A VARIATIONAL FORMULA FOR RISK-SENSITIVE CONTROL OF DIFFUSIONS IN $\mathbb{R}^d$

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Abstract. We address the variational problem for the generalized principal eigenvalue on $\mathbb{R}^d$ of linear and semilinear elliptic operators associated with nondegenerate diffusions controlled through the drift. We establish the Collatz–Wielandt formula for potentials that vanish at infinity under minimal hypotheses, and also for general potentials under blanket geometric ergodicity assumptions. We also present associated results having the flavor of a refined maximum principle.

Key words. Principal eigenvalue; (Agmon) ground state; semilinear PDE; Donsker–Varadhan functional; Collatz–Wielandt formula; min-max formula

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1. Introduction. Since the seminal work of Donsker and Varadhan [13, 14], a lot of effort has been devoted to variational characterizations of principal eigenvalues of elliptic operators. More recently, the work of Berestycki, Nirenberg, and Varadhan [7] opened up the study of generalized eigenvalues in unbounded domains (see also [9]), while advances in nonlinear Perron–Frobenius theory [15, 16] made possible the extension of the classical Collatz–Wielandt formula for the Perron–Frobenius eigenvalue of irreducible nonnegative matrices to more abstract settings. See also [17, Chapter 3] for a Collatz–Wielandt formula for symmetric second-order operators in bounded domains.

The motivation for this work is the infinite horizon risk-sensitive control problem on the entire domain, which seeks to minimize the asymptotic growth rate of the expected "exponential of integral" cost, and which, under suitable assumptions, coincides with the generalized principal eigenvalue of the associated semilinear elliptic operator (for some recent results see [1, 3]). Recall the celebrated formula of Donsker and Varadhan: for a uniformly elliptic nondivergence form operator $\mathcal{L}$ on a smooth bounded domain $D \subset \mathbb{R}^d$, the principal eigenvalue $\lambda_1(\mathcal{L}, D)$ can be expressed as

$$\lambda_1(\mathcal{L}, D) = \sup_{\varphi \in C^{2,+}(D)} \inf_{\mu \in \mathcal{P}(\bar{D})} \int_D \frac{\mathcal{L}\varphi(x)}{\varphi(x)} \, \mu(\mathrm{d}x),$$

where $\mathcal{P}(\bar{D})$ denotes the set of Borel probability measures on $\bar{D}$, and $C^{2,+}(D)$ the space of positive functions in $C^2(D) \cap C(\bar{D})$. Taking the supremum over measures, followed by the infimum over the function space, also results in equality, and this forms an extension of the classical Collatz–Wielandt formula. For versions of this formula for nonlinear operators on a bounded domain see [6, 18].

The Collatz–Wielandt formula for a reflected controlled diffusion on a bounded domain has been studied in [5] with the aid of nonlinear versions of the Krein–Rutman theorem. Establishing this min-max formula over $\mathbb{R}^d$ is quite challenging not only due to the lack of compactness but also because the generalized principal eigenvalue of an operator does not enjoy all the structural properties of eigenvalues over bounded domains. We take a different approach which is based on the stochastic representation

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of principal eigenfunctions to obtain several variational formulations of the principal eigenvalues. For potentials that vanish at infinity, we exhibit the Collatz–Wielandt formula under minimal assumptions (see Theorem 2.6). For more general potentials, we impose blanket geometric ergodicity assumptions to handle the lack of compactness (see Assumptions 2.9 and 2.10) and establish the formula in Theorem 2.13. We then continue with two results in the flavor of a refined maximum principle (see Theorems 2.15 and 2.16) and conclude the study with some characterizations of the generalized principal eigenvalue (Theorems 2.17 to 2.19). The proofs of these results are in section 3.

We would also like to mention the recent work in [2] which studies the maximization of the risk-sensitive average reward on the whole space, without employing any blanket ergodicity assumptions. The approach in [2] leads to a concave maximization problem, which is quite different from the "sup–inf" and "inf–sup" formulas in the current paper.

2. Assumptions and main results.

2.1. The controlled diffusion model. Consider a controlled diffusion process $X = \{X_t,\, t \geq 0\}$ which takes values in the d-dimensional Euclidean space $\mathbb{R}^d$ and is governed by the Itô equation

$$(2.1) \qquad \mathrm{d}X_t \;=\; b(X_t, U_t)\, \mathrm{d}t + \sigma(X_t)\, \mathrm{d}W_t\,.$$

All random processes in (2.1) live in a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. The process W is a d-dimensional standard Wiener process independent of the initial condition X_0. The control process U takes values in a compact, metrizable set $\mathbb{U}$, and $U_t(\omega)$ is jointly measurable in $(t, \omega) \in [0, \infty) \times \Omega$. The set $\mathfrak{U}$ of *admissible controls* consists of the control processes U that are *nonanticipative*: for $s < t$, $W_t - W_s$ is independent of

$$\mathfrak{F}_s \;:=\; \text{the completion of } \cap_{y>s}\, \sigma\{X_0, U_r, W_r,\ r \leq y\} \text{ relative to } (\mathfrak{F}, \mathbb{P})\,.$$

We impose the following standard assumptions on the drift b and the diffusion matrix σ to guarantee existence and uniqueness of solutions.

(A1) *Local Lipschitz continuity:* The functions $b\colon \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d \times d}$ are continuous and satisfy

$$|b(x, u) - b(y, u)| + \|\sigma(x) - \sigma(y)\| \;\leq\; C_R\, |x - y| \qquad \forall\, x, y \in B_R\,,\ \forall\, u \in \mathbb{U}\,,$$

for some constant $C_R > 0$ depending on $R > 0$.

(A2) *Affine growth condition:* For some $C_0 > 0$, we have

$$\sup_{u \in \mathbb{U}} \langle b(x, u), x \rangle^+ + \|\sigma(x)\|^2 \;\leq\; C_0\big(1 + |x|^2\big) \qquad \forall\, x \in \mathbb{R}^d\,.$$

(A3) *Nondegeneracy:* For each $R > 0$, it holds that

$$\sum_{i,j=1}^{d} a^{ij}(x)\xi_i \xi_j \geq C_R^{-1} |\xi|^2 \qquad \forall x \in B_R\,,$$

and for all $\xi = (\xi_1, \ldots, \xi_d)^{\mathsf{T}} \in \mathbb{R}^d$, where $a = \frac{1}{2}\sigma\sigma^{\mathsf{T}}$.

It is well known that under (A1)–(A3), for any admissible control there exists a unique solution of (2.1) [4, Theorem 2.2.4]. We define the family of operators $\mathcal{L}_u \colon C^2(\mathbb{R}^d) \to C(\mathbb{R}^d)$, where $u \in \mathbb{U}$ plays the role of a parameter, by

$$\mathcal{L}_u f(x) \coloneqq a^{ij}(x)\,\partial_{ij}f(x) + b^i(x,u)\,\partial_i f(x)\,, \quad u \in \mathbb{U}\,,\ x \in \mathbb{R}^d\,.$$

Here we adopt the notation $\partial_i \coloneqq \frac{\partial}{\partial x_i}$ and $\partial_{ij} \coloneqq \frac{\partial^2}{\partial x_i \partial x_j}$ for $i,j \in \{1,\dots,d\}$, and we often use the standard summation rule that repeated subscripts and superscripts are summed from 1 through d. Let $c(x,u)$ be a function in $C(\mathbb{R}^d \times \mathbb{U}, \mathbb{R})$ that is locally Lipschitz in x uniformly with respect to $u \in \mathbb{U}$ and is bounded below in $\mathbb{R}^d$. We consider the following semilinear operator:

$$(2.2) \qquad \mathcal{G}f(x) \coloneqq a^{ij}(x)\,\partial_{ij}f(x) + \min_{u \in \mathbb{U}}\big[b^i(x,u)\,\partial_i f(x) + c(x,u)f(x)\big]\,, \quad x \in \mathbb{R}^d\,.$$

We remark that as far as the results of the paper are concerned, local Lipschitz continuity of $x \mapsto c(x,u)$ may be relaxed to local Hölder continuity.

2.2. Statements of the main results.. Let D be a smooth bounded domain. Without any loss of generality we assume that $0 \in D$. The principal eigenvalue of $\mathcal{G}$ with Dirichlet boundary condition is defined as follows:

$$(2.3) \qquad \lambda_D(\mathcal{G}) \coloneqq \inf\big\{\lambda : \exists\,\psi \in C(\bar{D}) \cap C^2(D),\ \psi > 0 \text{ in } D,$$
$$\text{satisfying } \mathcal{G}\psi - \lambda\psi \le 0 \text{ in } D\big\}\,.$$

It is then known from [18, Theorem 1.1] that there exists a unique $\Psi = \Psi_D \in C(\bar{D}) \cap C^2(D)$ with $\Psi(0) = 1$, $\Psi > 0$ in D, which satisfies

$$(2.4) \qquad \mathcal{G}\Psi = \lambda_D(\mathcal{G})\Psi \quad \text{in } D\,, \quad \Psi = 0 \quad \text{on } \partial D\,.$$

By $C^{2,+}(D)$ we denote the set of functions in $C^2(D) \cap C(\bar{D})$ that are positive in D, and $C_0^{2,+}(D)$ denotes the collection of functions in $C^{2,+}(D)$ that vanish on ∂D. Our first result establishes a Collatz–Wielandt formula for λ_D. The representation (2.6) below can also be found in [6], where it plays a crucial role in obtaining necessary and sufficient conditions for the solvability of certain Dirichlet problems.

