matrices in Banach algebras $\mathcal{B}_{r,\alpha}$ with $1 \leq r \leq \infty$ and $\alpha > d(1-1/r)$ admit norm-controlled inversions in $\mathcal{B}(\ell^2)$. In this section, we show that a polynomial can be selected to be norm-controlled function h in (1.7) if the counting measure μ on the graph \mathcal{G} is normal.

Theorem 5.1. Let $1 \le r \le \infty$, r' = r/(r-1), $\alpha > d/r'$, \mathcal{G} be a connected simple graph with Beurling dimension d and normal counting measure μ , and let $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ be invertible in $\mathcal{B}(\ell^2)$. Then there exists an absolute constant C, independent of A, such that

$$||A^{-1}||_{\mathcal{B}_{r,\alpha}} \leq C||A^{-1}||_{\mathcal{B}(\ell^{2})}(||A^{-1}||_{\mathcal{B}(\ell^{2})}||A||_{\mathcal{B}_{r,\alpha}})^{(\alpha+d/r)/\min(\alpha-d/r',1)} \times \begin{cases} 1 & \text{if } \alpha \neq d/r' + 1\\ \left(\ln\left(||A^{-1}||_{\mathcal{B}(\ell^{2})}||A||_{\mathcal{B}_{r,\alpha}} + 1\right)\right)^{(d+1)/r'} & \text{if } \alpha = d/r' + 1. \end{cases}$$
(5.1)

For invertible matrices $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ with $\alpha > d/r' + 1$, it follows from Theorem 5.1 that

$$||A^{-1}||_{\mathcal{B}_{r,\alpha}} \le C||A^{-1}||_{\mathcal{B}(\ell^2)} (||A^{-1}||_{\mathcal{B}(\ell^2)} ||A||_{\mathcal{B}_{r,\alpha}})^{\alpha+d/r}.$$
(5.2)

A weak version of the above estimate, with the exponent $\alpha + d/r$ in (5.2) replaced by a larger exponent $2\alpha + 2 + 2/(\alpha - 2)$, is established in [25] for matrices in the Jaffard algebra $\mathcal{J}_{\alpha}(\mathcal{Z}) = \mathcal{B}_{\infty,\alpha}(\mathcal{Z})$.

The estimate (5.2) on norm-controlled inversion is almost optimal, as shown in the following example that for any $\epsilon > 0$ there does not exist an absolute constant C_{ϵ} such that

$$||A^{-1}||_{\mathcal{B}_{r,\alpha}} \le C_{\epsilon} ||A^{-1}||_{\mathcal{B}(\ell^2)} (||A^{-1}||_{\mathcal{B}(\ell^2)} ||A||_{\mathcal{B}_{r,\alpha}})^{\alpha+d/r-1-\epsilon}.$$
(5.3)

Example 5.2. Let $1 \le r \le \infty, \alpha > 1 - 1/r$ and $\mathcal{G} = \mathcal{Z}^d$ with d = 1. For sufficiently small $\gamma \in (0,1)$, define $A_{\gamma} = (a_{\gamma}(i,j))_{i,j \in \mathbb{Z}}$ by

$$a_{\gamma}(i,j) = \begin{cases} 1 & \text{if } j = i \\ -e^{-\gamma} & \text{if } j = i+1 \\ 0 & \text{elsewhere.} \end{cases}$$
 (5.4)

Then

$$||A_{\gamma}||_{\mathcal{B}_{r,\alpha}} = (1 + 2^{\alpha r} e^{-\gamma r})^{1/r} \in [2^{\alpha - 1}, 2^{\alpha + 1}].$$
 (5.5)

Observe that A_{γ} is invertible in $\mathcal{B}(\ell^2)$ and its inverse is given by $B_{\gamma} = (b_{\gamma}(i,j))_{i,j \in \mathbb{Z}}$, where

$$b_{\gamma}(i,j) = \begin{cases} e^{-(j-i)\gamma} & \text{if } j \ge i \\ 0 & \text{elsewhere.} \end{cases}$$

Therefore for sufficiently small $\gamma \in (0,1)$, we have

$$||A^{-1}||_{\mathcal{B}(\ell^2)} = (1 - e^{-\gamma})^{-1} \in [\gamma^{-1}, 2\gamma^{-1}]$$
(5.6)

and

$$||A^{-1}||_{\mathcal{B}_{r,\alpha}} = \begin{cases} \left(\sum_{n=0}^{\infty} (n+1)^{\alpha r} e^{-nr\gamma}\right)^{1/r} & \text{if } 1 \le r < \infty \\ \sup_{n \ge 0} (n+1)^{\alpha} e^{-n\gamma} & \text{if } r = \infty \end{cases}$$

$$\in \gamma^{-\alpha - 1/r} [1, 2] \times \begin{cases} r^{-\alpha - 1/r} (\Gamma(\alpha r + 1))^{1/r} & \text{if } 1 \le r < \infty \\ (\alpha/e)^{\alpha} & \text{if } r = \infty, \end{cases}$$
(5.7)

where $\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx$ is the Gamma function. Hence for sufficiently small γ , the left hand side of (5.3) is of order $\gamma^{-\alpha-1/r}$ and the right hand side of (5.3) is of order $\gamma^{-\alpha-1/r+\epsilon}$ for $\alpha > 1 + d/r'$. This proves (5.3).

