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This paper is a mathematical analysis of conduction effects at interfaces between insulators. Motivated by work of Haldane and Raghu (2008), we continue the study of a linear PDE initiated by Fefferman, Lee-Thorp, and Weinstein (2016). This PDE is induced by a continuous honeycomb Schrödinger operator with a line defect.

This operator exhibits remarkable connections between topology and spectral theory. It has essential spectral gaps about the Dirac point energies of the honeycomb background. In a perturbative regime, Fefferman, Lee-Thorp, and Weinstein constructed edge states: time-harmonic waves propagating along the interface, localized transversely. At leading order, these edge states are adiabatic modulations of the Dirac-point Bloch modes. Their envelopes solve a Dirac equation that emerges from a multiscale procedure.

We develop a scattering-oriented approach that derives *all* possible edge states, at arbitrary precision. The key component is a resolvent estimate connecting the Schrödinger operator to the emerging Dirac equation. We discuss topological implications via the computation of the spectral flow, or edge index.

1. Introduction and results

A central branch of condensed matter physics studies energy propagation between dissimilar media. In favorable conditions, the interface acts like a unidirectional channel for electronic transport: the material is conducting in the edge direction but remains insulating transversely. In experiments, this property is remarkably robust: it persists even if the interface becomes bent, sharp or disordered. The first theoretical investigations concerned the quantum Hall effect [Ando et al. 1975; von Klitzing et al. 1980; Halperin 1982; Thouless et al. 1982; Hatsugai 1993]. The research has since focused on topological insulators

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[Kane and Mele 2005a; 2005b; Fu et al. 2007; Moore and Balents 2007; Hsieh et al. 2008; Roy 2009; Zhang et al. 2009; Jotzu et al. 2014], together with their applications in electronics, photonics, acoustics, mechanics and geophysics [Khanikaev et al. 2007; Yu et al. 2008; Wang et al. 2008; Singha et al. 2011; Rechtsman et al. 2013; Nash et al. 2015; Brendel et al. 2017; Delplace et al. 2017; Ozawa et al. 2018; Perrot et al. 2018].

Energy transport along the interface may be interpreted as a bifurcation phenomenon. In certain periodic materials, the introduction of an edge forces Bloch modes to bifurcate into edge states: time-harmonic waves propagating along rather than across the edge. This seemingly goes back to Tamm [1932], who looked at bifurcations from local extrema in the band spectrum. Shockley [1939] next studied bifurcations from linear crossings in the band spectrum on a one-dimensional example. In contrast with Tamm's work, Shockley's analysis applies to insulators with narrow energy gaps. It was later discovered that Shockley's states may be topologically protected: *they may persist against large local perturbations.*

Honeycomb structures are invariant under $\frac{2\pi}{3}$-rotation and spatial inversion. These symmetries generate Dirac points: conical degeneracies in the band spectrum. Impurities breaking spatial inversion split the dispersion surfaces away and open energy gaps: the material transits from a metal to an insulator. Here we analyze interface effects at the junction of two such insulators.

Motivated by [Haldane and Raghu 2008; Raghu and Haldane 2008], Fefferman, Lee-Thorp and Weinstein [Fefferman et al. 2016b] introduced a PDE that models parity-breaking perturbations of a *continuous* honeycomb lattice (see Section 1A–1B). The perturbed operator exhibits (a) an edge that separates two asymptotically periodic near-honeycomb structures; (b) gaps in the essential spectrum centered at Dirac point energies of the honeycomb background. Under a spectral condition on the unperturbed operator (see [Fefferman et al. 2016b, §1.3] and Section 1C), Fefferman, Lee-Thorp and Weinstein designed edge states as adiabatic modulations of the Dirac-point Bloch modes. Their envelopes are eigenvectors of a Dirac operator produced via a multiscale procedure. See [Fefferman et al. 2016b, Theorem 7.3].

Here, we follow instead a scattering approach. We recover the results of [Fefferman et al. 2016a; 2016b]. In addition, we obtain

- a resolvent estimate connecting the initial PDE to the emerging Dirac equation,
- the complete characterization of edge states in the energy gap,
- full expansions of the edge states at all order in the size of the perturbation.

See Sections 1E and 3C for precise statements.