THEOREM 2.1. *Let $D \subset \mathbb{R}^d$ be a smooth bounded domain. Then*

$$(2.5) \qquad \lambda_D(\mathcal{G}) = \sup_{\psi \in C_0^{2,+}(D)} \inf_{\mu \in \mathcal{P}(D)} \int_D \frac{\mathcal{G}\psi}{\psi}\,\mathrm{d}\mu$$

$$(2.6) \qquad = \inf_{\psi \in C^{2,+}(D)} \sup_{\mu \in \mathcal{P}(D)} \int_D \frac{\mathcal{G}\psi}{\psi}\,\mathrm{d}\mu\,,$$

where $\mathcal{P}(A)$ denotes the set of all Borel probability measures on the set A.

Remark 2.2. The function space $C_0^{2,+}(D)$ in the representation formula (2.5) cannot, in general, be enlarged to $C^{2,+}(D)$. To see this consider any smooth domain D_1 strictly containing D. Let $\lambda_1 = \lambda_{D_1}$ be the Dirichlet principal eigenvalue of $\mathcal{G}$ in D_1, and let Ψ_1 denote the corresponding (positive) principal eigenfunction. It is known that $\lambda_1 > \lambda_D(\mathcal{G})$ [18, Remark 3]. Take $\psi = \Psi + \Psi_1$. Then by the concavity of $\mathcal{G}$, we have

$$\inf_{\mu \in \mathcal{P}(D)} \int_D \frac{\mathcal{G}\psi}{\psi} \ge \min_{\bar{D}} \frac{\lambda_1\Psi_1 + \lambda_D(\mathcal{G})\Psi}{\Psi_1 + \Psi} > \lambda_D\,.$$

Our next goal is to establish a similar characterization for the generalized principal eigenvalue of $\mathcal{G}$ in $\mathbb{R}^d$. To begin with, we consider the uncontrolled problem. In this case, we have a linear operator of the form

$$(2.7) \qquad \mathcal{L}f(x) \;\coloneqq\; a^{ij}(x)\,\partial_{ij}f(x) + b^i(x)\,\partial_i f(x) + c(x)f(x)\,, \quad x \in \mathbb{R}^d\,.$$

Here, we assume that b, c are locally bounded, Borel measurable functions, and that a is continuous and satisfies (A3). We recall the definition of the principal eigenvalue of $\mathcal{L}$ from [9], denoted as $\lambda^*(\mathcal{L})$.

$$(2.8) \qquad \begin{aligned} \lambda^*(\mathcal{L}) \;\coloneqq\; \inf\big\{ \lambda \in \mathbb{R} : \exists\,\psi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d),\ \psi > 0, \\ \text{satisfying } \mathcal{L}\psi - \lambda\psi \le 0 \text{ a.e. in } \mathbb{R}^d \big\}\,. \end{aligned}$$

Note the analogy between (2.3) and (2.8).

We start by showing that if $\mathcal{L}$ has smooth coefficients, and $\lambda^*(\mathcal{L}) < \infty$, then

$$(2.9) \qquad \lambda^*(\mathcal{L}) \;=\; \inf_{\psi \in C^{2,+}(\mathbb{R}^d)}\ \sup_{\mu \in \mathcal{P}(\mathbb{R}^d)}\ \int_{\mathbb{R}^d} \frac{\mathcal{L}\psi}{\psi}\,\mathrm{d}\mu\,.$$

This is essentially in (1.12)–(1.13) of [7]. We can prove this from the definition of $\lambda^*(\mathcal{L})$ and the existence of an eigenfunction, or we can use the following argument. If not, then there exist $\psi \in C^{2,+}(\mathbb{R}^d)$ and $\epsilon > 0$ such that $\mathcal{L}\psi < (\lambda^*(\mathcal{L}) - \epsilon)\psi$ on $\mathbb{R}^d$. Let λ_n denote the principal eigenvalue of $\mathcal{L}$ in B_n, and choose n large enough so that $\lambda_n > \lambda^*(\mathcal{L}) - \epsilon$. With ψ_n denoting the principal eigenfunction on B_n we have

$$\mathcal{L}(\psi - \psi_n) - \lambda_n(\psi - \psi_n) \;\le\; (\lambda^*(\mathcal{L}) - \epsilon - \lambda_n)\psi\,.$$

Scaling ψ so that it touches ψ_n at some point from above, and applying the strong maximum principle, we obtain $\psi = \psi_n$ on B_n, which is not possible since ψ_n vanishes on ∂B_n. The analogous result holds for the semilinear operator $\mathcal{G}$.

We next show that the Collatz–Wielandt formula in (2.5) does not hold, in general, for $\lambda^*(\mathcal{L})$. Consider the generalized eigenvalues $\lambda'(\mathcal{L})$ and $\lambda''(\mathcal{L})$ defined by

$$\begin{aligned} \lambda'(\mathcal{L}) \;\coloneqq\; \sup\big\{ \lambda \in \mathbb{R} : \exists\,\psi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d),\ \psi > 0, \\ \text{satisfying } \mathcal{L}\psi - \lambda\psi \ge 0 \text{ a.e. in } \mathbb{R}^d \big\}\,, \end{aligned}$$

$$\begin{aligned} \lambda''(\mathcal{L}) \;\coloneqq\; \inf\big\{ \lambda \in \mathbb{R} : \exists\,\psi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d),\ \inf_{\mathbb{R}^d}\psi > 0, \\ \text{satisfying } \mathcal{L}\psi - \lambda\psi \le 0 \text{ a.e. in } \mathbb{R}^d \big\}\,. \end{aligned}$$

It is known that, in general, $\lambda^*(\mathcal{L}) \le \lambda'(\mathcal{L})$ (see [9, Theorem 1.7]). But this inequality might be strict [8].

Example 2.3. We borrow this example from [8]. Consider the operator $\mathcal{L}\phi \coloneqq \phi'' - \phi'$, with $d = 1$. If $\psi \in L^\infty(\mathbb{R})$ satisfies $\mathcal{L}\psi - \lambda\psi \ge 0$, then applying the Itô–Krylov formula we obtain $\psi(x) \le \mathrm{e}^{-\lambda t}\|\psi\|_\infty$ for all $t \ge 0$. Taking logarithms, it follows that $\lambda \le 0$. On the other hand, for $\psi = 1$ we have $\mathcal{L}\psi = 0$, and therefore we obtain

$$\sup_{\psi \in C^{2,+}_b(\mathbb{R})}\ \inf_{\mu \in \mathcal{P}(\mathbb{R})}\ \int_{\mathbb{R}} \frac{\mathcal{L}\psi}{\psi}\,\mathrm{d}\mu \;\ge\; 0 \;=\; \lambda'(\mathcal{L})\,.$$

For $R > 0$, with $\phi_R(x) = \cos(\frac{\pi}{2R}x)\exp(\frac{x}{2})$, we have

$$\mathcal{L}\phi_R = -\left(\frac{1}{4} + \frac{\pi^2}{4R^2}\right)\phi_R \quad \text{in } [-R, R].$$

Using [8, Proposition 3.1] we deduce that $\lambda^*(\mathcal{L}) = \lim_{R\to\infty} -(\frac{1}{4} + \frac{\pi^2}{4R^2}) = -\frac{1}{4}$. Thus we obtain

$$\lambda^*(\mathcal{L}) < \sup_{\psi \in C_b^{2,+}(\mathbb{R})} \inf_{\mu \in \mathcal{P}(\mathbb{R})} \int_{\mathbb{R}} \frac{\mathcal{L}\psi}{\psi}\, \mathrm{d}\mu\,.$$

In analogy to (2.8) we define the principal eigenvalue on $\mathbb{R}^d$ of the semilinear operator $\mathcal{G}$ as follows

(2.10)
$$\lambda^*(\mathcal{G}) := \inf\big\{\lambda \in \mathbb{R} : \exists\, \psi \in C^2(\mathbb{R}^d),\ \psi > 0,$$
$$\text{satisfying } \mathcal{G}\psi - \lambda\psi \leq 0 \text{ a.e. in } \mathbb{R}^d\big\}\,.$$

As in the case of the linear operator, we have the following characterization of the principal eigenvalue for the semilinear operator.

LEMMA 2.4. *Let λ_n be the principal eigenvalue of $\mathcal{G}$ in B_n i.e., for some positive $\Psi_n \in C^2(B_n) \cap C(\bar{B}_n)$ we have*

$$\mathcal{G}\Psi_n = \lambda_n \Psi_n \quad \text{in } B_n\,, \quad \Psi_n = 0 \quad \text{on } \partial B_n\,.$$

Then $\lim_{n\to\infty} \lambda_n = \lambda^*(\mathcal{G})$.

Proof. In view of (2.3), we note that $\lambda_n \leq \lambda_{n+1}$ for all n. By the definition in (2.10) we have $\lambda_n \leq \lambda^*(\mathcal{G})$. Thus, $\lim_{n\to\infty} \lambda_n = \hat{\lambda} \leq \lambda^*(\mathcal{G})$. Using a standard argument of elliptic PDEs, we can find a positive $\hat{\Phi} \in C^2(\mathbb{R}^d)$ satisfying

$$\mathcal{G}\hat{\Phi} = \hat{\lambda}\,\hat{\Phi} \quad \text{in } \mathbb{R}^d\,.$$

See, for instance, [1, 10]. By (2.10), we then have $\lambda^*(\mathcal{G}) \leq \hat{\lambda}$. Therefore, $\lim_{n\to\infty} \lambda_n = \lambda^*(\mathcal{G})$. $\qquad\square$

Next, consider the extremal operator $\mathcal{H}$ defined by

(2.11) $$\mathcal{H}f(x) := a^{ij}(x)\,\partial_{ij}f(x) + \max_{u\in\mathbb{U}} \big[b^i(x,u)\,\partial_i f(x) + c(x,u)f(x)\big]\,.$$

This operator corresponds to the maximization problem of the risk-sensitive ergodic average. The principal eigenvalue $\lambda_D(\mathcal{H})$ is defined in the same fashion as in (2.3). We show that (2.6) holds for the operator $\mathcal{H}$ with the "inf" and "sup" in reverse order. This result is also used in Theorem 2.6 below.

