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On uniqueness of solutions to viscous HJB equations with a subquadratic nonlinearity in the gradient

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ABSTRACT
Uniqueness of positive solutions to viscous Hamilton–Jacobi–Bellman (HJB) equations of the form $-\Delta u(x)+\frac{1}{\gamma}|Du(x)|^{\gamma}=f(x)-\lambda$, with f a coercive function and λ a constant, in the subquadratic case, that is, $\gamma\in(1,2)$, appears to be an open problem. Barles and Meireles [*Comm. Partial Differential Equations* 41 (2016)] show uniqueness in the case that $f(x)\approx|x|^{\beta}$ and $|Df(x)|\lesssim|x|^{(\beta-1)_+}$ for some $\beta>0$, essentially matching earlier results of Ichihara, who considered more general Hamiltonians but with better regularity for *f*. Without enforcing this assumption, to our knowledge, there are no results on uniqueness in the literature. In this short article, we show that the equation has a unique positive solution for any locally Lipschitz continuous, coercive f which satisfies $|Df(x)|\le\kappa(1+|f(x)|^{2-1/\gamma})$ for some positive constant κ. Since $2-\frac{1}{\gamma}>1$, this assumption imposes very mild restrictions on the growth of the potential *f*. We also show that this solution fully characterizes optimality for the associated ergodic problem. Our method involves the study of an infinite dimensional linear program for elliptic equations for measures, and is very different from earlier approaches. It also applies to the larger class of Hamiltonians studied by Ichihara, and we show that it is well suited to provide optimality results for the associated ergodic control problem, even in a pathwise sense, and without resorting to the parabolic problem.

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1. Introduction

We consider the viscous Hamilton–Jacobi–Bellman (HJB) equation

$$-\Delta u(x)+\frac{1}{\gamma}|Du(x)|^{\gamma}\;=\;f(x)-\lambda, \tag{EP}$$

for $(u,\lambda)\in C^2(\mathbb{R}^d)\times\mathbb{R}$, with $\gamma>1$. Here, $f\in\mathcal{W}^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$ and is coercive. By coercive, sometimes also called *inf-compact*, we refer to a function f whose sublevel sets $\{x\in\mathbb{R}^d: f(x)\le r\}$ are compact (or empty) for every $r\in\mathbb{R}$. As shown in [1], (EP) has a classical solution u for any $\lambda\le\lambda^*$, where

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$$\lambda^* \;:=\; \sup\left\{\lambda \in \mathbb{R}^d \;: \text{(EP) has a subsolution}\right\}. \tag{1.1}$$

This equation which has a long history in the literature, has been studied in [2, 3] for somewhat more general Hamiltonians, and was recently revisited by Barles in [1]. What is of interest here, is to characterize the solutions of (EP) which are bounded from below, that is, without loss of generality, the positive solutions. Naturally, when we refer to this equation having a unique positive solution, we mean that the solution is unique up to an additive constant. In the superquadratic case ($\gamma \geq 2$), [1, Theorem 2.6] shows that (EP) has a unique positive solution for any coercive f, and in addition, for this solution, $\lambda = \lambda^*$. In the subquadratic case ($\gamma \in (1,2)$), [1] adopts assumption (H2) in [2], which states that f satisfies a bound of the form

$$c^{-1}|x|^\beta - c \;\leq\; f(x) \;\leq\; c\Big(1+|x|^\beta\Big), \text{ and } |Df(x)| \;\leq\; c^{-1}\Big(1+|x|^{(\beta-1)_+}\Big) \tag{1.2}$$

for some positive constants β and c for all $x \in \mathbb{R}^d$. Without enforcing this assumption, to our knowledge, there are no results on uniqueness in the literature, and therefore also no verification of optimality results. Note that [3] introduces an additional stable drift to study the subquadratic case.

There is substantial literature on viscous HJB equations, other than [1–3] mentioned above. It is not our intent to review this literature, since it does not address the problem studied in this article, but we should at least mention [4–11].

We adopt the following assumption for $\gamma \in (1,2)$.

(A1) *The function f is locally Lipschitz continuous and coercive, and there exists a constant κ_0 such that*

$$|Df(x)| \;\leq\; \kappa_0\Big(1+|f(x)|^{2-\frac{1}{\gamma}}\Big) \quad \forall\; x \in \mathbb{R}^d.$$

We show that, under (A1), there exists a unique positive solution u to (EP). In addition, this solution fully characterizes the ergodic control problem in the sense that a stationary Markov control is optimal if and only if it agrees a.e. on $\mathbb{R}^d$ with the function ξ_u in (2.18) (see Theorem 4.1). The method we follow covers the more general Hamiltonians studied in [2, 3], and also improves the existing results for the superquadratic case. This is discussed in Section 3.

1.1. Brief summary of the method

Consider the operator $\mathcal{A} : C^2(\mathbb{R}^d) \to C(\mathbb{R}^d \times \mathbb{R}^d)$ defined by

$$\mathcal{A}g(x,\xi) \;:=\; -\Delta g(x) + \xi \cdot Dg(x), \quad (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{1.3}$$

Let $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ denote the space of probability measures on the Borel σ-algebra of $\mathbb{R}^d \times \mathbb{R}^d$, denoted as $\mathfrak{B}(\mathbb{R}^d \times \mathbb{R}^d)$, endowed with the Prokhorov topology. We say that $\mu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ is *infinitesimally invariant* for the operator $\mathcal{A}$ if $\int \mathcal{A}g \,\mathrm{d}\mu = 0$ for all $g \in C_c^2(\mathbb{R}^d)$, the latter denoting the functions in $C^2(\mathbb{R}^d)$ with compact support, and denote the set of these probability measures by $\mathcal{M}$. Let

$$F(x,\xi) \;:=\; f(x) + \frac{1}{\gamma^*}|\xi|^{\gamma^*} \;,\; \text{with } \gamma^* \;:=\; \frac{\gamma}{\gamma-1}. \tag{1.4}$$

For $\mu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ we use the simple notation

$$\mu(F) \;:=\; \int_{\mathbb{R}^d \times \mathbb{R}^d} F(x,\xi)\; \mu(\mathrm{d}x, \mathrm{d}\xi),$$

and define

$$\mathcal{M}_F \;:=\; \{\mu \in \mathcal{M}\;:\mu(F) < \infty\}. \tag{1.5}$$

In other words, $\mathcal{M}_F$ is the subset of $\mathcal{M}$ consisting of those probability measures under which F is integrable. It is simple to show that $\mathcal{M}_F$ is always nonempty. Thus, since F is coercive, the set

$$\mathcal{M}_{F,r} \;:=\; \{\mu \in \mathcal{M}\;:\mu(F) \le r\}$$

is compact for all $r > 0$ sufficiently large. Clearly, it is also convex.