The full identification of edge states represents the most significant advance. It allows for topological interpretation of the results. In Section 1G, we compute the signed number of eigenvalues that move across Dirac point energies when the edge-parallel quasimomentum runs from 0 to 2π. This is a topological invariant of the system — called spectral flow or edge index — and it vanishes here. This calculation confirms numerical simulations [Raghu and Haldane 2008; Fefferman et al. 2016a; Lee-Thorp et al. 2019]. It corroborates the prediction of the Kitaev table [Kitaev 2009; Ryu et al. 2010], combined with the bulk-edge correspondence: breaking spatial inversion while keeping time-reversal invariance does not create protected edge states.

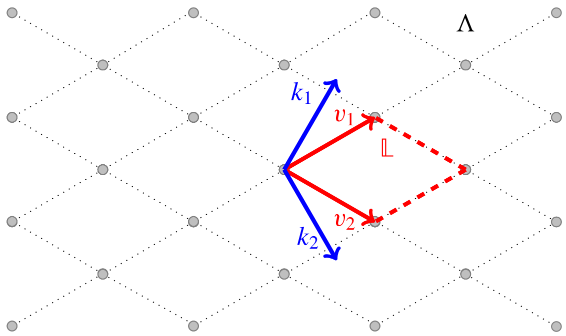

Figure 1. The equilateral lattice with its generating vectors v_1, v_2 and dual vectors k_1, k_2 together with the fundamental cell $\mathbb{L}$.

In the last part of the work, we consider a magnetic analog of the operator studied in [Fefferman et al. 2016a; 2016b], similar to those of [Raghu and Haldane 2008; Haldane and Raghu 2008; Lee-Thorp et al. 2019]. It models time-reversal breaking instead of parity breaking. We show that the corresponding spectral flow equals either 2 or -2. This confirms the existence of at least two topologically protected, unidirectionally propagating waves along the edge; see [Haldane and Raghu 2008] and the Kitaev table [Kitaev 2009; Ryu et al. 2010], as well as the numerical results [Raghu and Haldane 2008; Lee-Thorp et al. 2019].

1A. ***Periodic operators and Dirac points.*** We start with a description of honeycomb potentials as in [Fefferman and Weinstein 2012]. Let Λ be the equilateral $\mathbb{Z}^2$-lattice. It is generated by two vectors v_1 and v_2, given in canonical coordinates by

$$v_1 = a\begin{bmatrix}\sqrt{3}\\ 1\end{bmatrix}, \quad v_2 = a\begin{bmatrix}\sqrt{3}\\ -1\end{bmatrix}, \tag{1-1}$$

where $a > 0$ is a constant such that $\mathrm{Det}[v_1, v_2] = 1$. The dual basis k_1, k_2 consists of two vectors in $(\mathbb{R}^2)^*$ which satisfy $\langle k_i, v_j\rangle = \delta_{ij}$. (See Figure 1.) The dual lattice is $\Lambda^* = \mathbb{Z}k_1 \oplus \mathbb{Z}k_2$. The corresponding fundamental cell and dual fundamental cell are

$$\mathbb{L} \stackrel{\text{def}}{=} \{sv_1 + s'v_2 : s, s' \in [0,1)\}, \quad \mathbb{L}^* \stackrel{\text{def}}{=} \{\tau k_1 + \tau' k_2 : \tau, \tau' \in [0, 2\pi)\}. \tag{1-2}$$

Definition 1.1. We say that $V \in C^\infty(\mathbb{R}^2, \mathbb{R})$ is a honeycomb potential if:

- V is Λ-periodic: $V(x+w) = V(x)$ for $w \in \Lambda$.
- V is even: $V(x) = V(-x)$.
- V is invariant under the $\frac{2\pi}{3}$-rotation

$$V(Rx) = V(x), \quad R \stackrel{\text{def}}{=} \frac{1}{2}\begin{bmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{bmatrix}.$$

A simple example of honeycomb potential is the periodization of a radial function over the lattice

$$\left(\frac{v_1+v_2}{3} + \Lambda\right) \cup \left(\frac{2v_1+2v_2}{3} + \Lambda\right);$$