THEOREM 2.5. *Let $\lambda_D(\mathcal{H})$ be the Dirichlet principal eigenvalue of $\mathcal{H}$, where D is a bounded smooth domain or $\mathbb{R}^d$. Then we have*

(2.12) $$\sup_{\mu\in\mathcal{P}(D)} \inf_{\psi\in C^{2,+}(D)} \int_D \frac{\mathcal{H}\psi}{\psi}\, \mathrm{d}\mu = \lambda_D(\mathcal{H})\,.$$

We return to the operators $\mathcal{L}$ and $\mathcal{G}$ to state the main results. Note that the process associated with the operator $\mathcal{L}$ in Example 2.3 is transient. Our first result establishes a Collatz–Wielandt formula for $\lambda^*(\mathcal{L})$, when the underlying process is

recurrent, and c is bounded. We let $C_b^{2,+}(\mathbb{R}^d) \coloneqq C_b^+(\mathbb{R}^d) \cap C^2(\mathbb{R}^d)$, where $C_b^+(\mathbb{R}^d)$ denotes the set of positive bounded functions on $\mathbb{R}^d$. Also, $C^{2,+}(\mathbb{R}^d)$ denotes the class of positive functions in $C^2(\mathbb{R}^d)$.

THEOREM 2.6. *Consider the linear operator $\mathcal{L}$ in (2.7), and assume that b and c are locally Hölder continuous, and c is a function that vanishes at infinity. Suppose that the process X is recurrent. Then, if $\lambda^*(\mathcal{L}) > 0$, we have*

$$
\begin{aligned}
\lambda^*(\mathcal{L}) &= \sup_{\psi \in C_b^{2,+}(\mathbb{R}^d)} \inf_{\nu \in \mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{\mathcal{L}\psi}{\psi} \, \mathrm{d}\nu \\
(2.13) \qquad &= \inf_{\psi \in C_b^{2,+}(\mathbb{R}^d)} \sup_{\nu \in \mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{\mathcal{L}\psi}{\psi} \, \mathrm{d}\nu \\
&= \sup_{\nu \in \mathcal{P}(\mathbb{R}^d)} \inf_{\psi \in C_b^{2,+}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{\mathcal{L}\psi}{\psi} \, \mathrm{d}\nu \,.
\end{aligned}
$$

In general, i.e., independent of the sign of $\lambda^*(\mathcal{L})$, (2.13) holds if we replace $C_b^{2,+}(\mathbb{R}^d)$ with $C^{2,+}(\mathbb{R}^d)$ in the second and third equalities. Moreover, the first equality also holds for $\lambda^*(\mathcal{L}) \le 0$. The analogous result for the first two equalities holds for the semilinear operator $\mathcal{G}$ in (2.2), provided that $\sup_{(x,u) \in B_n^c \times \mathbb{U}} c(x,u) \to 0$ as $n \to \infty$, and under the assumption that the process X is recurrent under any stationary Markov control.

Given a set A, the first exit time from A is denoted by

$$
\tau(A) = \inf\{t > 0 : X_t \notin A\} \,.
$$

For the first hitting time to the ball B_r we use the abbreviated notation $\breve{\tau}_r = \tau(B_r^c)$. We also let $\tau_r \coloneqq \tau(B_r)$.

Remark 2.7. Suppose $\lambda^*(\mathcal{L}) < 0$, $c \in C_0(\mathbb{R}^d)$, a and b are bounded, and the diffusion is geometrically ergodic. Then there is no $\psi \in C_b^{2,+}(\mathbb{R}^d)$ satisfying

$$
\sup_{\mu \in \mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{\mathcal{L}\psi}{\psi} \, \mathrm{d}\mu = \sup_{\mathbb{R}^d} \frac{\mathcal{L}\psi}{\psi} < 0 \,.
$$

Otherwise, we would have $\mathcal{L}\psi + 2\delta\psi \le 0$ for some $\delta > 0$. Applying Itô's formula and the fact $\lim_{|x| \to \infty} c(x) = 0$ we obtain

$$
\psi(x) \ge \mathbb{E}_x\left[e^{\delta \breve{\tau}_r} \psi(X_{\breve{\tau}_r}) \right] \quad \text{for large enough } r \,.
$$

But the right-hand side is unbounded, resulting in ψ being unbounded. This contradicts the fact $\psi \in C_b^{2,+}(\mathbb{R}^d)$. Thus in this case

$$
\lambda^*(\mathcal{L}) < \inf_{\psi \in C_b^{2,+}(\mathbb{R}^d)} \sup_{\mu \in \mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{\mathcal{L}\psi}{\psi} \, \mathrm{d}\mu \,.
$$

On the other hand, if X is null-recurrent and $c \in C_0(\mathbb{R}^d)$, then $\lambda^*(\mathcal{L})$ cannot be nonzero if the principal eigenfunction is bounded. For if Ψ^* is bounded, then applying Itô's formula it is easy to see that

$$
\mathbb{E}_x\left[\int_0^T \left(c(X_t)\Psi^*(X_t) - \lambda^*(\mathcal{L})\Psi^*(X_t) \right) \mathrm{d}t \right] = 0 \,.
$$

Note that $\inf_{\mathbb{R}^d} \Psi^* > 0$ by [1, Lemma 2.1], since $\lambda^*(\mathcal{L}) < 0$. Now divide both sides by T and let $T \to \infty$ to assert that $\lambda^*(\mathcal{L}) = 0$.

Remark 2.8. Theorem 2.6 offers a variational formula for the principal eigenvalue in the spirit of [14]. If we define $\mathcal{L}_0 f(x) := a^{ij}(x)\, \partial_{ij} f(x) + b^i(x)\, \partial_i f(x)$, and the rate function

$$I(\nu) := -\inf_{f \in C^{2,+}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{\mathcal{L}_0 f}{f}\, d\nu\,,$$

then

$$\lambda^*(\mathcal{L}) = \sup_{\nu \in \mathcal{P}(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} c(x)\, \nu(dx) - I(\nu) \right)\,.$$

Assumption 2.9. The following hold:
(i) There exist an inf-compact function $\ell \in C(\mathbb{R}^d)$, and a positive function $\mathcal{V} \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$, satisfying $\inf_{\mathbb{R}^d} \mathcal{V} > 0$, such that

$$\sup_{u \in \mathbb{U}} \mathcal{L}_u \mathcal{V} \leq \kappa_1 \mathbb{1}_{\mathcal{K}} - \ell \mathcal{V} \quad \text{in } \mathbb{R}^d$$

for some constant κ_1 and a compact set $\mathcal{K}$.
(ii) The function $x \mapsto \beta \ell(x) - \max_{u \in \mathbb{U}} c(x,u)$ is inf-compact for some $\beta \in (0,1)$.

As noted in [3], Assumption 2.9 does not hold for diffusions with bounded a, and b. Therefore, to treat this case, we consider an alternate set of conditions.

Assumption 2.10. The following hold:
(i) There exist a positive function $\mathcal{V} \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$, satisfying $\inf_{\mathbb{R}^d} \mathcal{V} > 0$, and a constant $\gamma > 0$ such that

$$(2.14) \qquad \sup_{u \in \mathbb{U}} \mathcal{L}_u \mathcal{V} \leq \kappa_1 \mathbb{1}_{\mathcal{K}} - \gamma \mathcal{V} \quad \text{in } \mathbb{R}^d$$

for some constant κ_1 and a compact set $\mathcal{K}$.
(ii) $\|c^-\|_\infty + \limsup_{|x| \to \infty} \max_{u \in \mathbb{U}} c(x,u) < \gamma$.

The eigenvalue $\lambda^*(\mathcal{G})$ in (2.10) represents the optimal risk-sensitive ergodic cost [1, 3, 10, 11]. In order to define this control problem, we need to introduce some additional notation. For an admissible control U, the risk-sensitive criterion is defined as

$$\mathcal{E}(U) = \inf_{x \in \mathbb{R}^d} \limsup_{T \to \infty} \frac{1}{T} \log \mathbb{E}^U_x \left[e^{\int_0^T c(X_s, U_s)\, ds} \right].$$

The optimal value is defined as $\Lambda^* = \inf_{U \in \mathfrak{U}} \mathcal{E}(U)$.

Notation 2.11. For a continuous function $g : \mathbb{R}^d \to (0,\infty)$ which is bounded below away from 0, we let $\mathcal{O}(g)$ denote the space of continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ satisfying $\sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{g(x)} < \infty$, and $\mathfrak{o}(g)$ its subset consisting of those functions which satisfy $\limsup_{R \to \infty} \sup_{x \in B_R^c} \frac{|f(x)|}{g(x)} = 0$.

We borrow the following result from [3].