Consider the minimization problem

$$\overline{\lambda} \;:=\; \inf_{\mu \in \mathcal{M}} \mu(F). \tag{LP}$$

The lower semicontinuity of $\mu \mapsto \mu(F)$ then implies that the infimum of (LP) is attained in $\mathcal{M}$. We let $\mathcal{M}_F^*$ denote the set of points in $\mathcal{M}$ which attain this infimum.

Our approach to the proof of uniqueness of positive solutions of (EP) is as follows: First, we show that if $(u,\lambda) \in C^2(\mathbb{R}^d) \times \mathbb{R}$ is any pair solving (EP), with u a positive function, then $\lambda = \overline{\lambda}$ and some measure $\mu \in \mathcal{M}$ taking the form $\mu(\mathrm{d}x, \mathrm{d}\xi) = \nu(\mathrm{d}x)\delta_{Du(x)}(\mathrm{d}\xi)$, with $\nu \in \mathcal{P}(\mathbb{R}^d)$ and $\delta_{Du(x)}$ denoting the Dirac mass at $Du(x)$, attains the infimum in (LP), that is, it belongs to $\mathcal{M}_F^*$. Next, we show that $\mathcal{M}_F^*$ is a singleton, thus establishing the uniqueness of a positive solution to (EP).

1.2. Notation

The standard Euclidean norm in $\mathbb{R}^d$ is denoted by $|\cdot|$, and $\mathbb{N}$ stands for the set of natural numbers. The closure, the boundary and the complement of a set $A \subset \mathbb{R}^d$ are denoted by $\overline{A}, \partial A$, and A^c, respectively. The open ball of radius r in $\mathbb{R}^d$, centered at $x \in \mathbb{R}^d$, is denoted by $B_r(x)$, and B_r is the ball centered at 0. We use $a_\pm := \max(\pm a, 0)$ for $a \in \mathbb{R}$.

For a Borel space Y, $\mathcal{P}(Y)$ denotes the set of probability measures on its Borel σ-algebra, and δ_y denotes the Dirac mass at $y \in Y$. For $\mu \in \mathcal{P}(Y)$ and a measurable function $g : Y \to \mathbb{R}$ which is integrable under μ, we often use the simplifying notation $\mu(g) := \int_Y f \;\mathrm{d}\mu$.

2. Main results

Throughout this section we assume $\gamma \in (1,2)$, unless otherwise explicitly mentioned. Also, without loss of generality we assume that $f \ge 1$, and we scale a solution of (EP), which is bounded from below, by an additive constant so that $\inf_{\mathbb{R}^d} \; u = 1$.

We start with the very useful gradient estimate in [2, Theorem B.1], stated under weaker regularity in [1, Theorem A.2] for (EP). It appears that the Bernstein approach for this estimate originates in [11, Theorem A.1]. We need to use this estimate on small balls, so we scale it as follows.

Corollary 2.1. *There exists a constant C such that for any solution u of (EP) we have*

$$\sup_{z\in B_r(x)} |Du(z)| \;\le\; C\left(r^{-\frac{1}{\gamma-1}} + \sup_{z\in B_{2r}(x)} (f(z)-\lambda)_+^{1/\gamma} + \sup_{z\in B_{2r}(x)} |Df(z)|^{1/(2\gamma-1)}\right) \tag{2.1}$$

for all $x\in\mathbb{R}^d$, and for all $r>0$. In particular, under (A1), *with perhaps a different constant C_0, we have*

$$\sup_{z\in B_r(x)} |Du(z)| \;\le\; C_0\left(r^{-\frac{1}{\gamma-1}} + \sup_{z\in B_{2r}(x)} (f(y)-\lambda)_+^{1/\gamma}\right) \quad \forall x\in\mathbb{R}^d, \forall r>0. \tag{2.2}$$

Proof. Fix any $x\in\mathbb{R}^d$. For $r>0$, let $u_r(y) := r^{\frac{2-\gamma}{\gamma-1}}u(x+ry)$ and $f_r(y) := r^{\gamma^*}(f(x+ry)-\lambda)$. The function u_r satisfies

$$-\Delta u_r(y) + \frac{1}{\gamma}|Du_r(y)|^\gamma \;=\; f_r(y). \tag{2.3}$$

By [2, Theorem B.1], there exists a constant C such that any solution u_r of (2.3) satisfies

$$\sup_{y\in B_1(x)} |Du_r(y)| \;\le\; C\left(1 + \sup_{y\in B_2(x)} (f_r(y))_+^{1/\gamma} + \sup_{y\in B_2(x)} |Df_r(y)|^{1/(2\gamma-1)}\right) \quad \forall x\in\mathbb{R}^d.$$

from which (2.1) follows.

We continue by proving a useful lower bound for positive solutions of (EP). Define

$$\Gamma_x \;:=\; [f(x)]^{-\frac{1}{\gamma^*}}, \quad x\in\mathbb{R}^d.$$

Lemma 2.1. *Assume* (A1). *Then, for every positive solution u of* (EP), *the following hold.*

(a) *There exist positive constants r and κ such that*

$$\inf_{y\in B_r} u(x+\Gamma_x\, y) \;\ge\; \kappa\, [f(x)]^{\frac{\gamma^*-2}{\gamma^*}} \quad \forall x\in\mathbb{R}^d. \tag{2.4}$$

(b) *There exists a positive constant M_0 such that*

$$\frac{|Du(x)|^2}{u(x)} \;\le\; M_0\, f(x) \quad \forall x\in\mathbb{R}^d. \tag{2.5}$$

Proof. Note that by (A1) there exists some $R>0$ such that

$$|Df(x_n)| \;\le\; 2\kappa_0\big[f(x_n)\big]^{\frac{\gamma^*+1}{\gamma^*}} \quad \forall x\in B_R^c. \tag{2.6}$$