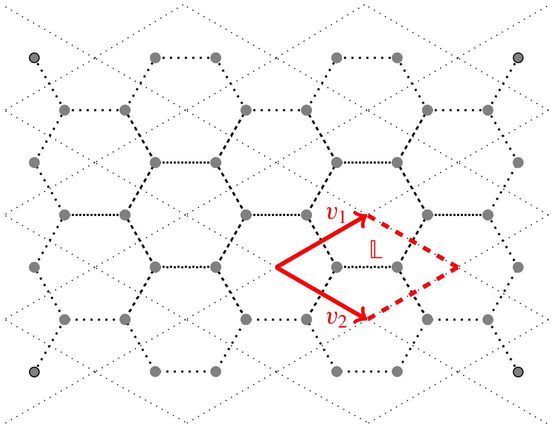

Figure 2. If each gray circle supports the same radial function (with respect to the center of the circle), the resulting potential has the honeycomb symmetry.

see Figure 2. Given a honeycomb potential V, we will study spatially delocalized perturbations of the (unbounded) Schrödinger operator

$$P_0 \stackrel{\text{def}}{=} -\Delta + V : L^2(\mathbb{R}^2, \mathbb{C}) \to L^2(\mathbb{R}^2, \mathbb{C}),$$

with domain $H^2(\mathbb{R}^2, \mathbb{C})$. This operator is periodic with respect to Λ. This allows us to apply Floquet–Bloch theory; see [Reed and Simon 1978, §XIII]: P_0 leaves the space

$$L^2_\xi \stackrel{\text{def}}{=} \{u \in L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{C}) : u(x+w) = e^{i\langle \xi, w\rangle} u(x),\ w \in \Lambda\}, \quad \xi \in \mathbb{R}^2,$$

invariant. The space L^2_ξ is Hilbertian when equipped with the Hermitian form

$$\langle f, g\rangle_{L^2_\xi} \stackrel{\text{def}}{=} \int_{\mathbb{L}} \overline{f(x)} g(x)\, dx.$$

Let $P_0(\xi)$ be formally equal to $P_0 = -\Delta + V$, but acting on L^2_ξ. It has compact resolvent and discrete spectrum — denoted below by $\Sigma_{L^2_\xi}(P_0(\xi))$ — depending on ξ:

$$\lambda_{0,1}(\xi) \leq \lambda_{0,2}(\xi) \leq \cdots \leq \lambda_{0,j}(\xi) \leq \cdots.$$

The maps $\xi \in \mathbb{R}^2 \mapsto \lambda_{0,j}(\xi)$ are called dispersion surfaces of P_0. The L^2-spectrum of P_0 consists of the ranges of the dispersion surfaces: it equals

$$\Sigma_{L^2}(P_0) = \bigcup_{\xi \in \mathbb{R}^2} \Sigma_{L^2_\xi}(P_0(\xi)) = \{\lambda_{0,j}(\xi) : j \geq 1,\ \xi \in \mathbb{R}^2\}.$$

We now discuss Dirac points. Roughly speaking, they correspond to the conical degeneracies in the band spectrum of P_0.

Definition 1.2. A pair $(\xi_\star, E_\star) \in \mathbb{R}^2 \times \mathbb{R}$ is a Dirac point of $P_0 = -\Delta + V$ if:

(i) $E_\star$ is an $L^2_{\xi_\star}$-eigenvalue of $P_0(\xi_\star)$ of multiplicity 2;

(ii) There exists an orthonormal basis $\{\phi_1, \phi_2\}$ of $\ker_{L^2_{\xi_\star}}(P_0(\xi_\star) - E_\star)$ such that

$$\phi_1(Rx) = e^{2i\pi/3}\phi_1(x), \quad \phi_2(x) = \overline{\phi_1(-x)}, \quad \phi_2(Rx) = e^{-2i\pi/3}\phi_2(x). \tag{1-3}$$

(iii) There exist $j_\star \geq 1$ and $\nu_F > 0$ such that for ξ close to $\xi_\star$,

$$\lambda_{0,j_\star}(\xi) = E_\star - \nu_F \cdot |\xi - \xi_\star| + O(\xi - \xi_\star)^2,$$
$$\lambda_{0,j_\star+1}(\xi) = E_\star + \nu_F \cdot |\xi - \xi_\star| + O(\xi - \xi_\star)^2.$$