THEOREM 2.12 ([3, Theorem 4.1]). *Suppose that either Assumption* 2.9 *or* 2.10 *holds. Then* $\Lambda^* = \lambda^*(\mathcal{G})$*, and for some function* $\Phi^* \in C^{2,+}(\mathbb{R}^d) \cap \mathcal{O}(\mathcal{V}^\beta)$*, for some* $\beta \in (0,1)$*, we have*

$$(2.15) \qquad \mathcal{G}\Phi^*(x) = \min_{u \in \mathbb{U}} \left[\mathcal{L}_u \Phi^*(x) + c(x,u)\Phi^*(x) \right] = \lambda^*(\mathcal{G})\Phi^*(x), \quad x \in \mathbb{R}^d\,.$$

In addition, we have the following:

(i) *Any measurable selector* $v_* \colon \mathbb{R}^d \to \mathbb{U}$ *from the minimizer of* (2.15) *is an optimal Markov control with respect to the risk-sensitive criterion.*

(ii) *The function* Φ^* *has the stochastic representation*

$$(2.16) \qquad \Phi^*(x) \;=\; \mathbb{E}_x^{v_*}\!\left[\mathrm{e}^{\int_0^{\breve{\tau}_r}(c(X_s, v_*(X_s))-\lambda^*(\mathcal{G}))\,\mathrm{d}s}\Phi^*(X_{\breve{\tau}_r})\right] \quad \forall\, x \in B_r^c\,,$$

for any $r > 0$.

(iii) Φ^* *is the unique (up to a multiplicative constant) positive solution of* (2.15) *in* $C^2(\mathbb{R}^d)$.

Proof. For the proof of this and related statements we refer the reader to [3, Theorems 4.1–4.3]. We provide a short proof of the fact that $\Phi^* \in \mathcal{O}(\mathcal{V}^\beta)$ for the convenience of the reader. We consider Assumption 2.10. Choose r large enough so that for some suitable $\beta \in (0,1)$ we have $\max_{u\in\mathbb{U}} c(x,u) \leq \beta\gamma$ for $x \in B_r^c$. Without loss of generality we may assume $\mathcal{K} \subset B_r$. From the proof of [3, Theorem 4.2] it follows that for $x \in B_r^c$ we have

$$\Phi^*(x) \;=\; \mathbb{E}_x^{v_*}\!\left[\mathrm{e}^{\int_0^{\breve{\tau}_r}(c(X_s, v(X_s))-\lambda^*(\mathcal{G}))\,\mathrm{d}s}\,\Phi^*(X_{\breve{\tau}_r})\right],$$

which in turn, gives (since $\lambda^*(\mathcal{G}) = \Lambda^* \geq 0$)

$$\Phi^*(x) \;\leq\; \mathbb{E}_x^{v}\!\left[\mathrm{e}^{\beta\gamma\breve{\tau}_r}\Phi^*(X_{\breve{\tau}_r})\right] \;\leq\; \mathbb{E}_x^{v}\!\left[\mathrm{e}^{\gamma\breve{\tau}_r}\big(\Phi^*(X_{\breve{\tau}_r})\big)^{1/\beta}\right]^{\beta} \;\leq\; \left[\max_{\partial B_r}\frac{\Phi^*}{\mathcal{V}^\beta}\right]\big(\mathcal{V}(x)\big)^{\beta},$$

where in the last inequality we use (2.14). The proof under Assumption 2.9 is exactly analogous. □

We next state the Collatz–Wielandt formula for $\lambda^*(\mathcal{G})$.

THEOREM 2.13. *Grant either* Assumption 2.9 *or* 2.10. *Then*

$$(2.17) \qquad\qquad \lambda^*(\mathcal{G}) \;=\; \sup_{\psi\in C^{2,+}(\mathbb{R}^d)\cap\mathfrak{o}(\mathcal{V})}\ \inf_{\mu\in\mathcal{P}(\mathbb{R}^d)}\ \int_{\mathbb{R}^d}\frac{\mathcal{G}\psi}{\psi}\,\mathrm{d}\mu$$

$$(2.18) \qquad\qquad\qquad =\; \inf_{\psi\in C^{2,+}(\mathbb{R}^d)}\ \sup_{\mu\in\mathcal{P}(\mathbb{R}^d)}\ \int_{\mathbb{R}^d}\frac{\mathcal{G}\psi}{\psi}\,\mathrm{d}\mu\,.$$

Remark 2.14. The class of test functions ψ in the representation (2.17) cannot, in general, be enlarged to $C^{2,+}(\mathbb{R}^d)$. For a linear operator $\mathcal{L}$, it is known from [9, Theorem 1.4] (see also Theorem 2.18 below) that, for any $\lambda \geq \lambda^*(\mathcal{L})$, there exists $\Psi \in C^{2,+}(\mathbb{R}^d)$ satisfying

$$\mathcal{L}\Psi \;=\; \lambda\Psi\,.$$

Thus we obtain

$$\sup_{\psi\in C^{2,+}(\mathbb{R}^d)}\ \inf_{\mu\in\mathcal{P}(\mathbb{R}^d)}\ \int_{\mathbb{R}^d}\frac{\mathcal{L}\psi}{\psi}\,\mathrm{d}\mu \;=\; \infty\,.$$

The proof of Theorem 2.13 gives us the following maximum principle for the semilinear operator $\mathcal{G}$ in $\mathbb{R}^d$. This should be compared with [9, Theorem 1.6].

THEOREM 2.15. *Let either* Assumption 2.9 *or* 2.10 *hold. Let* $\varphi \in C^2(\mathbb{R}^d) \cap \mathfrak{o}(\mathcal{V})$ *satisfy* $\mathcal{G}\varphi - \lambda^*(\mathcal{G})\varphi \geq 0$ *in* $\mathbb{R}^d$, *and* $\varphi(x_0) > 0$ *for some* $x_0 \in \mathbb{R}^d$. *Then* $\varphi = \kappa\Phi^*$ *for some* $\kappa > 0$.

The following theorem could be seen as a refined maximum principle in $\mathbb{R}^d$.

THEOREM 2.16. *Let either Assumption 2.9 or 2.10 hold. Also suppose that $\lambda^*(\mathcal{G}) <$ 0. Then for any $\varphi \in C^2(\mathbb{R}^d) \cap \mathfrak{o}(\mathcal{V})$ satisfying $\mathcal{G}\varphi \geq 0$ in $\mathbb{R}^d$ we have either $\varphi < 0$ or $\varphi = 0$ in $\mathbb{R}^d$.*

The next result provides another characterization of $\lambda^*(\mathcal{G})$.

THEOREM 2.17. *Under either Assumption 2.9 or 2.10, we have*

$$\lambda^*(\mathcal{G}) = \lambda''(\mathcal{G}) = \inf \left\{ \lambda \in \mathbb{R} : \exists \psi \in C^2(\mathbb{R}^d), \inf_{\mathbb{R}^d} \psi > 0, \right.$$

$$\left. satisfying \ \mathcal{G}\psi - \lambda\psi \leq 0 \ a.e. \ in \ \mathbb{R}^d \right\}.$$

We next prove the existence of infinitely many generalized eigenvalues for the semilinear operator $\mathcal{G}$. For the linear operator $\mathcal{L}$, it has been recently shown in [9, Theorem 1.4] that for any $\lambda \geq \lambda^*(\mathcal{L})$ there exists a positive $\Psi \in \mathcal{W}^{2,d}_{\text{loc}}(\mathbb{R}^d)$ satisfying $\mathcal{L}\Psi = \lambda\Psi$. Our next result is in the same spirit but for the semilinear operator $\mathcal{G}$.

THEOREM 2.18. *For any $\lambda \geq \lambda^*(\mathcal{G})$ there exists a positive $\Phi_\lambda \in C^2(\mathbb{R}^d)$ satisfying*

$$\mathcal{G}\Phi_\lambda = \lambda\Phi_\lambda \quad in \ \mathbb{R}^d.$$

It is straightforward to show that we have an analogous version of all the preceding results for $\mathcal{H}$ (see (2.11)). Let $\lambda^*(\mathcal{H})$ be the corresponding principal eigenvalue defined as in (2.10). It is clear from the definition that $\lambda^*(\mathcal{G}) \leq \lambda^*(\mathcal{H})$. We present the following result. A similar result is known for Dirichlet problems in bounded domains [18, Theorem 1.8].

THEOREM 2.19. *Let either Assumption 2.9 or 2.10 hold, and suppose $\lambda^*(\mathcal{G}) < \lambda^*(\mathcal{H})$. Then, for any $\lambda \in (\lambda^*(\mathcal{G}), \lambda^*(\mathcal{H}))$, there exists no nontrivial solution of $\mathcal{G}\varphi - \lambda\varphi = 0$ for some $\varphi \in C^2(\mathbb{R}^d) \cap \mathfrak{o}(\mathcal{V})$.*

3. Proofs. We begin with the proof of Theorem 2.1. Let us point out that we use several results from [18], which deals with operators that are convex in $(\nabla^2\psi, \nabla\psi, \psi)$. Since $\mathcal{G}$ is concave in $(\nabla^2\psi, \nabla\psi, \psi)$, we can apply the results of [18] with suitable modification.