Choose r positive and small enough such that $r\le\frac{\gamma^*}{8\kappa_0}$. We claim that the assertion in part (a) holds for this r. To prove this, we use contradiction. Suppose that

$$\inf_{y\in B_r} u(x_n+\Gamma_{x_n}y)\big[f(x_n)\big]^{\frac{2-\gamma^*}{\gamma^*}} \xrightarrow[n\to\infty]{} 0 \tag{2.7}$$

along some sequence $\{x_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^d$, such that $|x_n|\to\infty$ as $n\to\infty$. We write (EP) as

$$-\Delta u(x)+b(x)\cdot Du(x) \;=\; f(x)-\lambda,$$

with $b(x):=\frac{1}{\gamma}|Du|^{\gamma-2}Du(x)$. Simplifying the notation, let $\Gamma_n\equiv\Gamma_{x_n}=[f(x_n)]^{-\frac{1}{\gamma^*}}$, and define the sequence of scaled functions

$$u_n(y) \;:=\; \frac{u(x_n+\Gamma_n y)}{\big[f(x_n)\big]^{\frac{\gamma^*-2}{\gamma^*}}},\quad b_n(y) \;:=\; \Gamma_n\; b(x_n+\Gamma_n y),\ \text{and}\ f_n(y) \;:=\; \frac{f(x_n+\Gamma_n y)-\lambda}{f(x_n)},$$

for $y\in\mathbb{R}^d$ and $n\in\mathbb{N}$. Then we obtain from (EP) that

$$-\Delta u_n(y)+b_n(y)\cdot Du_n(y) \;=\; f_n(y). \tag{2.8}$$

Integrating (2.6), we obtain

$$\big[f(x_n+\Gamma_n y)\big]^{-\frac{1}{\gamma^*}}-\big[f(x_n)\big]^{-\frac{1}{\gamma^*}} \;\geq\; -\frac{2\kappa_0}{\gamma^*}\;|y|\;\Gamma_n. \tag{2.9}$$

Computing also the lower bound inherited from (2.6), and combining it with (2.9), we obtain

$$\left(\frac{2}{3}\right)^{\gamma^*} f(x_n) \;\leq\; f(x_n+\Gamma_n y) \;\leq\; 2^{\gamma^*}f(x_n)\quad \forall y\in B_{4r},\quad \forall x_n\in B_R^c. \tag{2.10}$$

This shows that f_n and f_n^{-1} are bounded in B_{4r} uniformly in $n\in N$. To establish a bound for b_n on B_{2r}, it is enough to show that, for some constant C, we have

$$|Du(x_n+\Gamma_n y)| \;\leq\; C\big(1+\Gamma_n^{1-\gamma^*}\big) \;=\; C\Big(1+\big[f(x_n)\big]^{1/\gamma}\Big)\quad \forall y\in B_{2r}. \tag{2.11}$$

By (2.2) we have

$$\begin{aligned}\sup_{y\in B_{2r}}\;|Du(x_n+\Gamma_n y)| \;&\leq\; C_0\left((2\Gamma_n r)^{-\frac{1}{\gamma-1}}+\sup_{y\in B_{4r}}\;\big(f(x_n+\Gamma_n y)-\lambda\big)_+^{1/\gamma}\right)\\ &=\; C_0\left((2r)^{-\frac{1}{\gamma-1}}\big[f(x_n)\big]^{\frac{1}{\gamma}}+\sup_{y\in B_{4r}}\;\big(f(x_n+\Gamma_n y)-\lambda\big)_+^{1/\gamma}\right).\end{aligned} \tag{2.12}$$

Thus (2.11) follows by (2.10) and (2.12).

Therefore, since, as we have shown, f_n, f_n^{-1} and b_n are bounded in B_{2r} uniformly in $n\in N$, then, by using for example [12, Lemma 3.6], we see that equation (2.8) contradicts the hypothesis in (2.7) that $\inf_{y\in B_r} u_n(y)\to 0$. This completes the proof of part (a).

Moving to part (b), let r be as chosen in the proof of part (a). We have shown above that

$$\sup_{x\in\mathbb{R}^d}\;\sup_{y\in B_{4r}}\;\frac{f(x+\Gamma_x y)-\lambda}{f(x)}\;<\infty. \tag{2.13}$$

On the other hand, using the estimate (2.12) on $B_{r/2}$, with r and R as in part (a), we have

$$\begin{aligned}\frac{|Du(x)|^2}{u(x)f(x)} &\le \frac{C_0}{u(x)f(x)}\left((r/2)^{-\frac{1}{\gamma-1}}[f(x)]^{\frac{1}{\gamma}} + \sup_{y\in B_r}\left(f(x+\Gamma_x y)-\lambda\right)_+^{1/\gamma}\right)^2 \\ &\le 2C_0\left((r/2)^{-\frac{1}{\gamma-1}}\frac{[f(x)]^{\frac{\gamma^*-2}{\gamma^*}}}{u(x)} + \sup_{y\in B_r}\frac{[f(x+\Gamma_x y)]^{\frac{\gamma^*-2}{\gamma^*}}}{u(x)}\frac{f(x+\Gamma_x y)}{f(x)}\right)\end{aligned}$$

for all $x\in\mathbb{R}^d$. Therefore, (2.5) follows by (2.4) and (2.13). This completes the proof. □

Remark 2.1. The estimate in Lemma 2.1 is not suitable for the superquadratic case. A different scaling can be used when $\gamma\ge 2$. First, we replace (A1) by

$$|Df(x)| \le \kappa_0\left(1+|f(x)|^{\frac{4\gamma-3}{3\gamma-2}}\right) \quad \forall x\in\mathbb{R}^d. \tag{2.14}$$

Then, under (2.14), we obtain

$$\inf_{y\in B_r} u(x+\Gamma_x\, y) \ge \kappa\, [f(x)]^{\frac{\gamma}{3\gamma-2}} \quad \forall x\in\mathbb{R}^d.$$

for some positive constants r and κ. To prove this, we use $\Gamma_x=[f(x)]^{\frac{1-\gamma}{3\gamma-2}}$, and follow the proof of Lemma 2.1.

To continue, we need the following notation.

Notation 2.1. For $r>0$, we let χ_r be a concave $C^2(\mathbb{R})$ function such that $\chi_r(t)=t$ for $t\le r$, and $\chi_r'(t)=0$ for $t\ge 3r$. Then χ_r' and $-\chi''_r$ are nonnegative, and the latter is supported on $[r,3r]$. In addition, we select χ_r so that

$$|\chi''_r(t)| \le \frac{1}{t} \quad \forall t>0. \tag{2.15}$$

This is always possible. For example, we can specify χ''_r as

$$\chi''_r(t) = \begin{cases} \dfrac{t-r}{r^2} & \text{if } r\le t\le \dfrac{3r}{2}, \\ \dfrac{1}{2r} & \text{if } \dfrac{3r}{2}\le t\le\dfrac{5r}{2}, \\ \dfrac{3}{r}-\dfrac{t}{r^2} & \text{if } \dfrac{5r}{2}\le t\le 3r. \end{cases}$$

Recall the definitions in (1.4), (1.5) and (LP).