To prove Theorem 5.1, we need a distribution property for fusion vertices of a maximal N-disjoint set.

Proposition 5.3. Let $\mathcal{G} := (V, E)$ be a connected simple graph with Beurling dimension d and normal counting measure μ , and let $V_N, 1 \leq N \leq \operatorname{diam}(\mathcal{G})$, be maximal N-disjoint sets of fusion vertices. Then for all $\lambda \in V$,

$$\#\{\lambda_m \in V_N, \rho(\lambda_m, \lambda) \le NR\} \le \frac{D_1(\mathcal{G})}{D_2(\mathcal{G})} (R+1)^d, \ 0 \le R \le \frac{\operatorname{diam}(\mathcal{G})}{N} + 1, \tag{5.8}$$

and

$$\#\{\lambda_m \in V_N, \rho(\lambda_m, \lambda) \le NR\} \ge \frac{D_2(\mathcal{G})}{D_1(\mathcal{G})} \left(\frac{R-2}{3}\right)^d, \ 3 \le R \le \frac{\operatorname{diam}(\mathcal{G})}{N} + 1. \tag{5.9}$$

Proof. Take a vertex $\lambda \in V$ and a nonnegative integer R. Set

$$E = \{\lambda_m \in V_N : \ \rho(\lambda_m, \lambda) \le NR\}.$$

Then

$$\cup_{\lambda_m \in E} B(\lambda_m, N) \subset B(\lambda, N(R+1)).$$

This, together with (2.2), (2.3) and (4.7), implies that

$$\begin{split} D_2(\mathcal{G})(N+1)^d \# E &\leq \sum_{\lambda_m \in E} \mu(B(\lambda_m, N)) \\ &= \mu \big(\cup_{\lambda_m \in E} B(\lambda_m, N) \big) \leq D_1(\mathcal{G})(N(R+1)+1)^d. \end{split}$$

Hence the upper bound estimate in (5.8) follows.

Take a vertex $\lambda \in V$ and an integer $R \geq 3$. Applying the covering property (4.8), we have

$$B(\lambda, (R-2)N) \subset \bigcup_{\lambda_m \in E} B(\lambda_m, 2N).$$

This together with (2.2) and (2.3) implies that

$$D_2(\mathcal{G})((R-2)N+1)^d \le D_1(\mathcal{G})(2N+1)^d \# E.$$

Hence the lower bound estimate in (5.9) follows. \square

Let $V_N, N \geq 1$, be maximal N-disjoint sets of fusion vertices. Let $\mathcal{B}_{r,\alpha;N}$ contain all matrices $B := (b(\lambda_m, \lambda_{m'}))_{\lambda_m, \lambda_{m'} \in V_N}$ with $\|B\|_{\mathcal{B}_{r,\alpha;N}} < \infty$, where $h_{B,N}(n) = \sup_{\rho(\lambda_m, \lambda_{m'}) \geq N_N} |b(\lambda_m, \lambda_{m'})|, n \geq 0$, and

$$||B||_{\mathcal{B}_{r,\alpha;N}} = \begin{cases} \left(\sum_{n=0}^{\infty} (h_{B,N}(n))^r (n+1)^{\alpha r + d - 1} \right)^{1/r} & \text{if } 1 \le r < \infty \\ \sup_{n \ge 0} h_{B,N}(n) (n+1)^{\alpha} & \text{if } r = \infty, \end{cases}$$

cf. the Beurling class $\mathcal{B}_{r,\alpha}(\mathcal{G})$ in (3.2). Clearly $\|\cdot\|_{\mathcal{B}_{r,\alpha;N}}$ is a norm. The next proposition states that $\mathcal{B}_{r,\alpha;N}$ are Banach algebras.

Proposition 5.4. Let \mathcal{G}, r, α be as in Theorem 5.1, and let $V_N, N \geq 1$, be a maximal N-disjoint set of fusion vertices. Then there exists an absolute constant C, independent of integers $N \geq 1$, such that

$$||AB||_{\mathcal{B}_{r,\alpha;N}} \le C||A||_{\mathcal{B}_{r,\alpha;N}} ||B||_{\mathcal{B}_{r,\alpha;N}} \quad \text{for all } A, B \in \mathcal{B}_{r,\alpha;N}.$$
 (5.10)

The above lemma can be proved by following the argument used in Proposition 3.4. We omit the detailed proof here.