Proof of Theorem 2.1. Throughout the proof, we let $\lambda_D \equiv \lambda_D(\mathcal{G})$. We claim that for any $\psi \in C^{2,+}_0(D)$ we have

(3.1)
$$\inf_D \frac{\mathcal{G}\psi}{\psi} \leq \lambda_D.$$

Arguing by contradiction, suppose that for some $\lambda > \lambda_D$ it holds that

$$\inf_D \frac{\mathcal{G}\psi}{\psi} = \lambda.$$

Thus we have

$$\mathcal{G}\psi \geq \lambda\psi \quad in \ D, \quad \psi > 0 \quad in \ D, \quad \psi = 0 \quad on \ \partial D.$$

Then $\psi = t\Psi$ for some $t > 0$ by [18, Theorem 1.2], where Ψ denotes the principal eigenvector in (2.4). This implies that $\mathcal{G}\Psi \geq \lambda\Psi$, thus leading to a contradiction, and establishing the claim in (3.1).

By (3.1), we obtain

$$\sup_{\psi\in C^{2,+}(D)}\ \inf_{\mu\in\mathcal{P}(D)}\ \int_D \frac{\mathcal{G}\psi}{\psi}\,\mathrm{d}\mu \ =\ \sup_{\psi\in C^{2,+}(D)}\ \inf_D \frac{\mathcal{G}\psi}{\psi}\ \le\ \lambda_D\,.$$

On the other hand, choosing Ψ as a test function, we have from (2.4) that

$$\sup_{\psi\in C^{2,+}(D)}\ \inf_{\mu\in\mathcal{P}(D)}\ \int_D \frac{\mathcal{G}\psi}{\psi}\,\mathrm{d}\mu \ \ge\ \inf_{\mu\in\mathcal{P}(D)}\ \int_D \frac{\mathcal{G}\Psi}{\Psi}\,\mathrm{d}\mu \ =\ \lambda_D\,.$$

This proves (2.5).

Now we consider (2.6). We clearly have

$$\inf_{\psi\in C^{2,+}(D)}\ \sup_{\mu\in\mathcal{P}(D)}\ \int_D \frac{\mathcal{G}\psi}{\psi}\,\mathrm{d}\mu \ \le\ \sup_{\mu\in\mathcal{P}(D)}\ \int_D \frac{\mathcal{G}\Psi}{\Psi}\,\mathrm{d}\mu \ =\ \lambda_D\,.$$

To get the opposite inequality, we apply the characterization in (2.3). Note that it follows from (2.3) that for any $\psi\in C^{2,+}(D)$ we have $\sup_D \frac{\mathcal{G}\psi}{\psi}\ge\lambda_D$, and hence

$$\inf_{\psi\in C^{2,+}(D)}\ \sup_{\mu\in\mathcal{P}(D)}\ \int_D \frac{\mathcal{G}\psi}{\psi}\,\mathrm{d}\mu \ =\ \inf_{\psi\in C^{2,+}(D)}\ \sup_D \frac{\mathcal{G}\psi}{\psi}\ \ge\ \lambda_D\,.$$

This establishes (2.6) and completes the proof. $\square$

Remark 3.1. Equations (2.5) and (2.6) hold for a more general class of operators. More precisely, if $\mathcal{G}(\nabla^2\psi,\nabla\psi,\psi,x)$ is a general nonlinear elliptic operator that is concave in the first three arguments and satisfies the assumptions (H_0)–(H_2) in [18], we still have a Collatz–Wielandt formula for the eigenvalue λ_D. The proof follows from the arguments in the proof of Theorem 2.1. In particular, if we consider the operator

$$\mathcal{G}(\nabla^2\psi,\nabla\psi,\psi,x) \ =\ a^{ij}(x)\,\partial_{ij}\psi(x) + b^i(x)\,\partial_i\psi(x) + c(x)\psi\,,$$

where b, and c are bounded Borel measurable functions, and a is continuous and satisfies (A3), then we have the Collatz–Wielandt representation in (2.5)-(2.6) for its principal eigenvalue.

Proof of Theorem 2.5. Using the eigenvalue equation for $\mathcal{H}$, analogous to (2.4), it is easily seen that

$$\sup_{\mu\in\mathcal{P}(D)}\ \inf_{\psi\in C^{2,+}(D)}\ \int_D \frac{\mathcal{H}\psi}{\psi}\,\mathrm{d}\mu \ \le\ \lambda_D(\mathcal{H})\,.$$

To show the reverse inequality we consider a smooth domain $D_n \Subset D$. Define $\tilde{\mathcal{H}}$ as

$$\tilde{\mathcal{H}}\psi \ =\ \mathcal{H}\psi + \langle\nabla\psi, a\nabla\psi\rangle\,.$$

Note that $\psi \in C^2(D) \mapsto \tilde{\mathcal{H}}\psi$ is convex. Therefore,

$$
\begin{aligned}
\sup_{\mu \in \mathcal{P}(D)} \inf_{\psi \in C^{2,+}(D)} \int_D \frac{\mathcal{H}\psi}{\psi}\, \mathrm{d}\mu &\geq \sup_{\mu \in \mathcal{P}(\bar{D}_n)} \inf_{\psi \in C^{2,+}(D)} \int_D \frac{\mathcal{H}\psi}{\psi}\, \mathrm{d}\mu \\
&= \sup_{\mu \in \mathcal{P}(\bar{D}_n)} \inf_{\psi \in C^2(D)} \int_D \tilde{\mathcal{H}}\psi\, \mathrm{d}\mu \\
&= \inf_{\psi \in C^2(D)} \sup_{\mu \in \mathcal{P}(\bar{D}_n)} \int_D \tilde{\mathcal{H}}\psi\, \mathrm{d}\mu \\
&= \inf_{\psi \in C^{2,+}(D)} \sup_{\mu \in \mathcal{P}(\bar{D}_n)} \int_D \frac{\mathcal{H}\psi}{\psi}\, \mathrm{d}\mu \\
&= \inf_{\psi \in C^{2,+}(D)} \max_{\bar{D}_n} \frac{\mathcal{H}\psi}{\psi},
\end{aligned}
$$

(3.2)

where in the third line we used Sion's minimax theorem [19]. In view of [18, Theorem 1.1] we have

$$
\max_{\bar{D}_n} \frac{\mathcal{H}\psi}{\psi} \geq \lambda_{D_n}(\mathcal{H}) \qquad \forall\, \psi \in C^{2,+}(D),
$$

and therefore, combining with (3.2), we obtain

$$
\sup_{\mu \in \mathcal{P}(D)} \inf_{\psi \in C^{2,+}(D)} \geq \lambda_{D_n}(\mathcal{H}).
$$

Now let $n \to \infty$ so that $D_n \uparrow D$, and use the fact $\lambda_{D_n}(\mathcal{H}) \to \lambda_D(\mathcal{H})$ [18, Proposition 4.10] to deduce that

$$
\sup_{\mu \in \mathcal{P}(D)} \inf_{\psi \in C^{2,+}(D)} \int_D \frac{\mathcal{H}\psi}{\psi}\, \mathrm{d}\mu \geq \lambda_D(\mathcal{H}),
$$

thus establishing (2.12). □

Proof of Theorem 2.6. We distinguish two cases. First, suppose that $\lambda^*(\mathcal{L}) > 0$. In this case we claim that the principal eigenfunction Φ^* is bounded. For this, we argue as follows. The principal eigenfunction is obtained as a limit of the Dirichlet principal eigenfunction over balls. Again, if (λ_n, Ψ_n) is the principal eigenpair in $B_n(0)$, then it follows from [3, Lemma 2.2] that

$$
\Psi_n(x) = \mathbb{E}_x\!\left[\mathrm{e}^{\int_0^{\breve{\tau}_r}[c(X_s) - \lambda_n]\,\mathrm{d}s}\, \Psi_n(X_{\breve{\tau}_r})\, \mathbb{1}_{\{\breve{\tau}_r < \tau_n\}}\right], \quad x \in B_n \setminus B_r^c(0).
$$

for any $n > r > 0$. Since $\lambda_n \to \lambda^*(\mathcal{L})$, and c vanishes at infinity, it follows that for some large enough r and $n_0 \in \mathbb{N}$, we have $\sup_{x \in B_r^c} c(x) - \lambda_n < 0$ for all $n \geq n_0$. This shows that $\sup_{B_n} \Psi_n = \sup_{B_r} \Psi_n$ for all $n \geq n_0$. Thus, the limit of Ψ_n is also bounded, proving the claim.

Therefore, we have

$$
\lambda^*(\mathcal{L}) \leq \underline{\lambda}^* := \sup_{\psi \in C_b^{2,+}(\mathbb{R}^d)} \inf_{\mu \in \mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{\mathcal{L}\psi}{\psi}\, \mathrm{d}\mu.
$$

Suppose $\psi \in C_b^{2,+}(\mathbb{R}^d)$ satisfies $\mathcal{L}\psi - (\lambda^*(\mathcal{L}) + \epsilon)\psi \geq 0$ for some $\epsilon > 0$. Let $r > 0$ be such that $\sup_{x \in B_r^c} c(x) < \lambda^*(\mathcal{L}) + \varepsilon$. It is fairly straightforward to show that ψ

satisfies

$$\psi(x) \;\leq\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\breve{\tau}_r}(c(X_s)-\lambda^*(\mathcal{L})-\epsilon)\,\mathrm{d}s}\psi(X_{\breve{\tau}_r})\Bigr] \qquad \forall\,x \in B_r^c\,.$$

Indeed, since

$$\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\tau_R}(c(X_s)-\lambda^*(\mathcal{L})-\epsilon)\,\mathrm{d}s}\psi^*(X_{\tau_R})\,\mathbb{1}_{\{\tau_R<\tau_r\}}\Bigr] \;\leq\; \|\psi\|_\infty\,\mathbb{P}_x\bigl(\tau_R < \tau_r\bigr) \qquad \forall\,x \in B_r^c\,,$$

it follows that this quantity tends to 0 as $R \to \infty$.