When V is a honeycomb potential, [Fefferman and Weinstein 2012] showed that $P_0 = -\Delta + V$ generically admits Dirac points $(\xi_\star, E_\star)$. We refer to that paper for details and to Section 2C for a review of their results. Because of (1-3), $(\xi_\star, E_\star)$ must satisfy

$$\xi_\star \in \{\xi_\star^A, \xi_\star^B\} \bmod 2\pi\Lambda^*, \quad \xi_\star^A \stackrel{\text{def}}{=} \tfrac{2\pi}{3}(2k_1 + k_2), \quad \xi_\star^B \stackrel{\text{def}}{=} \tfrac{2\pi}{3}(k_1 + 2k_2). \tag{1-4}$$

See Figure 3. Symmetries impose that $(\xi_\star^A, E_\star)$ is a Dirac point of P_0 if and only if $(\xi_\star^B, E_\star)$ is a Dirac point of P_0. We call the pair (ϕ_1, ϕ_2) of (1-3) a Dirac eigenbasis.

As observed in [Fefferman and Weinstein 2012], Dirac points are stable against small perturbations preserving spatial inversion (parity) and time-reversal symmetry (conjugation). Conversely, breaking parity (while keeping conjugation invariance) generically opens spectral gaps about Dirac point energies. For $\delta \neq 0$, we introduce the operator

$$\begin{gathered} P_\delta \stackrel{\text{def}}{=} P_0 + \delta W = -\Delta + V + \delta W, \quad \text{where} \\ W \in C^\infty(\mathbb{R}^2, \mathbb{R}), \qquad W(x+w) = W(x), \quad w \in \Lambda, \qquad W(-x) = -W(x). \end{gathered} \tag{1-5}$$

We will assume in the rest of the paper that the nondegeneracy condition

$$\vartheta_\star \stackrel{\text{def}}{=} \langle \phi_1, W\phi_1 \rangle_{L^2_{\xi_\star}} \neq 0 \tag{1-6}$$

holds. This condition is generic in the sense that it excludes only a hyperplane of potentials W in the space of odd, smooth, Λ-periodic functions. Under (1-6), if $(\xi_\star, E_\star)$ is a Dirac point of P_0, then the operator $P_\delta(\xi_\star)$ (equal to P_δ, but acting on $L^2_{\xi_\star}$) admits an $L^2_{\xi_\star}$-spectral gap centered at $E_\star$:

$$\operatorname{dist}(\Sigma_{L^2_{\xi_\star}}(P_\delta(\xi_\star)), E_\star) = \vartheta_F \cdot \delta + O(\delta^2), \quad \vartheta_F \stackrel{\text{def}}{=} |\vartheta_\star|.$$

This gap has width $2\vartheta_F \cdot \delta + O(\delta^2)$; see Figure 3. This is a simple fact proved via perturbation analysis; see, e.g., [Fefferman and Weinstein 2012, Remark 9.2] or Section 4B. Whether this $L^2_{\xi_\star}$-spectral gap extends to a *global* L^2-gap of P_δ depends on the global behavior of the dispersion surfaces of P_0; see [Fefferman et al. 2016b, §1.3 and §8]. When it does, the operators $P_{\pm\delta}$ describe insulators at energy $E_\star$ with a narrow gap centered at $E_\star$. These materials are parity-breaking perturbations of the *metal* modeled by P_0.

1B. ***Edges and the model.*** We now describe the model of Fefferman, Lee-Thorp, and Weinstein [Fefferman et al. 2016a; 2016b] for honeycomb operators with an edge. Fix $v = a_1 v_1 + a_2 v_2 \in \Lambda$, with $a_1, a_2 \in \mathbb{Z}$

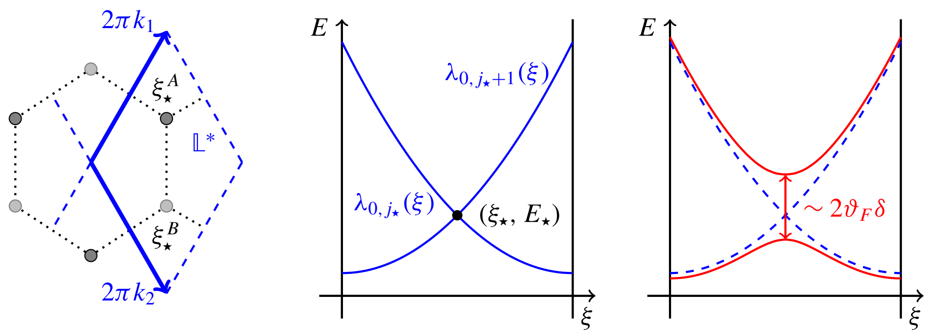

Figure 3. The picture on the left represents the Dirac points $\xi_\star^A$ and $\xi_\star^B$ inside a dual fundamental cell $\mathbb{L}^*$. The two pictures on the right represent the bifurcation of a Dirac point $(\xi_\star, E_\star)$ to an open gap on a one-dimensional section of the Brillouin zone.