To prove Theorem 5.1, we also need a technical lemma.

Lemma 5.5. Let $\mathcal{G}, A, r, r', \alpha$ be as in Theorem 5.1, and let $V_N, N \geq 2$, be maximal N-disjoint sets of fusion vertices. Then there exists an absolute constant C_0 independent of A such that

$$\|\Psi_{\lambda_{m}}^{4N}c\|_{2} \leq 2\|A^{-1}\|_{\mathcal{B}(\ell^{2})} \Big(\|\Psi_{\lambda_{m}}^{4N}Ac\|_{\ell^{2}} + \sum_{\lambda_{m'}\in V_{N}} \|\chi_{\lambda_{m}}^{5N}[\Psi_{\lambda_{m}}^{4N}, A]\chi_{\lambda_{m'}}^{4N}\|_{\mathcal{S}} \|\Psi_{\lambda_{m'}}^{4N}c\|_{2} \Big)$$

$$(5.11)$$

for all vertices $\lambda_m \in V_N$, sequences $c \in \ell^2$ and integers N satisfying

$$N^{\alpha - d/r'} \ge 2C_0 \|A^{-1}\|_{\mathcal{B}(\ell^2)} \|A\|_{\mathcal{B}_{r,\alpha}},\tag{5.12}$$

where $[\Psi_{\lambda_m}^{4N}, A] = \Psi_{\lambda_m}^{4N} A - A \Psi_{\lambda_m}^{4N}$ and C_0 is the constant in (4.21). Moreover, there exists an absolute constant C independent of $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ such that

$$\left(\sum_{n=0}^{\infty} (h_{A,N}(n))^{r} (n+1)^{\alpha r+d-1}\right)^{1/r} \leq C \|A\|_{\mathcal{B}_{r,\alpha}} N^{-\min(\alpha-d/r',1)}
\times \begin{cases} 1 & \text{if } \alpha \neq 1 + d/r' \\ (\ln N)^{1/r'} & \text{if } \alpha = 1 + d/r' \end{cases}$$
(5.13)

if $1 \le r < \infty$, and

$$\sup_{n\geq 0} h_{A,N}(n)(n+1)^{\alpha} \leq C \|A\|_{\mathcal{B}_{\infty,\alpha}} N^{-\min(\alpha-d,1)} \begin{cases} 1 & \text{if } \alpha \neq d+1\\ \ln N & \text{if } \alpha = d+1 \end{cases}$$
 (5.14)

if $r = \infty$, where

$$h_{A,N}(n) = \sup_{\rho(\lambda_m, \lambda_{m'}) > Nn} \|\chi_{\lambda_m}^{5N}[\Psi_{\lambda_m}^{4N}, A]\chi_{\lambda_{m'}}^{4N}\|_{\mathcal{S}}, \ n \ge 0.$$
 (5.15)

Proof. We follow the argument in [48,54] where $\mathcal{G} = \mathbb{Z}^d$ with d = 1. Take $\lambda_m \in V_N$, $c := (c(\lambda))_{\lambda \in V} \in \ell^2$, and let A_N be as in (3.13). By the invertibility on A, we have

$$\begin{split} \|\Psi_{\lambda_m}^{4N}c\|_2 &\leq \|A^{-1}\|_{\mathcal{B}(\ell^2)} \|A\Psi_{\lambda_m}^{4N}c\|_2 \\ &\leq \|A^{-1}\|_{\mathcal{B}(\ell^2)} \|\Psi_{\lambda_m}^{4N}Ac\|_2 + \|A^{-1}\|_{\mathcal{B}(\ell^2)} \|[\Psi_{\lambda_m}^{4N},A]c\|_2. \end{split}$$
(5.16)

By the covering property in Proposition 4.2, $\Psi^{4N} := \sum_{\lambda_{m'} \in V_N} \Psi^{4N}_{\lambda_{m'}}$ is a diagonal matrix with bounded inverse, and

$$\|(\Psi^{4N})^{-1}\|_{\mathcal{B}(\ell^2)} \le 1.$$
 (5.17)