Lemma 2.2. *Assume* (A1). *For any positive solution* $u\in C^2(\mathbb{R}^d)$ *of* (EP) *with eigenvalue* λ *and* $\mu\in\mathcal{M}_F$, *we have*

$$\mu(F)-\lambda = \int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{\gamma^*}|\xi|^{\gamma^*}-\xi\cdot Du(x)+\frac{1}{\gamma}|Du(x)|^{\gamma}\right)\mu(\mathrm{d}x,\mathrm{d}\xi) \ge 0. \tag{2.16}$$

In particular, $\lambda\le\overline{\lambda}$.

Proof. Since u is coercive by Lemma 2.1 (a), it follows that $\chi_r(u)-r-1$ is compactly supported. Thus we have

$$\int \big(\Delta\chi_r(u) - \xi\cdot D\chi_r(u)\big)\ \mathrm{d}\mu \;=\; 0 \quad \forall \mu\in\mathcal{M},$$

by the definition of $\mathcal{M}$. On the other hand, we have

$$\begin{aligned}\Delta\chi_r(u) - \xi\cdot D\chi_r(u) \;&=\; \chi''_r(u)|Du|^2 + \chi'_r(u)(\Delta u - \xi\cdot Du)\\ &=\; \chi''_r(u)|Du|^2 + \chi'_r(u)\left(\lambda + \frac{1}{\gamma}|Du|^\gamma - f - \xi\cdot Du\right)\\ &=\; \chi''_r(u)|Du|^2 + \chi'_r(u)\left(\lambda - f - \frac{1}{\gamma^*}|\xi|^{\gamma^*}\right)\\ &\quad + \chi'_r(u)\left(\frac{1}{\gamma^*}|\xi|^{\gamma^*} - \xi\cdot Du + \frac{1}{\gamma}|Du|^\gamma\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\int \chi'_r(u)\left(f + \frac{1}{\gamma^*}|\xi|^{\gamma^*} - \lambda\right)\ \mathrm{d}\mu &= \int \chi''_r(u)|Du|^2\ \mathrm{d}\mu\\ &\quad + \int \chi'_r(u)\left(\frac{1}{\gamma^*}|\xi|^{\gamma^*} - \xi\cdot Du + \frac{1}{\gamma}|Du|^\gamma\right)\ \mathrm{d}\mu\end{aligned}$$

for all $\mu\in\mathcal{M}$. By Lemma 2.1 we have $\frac{|Du|^2}{u}\le M_0 f$ for some constant M_0. Using this together with (2.15), we obtain

$$\begin{aligned}\int \chi''_r(u)|Du|^2\ \mathrm{d}\mu \;&\le\; \int \mathbb{1}_{\{x:u(x)\ge r\}}\frac{|Du(x)|^2}{u(x)}\ \mathrm{d}\mu\\ &\le\; M_0\int \mathbb{1}_{\{x:u(x)\ge r\}}\ f(x)\ \mathrm{d}\mu \underset{r\to\infty}{\longrightarrow} 0,\end{aligned}$$

since $\int f\,\mathrm{d}\mu<\infty$ by the definition of $\mathcal{M}_F$. Thus letting $r\nearrow\infty$, and applying the monotone convergence theorem, we obtain (2.16), thus completing the proof. □

It is convenient to express the operator $\mathcal{A}$ in (1.3) in terms of a family of operators $\{\mathcal{L}_\xi\}_{\xi\in\mathbb{R}^d}$ defined by

$$\mathcal{L}_\xi g(x) \;:=\; -\Delta g(x) + \xi\cdot Dg(x), \quad g\in C^2(\mathbb{R}^d). \tag{2.17}$$

It is clear from the Legendre–Fenchel transform

$$\max_{\xi\in\mathbb{R}^d}\left(\xi\cdot p - \frac{1}{\gamma^*}|\xi|^{\gamma^*}\right) \;=\; \frac{1}{\gamma}|p|^\gamma,$$

that, for any solution u of (EP), we have

$$\max_{\xi\in\mathbb{R}^d}\left[\mathcal{L}_\xi u(x) - \frac{1}{\gamma^*}|\xi|^{\gamma^*}\right] \;=\; -\Delta u(x) + \frac{1}{\gamma}|Du(x)|^\gamma \;=\; f(x) - \lambda,$$

and that the maximum is realized at $\xi(x) = |Du(x)|^{\gamma-2}Du(x)$.

The next lemma applies to any $\gamma>1$.

Lemma 2.3. *Let $\gamma>1$. Let u be a coercive solution of (EP) with eigenvalue λ. Define*

$$\xi_u(x) \;:=\; |Du(x)|^{\gamma-2}\ Du(x). \tag{2.18}$$

Then, there exists a Borel probability measure ν_u *on* $\mathfrak{B}(\mathbb{R}^d)$, *such that*

$$\mu_u(\mathrm{d}x, \mathrm{d}\xi) \;:=\; \nu_u(\mathrm{d}x)\delta_{\xi_u(x)}(\mathrm{d}\xi) \;\in\; \mathcal{M}, \tag{2.19}$$

and

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} F(x,\xi)\;\mu_u(\mathrm{d}x,\mathrm{d}\xi) \;\le\; \lambda. \tag{2.20}$$

In particular, $\overline{\lambda} \le \lambda$.

Proof. Using the definition in (2.17), we write

$$\mathcal{L}_{\xi_u(x)}u(x) \;=\; -\Delta u(x) + |Du(x)|^{\gamma-2}Du(x) \;=\; F(x,\xi)-\lambda. \tag{2.21}$$

Since u is coercive, we can apply [13, Theorem 1.2] to assert the existence of a unique $\nu_u \in \mathcal{P}(\mathbb{R}^d)$ which satisfies

$$\int_{\mathbb{R}^d} \mathcal{L}_{\xi_u(x)}f(x)\;\nu_u(\mathrm{d}x) \;=\; 0 \quad \forall f\in C_c^2(\mathbb{R}^d). \tag{2.22}$$

Thus (2.19) follows from (2.22) and the definition of μ_u, whereas (2.20) follows by integrating (2.21) with respect to ν_u using a cutoff function as in the proof of Lemma 2.2, which shows that $\int_{\mathbb{R}^d}\mathcal{L}_{\xi_u(x)}g(x)\;\mathrm{d}\mu_u \le 0$ for any positive function $g\in C^2(\mathbb{R}^d)$. □

Remark 2.2. Lemma 2.3 can also be established by a simple probabilistic argument. Viewing (2.19) as a Foster–Lyapunov equation, it is well known that the coercivity of u implies that the diffusion with extended generator $\mathcal{L}_{\xi_u(x)}$ is positive recurrent. The measure ν_u can then be specified as the unique invariant probability measure of this diffusion.