relatively prime, representing the direction of an edge $\mathbb{R}v$. We introduce $v' \in \Lambda$ and $k, k' \in \Lambda^*$ such that

$$\begin{gathered} v' \stackrel{\text{def}}{=} b_1 v_1 + b_2 v_2, \quad a_1 b_2 - a_2 b_1 = 1, \quad b_1, b_2 \in \mathbb{Z}, \\ k \stackrel{\text{def}}{=} b_2 k_1 - b_1 k_2, \quad k' \stackrel{\text{def}}{=} -a_2 k_1 + a_1 k_2. \end{gathered} \tag{1-7}$$

The pairs (v, v') and (k, k') are dual to one another and span Λ and Λ^*. See Section 2E.

Recall that $P_{\pm\delta} = -\Delta + V \pm \delta W$. Fefferman, Lee-Thorp, and Weinstein [Fefferman et al. 2016a; 2016b] analyzed an operator $\mathscr{P}_\delta$ that describes an adiabatic transition from $P_{-\delta}$ to P_δ transversely to the edge $\mathbb{R}v$. Specifically,

$$\mathscr{P}_\delta \stackrel{\text{def}}{=} P_0 + \delta \cdot \kappa_\delta \cdot W = -\Delta + V + \delta \cdot \kappa_\delta \cdot W.$$

Above, the function $\kappa_\delta \in C^\infty(\mathbb{R}^2, \mathbb{R})$ is an adiabatic modulation of a domain wall $\kappa \in C^\infty(\mathbb{R}, \mathbb{R})$ along $\mathbb{R}v$:

$$\kappa_\delta(x) = \kappa(\delta\langle k', x\rangle), \quad \exists L > 0, \quad \kappa(t) = \begin{cases} -1 & \text{when } t \le -L, \\ 1 & \text{when } t \ge L. \end{cases} \tag{1-8}$$

The operator $\mathscr{P}_\delta$ is a Schrödinger operator with potential represented in Figure 9. It models the soft junction of two insulators modeled by $P_{\pm\delta}$ along the interface $\mathbb{R}v$.

Although $\mathscr{P}_\delta$ is not periodic with respect to Λ, it is periodic with respect to $\mathbb{Z}v$ because $\langle k', v\rangle = 0$. For every $\zeta \in \mathbb{R}$, $\mathscr{P}_\delta$ acts as an unbounded operator on

$$L^2[\zeta] \stackrel{\text{def}}{=} \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{C}) : u(x+v) = e^{i\zeta} u(x), \ \int_{\mathbb{R}^2/\mathbb{Z}v} |u(x)|^2\, dx < \infty \right\}, \tag{1-9}$$

with domain $H^2[\zeta]$ — defined according to (1-9). Let $\mathscr{P}_\delta[\zeta]$ be the resulting operator.

We continue the analysis of [Fefferman et al. 2016a; 2016b]: we study the electronic properties of the material modeled by $\mathscr{P}_\delta$. We investigate whether energy propagates along the edge $\mathbb{R}v$. This boils down to studying *edge states* of $\mathscr{P}_\delta$. These are time-harmonic waves propagating along $\mathbb{R}v$ and localized transversely to $\mathbb{R}v$. Mathematically, they are the $L^2[\zeta]$-eigenvectors of $\mathscr{P}_\delta[\zeta]$. Such states correspond to diffusionless electronic channels along $\mathbb{R}v$; they have great potential in technological applications.

1C. ***The no-fold condition of Fefferman, Lee-Thorp, and Weinstein.*** We set $\zeta_\star = \langle \xi_\star, v \rangle$ and $\zeta_\star^J = \langle \xi_\star^J, v \rangle$. Thanks to (1-4),

$$\zeta_\star^A = \tfrac{2\pi}{3}(2a_1 + a_2), \qquad \zeta_\star^B = \tfrac{2\pi}{3}(a_1 + 2a_2). \tag{1-10}$$

Hence, $\zeta_\star \in \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\} \bmod 2\pi\mathbb{Z}$. Recall the no-fold condition [Fefferman et al. 2016b, §1.3].