Thus $C\Phi^* \geq \psi$ for some positive constant C, and this is clearly impossible by the strong maximum principle. It follows that $\lambda^*(\mathcal{L}) = \underline{\lambda}^*$.

On the other hand, as shown in (2.9), we have

(3.3) $$\lambda^*(\mathcal{L}) \;=\; \bar{\lambda}^* \;:=\; \inf_{\psi\in C_b^{2,+}(\mathbb{R}^d)}\;\sup_{\mu\in\mathcal{P}(\mathbb{R}^d)}\;\int_{\mathbb{R}^d}\frac{\mathcal{L}\psi}{\psi}\,\mathrm{d}\mu\,,$$

and the proof of Theorem 2.5 gives us

(3.4) $$\lambda^*(\mathcal{L}) \;=\; \sup_{\mu\in\mathcal{P}(\mathbb{R}^d)}\;\inf_{\psi\in C_b^{2,+}(\mathbb{R}^d)}\;\int_{\mathbb{R}^d}\frac{\mathcal{L}\psi}{\psi}\,\mathrm{d}\mu\,.$$

Next, suppose that $\lambda^*(\mathcal{L}) = 0$. Note that by definition we have

$$\sup_{\psi\in C_b^{2,+}(\mathbb{R}^d)}\;\inf_{\mathbb{R}^d}\;\frac{\mathcal{L}\psi}{\psi} \;\leq\; \lambda'(\mathcal{L})\,.$$

Again by [9, Theorem 1.9 (iii)] we have $\lambda^*(\mathcal{L}) = \lambda'(\mathcal{L}) = 0$. This gives us

$$\sup_{\psi\in C_b^{2,+}(\mathbb{R}^d)}\;\inf_{\mu\in\mathcal{P}(\mathbb{R}^d)}\;\int_{\mathbb{R}^d}\frac{\mathcal{L}\psi}{\psi}\mathrm{d}\mu \;\leq\; 0\,.$$

Since $\lambda'(\mathcal{L}) = 0$, then by definition, for any $\varepsilon > 0$, there exists $\psi \in C_b^{2,+}(\mathbb{R}^d)$ satisfying $\mathcal{L}\psi \geq -\varepsilon\psi$, which implies that

$$\frac{\mathcal{L}\psi}{\psi} \;\geq\; -\varepsilon\,.$$

Thus

$$\inf_{\mu\in\mathcal{P}(\mathbb{R}^d)}\;\int_{\mathbb{R}^d}\frac{\mathcal{L}\psi}{\psi}\mathrm{d}\mu \;\geq\; -\varepsilon\,,$$

and since $\varepsilon > 0$ is arbitrary, we obtain

$$\sup_{\psi\in C_b^{2,+}(\mathbb{R}^d)}\;\inf_{\mu\in\mathcal{P}(\mathbb{R}^d)}\;\int_{\mathbb{R}^d}\frac{\mathcal{L}\psi}{\psi}\mathrm{d}\mu \;=\; 0\,.$$

Lastly, consider the case $\lambda^*(\mathcal{L}) < 0$. Then it is easy to show that Φ^* is bounded away from 0. Hence $\lambda^*(\mathcal{L}) = \lambda''(\mathcal{L})$. By (A1)–(A3) and [9, Theorem 1.7 (iii)], we have $\lambda''(\mathcal{L}) \geq \lambda'(\mathcal{L})$. Since $\lambda^*(\mathcal{L}) = \lambda''(\mathcal{L})$, we have $\lambda^*(\mathcal{L}) \geq \lambda'(\mathcal{L})$, which implies that $\lambda^*(\mathcal{L}) \geq \underline{\lambda}^*$. On the other hand, we have $\underline{\lambda}^* \geq \lambda^*(\mathcal{L})$ [9, Theorem 1.7 (ii)]. Thus $\underline{\lambda}^* = \lambda^*(\mathcal{L})$.

We leave it to the reader to verify that (3.3) and (3.4) hold if we replace $C_b^{2,+}(\mathbb{R}^d)$ with $C^{2,+}(\mathbb{R}^d)$, and that this is also true in the case $\lambda^*(\mathcal{L}) \geq 0$.

It remains to consider $\mathcal{G}$. Suppose $\lambda^*(\mathcal{G}) > 0$. As before, the corresponding principal eigenfunction is bounded. Therefore, the second equality follows from the proof of (2.9). Moreover,

$$\lambda^*(\mathcal{G}) \;\leq\; \sup_{\psi \in C_b^{2,+}(\mathbb{R}^d)} \; \inf_{\mu \in \mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{\mathcal{G}\psi}{\psi}\, \mathrm{d}\mu \,.$$

Let v_* be a minimizing selector of

$$\mathcal{G}\Phi^* \;=\; \lambda^*(\mathcal{G})\Phi^* \,,$$

and recall that the associated process is recurrent. Then, denoting the corresponding generator by $\mathcal{L}_{v_*}$ and applying the previous result, we note that

$$\lambda^*(\mathcal{G}) \;=\; \lambda^*(\mathcal{L}_{v_*}) \;\geq\; \sup_{\psi \in C_b^{2,+}(\mathbb{R}^d)} \; \inf_{\mu \in \mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{\mathcal{L}_{v_*}\psi}{\psi}\, \mathrm{d}\mu$$

$$\geq\; \sup_{\psi \in C_b^{2,+}(\mathbb{R}^d)} \; \inf_{\mu \in \mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{\mathcal{G}\psi}{\psi}\, \mathrm{d}\mu \,.$$

Combining, we have

$$\lambda^*(\mathcal{G}) \;=\; \sup_{\psi \in C_b^{2,+}(\mathbb{R}^d)} \; \inf_{\mu \in \mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{\mathcal{G}\psi}{\psi}\, \mathrm{d}\mu \,.$$

This completes the proof.   □

We need the following lemma for the proofs of Theorems 2.13 and 2.15.

LEMMA 3.2. *Grant Assumption 2.9 or Assumption 2.10. Suppose that the function* $\psi \in \mathcal{W}_{\mathrm{loc}}^{2,d,+}(\mathbb{R}^d) \cap \mathfrak{o}(\mathcal{V})$ *satisfies*

$$(3.5) \qquad\qquad \mathcal{L}_v\psi + c(x, v(x))\psi \;\geq\; \lambda\psi \quad a.e. \ in \ \mathbb{R}^d$$

for some Markov control v, *and* $\lambda \in \mathbb{R}$. *Then there exists* $r_\circ > 0$ *not depending on* ψ *such that*

$$(3.6) \qquad\qquad \psi(x) \;\leq\; \mathbb{E}_x^v\!\left[\mathrm{e}^{\int_0^{\breve{\tau}_r}(c(X_s, v(X_s)) - \lambda)\, \mathrm{d}s}\, \psi(X_{\breve{\tau}_r}) \right] \quad for \ x \in B_r^c \,,$$

for all $r \geq r_\circ$. *In addition, if* (3.5) *holds with equality, then we have equality in* (3.6).

Proof. We only consider the case of Assumption 2.9. The proof under Assumption 2.10 is completely analogous. Choose r large enough so that $\max_{u \in \mathbb{U}}(c(x, u) - \lambda) < \ell(x)$ in B_r^c and $\mathcal{K} \subset B_r$. For any $R > r > 0$, we have

$$(3.7) \quad \begin{aligned} \psi(x) \;\leq\; & \mathbb{E}_x^v\!\left[\mathrm{e}^{\int_0^{\breve{\tau}_r}(c(X_s, v(X_s)) - \lambda)\, \mathrm{d}s}\, \psi(X_{\breve{\tau}_r}) \mathbb{1}_{\{\breve{\tau}_r < \tau_R\}} \right] \\ & + \mathbb{E}_x^v\!\left[\mathrm{e}^{\int_0^{\tau_R}(c(X_s, v(X_s)) - \lambda)\, \mathrm{d}s}\, \psi(X_{\tau_R}) \mathbb{1}_{\{\breve{\tau}_r > \tau_R\}} \right]. \end{aligned}$$

We first estimate the limit of the second term of (3.7) as $R \to \infty$.

$$\mathbb{E}_x^v \left[e^{\int_0^{\tau_R} (c(X_s, v(X_s)) - \lambda) \, ds} \psi(X_{\tau_R}) \mathbb{1}_{\{\breve{\tau}_r > \tau_R\}} \right]$$

(3.8)
$$\leq \left(\max_{\partial B_R} \frac{|\psi|}{\mathcal{V}} \right) \mathbb{E}_x^v \left[e^{\int_0^{\tau_R} (c(X_s, v(X_s)) - \lambda) \, ds} \mathcal{V}(X_{\tau_R}) \mathbb{1}_{\{\breve{\tau}_r > \tau_R\}} \right]$$

$$\leq \left(\max_{\partial B_R} \frac{|\psi|}{\mathcal{V}} \right) \mathbb{E}_x^v \left[e^{\int_0^{\tau_R} \ell(X_s) \, ds} \mathcal{V}(X_{\tau_R}) \mathbb{1}_{\{\breve{\tau}_r > \tau_R\}} \right]$$

$$\leq \left(\max_{\partial B_R} \frac{|\psi|}{\mathcal{V}} \right) \mathcal{V}(x) \xrightarrow[R \to \infty]{} 0 \,,$$

where in the last line we use the fact that $\psi \in \mathfrak{o}(\mathcal{V})$. Thus, letting $R \to \infty$ in (3.7), and using the monotone convergence theorem, we obtain (3.6). The last sentence is evident from (3.7) and (3.8). This completes the proof. □

We continue with the proof of Theorem 2.13.