We now discuss some properties of the set $\mathcal{M}$ of infinitesimally invariant measures which are needed for the proof of Theorem 2.1 below. It is clear that every $\mu\in\mathcal{M}$ can be disintegrated into a probability measure $\nu(\mathrm{d}x)\in\mathcal{P}(\mathbb{R}^d)$ and a Borel measurable probability kernel $\eta(x,\mathrm{d}\xi)$ on $\mathbb{R}^d\times\mathfrak{B}(\mathbb{R}^d)$. We denote this disintegration by $\mu=\nu\circledast\eta$. For $\nu\circledast\eta\in\mathcal{M}$, define $\overline{\eta}(x) := \int_{\mathbb{R}^d}\xi\;\eta(x,\mathrm{d}\xi)$. Then $\overline{\eta}:\mathbb{R}^d\to\mathbb{R}^d$ is a Borel measurable map. It is straightforward to verify that $\nu\circledast\delta_{\overline{\eta}}$ is in $\mathcal{M}$. Since, by convexity, we have

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}|\xi|^{\gamma}\mu(\mathrm{d}x,\mathrm{d}\xi) \;\ge\; \int_{\mathbb{R}^d}|\overline{\eta}(x)|^{\gamma}\;\nu(\mathrm{d}x),$$

it is clear that the infimum in (LP) is attained at some $\mu\in\mathcal{M}$ whose disintegration results in a kernel $\eta(x,\,\cdot\,)$ which is Dirac for each $x\in\mathbb{R}^d$. Then, η can be represented as a Borel measurable map $v:\mathbb{R}^d\to\mathbb{R}^d$, and vice-versa. We denote the class of such measures as $\overline{\mathcal{M}}$, and abusing the notation we represent them as $\mu=\nu\circledast v$. Consider such a $\mu=\nu\circledast v$ in $\overline{\mathcal{M}}\cap\mathcal{M}_F$. It follows by (2.16) that $\int_{\mathbb{R}^d}|v(x)|^{\gamma^*}\;\eta(\mathrm{d}x)\;<\;\infty$. Since $\gamma^*>2$, this implies that $\int_{\mathbb{R}^d}|v(x)|^2\;\eta(\mathrm{d}x)\;<\;\infty$, and thus ν has density $\varrho\in L^{d/(d-1)}(\mathbb{R}^d)$ with respect to the Lebesgue measure by [14, Theorem 1.1].

We continue with our main theorem. Recall the definition of λ^* in (1.1), and that $\mathcal{M}_F^*$ denotes the subset of $\mathcal{M}$ consisting of points that attain the infimum in (LP).

Theorem 2.1. *Assume* (A1). *The following hold.*

(a) *For any positive solution u of (EP) with eigenvalue λ we have*

$$\mu_u(F) \;=\; \lambda \;=\; \overline{\lambda} \;=\; \lambda^*,$$

with μ_u as in Lemma 2.3.

(b) *The set $\mathcal{M}_F^*$ is a singleton.*

(c) *There exists a unique positive solution of (EP).*

Proof. By Lemma 2.1 every positive solution of (EP) is coercive. Therefore, the first two equalities in part (a) follow by Lemmas 2.2 and 2.3. By [1, Theorem 2.6], there exists a solution with eigenvalue λ^* which is bounded from below. This of course implies $\overline{\lambda} = \lambda^*$ thus completing the proof of part (a).

Let ν_u and ξ_u be as in Lemma 2.3. Let $\mu = \nu \circledast v$ be any element of $\mathcal{M}_F^* \cap \overline{\mathcal{M}}$. By the discussion in the paragraph preceding the theorem, ν has a density $\varrho \in L^{d/(d-1)}(\mathbb{R}^d)$ with respect to the Lebesgue measure. Let ϱ_u denote the density of ν_u, which, as well known, is strictly positive. Let

$$\zeta \;:=\; \frac{\rho}{\rho+\rho_u}\;,\quad \zeta_u \;:=\; \frac{\rho_u}{\rho+\rho_u}\;,\quad \overline{v} \;:=\; \zeta v + \zeta_u \xi_u\;,\quad \text{and } \overline{\nu} \;:=\; \frac{1}{2}(\nu+\nu_u).$$

It is straightforward to verify that $\overline{\nu} \circledast \overline{v} \in \overline{\mathcal{M}} \cap \mathcal{M}_F$.

By optimality, we have

$$\begin{aligned}
0 &\leq \int_{\mathbb{R}^d} |\overline{v}|^{\gamma^*}\, \mathrm{d}\overline{\nu} - \frac{1}{2}\int_{\mathbb{R}^d} |v|^{\gamma^*}\, \mathrm{d}\nu - \frac{1}{2}\int_{\mathbb{R}^d} |\xi_u|^{\gamma^*}\, \mathrm{d}\nu_u \\
&= \int_{\mathbb{R}^d} |\zeta\; v + \zeta_u\; \xi_u|^{\gamma^*}\, \mathrm{d}\overline{\nu} - \frac{1}{2}\int_{\mathbb{R}^d} |v|^{\gamma^*}\, \mathrm{d}\nu - \frac{1}{2}\int_{\mathbb{R}^d} |\xi_u|^{\gamma^*}\, \mathrm{d}\nu_u \\
&= \int_{\mathbb{R}^d} \left(|\zeta\; v + \zeta_u\; \xi_u|^{\gamma^*} - \zeta\; |v|^{\gamma^*} - \zeta_u\; |\xi_u|^{\gamma^*}\right) \mathrm{d}\overline{\nu} \;\leq\; 0
\end{aligned} \tag{2.23}$$

by convexity. Thus $\overline{\nu} \circledast \overline{v} \in \mathcal{M}_F^*$. Since ρ_u is strictly positive, (2.23) implies that $v = \xi_u$ on the support of ϱ. It is clear that if v is modified outside the support of ϱ, then the modified measure is also infinitesimally invariant for $\mathcal{A}$. Therefore $\nu \circledast \xi_u \in \mathcal{M}_F^*$. The uniqueness of a probability measure satisfying (2.22) then implies that $\nu = \nu_u$, which in turn implies (since $v = \xi_u$ on the support of ν) that $v = \xi_u$ a.e. in $\mathbb{R}^d$. This completes the proof of part (b).

Turning to part (c), existence of a positive solution follows from [1, Theorem 2.6]. By part (b), for any positive solutions u and w, we have $\xi_u = \xi_w$ a.e. in $\mathbb{R}^d$, implying that $Du = Dw$ on $\mathbb{R}^d$. Thus the solution is unique up to an additive constant. This completes the proof. □

3. More general Hamiltonians

In this section we consider viscous equations taking the form

$$-\Delta u(x) + H(x, Du) \;=\; f(x) - \lambda, \tag{3.1}$$

with more general Hamiltonians H. We adopt the following assumptions.