Definition 1.3. The no-fold condition holds along the edge $\mathbb{R}v$ at $\zeta_\star$ if

$$\forall j \geq 1,\ \forall \tau \in \mathbb{R}, \quad \lambda_{0,j}(\zeta_\star k + \tau k') = E_\star \quad \Longrightarrow \quad j \in \{j_\star, j_\star + 1\} \quad \text{and} \quad \tau = \langle \xi_\star, v' \rangle \bmod 2\pi.$$

The essential spectrum of $\mathscr{P}_\delta[\zeta_\star]$ is obtained from the (essential) spectra of the bulk operators $P_{\pm\delta}[\zeta_\star]$ (the operators formally equal to $P_{\pm\delta}$, but acting on $L^2[\zeta_\star]$). These are conjugated under spatial inversion. Therefore they have the same spectrum. From Floquet–Bloch theory,

$$\Sigma_{L^2[\zeta_\star],\mathrm{ess}}(\mathscr{P}_\delta[\zeta_\star]) = \Sigma_{L^2[\zeta_\star]}(P_\delta[\zeta_\star]) = \bigcup_{\xi \in \zeta_\star k + \mathbb{R}k'} \Sigma_{L^2_\xi}(P_\delta(\xi)).$$

If $(\xi_\star, E_\star)$ is a Dirac point of P_0 and $\vartheta_\star \neq 0$, then for small δ, $P_{\pm\delta}(\xi)$ has an L^2_ξ-spectral gap centered at $E_\star$ when ξ is $O(\delta)$-away from $\xi_\star$ — see, e.g., Section 4B. The no-fold condition requires this gap to extend to an $L^2[\zeta_\star]$-spectral gap of $P_{\pm\delta}[\zeta_\star]$.

The no-fold condition holds for $|V|_\infty$ sufficiently small and the zigzag edge $a_1 = 1,\ a_2 = 0$ [Fefferman et al. 2016b, Theorem 8.2]. It holds for $|V|_\infty$ sufficiently large and edges satisfying $a_1 \neq a_2 \bmod 3$ [Fefferman et al. 2018, Corollary 6.3]. It may fail in physically relevant cases. See, e.g., the case of certain low-contrast potentials and the zigzag edge [Fefferman et al. 2016b, Theorem 8.4] and armchair-type edges $v = a_1 v_1 + a_2 v_2$, where $a_1 - a_2 = 0 \bmod 3$ [Fefferman et al. 2018, Remark 6.5] or Section 2E. In particular, if the no-fold condition holds, (1-10) and $a_1 - a_2 \neq 0 \bmod 3$ prescribe the possible values of $\zeta_\star$:

$$\zeta_\star \in \{\zeta_\star^A, \zeta_\star^B\} = \left\{\tfrac{2\pi}{3}, \tfrac{4\pi}{3}\right\} \bmod 2\pi\mathbb{Z}.$$

1D. ***The multiscale approach of [Fefferman et al. 2016b] and the Dirac operator.*** Let $(\xi_\star, E_\star)$ be a Dirac point of P_0 and (ϕ_1, ϕ_2) be a Dirac eigenbasis (see Definition 1.2). The map

$$\eta \in \mathbb{R}^2 \mapsto 2\langle \phi_1, (\eta \cdot D_x)\phi_2 \rangle \in \mathbb{C} \tag{1-11}$$

is linear. Because of rotational invariance of $P_0 = -\Delta + V$, the map (1-11) acts (as an application from $\mathbb{C}$ to $\mathbb{C}$) like a complex multiplication:

$$\exists \nu_\star \in \mathbb{C} \setminus \{0\}, \quad \forall \eta \in \mathbb{R}^2 \equiv \mathbb{C}, \qquad \nu_\star \eta = 2\langle \phi_1, (\eta \cdot D_x)\phi_2 \rangle_{L^2_{\xi_\star}}.$$

See Section 2C. Recall that $\vartheta_\star = \langle \phi_1, W\phi_1 \rangle_{L^2_{\xi_\star}} \neq 0$ and that κ satisfies (1-8). In this section, we review the role of the (unbounded) Dirac operator

$$\not{D}_\star = \begin{bmatrix} 0 & \nu_\star k' \\ \overline{\nu_\star k'} & 0 \end{bmatrix} D_t + \vartheta_\star \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \kappa : L^2(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$$

in the analysis of Fefferman, Lee-Thorp, and Weinstein [Fefferman et al. 2016b].