Therefore

$$\begin{split} \|[\Psi_{\lambda_{m}}^{4N}, A]c\|_{2} &\leq \|\chi_{\lambda_{m}}^{5N}[\Psi_{\lambda_{m}}^{4N}, A]c\|_{2} + \|(I - \chi_{\lambda_{m}}^{5N})A\chi_{\lambda_{m}}^{4N}\Psi_{\lambda_{m}}^{4N}c\|_{2} \\ &\leq \sum_{\lambda_{m'} \in V_{N}} \|\chi_{\lambda_{m}}^{5N}[\Psi_{\lambda_{m}}^{4N}, A]\chi_{\lambda_{m'}}^{4N}\|_{\mathcal{B}(\ell^{2})} \|(\Psi^{4N})^{-1}\|_{\mathcal{B}(\ell^{2})} \|\Psi_{\lambda_{m'}}^{4N}c\|_{2} \\ &+ \|(I - \chi_{\lambda_{m}}^{5N})A\chi_{\lambda_{m}}^{4N}\|_{\mathcal{S}} \|\Psi_{\lambda_{m}}^{4N}c\|_{2} \\ &\leq \sum_{\lambda_{m'} \in V_{N}} \|\chi_{\lambda_{m}}^{5N}[\Psi_{\lambda_{m}}^{4N}, A]\chi_{\lambda_{m'}}^{4N}\|_{\mathcal{S}} \|\Psi_{\lambda_{m'}}^{4N}c\|_{2} \\ &+ \|A - A_{N}\|_{\mathcal{S}} \|\Psi_{\lambda_{m}}^{4N}c\|_{2}, \end{split} \tag{5.18}$$

where the last inequality follows from (5.17) and Proposition 3.3, cf. [48,54]. Combining (5.16) and (5.18), and then using (4.21) and (5.12), we complete the proof of the upper bound estimate (5.11) for $\|\Psi_{\lambda_m}^{4N}c\|_2$.

Write $A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in V}$ and define $h_A(n) = \sup_{\rho(\lambda, \lambda') \geq n} |a(\lambda, \lambda')|$. For $\lambda_m, \lambda_{m'} \in V_N$ with $\rho(\lambda_m, \lambda_{m'}) \geq 16N$, we obtain from (2.2) and the supporting property for $\Psi_{\lambda_m}^{4N}$ that

$$\|\chi_{\lambda_m}^{5N}[\Psi_{\lambda_m}^{4N}, A]\chi_{\lambda_{m'}}^{4N}\|_{\mathcal{S}} = \|\Psi_{\lambda_m}^{4N}A\chi_{\lambda_{m'}}^{4N}\|_{\mathcal{S}} \le Ch_A(\rho(\lambda_m, \lambda_{m'})/2)N^d.$$
 (5.19)

For $\lambda_m, \lambda_{m'} \in V_N$ with $\rho(\lambda_m, \lambda_{m'}) < 16N$,

$$\|\chi_{\lambda_{m}}^{5N}[\Psi_{\lambda_{m}}^{4N}, A]\chi_{\lambda_{m'}}^{4N}\|_{\mathcal{S}}$$

$$= \|\left(\chi_{[0,5N]}(\rho(\lambda, \lambda_{m}))a(\lambda, \lambda')\chi_{[0,4N]}(\rho(\lambda', \lambda_{m'}))\right) \times \left(\psi_{0}\left(\frac{\rho(\lambda, \lambda_{m})}{4N}\right) - \psi_{0}\left(\frac{\rho(\lambda', \lambda_{m})}{4N}\right)\right)\right)_{\lambda, \lambda' \in V}\|_{\mathcal{S}}$$

$$\leq CN^{-1} \sum_{n=0}^{25N} h_{A}(n)(n+1)^{d}$$

$$\leq C\|A\|_{\mathcal{B}_{r,\alpha}} \begin{cases} N^{-1} & \text{if } \alpha > 1 + d/r' \\ N^{-1}(\ln N)^{1/r'} & \text{if } \alpha = 1 + d/r' \\ N^{-\alpha + d/r'} & \text{if } \alpha < 1 + d/r', \end{cases}$$

$$(5.20)$$

where the first inequality follows from (2.4), and the last estimate is obtained by applying a Hölder inequality, cf. (4.20). Combining (5.19) and (5.20) proves (5.13) and (5.14).

Now we start the proof of Theorem 5.1.