(A2) *The function* f *is in* $C^2(\mathbb{R}^d)$ *and is coercive. The Hamiltonian* H *satisfies the following.*

i. $H \in C^2(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}))$, *and* $p \mapsto H(x,p)$ *is strictly convex for all* $x \in \mathbb{R}^d$.
ii. *There exist constants* $h_0 > 0$ *and* $\gamma > 1$, *such that*

$$h_0^{-1}|p|^\gamma - h_0 \;\leq\; H(x,p) \;\leq\; h_0\big(|p|^\gamma + 1\big)\;, \quad |D_x H(x,p)| \;\leq\; h_0^{-1}\big(1 + |p|^\gamma\big).$$

The hypothesis (A2) is equivalent to (A2′) below for the Lagrangian L, which is related to H via the Fenchel–Legendre transform, that is,

$$H(x,p) \;=\; \sup_{\xi \in \mathbb{R}^d} \;\big(\xi \cdot p - L(x,\xi)\big).$$

(A2′) *The function* f *is in* $C^2(\mathbb{R}^d)$ *and is coercive. The Lagrangian* L *satisfies the following.*

i. $L \in C^2(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}))$, and $\xi \mapsto L(x,\xi)$ is strictly convex for all $x \in \mathbb{R}^d$.
ii. *There exist constants* $l_0 > 0$ *and* $\gamma^* > 1$, *such that*

$$l_0^{-1}|\xi|^{\gamma^*} - l_0 \;\leq\; L(x,\xi) \;\leq\; l_0\Big(|\xi|^{\gamma^*} + 1\Big)\;, \quad |D_x L(x,\xi)| \;\leq\; l_0^{-1}\Big(1 + |\xi|^{\gamma^*}\Big).$$

In addition, under (A2) or (A2′), there exists positive constants h_1 and l_1 such that

$$\begin{aligned} h_1|p|^{\gamma-1} - h_1^{-1} \;&\leq\; |D_p H(x,p)| \;\leq\; h_1^{-1}\big(|p|^{\gamma-1} + 1\big)\;, \\ l_1|\xi|^{\gamma^*-1} - l_1^{-1} \;&\leq\; |D_\xi L(x,\xi)| \;\leq\; l_1^{-1}\Big(|\xi|^{\gamma^*-1} + 1\Big)\;, \end{aligned} \tag{3.2}$$

for all$(x,p,\xi) \in \mathbb{R}^{3d}$, and

$$H(x,p) + L(x,\xi) \;\geq\; \xi \cdot p,$$

with equality if and only if $\xi = D_p H(x,p)$ or $p = D_\xi L(x,\xi)$.

The model above is slightly more general than the model in [2, 3]. A more restrictive assumption on H is used in [2], while H does not depend on x in [3]. For the properties mentioned above see [2, Theorem 3.4] and [3, Proposition 4.1].

As mentioned earlier, [2] imposes the assumptions in (1.2) for f, for both the subquadratic and superquadratic cases. Barles in [1] uses (1.2) only for the subquadratic case, while [3] does not consider unbounded f for the subquadratic case. Analogous is the model in [5, Section 4.6].

The results for this Hamiltonian are essentially the same as those in Section 2. We need the following ramification of [2, Theorem B.1] analogous to Corollary 2.1 valid for solutions of (3.1).

Corollary 3.1. Assume (A2). *Then, there exists a constant* C *such that any solution* u *of (3.1) satisfies (2.1).*

Proof. A closer inspection of the proof of [2, Theorem B.1] reveals that the following is established. Let $g \in C^2(\mathbb{R}^d)$ be a coercive function. There exists a function $C : (0,\infty)^2 \to (0,\infty)$ such that if $\varphi \in C^3(\mathbb{R}^d)$ satisfies

$$\begin{aligned} -D(\Delta\varphi) &\le c_1\bigl(1+|D\varphi|^\gamma+|Dg|\bigr)\,, \\ |D\varphi|^\gamma &\le c_2\bigl(|\Delta\varphi|+|g|\bigr)\,, \end{aligned} \tag{3.3}$$

for a pair of positive constants (c_1, c_2), then

$$\sup_{y\in B_1(x)} |Du(y)| \le C(c_1,c_2)\left(1+\sup_{y\in B_2(x)}(g(y))_+^{1/\gamma}+\sup_{y\in B_2(x)}|Dg(y)|^{1/(2\gamma-1)}\right) \tag{3.4}$$

for all $x\in\mathbb{R}^d$. We use scaling. With u a solution of (3.1), we define $u_r(y) := r^{\frac{2-\gamma}{\gamma-1}}u(x+ry)$. Using (A2) (ii) and (3.2), we deduce that u_r and $g\equiv r^{\gamma^*}(f(x+ry)-\lambda)$ satisfy (3.3) for all $r\in(0,1]$ and for constants c_1 and c_2 which do not depend on r. The result then follows by (3.4).

For the model in (3.1), we define

$$F(x,\xi) := f(x)+L(x,\xi), \tag{3.5}$$

and $\mathcal{M}_F$ and $\overline{\lambda}$ as in (1.5) and (LP), respectively, relative to F in (3.5). We also let

$$\xi_u(x) := D_pH(x,p). \tag{3.6}$$

Recall that $\mathcal{M}_F^*$ is the set of measures in $\mathcal{M}$ that attain the infimum in (LP).

Theorem 3.1. *Assume* (A1)–(A2) *and* $\gamma\in(1,2)$. *Then*

(a) *The conclusions of Lemma 2.1 hold.*
(b) *For any positive solution $u\in C^2(\mathbb{R}^d)$ of (3.1) with eigenvalue λ and $\mu\in\mathcal{M}_F$, we have*

$$\mu(F)-\lambda = \int_{\mathbb{R}^d\times\mathbb{R}^d}\left(L(x,\xi)-\xi\cdot Du(x)+\frac{1}{\gamma}|Du(x)|^\gamma\right)\,\mu(\mathrm{d}x,\mathrm{d}\xi) \ge 0, \tag{3.7}$$

and there exists a Borel probability measure ν_u on $\mathfrak{B}(\mathbb{R}^d)$, such that, with ξ_u as defined in (3.6), we have

$$\mu_u(\mathrm{d}x,\mathrm{d}\xi) := \nu_u(\mathrm{d}x)\delta_{\xi_u(x)}(\mathrm{d}\xi) \in \mathcal{M}, \tag{3.8}$$

and

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}F(x,\xi)\,\mu_u(\mathrm{d}x,\mathrm{d}\xi) = \overline{\lambda}.$$

In particular, $\lambda=\overline{\lambda}$.
(c) *We have $\mathcal{M}_F^*=\{\mu_u\}$, with μ_u as in (3.8).*
(d) *There exists at most one positive solution of (3.1).*

Proof. Part (a) follows as in Lemma 2.1 with a slight modification. Instead of (2.8), we use the inequality

$$-\Delta u_n(y)+\xi_{n,u}(y)\cdot Du_n(y) \ge f_n(y),$$

with $\xi_{n,u}(y) := \Gamma_n\,\xi_u(x_n+\Gamma_n y)$, and ξ_u as in (3.6). Then we apply (3.2) and Corollary 3.1, and follow the proof of Lemma 2.1.