Proof of Theorem 2.13. Throughout this proof $\lambda^* \equiv \lambda^*(\mathcal{G})$. Since $\Phi^* \in \mathcal{O}(\mathcal{V}^\beta)$ by Theorem 2.12, it follows that

(3.9)
$$\lambda^* \leq \sup_{\psi \in C^{2,+}(\mathbb{R}^d) \cap \mathfrak{o}(\mathcal{V})} \inf_{\mu \in \mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{\mathcal{G}\psi}{\psi} \, d\mu \,.$$

We claim that for any $\psi \in C^{2,+}(\mathbb{R}^d) \cap \mathfrak{o}(\mathcal{V})$ we have

(3.10)
$$\inf_{\mathbb{R}^d} \frac{\mathcal{G}\psi}{\psi} \leq \lambda^* \,.$$

Indeed, suppose to the contrary that for some $\lambda > \lambda^*$ it holds that

$$\inf_{\mathbb{R}^d} \frac{\mathcal{G}\psi}{\psi} = \lambda \,.$$

This implies that

$$\mathcal{G}\psi - \lambda\psi \geq 0 \quad \text{in } \mathbb{R}^d \,.$$

Let v_* be a measurable selector of the HJB in Theorem 2.12. To simplify the notation we let $c_{v_*}(x) := c(x, v_*(x))$. Then we have

(3.11)
$$\mathcal{L}_{v_*}\psi + (c_{v_*} - \lambda)\psi \geq 0 \,.$$

By Lemma 3.2 and Theorem 2.12 we have

(3.12)
$$\psi(x) \leq \mathbb{E}_x^{v_*} \left[e^{\int_0^{\breve{\tau}_r} (c_{v_*}(X_s) - \lambda) \, ds} \psi(X_{\breve{\tau}_r}) \right], \quad x \in B_r^c \,,$$

and

(3.13)
$$\Phi^*(x) = \mathbb{E}_x^{v_*} \left[e^{\int_0^{\breve{\tau}_r} (c_{v_*}(X_s) - \lambda^*) \, ds} \Phi^*(X_{\breve{\tau}_r}) \right], \quad x \in B_r^c \,,$$

respectively. Let $\kappa = \max_{\bar{B}_r} \frac{\psi}{\Phi^*}$. Then from (3.12) and (3.13) we see that $\psi \leq \kappa \Phi^*$ in $\mathbb{R}^d$, and for some $|x_0| \leq r$ we have $\varphi(x_0) - \kappa \Phi^*(x_0) = 0$. Since

$$\mathcal{L}_{v_*}\Phi^* + (c_{v_*} - \lambda^*)\Phi^* = 0 \,,$$

using (3.11) we obtain

$$\mathcal{L}_{v_*}(\kappa\Phi^* - \psi) - (c_{v_*} - \lambda^*)^-(\kappa\Phi^* - \psi) \leq 0 \quad \text{in } \mathbb{R}^d.$$

It follows by the strong maximum principle that $\kappa\Phi^* = \psi$, and this contradicts (3.11) since $\lambda > \lambda^*$. This proves (3.10).

Now using (3.10) we obtain

$$\sup_{\psi \in C^{2,+}(\mathbb{R}^d) \cap \mathfrak{o}(\mathcal{V})} \quad \inf_{\mu \in \mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{\mathcal{G}\psi}{\psi} \, \mathrm{d}\mu \leq \sup_{\psi \in C^{2,+}(\mathbb{R}^d) \cap \mathfrak{o}(\mathcal{V})} \inf_{\mathbb{R}^d} \frac{\mathcal{G}\psi}{\psi} \leq \lambda^*.$$

Hence, using (3.9) we obtain (2.17) .

From (2.10) it is easily seen that

$$\sup_{\mathbb{R}^d} \frac{\mathcal{G}\psi}{\psi} \geq \lambda^* \quad \text{for any } \psi \in C^{2,+}(\mathbb{R}^d),$$

and therefore

$$\sup_{\mu \in \mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{\mathcal{G}\psi}{\psi} \, \mathrm{d}\mu \geq \lambda^* \quad \text{for any } \psi \in C^{2,+}(\mathbb{R}^d).$$

This gives us

$$\inf_{\psi \in C^{2,+}(\mathbb{R}^d)} \sup_{\mu \in \mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{\mathcal{G}\psi}{\psi} \, \mathrm{d}\mu \geq \lambda^*.$$

Now choosing $\psi = \Phi^*$ in the above display, we get equality, which proves (2.18). □

The function space used in the representation (2.17) can be extended to $\mathcal{A}_{\mathcal{V}} := C^{2,+}(\mathbb{R}^d) \cap \mathfrak{O}(\mathcal{V})$, provided we impose certain assumptions on the Lyapunov function $\mathcal{V}$. This is the subject of the following theorem.

THEOREM 3.3. *Suppose that any one of the following is true.*
(a) *Assumption 2.9 (i) holds with an inf-compact function $\mathcal{V}$ and the function*

$$x \mapsto \ell(x) - \frac{\langle \nabla\mathcal{V}(x), a(x)\nabla\mathcal{V}\rangle}{\mathcal{V}^2(x)\log\mathcal{V}(x)} - \max_{u \in \mathbb{U}} c(x,u) \quad \text{is inf-compact.}$$

(b) *Assumption 2.10 holds with an inf-compact function $\mathcal{V}$ and*

$$\lim_{|x| \to \infty} \frac{\langle \nabla\mathcal{V}(x), a(x)\nabla\mathcal{V}\rangle}{\mathcal{V}^2(x)\log\mathcal{V}(x)} = 0.$$

Then we have

$$\lambda^*(\mathcal{G}) = \sup_{\psi \in \mathcal{A}_{\mathcal{V}}} \inf_{\mu \in \mathcal{P}(\mathbb{R}^d)} \int_D \frac{\mathcal{G}\psi}{\psi} \, \mathrm{d}\mu.$$

Proof. From [3, Theorems 4.1 and 4.2] we note that parts (i)–(ii) of Theorem 2.12 hold under the above assumptions. Using (2.16) it is easily seen that $\Phi^* \in \mathfrak{O}(\mathcal{V})$. Now define

$$\tilde{\mathcal{V}} := \mathcal{V}\log\mathcal{V} \quad \text{and} \quad \tilde{\ell} := \ell - \frac{\langle \nabla\mathcal{V}(x), a(x)\nabla\mathcal{V}\rangle}{\mathcal{V}^2(x)\log\mathcal{V}(x)}.$$

Then an easy calculation gives

$$\max_{u \in \mathbb{U}} \mathcal{L}_u \tilde{\mathcal{V}} \leq \kappa_1 (\log \mathcal{V}) \mathbb{1}_{\mathcal{K}} - \ell \tilde{\mathcal{V}} + \kappa_1 \mathbb{1}_{\mathcal{K}} - \ell \mathcal{V} + \frac{1}{\mathcal{V}} \langle \nabla \mathcal{V}, a \nabla \mathcal{V} \rangle$$

$$\leq \kappa \Big(\max_{\mathcal{K}} \mathcal{V} + 1 \Big) \mathbb{1}_{\mathcal{K}} - \tilde{\ell} \tilde{\mathcal{V}}.$$

Therefore, $\tilde{\mathcal{V}}$ can be used as a new Lyapunov function pay-off function $\tilde{\ell}$. Again, $\mathcal{V}$ being inf-compact, we have $\mathcal{O}(\mathcal{V}) \subset \mathfrak{o}(\tilde{\mathcal{V}})$. Hence for any function ψ satisfying (3.11) the estimate in (3.12) holds. Then rest of the proof follows from Theorem 2.13. □

The proof of Theorem 2.15, which follows, uses an argument similar to the one used in the proof of Theorem 2.13.

Proof of Theorem 2.15. It is given that $\varphi(x_0) > 0$. Without loss of generality we may assume that $x_0 = 0$ and $\varphi > 0$ in $B_\delta(0)$ for some $\delta > 0$. Choose a stable optimal Markov policy v_* from (2.15) as in the proof of Theorem 2.13. By Lemma 3.2 we have the stochastic representation in (3.12) for all large enough $r > 0$. Let $\kappa = \max_{\bar{B}_r} \frac{\varphi^+}{\Phi^*}$. Note that $\kappa > 0$ since $\varphi > 0$ in $B_\delta(0)$. It now follows from (2.16) and (3.12) that $\varphi \leq \kappa \Phi^*$ in $\mathbb{R}^d$, and for some $y_0 \in \bar{B}_r$ we have $\varphi(y_0) = \kappa \Phi^*(y_0)$. Combining the inequalities

$$\mathcal{L}_{v_*} \Phi^* + \big(c_{v_*} - \lambda^*(\mathcal{G}) \big) \Phi^* = 0 \quad \text{and} \quad \mathcal{L}_{v_*} \varphi + \big(c_{v_*} - \lambda^*(\mathcal{G}) \big) \varphi \geq 0 \,,$$

we obtain

$$\mathcal{L}_{v_*} (\kappa \Phi^* - \varphi) - \big(c_{v_*} - \lambda^*(\mathcal{G}) \big)^- (\kappa \Phi^* - \varphi) \leq 0 \quad \text{in } \mathbb{R}^d \,.$$

Therefore, $\kappa \Phi^* = \varphi$ in $\mathbb{R}^d$ by the strong maximum principle. This completes the proof. □

We continue with the proof of Theorem 2.16.