Proof of Theorem 5.1. Let $N \geq 2$ be chosen later. Define $V_{A,N} := (V_{A,N}(\lambda_m, \lambda_{m'}))_{\lambda_m, \lambda_{m'} \in V_N}$ and write $(V_{A,N})^l = (V_{A,N}^l(\lambda_m, \lambda_{m'}))_{\lambda_m, \lambda_{m'} \in V_N}$, where

$$V_{A,N}(\lambda_m, \lambda_{m'}) = 2\|A^{-1}\|_{\mathcal{B}(\ell^2)} \|\chi_{\lambda_m}^{5N}[\Psi_{\lambda_m}^{4N}, A]\chi_{\lambda_{m'}}^{4N}\|_{\mathcal{S}}$$

Then we obtain from Proposition 5.4 and Lemma 5.5 that

$$||V_{A,N}||_{\mathcal{B}_{r,\alpha;N}} \le D_3 ||A^{-1}||_{\mathcal{B}(\ell^2)} ||A||_{\mathcal{B}_{r,\alpha}} N^{-\min(\alpha - d/r',1)}$$

$$\times \begin{cases} 1 & \text{if } \alpha \ne 1 + d/r' \\ (\ln N)^{1/r'} & \text{if } \alpha = 1 + d/r', \end{cases}$$
(5.21)

and

$$\|(V_{A,N})^l\|_{\mathcal{B}_{r,\alpha;N}} \le D_4^{l-1} \|(V_{A,N})\|_{\mathcal{B}_{r,\alpha;N}}^l \tag{5.22}$$

where D_3, D_4 are absolute constants independent of matrices A and integers N and l. Let $N_2 \ge 2$ be the minimal integer satisfying (5.12) and

$$1 \ge 4D_3 D_4 \|A^{-1}\|_{\mathcal{B}(\ell^2)} \|A\|_{\mathcal{B}_{r,\alpha}} N_2^{-\min(\alpha - d/r', 1)}$$

$$\times \begin{cases} 1 & \text{if } \alpha \ne 1 + d/r' \\ (\ln N_2)^{1/r'} & \text{if } \alpha = 1 + d/r'. \end{cases}$$
(5.23)

Then

$$N_{2} \leq C (\|A^{-1}\|_{\mathcal{B}(\ell^{2})} \|A\|_{\mathcal{B}_{r,\alpha}})^{1/\min(\alpha - d/r',1)} \times \begin{cases} 1 & \text{if } \alpha \neq 1 + d/r' \\ \left(\ln (\|A^{-1}\|_{\mathcal{B}(\ell^{2})} \|A\|_{\mathcal{B}_{r,\alpha}} + 1\right)\right)^{1/r'} & \text{if } \alpha = 1 + d/r'. \end{cases}$$
(5.24)

Let $W_{A,N_2} = \sum_{l=1}^{\infty} (V_{A,N_2})^l$. By (5.21), (5.22) and (5.23), we have

$$\|(V_{A,N_2})^l\|_{\mathcal{B}_{r,\alpha;N_2}} \le C2^{-l}, \ l \ge 1,$$
 (5.25)

which implies that

$$||W_{A,N_2}||_{\mathcal{B}_{r,\alpha;N_2}} \le C.$$
 (5.26)

For any $\lambda_m \in V_{N_2}$ and $c \in \ell^2(G)$, applying (5.11) repeatedly we obtain

$$\|\Psi_{\lambda_{m}}^{4N_{2}}c\|_{2} \leq 2\|A^{-1}\|_{\mathcal{B}(\ell^{2})}\|\Psi_{\lambda_{m}}^{4N_{2}}Ac\|_{2} + \sum_{\lambda_{m'}\in V_{N_{2}}} V_{A,N_{2}}(\lambda_{m},\lambda_{m'})\|\Psi_{\lambda_{m'}}^{4N_{2}}c\|_{2}$$

$$\leq \cdots$$

$$\leq 2\|A^{-1}\|_{\mathcal{B}(\ell^{2})}\|\Psi_{\lambda_{m}}^{4N_{2}}Ac\|_{2} + \sum_{\lambda_{m'}\in V_{N_{2}}} V_{A,N_{2}}^{k+1}(\lambda_{m},\lambda_{m'})\|\Psi_{\lambda_{m'}}^{4N_{2}}c\|_{2}$$

$$+2\|A^{-1}\|_{\mathcal{B}(\ell^{2})} \sum_{l=1}^{k} \sum_{\lambda_{m'}\in V_{N_{2}}} V_{A,N_{2}}^{l}(\lambda_{m},\lambda_{m'})\|\Psi_{\lambda_{m'}}^{4N_{2}}Ac\|_{2}, \quad k \geq 2. \quad (5.27)$$

Using the argument used to prove Proposition 3.3, we have

$$\sum_{\lambda_{m'} \in V_{N_2}} V_{A,N_2}^{k+1}(\lambda_m, \lambda_{m'}) \|\Psi_{\lambda_{m'}}^{4N_2} c\|_{2}$$

$$\leq \sum_{\lambda_{m'} \in V_{N_2}} V_{A,N_2}^{k+1}(\lambda_m, \lambda_{m'}) \|c\|_{2} \leq C \|(V_{A,N_2})^{k+1}\|_{\mathcal{B}_{r,\alpha;N_2}} \|c\|_{2}. \tag{5.28}$$