For part (b), we define

$$G(x,\xi,p) \ := \ H(x,p) - \xi\cdot p + L(x,\xi) \ \geq \ 0,$$

and write

$$\begin{aligned}\Delta u(x) - \xi\cdot Du(x) \ &= \ \lambda + H(x,Du) - f(x) - \xi\cdot Du(x)\\ &= \ \big(\lambda - L(x,\xi) - f(x)\big) + G(x,\xi,Du),\end{aligned}$$

and following the proof of Lemma 2.2, we obtain

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \big(L(x,\xi)+f(x)\big)\ \mu(\mathrm{d}x,\mathrm{d}\xi) - \lambda \ = \ \int_{\mathbb{R}^d\times\mathbb{R}^d} G(x,\xi,Du(x))\ \mu(\mathrm{d}x,\mathrm{d}\xi) \ \geq \ 0.$$

The remaining assertions in (b) follow by Lemma 2.3, with ξ_u as defined in (3.6).

Part (c) follows as in Lemma 2.3 with a slight difference. For $\gamma > 2$ we don't know a priori that $|\xi|^2$ is integrable under a measure in $\mathcal{M}_F$. So instead of the densities ζ and ζ_u we use the Radon–Nikodym derivatives. □

Part (d) follows from parts (b) and (c). This concludes the proof.

Remark 3.1. Theorem 3.1 does not address existence of a positive solution to (3.1). For Hamiltonians not depending on x, existence is asserted in [3]. In general, under some additional assumptions, we can show that there exists a positive solution to (3.1). In addition to (A1)–(A2), we also assume that for any bounded C^2 domain D, there exists a constant $\beta > 0$ satisfying the following: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|H(x,\xi) - \beta|\xi|^\gamma| \leq \varepsilon\Big(|\xi|^\gamma + (\mathrm{dist}(x,\partial D))^{-\gamma^*}\Big), \quad \text{whenever } \mathrm{dist}(x,\partial D) < \delta \text{ and } \xi\in\mathbb{R}^d. \tag{3.9}$$

This is same as [15, (2.23)]. It then follows from [15, Theorem 2.15] that there exists a unique $v_D \in C^2(D)$ and a constant c_D satisfying

$$-\Delta v_D(x) + H(x, Dv_D(x)) \ = \ f(x) - c_D \quad \text{in } D, \quad \text{and} \quad \lim_{x\to\partial D} v_D(x) \ = \ \infty. \tag{3.10}$$

Furthermore, c_D is characterized as follows:

$$c_D \ = \ \sup\ \big\{c\in\mathbb{R}\ :\exists v\in \mathrm{W}^{1,2}(D)\cap L^\infty(D) \text{ such that } -\Delta v(x) + H(x,Dv(x)) - f(x) + c \leq 0\big\}.$$

Thus c_D is monotone decreasing as a function of D. Denote by (v_n, c_n) the solution pair of (3.10) corresponding to $D = B_n$, and let $x_n \in \operatorname{Arg\,min}\, v_n$. It follows from the equation above that

$$-c_n - H(x_n, 0) + f(x_n) \ = \ -\Delta v_n(x_n) \ \leq \ 0,$$

which implies that

$$c_n \ \geq \ -\min_{\mathbb{R}^d}\ (f - h_0)_-.$$

Let $c = \lim_{n\to\infty} c_n$ which exists by the above estimate. It also clear that v_n attains its minimum in a compact set independent of n. Thus we can follow a standard argument (see [1, Theorem 2.6]) to show that $v_n - \min_{\mathbb{R}^d} v_n \to v$ as $n\to\infty$, and

$$-\Delta v(x) + H(x, Dv(x)) \;=\; f(x) - c \quad \text{in } \mathbb{R}^d.$$

Assumption (3.9) is satisfied by a large class of Hamiltonian. For instance, consider

$$H(x,\xi) \;=\; b(x)\cdot\xi + \frac{1}{\gamma}|\xi|^\gamma$$

for some bounded function b. Then we can choose $\beta = \frac{1}{\gamma}$ above. Note that (3.9) follows from the estimate

$$\begin{aligned} b(x)\cdot\xi \;&\le\; \varepsilon|\xi|^\gamma + C\varepsilon^{-\frac{1}{\gamma-1}} \\ &\le\; \varepsilon\Big(|\xi|^\gamma + C\varepsilon^{-\frac{\gamma}{\gamma-1}}\Big), \end{aligned}$$

where the constant C depends on $||b||_\infty$ and γ. Now choose $\delta = C^{-\frac{\gamma-1}{\gamma}}\varepsilon$.

3.1. Remarks on the superquadratic case

Cirant in [3] is adopting (A2), except that in his model the Hamiltonian does not depend on x, that is, $H(x,p) = H(p)$. He also assumes that f and Df have at most polynomial growth. He shows that there always exists a positive solution to (3.1), and that this has at least linear growth.

For this model we can establish that $\int |u|^2 \; D\mu < \infty$ for all $\mu \in \mathcal{M}_F$, and that therefore, (3.7) holds. However, the proof of this differs from the proof of Lemma 2.2. We choose instead a smooth concave function χ such that $\chi(s) = s$ for $s \le 0$, and $\chi(s) = 1$ for $s \ge 2$, and we scale it by defining $\chi_t(s) := t + \chi(s-t)$ for $t \in \mathbb{R}$. Since $\gamma \ge 2$, and Du has polynomial growth, while u has at least linear growth, we can follow the argument in the proof of [16, Theorem 4.1] to conclude that $\int |u|^2 \; D\mu < \infty$ for all $\mu \in \mathcal{M}_F$. In [3], the set of admissible controls are required to satisfy $\limsup_{T\to\infty} \frac{\mathbb{E}^x[u(X_t)]}{T} = 0$. This is an unnecessary restriction on the class of admissible controls, and can be avoided. Without assuming that $H(p)$ is strictly convex, which might result in non-uniqueness for u, the approach summarized above, shows that ξ_u is an optimal Markov control and the corresponding infinitesimal measure is a minimizer of (LP). Thus, we have a strong notion of optimality as explained in Section 4. Under the additional assumption that $H(p)$ is strictly convex, the positive solution u, and therefore also the optimal Markov control are unique.