Proof of Theorem 2.16. To the contrary, suppose that $\varphi(x_0) > 0$. Without loss of generality we may assume that $x_0 = 0$ and $\varphi > 0$ in $B_\delta(0)$ for some $\delta > 0$. Choosing an optimal stable control v_*, we deduce, as in the proof of Theorem 2.15, that for some positive κ we have $\kappa \Phi^* - \varphi \geq 0$ and the minimum value 0 is attained at some some point y_0. Denote $\xi = \frac{\varphi}{\kappa \Phi^*}$. An easy calculation gives

$$\mathcal{L}_{v_*} \xi + \langle b + 2a \nabla(\log \Phi^*), \nabla \xi \rangle + \lambda^*(\mathcal{G}) \xi \geq 0 \,, \quad \text{in } \mathbb{R}^d \,.$$

Note that $\xi \leq 1$ and $\xi(y_0) = 1$. Thus by the strong maximum principle we have $\xi = 1$, implying that $\varphi = \kappa \Phi^*$. But this is not possible as $\lambda^*(\mathcal{G}) < 0$. Hence we must have $\varphi \leq 0$. The result follows by another application of the strong maximum principle. □

To prove Theorem 2.17 we first consider an eigenvalue problem for a perturbed c. For Assumption 2.9, we define

$$c_m(x, u) = c(x, u) + \frac{1}{m} \ell(x) \quad \text{for } x \in \mathbb{R}^d, \ m \geq 1 \,.$$

For Assumption 2.10 and $m \geq 1$, we consider a smooth function $\zeta_m \colon \mathbb{R}^d \to [0, 1]$, satisfying $\zeta_m(x) = 1$ in B_m and $\zeta_m(x) = 0$ in B_{m+1}^c, and define

$$(3.14) \qquad c_m(x, u) := \zeta_m(x) c(x, u) + (1 - \zeta_m(x)) \Big(\delta + \limsup_{|z| \to \infty} \max_{u \in \mathbb{U}} c(z, u) \Big),$$

where δ is small enough to satisfy

$$\delta \;<\; \gamma - \|c^-\|_\infty - \limsup_{|z|\to\infty}\ \max_{u\in\mathbb{U}}\ c(z,u)\,.$$

Then, following an argument similar to [12, Lemmas 3.4 and 3.5], we can establish the following.

LEMMA 3.4. *Grant either Assumption* 2.9 *or* 2.10. *Then there exists a unique* $\Psi_m^* \in C^{2,+}(\mathbb{R}^d)$ *with* $\inf_{\mathbb{R}^d}\Psi_m^* > 0$, *satisfying*

$$\mathcal{G}_m\Psi_m^*(x) \;\coloneqq\; a^{ij}(x)\,\partial_{ij}\Psi_m^*(x) + \min_{u\in\mathbb{U}}\left[b^i(x,u)\,\partial_i\Psi_m^*(x) + c_m(x,u)\Psi_m^*(x)\right]$$
$$=\; \lambda^*(\mathcal{G}_m)\Psi_m^*(x) \qquad \forall\, x \in \mathbb{R}^d\,.$$

In addition, we have

$$\lambda^*(\mathcal{G}_m) \;=\; \inf_{U\in\mathfrak{U}}\ \limsup_{T\to\infty}\ \frac{1}{T}\ \log\mathbb{E}_x^U\left[e^{\int_0^T c_m(X_s,U_s)\,\mathrm{d}s}\right],$$

and $\lambda^*(\mathcal{G}_m) \to \lambda^*(\mathcal{G})$ *as* $m\to\infty$.

We are now ready to prove Theorem 2.17.

Proof of Theorem 2.17. Let

$$\lambda''(\mathcal{G}) \;=\; \inf\left\{\lambda : \exists\psi \in C^2(\mathbb{R}^d),\ \inf_{\mathbb{R}^d}\psi > 0,\ \text{satisfying } \mathcal{G}\psi - \lambda\psi \le 0 \text{ a.e. in } \mathbb{R}^d\right\}.$$

It then follows from (2.10) that $\lambda^*(\mathcal{G}) \le \lambda''(\mathcal{G})$. On the other hand, note that $c \le c_m$ for all m large, where c_m is the function in (3.14). Thus, using Lemma 3.4, we obtain

$$\mathcal{G}\Psi_m^* - \lambda^*(\mathcal{G}_m)\Psi_m^* \;\le\; 0 \quad \text{in } \mathbb{R}^d, \quad \inf_{\mathbb{R}^d}\Psi_m^* > 0\,.$$

Therefore, $\lambda''(\mathcal{G}) \le \lambda^*(\mathcal{G}_m)$ for all m, and letting $m\to\infty$ we obtain $\lambda^*(\mathcal{G}) = \lambda''(\mathcal{G})$. This concludes the proof. ☐

Next, we present the proof of Theorem 2.18.

Proof of Theorem 2.18. Since the existence of a solution is known when $\lambda = \lambda^*(\mathcal{G})$, we only consider the case $\lambda > \lambda^*(\mathcal{G})$. Recall λ_n from Lemma 2.4. Since $\lim_{n\to\infty}\lambda_n = \lambda^*(\mathcal{G})$, we have $\lambda > \lambda_n$ for all n. For each n, let f_n be a nonzero, nonnegative function supported in $B_{n+1}\setminus B_n$. Note that the principal eigenvalue of $\mathcal{G} - \lambda$, in the sense of (2.3), is $\lambda^*(\mathcal{G}) - \lambda < 0$. Therefore, by [18, Theorem 1.9], there exists a unique $\varphi_n \in C^2(B_{n+1}) \cap C(\bar{B}_{n+1})$ satisfying

$$(3.15) \qquad \mathcal{G}\varphi_n - \lambda\varphi_n \;=\; -f_n \quad \text{in } B_{n+1} \quad \text{and} \quad \varphi_n = 0 \quad \text{on } \partial B_{n+1}\,.$$

Moreover, $\varphi_n \ge 0$. Let v_n be a measurable selector of (3.15), i.e.,

$$a^{ij}(x)\,\partial_{ij}\varphi_n + b^i\big(x,v_n(x)\big)\,\partial_i\varphi_n + \big(c(x,v_n(x)) - \lambda\big)\varphi_n \;=\; -f_n \quad \text{in } B_{n+1}\,.$$

Applying Itô's formula, we obtain

$$\varphi_n(x) \;=\; \mathbb{E}_x^{v_n}\left[e^{\int_0^{t\wedge\tau_{n+1}}(c(X_s,v_n(X_s))-\lambda)\,\mathrm{d}s}\,\varphi_n\big(X_{t\wedge\tau_{n+1}}\big)\right]$$
$$+\; \mathbb{E}_x^{v_n}\left[\int_0^{t\wedge\tau_{n+1}} e^{\int_0^s(c(X_r,v_n(X_r))-\lambda)\,\mathrm{d}r}\,f_n(X_s)\,\mathrm{d}s\right]$$

for all $t \geq 0$ and $x \in B_{n+1}$. Since $f_n \gneqq 0$, this in particular, implies that $\varphi_n > 0$ in B_{n+1}. We normalize $\varphi_n(0) = 1$ by scaling f_n, and applying Harnack's inequality to (3.15), we deduce that for any compact set K we can find a constant C_K such that

$$\|\varphi_n\|_{\mathcal{W}^{2,p}(K)} < C_K \quad \forall n \text{ sufficiently large and } p \in (1, \infty) \,.$$

It is then standard to find a $\Psi \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p \geq 1$, such that $\varphi_n \to \Psi$ weakly in $\mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ and strongly in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^d)$ for some $\alpha \in (0,1)$. Therefore, we can pass to the limit in (3.15) to obtain

$$\mathcal{G}\Psi \;=\; \lambda\Psi \quad \text{in } \mathbb{R}^d, \quad \text{and } \Psi > 0 \,.$$

Using standard regularity theory from elliptic PDEs we assert that $\Psi \in C^2(\mathbb{R}^d)$. This completes the proof.   □

We conclude this section with the proof of Theorem 2.19.

Proof of Theorem 2.19. Let $\hat{v}$ be a measurable selector from the minimizer of $\mathcal{G}\varphi - \lambda\varphi = 0$. Since $\lambda \in (\lambda^*(\mathcal{G}), \lambda^*(\mathcal{H}))$, Lemma 3.2 asserts that φ has the stochastic representation in (3.6) with $v = \hat{v}$. Indeed, if c is bounded we have

$$
\begin{aligned}
\limsup_{|x|\to\infty} \left(\max_{u\in\mathbb{U}} c(x,u) - \lambda \right) &\leq \limsup_{|x|\to\infty} \max_{u\in\mathbb{U}} c(x,u) - \lambda^*(\mathcal{G}) \\
&\leq \limsup_{|x|\to\infty} \max_{u\in\mathbb{U}} c(x,u) + \|c^-\|_\infty \;<\; \gamma \,.
\end{aligned}
$$

In turn, the proof of Theorem 2.15 shows that either $\varphi < 0$ or $\varphi = 0$. But the first option implies that $\mathcal{H}(-\varphi) - \lambda(-\varphi) = 0$ which contradicts the definition of $\lambda^*(\mathcal{H})$ in (2.10). Hence $\varphi = 0$.   □

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