Taking limit in (5.27), we obtain from (5.25) and (5.28) that

$$\|\Psi_{\lambda_{m}}^{4N_{2}}c\|_{2} \leq 2\|A^{-1}\|_{\mathcal{B}(\ell^{2})}\|\Psi_{\lambda_{m}}^{4N_{2}}Ac\|_{2}$$

$$+2\|A^{-1}\|_{\mathcal{B}(\ell^{2})}\sum_{\lambda_{m'}\in V_{N_{2}}}W_{A,N_{2}}(\lambda_{m},\lambda_{m'})\|\Psi_{\lambda_{m'}}^{4N_{2}}Ac\|_{2},$$

$$(5.29)$$

where $W_{A,N_2} = (W_{A,N_2}(\lambda_m, \lambda_{m'}))_{\lambda_m, \lambda_{m'} \in V_{N_2}}$. Write $A^{-1} = (d(\lambda', \lambda))_{\lambda', \lambda \in V}$ and set $d_{\lambda} = (d(\lambda', \lambda))_{\lambda' \in V}, \lambda \in V$. Take $\lambda, \lambda' \in V$ and let $\lambda_m \in V_{N_2}$ be so chosen that

$$\rho(\lambda', \lambda_m) \le 2N_2. \tag{5.30}$$

The existence of such a fusion vertex λ_m follows from the covering property in Proposition 4.2. Applying (5.29) with c replaced by d_{λ} , we obtain

$$|d(\lambda',\lambda)| \leq \|\Psi_{\lambda_{m}}^{4N_{2}} d_{\lambda}\|_{2} \leq 2\|A^{-1}\|_{\mathcal{B}(\ell^{2})} \left| \psi_{0} \left(\frac{\rho(\lambda_{m},\lambda)}{4N_{2}} \right) \right|$$

$$+2\|A^{-1}\|_{\mathcal{B}(\ell^{2})} \sum_{\lambda_{m'} \in V_{N_{2}}} W_{A,N_{2}}(\lambda_{m},\lambda_{m'}) \left| \psi_{0} \left(\frac{\rho(\lambda_{m'},\lambda)}{4N_{2}} \right) \right|.$$
 (5.31)

Therefore

$$\sup_{\lambda',\lambda\in V} |d(\lambda',\lambda)| \leq 2\|A^{-1}\|_{\mathcal{B}(\ell^2)} \left(1 + \left(\sup_{\lambda_m,\lambda_{m'}\in V_{N_2}} |W_{A,N_2}(\lambda_m,\lambda_{m'})|\right)\right) \\
\times \left(\sup_{\lambda\in V} \sum_{\lambda_{m'}\in V_{N_2}} \chi_{B(\lambda_{m'},4N_2)}(\lambda)\right)\right) \\
\leq C\|A^{-1}\|_{\mathcal{B}(\ell^2)}, \tag{5.32}$$

where the last inequality follows from (5.26) and Proposition 4.2. For $n \ge 12$, it follows from (5.30) and (5.31) that

$$\sup_{\rho(\lambda',\lambda) \ge nN_2} |d(\lambda,\lambda')| \le 2\|A^{-1}\|_{\mathcal{B}(\ell^2)} g_{A,N_2}(n/2) \sup_{\lambda \in V} \sum_{\lambda_{m'} \in V_{N_2}} \chi_{B(\lambda_{m'},4N_2)}(\lambda)
\le C\|A^{-1}\|_{\mathcal{B}(\ell^2)} g_{A,N_2}(n/2),$$
(5.33)

where $g_{A,N_2}(n) = \sup_{\rho(\lambda_m,\lambda_{m'}) \geq Nn} |W_{A,N_2}(\lambda_m,\lambda_{m'})|$. Observe that

$$||A^{-1}||_{\mathcal{B}_{r,\alpha}} \le CN_2^{\alpha+d/r} \Big(\sum_{m>0} \Big(\sup_{\rho(\lambda,\lambda')\ge mN_2} |d(\lambda,\lambda')|\Big)^r (m+1)^{\alpha r+d-1}\Big)^{1/r}$$

for $1 \le r < \infty$, and

$$||A^{-1}||_{\mathcal{B}_{r,\alpha}} \le CN_2^{\alpha} \sup_{m\ge 0} \Big(\sup_{\rho(\lambda,\lambda')\ge mN_2} |d(\lambda,\lambda')|\Big) (m+1)^{\alpha}$$

for $r = \infty$. Combining the above two estimates with (5.26), (5.32) and (5.33), we obtain

$$||A^{-1}||_{\mathcal{B}_{r,\alpha}} \le C||A^{-1}||_{\mathcal{B}(\ell^2)} N_2^{\alpha + d/r}.$$
 (5.34)

Hence the desired estimate (5.1) follows from (5.24) and (5.34). \square

6. Norm-controlled powers

By (3.26), norms of powers $A^n, n \geq 1$, of a matrix $A \in \mathcal{B}_{r,\alpha}$ with $1 \leq r \leq \infty$ and $\alpha > d(1-1/r)$ are dominated by a subexponential function. In this section, we show that norms of powers $A^n, n \geq 1$, are controlled by a polynomial when the counting measure is normal.