4. Implications for the ergodic control problem

The problem (EP) is associated with an ergodic control problem for the diffusion $X = (X_t)_{t\ge 0}$ given by the Itô stochastic differential equation

$$\mathrm{d}X_t = -\xi_t \; \mathrm{d}t + \sqrt{2} \; \mathrm{d}W_t, \quad X_0 = x \in \mathbb{R}^d. \tag{4.1}$$

This equation is specified on a complete, filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P}, (\mathfrak{F}_t)_{t\ge 0})$, with $(W_t)_{t\ge 0}$ an $(\mathfrak{F}_t)$-adapted d-dimensional Brownian motion. An *admissible control* is an $\mathbb{R}^d$-valued $(\mathfrak{F}_t)$-progressively measurable process $\xi = (\xi_t)_{t\ge 0}$, such that $\mathbb{E}[\int_0^T |\xi_t|^{\gamma^*} \; \mathrm{d}t] < \infty$ for all $T > 0$, and we let $\mathcal{A}$ denote the class of such controls. The *running cost* function is given by F in (1.4).

In [2], optimality is established via the study of the parabolic problem. In view of the optimality results concerning (LP), we can state a stronger version of optimality. We state this result for the model in (EP), noting that an identical argument can be used to establish this for (3.1) under (A2). We let $\mathbb{E}^x_\xi$ denote the expectation operator for the diffusion in (4.1) controlled by $\xi \in \mathrm{A}$ with initial condition $X_0 = x$.

Theorem 4.1. *Assume* (A1). *Let $u \in C^2(\mathbb{R}^d)$ be the unique positive solution of* (EP) *in the subquadratic case, or as in Section 3.1 for the superquadratic case. With ξ_u as in (2.18), we have*

$$\liminf_{T\to\infty} \inf_{\xi\in \mathrm{A}\mathcal{A}} \mathbb{E}^x_\xi\left[\int_0^T F(X_t, \xi_t)\, \mathrm{d}t\right] \geq \bar{\lambda} = \limsup_{T\to\infty} \mathbb{E}^x_{\xi_u}\left[\int_0^T F(X_t, \xi_u(X_t))\, \mathrm{d}t\right]. \tag{4.2}$$

Moreover, (4.2) holds without the expectation operators in the a.s. pathwise sense. In addition a Markov control $v : \mathbb{R}^d \to \mathbb{R}^d$ is optimal, if and only if it agrees with ξ_u a.e. in $\mathbb{R}^d$.

Proof. The inequality in (4.2) follows from the fact that limit points of mean empirical measures in $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ are infinitesimal measures for the operator $\mathcal{A}$ (see Lemma 3.4.6 and Theorem 3.4.7 in [17]) together with the definition of $\bar{\lambda}$ in (LP). The equality follows by the ergodicity of the process under the control ξ_u and the fact that F is integrable under the invariant probability measure as asserted in Lemma 2.3. The pathwise results also follow from results in [17] referenced above. The verification part follows from Theorem 2.1. □

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References

[1] Barles, G., Meireles, J. (2016). On unbounded solutions of ergodic problems in $\mathbb{R}^m$ for viscous Hamilton-Jacobi equations. *Commun. Partial Differ. Equ.* 41(12):1985–2003.
[2] Ichihara, N. (2012). Large time asymptotic problems for optimal stochastic control with superlinear cost. *Stoch. Process. Appl.* 122(4):1248–1275.
[3] Cirant, M. (2014). On the solvability of some ergodic control problems in $\mathbb{R}^d$. *SIAM J. Control Optim.* 52(6):4001–4026.

[4] Barles, G., Souganidis, P. E. (2001). Space-time periodic solutions and long-time behavior of solutions to quasi-linear parabolic equations. *SIAM J. Math. Anal.* 32(6):1311–1323.

[5] Bensoussan, A., Frehse, J. (2002). Regularity results for nonlinear elliptic systems and applications, Applied Mathematical Sciences, vol. 151, Berlin: Springer.

[6] Bensoussan, A., Nagai, H. (1991). An ergodic control problem arising from the principal eigenfunction of an elliptic operator. *J. Math. Soc. Jpn.* 43(1):49–65.

[7] Barles, G., Porretta, A., Tchamba, T. T. (2010). On the large time behavior of solutions of the dirichlet problem for subquadratic viscous Hamilton-Jacobi equations. *J. Math. Pures Appl. (9)* 94(5):497–519.

[8] Fujita, Y., Ishii, H., Loreti, P. (2006). Asymptotic solutions of viscous Hamilton-Jacobi equations with Ornstein-Uhlenbeck operator. *Commun. Partial Differ. Equ.* 31(6):827–848.

[9] Ichihara, N., Sheu, S.-J. (2013). Large time behavior of solutions of Hamilton-Jacobi-Bellman equations with quadratic nonlinearity in gradients. *SIAM J. Math. Anal.* 45(1): 279–306.

[10] Bensoussan, A., Frehse, J. (1992). On Bellman equations of ergodic control in $\mathbf{R}^n$. *J. Reine Angew. Math.* 429:125–160.

[11] Lasry, J.-M., Lions, P.-L. (1989). Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The Model Problem. *Math. Ann*283(4):583–630.

[12] Arapostathis, A., Ghosh, M. K., Marcus, S. I. (1999). Harnack's inequality for cooperative weakly coupled elliptic systems. *Commun. Partial Differ. Equ.* 24(9-10):1555–1571.

[13] Bogachev, V. I., Rökner, M. (2000). A generalization of Khaśminskiĭ's theorem on the existence of invariant measures for locally integrable drifts. *Teor. Veroyatnost. i Primenen* 45(3):417–436.

[14] Bogachev, V. I., Krylov, N., Röckner, M. (1996). Regularity of invariant measures: the case of non-constant diffusion part. *J. Funct. Anal.* 138(1):223–242.

[15] Leonori, T., Porretta, A. (2016). Large solutions and gradient bounds for quasilinear elliptic equations. *Comm. Partial Differ. Equ.* 41(6):952–998. MR3509520

[16] Arapostathis, A., Biswas, A., Borkar, V. S., Suresh Kumar, K. (2019). A variational characterization of the risk-sensitive average reward for controlled diffusions in $\mathbb{R}^d$, arXiv e-prints 1903.08346 available at https://arxiv.org/abs/1903.08346.

[17] Arapostathis, A., Borkar, V. S., Ghosh, M. K. (2012). Ergodic control of diffusion processes, Encyclopedia of Mathematics and Its Applications, vol. 143. Cambridge: Cambridge University Press.