Theorem 6.1. Let $1 \le r \le \infty$, r' = r/(r-1), \mathcal{G} be a connected simple graph with Beurling dimension d and normal counting measure μ , and let $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ with $\alpha > d/r'$. Then there exists an absolute positive constant C, independent of matrices A and integers $n \ge 1$, such that

$$\frac{\|A^{n}\|_{\mathcal{B}_{r,\alpha}}}{\|A\|_{\mathcal{B}(\ell^{2})}^{n}} \leq Cn \left(\frac{n\|A\|_{\mathcal{B}_{r,\alpha}}}{\|A\|_{\mathcal{B}(\ell^{2})}}\right)^{(\alpha+d/r)/\min(\alpha-d/r',1)}
\times \begin{cases} 1 & \text{if } \alpha \neq d/r' + 1 \\ \left(\ln\left(\frac{n\|A\|_{\mathcal{B}_{r,\alpha}}}{\|A\|_{\mathcal{B}(\ell^{2})}} + 1\right)\right)^{(d+1)/r'} & \text{if } \alpha = d/r' + 1 \end{cases}$$
(6.1)

hold for all integers $n \geq 1$.

For matrices $A \in \mathcal{B}_{r,\alpha}$ with $\alpha > d/r' + 1$, we obtain from Theorem 6.1 that

$$||A^n||_{\mathcal{B}_{r,\alpha}} \le C \left(\frac{||A||_{\mathcal{B}_{r,\alpha}}}{||A||_{\mathcal{B}(\ell^2)}}\right)^{\alpha+d/r} n^{\alpha+d/r+1} ||A||_{\mathcal{B}(\ell^2)}^n, \ n \ge 1.$$
 (6.2)

As shown in (6.3) below, the estimate (6.2) on powers of matrices in the Beurling algebra $\mathcal{B}_{r,\alpha}$ is almost optimal. Let δ be the delta function with $\delta(0) = 1$ and $\delta(k) \neq 0$ for all nonzero integers k. Then for the matrix $A_1 = (\delta(i-j-1))_{i,j\in\mathbb{Z}}$, we have that $(A_1)^n = (a(i-j-n))_{i,j\in\mathbb{Z}}, n \geq 1$, and hence

$$\|(A_{1})^{n}\|_{\mathcal{B}_{r,\alpha}} = \begin{cases} \left(\sum_{k=0}^{n} (k+1)^{\alpha r}\right)^{1/r} & \text{if } 1 \leq r < \infty \\ (n+1)^{\alpha} & \text{if } r = \infty \end{cases}$$

$$\geq C \left(\frac{\|A_{1}\|_{\mathcal{B}_{r,\alpha}}}{\|A_{1}\|_{\mathcal{B}(\ell^{2})}}\right)^{\alpha+d/r} n^{\alpha+d/r} \|A_{1}\|_{\mathcal{B}(\ell^{2})}^{n}, \ n \geq 1, \tag{6.3}$$

where C is an absolute constant.

Let $\hat{a}(\xi) = \sum_{k \in \mathbb{Z}} a(k) e^{-ik\xi}$ satisfy $|\hat{a}(\xi)| \le 1$ for all $\xi \in \mathbb{R}$ and $\sup_{k \in \mathbb{Z}} |a(k)| (1+|k|)^{\alpha} < \infty$ for some $\alpha > 1$, and write

$$(\hat{a}(\xi))^n = \sum_{k \in \mathbb{Z}} a_n(k) e^{-ik\xi}, \ n \ge 1.$$

Then there exists a positive constant C independent of $n \geq 1$ such that

$$|a_n(k)| \le Cn^{\alpha+1}(1+|k|)^{-\alpha}, \ k \in \mathbb{Z}$$

by Theorem 6.1. Therefore for any $\epsilon > 0$, there exists a positive constant C_{ϵ} such that

$$\sum_{k \in \mathbb{Z}} |a_n(k)| \le C n^{1+\epsilon}, \ n \ge 1, \tag{6.4}$$

cf. [17,41,60,59] and references therein for various estimates. We remark that the above estimate for the Wiener norm of $(\hat{a}(\xi))^n, n \geq 1$, was established in [59], with the polynomial exponent $1 + \epsilon$ replaced by a smaller exponent $(1 - \mu/\nu)/2$, when

$$\hat{a}(\xi) = e^{-i\alpha\xi + i\xi^{\mu}q(\xi) - \gamma\xi^{\nu}(1+o(1))}$$

near the origin for some real polynomial q with $q(0) \neq 0$.

Let random variables $X_n, n \geq 1$, be a stationary Markov chain on a spatially distributed network, which is described by a connected simple graph $\mathcal{G} = (V, E)$. Then the probabilities $\Pr(X_{n+1} = \lambda \mid X_n = \lambda')$ of going from one vertex λ' at time n to another vertex λ at time n+1 is independent of $n \geq 1$,

$$\Pr(X_{n+1} = \lambda \mid X_n = \lambda') = p(\lambda, \lambda'), \ \lambda, \lambda' \in V \text{ and } n \ge 1.$$

Define the transition matrix of the above stationary Markov chain by $P = (p(\lambda, \lambda'))_{\lambda, \lambda' \in V}$. Then by Theorem 6.1, we have the following estimate on the probability $\Pr(X_m = \lambda | X_n = \lambda'), m > n \ge 1$, with the input vertex λ' and output vertex $\lambda \in V$.

Corollary 6.2. Let $\mathcal{G} := (V, E)$ be a connected simple graph with Beurling dimension d and normal counting measure μ , and let $X_n, n \geq 1$, be a stationary Markov chain on the graph \mathcal{G} with transition matrix $P \in \mathcal{B}_{\infty,\alpha}$ for some $\alpha > d+1$. Then there exists a positive constant C_{α} such that

$$\Pr(X_m = \lambda | X_n = \lambda') \le \min\left(C_\alpha(m - n)^{\alpha + 1}(\rho(\lambda, \lambda'))^{-\alpha}, 1\right) \tag{6.5}$$

for all $\lambda, \lambda' \in V$ and $m > n \ge 1$.

We finish this section with the proof of Theorem 6.1.

Proof of Theorem 6.1. Let $A \in \mathcal{B}_{r,\alpha}$, and write

$$A^{n} = \frac{1}{2\pi i} \int_{|z|=(1+1/n)\|A\|_{\mathcal{B}(\ell^{2})}} z^{n} (zI - A)^{-1} dz.$$

Then

$$||A^n||_{\mathcal{B}_{r,\alpha}} \le C||A||_{\mathcal{B}(\ell^2)}^n \int_{|z|=(1+1/n)||A||_{\mathcal{B}(\ell^2)}} ||(zI-A)^{-1}||_{\mathcal{B}_{r,\alpha}}|dz|.$$
 (6.6)

Observe that for $|z| = ||A||_{\mathcal{B}(\ell^2)} (1 + 1/n)$, we have

$$\|(zI - A)^{-1}\|_{\mathcal{B}(\ell^2)} \le |z|^{-1} \sum_{l=0}^{\infty} |z|^{-l} \|A\|_{\mathcal{B}(\ell^2)}^l \le n(\|A\|_{\mathcal{B}(\ell^2)})^{-1}, \tag{6.7}$$

and

$$||zI - A||_{\mathcal{B}_{r,\alpha}} \le |z| + ||A||_{\mathcal{B}_{r,\alpha}} \le C||A||_{\mathcal{B}_{r,\alpha}},$$
 (6.8)

where the last inequality holds by (3.24). By (6.7), (6.8) and Theorem 5.1, we get

$$\|(zI - A)^{-1}\|_{\mathcal{B}_{r,\alpha}} \le Cn(\|A\|_{\mathcal{B}(\ell^{2})})^{-1} \left(\frac{n\|A\|_{\mathcal{B}_{r,\alpha}}}{\|A\|_{\mathcal{B}(\ell^{2})}}\right)^{(\alpha + d/r)/\min(\alpha - d/r', 1)} \times \begin{cases} 1 & \text{if } \alpha \ne d/r' + 1\\ \left(\ln\left(\frac{n\|A\|_{\mathcal{B}_{r,\alpha}}}{\|A\|_{\mathcal{B}(\ell^{2})}} + 1\right)\right)^{(d+1)/r'} & \text{if } \alpha = d/r' + 1. \end{cases}$$
(6.9)

This together with (6.6) proves (6.1). \square

Acknowledgments

The authors would like to thank Professors Karlheinz Gröchenig, Andreas Klotz and Jose Luis Romero for their help and suggestion for the improvement of the manuscript, and also the reviewer for the constructive suggestions and comments. The authors are partially supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (NRF-2016R1D1A1B03930571), and the National Science Foundation (DMS-1412413 and DMS-1816313).

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