When $\vartheta_\star \neq 0$, [loc. cit.] produces arbitrarily accurate quasimodes of $\mathscr{P}_\delta[\zeta_\star]$ via a multiscale approach. These are pairs $(u_\delta, E_\delta) \in H^2_{\zeta_\star} \times \mathbb{R}$ satisfying

$$(\mathscr{P}_\delta[\zeta_\star] - E_\delta)u_\delta = O_{L^2[\zeta_\star]}(\delta^\infty), \quad E_\delta = E_\star + \delta E_1 + O(\delta^2).$$

They are power series in δ whose coefficients solve a hierarchy of equations of orders $1, \delta, \delta^2, \dots$. The operator $\not{D}_\star$ appears in the equation of order δ. This equation admits a solution if and only if E_1 is an eigenvalue of $\not{D}_\star$; see [loc. cit., §6].

The operator $\not{D}_\star$ has essential spectrum equal to $(-\infty, \vartheta_F] \cup [\vartheta_F, \infty)$. It has an odd number of eigenvalues $\{\vartheta_j\}_{j=-N}^N$ in $(-\vartheta_F, \vartheta_F)$, simple and symmetric about 0:

$$\vartheta_{-N} < \cdots < \vartheta_{-1} < \vartheta_0 = 0 < \vartheta_1 < \cdots < \vartheta_N, \quad \vartheta_{-j} = -\vartheta_j.$$

In particular, 0 is always an eigenvalue of $\not{D}_\star$. We refer to see Section 3B for details.

When the no-fold condition holds, [loc. cit.] uses a sophisticated Lyapounov–Schmidt reduction to prove that each eigenvalue ϑ_j of $\not{D}_\star$ seeds an $L^2[\zeta_\star]$-eigenvalue of $\mathscr{P}_\delta[\zeta_\star]$, with energy $E_\star + \delta\vartheta_j + O(\delta^2)$. They show that to leading order, the corresponding eigenvector equals the first term produced by the multiscale approach: it is

$$\alpha_1(\delta\langle k', x\rangle) \cdot \phi_1(x) + \alpha_2(\delta\langle k', x\rangle) \cdot \phi_2(x) + O_{H^2_{\zeta_\star}}(\delta^{1/2}), \quad (\not{D}_\star - \vartheta_j)\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = 0.$$

In other words, they validate mathematically the formal multiscale procedure at leading order. But some questions persist:

- Is the multiscale procedure rigorously valid at all orders?
- Do the eigenvalues of $\not{D}_\star$ seed *all* eigenvalues of $\mathscr{P}_\delta[\zeta_\star]$ near $E_\star$?
- How can the relation between $\mathscr{P}_\delta[\zeta_\star]$ and $\not{D}_\star$ be clarified?

The present work responds to these questions.

1E. *Results.* Our first result relates the resolvents of $\mathscr{P}_\delta[\zeta_\star]$ and $\not{D}_\star$. It requires the operator Π and its adjoint Π^*, defined as

$$\Pi : L^2(\mathbb{R}^2/\mathbb{Z}v, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2), \qquad (\Pi f)(t) \stackrel{\text{def}}{=} \int_0^1 f(sv + tv')\,ds,$$
$$\Pi^* : L^2(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}^2/\mathbb{Z}v, \mathbb{C}^2), \quad (\Pi^* g)(x) \stackrel{\text{def}}{=} g(\langle k', x\rangle),$$

and the dilation $\mathcal{U}_\delta$ defined as

$$\mathcal{U}_\delta : L^2(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2), \quad (\mathcal{U}_\delta f)(t) \stackrel{\text{def}}{=} f(\delta t).$$

Recall that V is a honeycomb potential — see Definition 1.1; $W \in C^\infty(\mathbb{R}^2, \mathbb{R})$ breaks spatial inversion — see (1-5); and $\kappa \in C^\infty(\mathbb{R}, \mathbb{R})$ is a domain wall function — see (1-8). We make the following assumptions:

(H1) $(\xi_\star, E_\star)$ is a Dirac point of $P_0 = -\Delta + V$ — see Definition 1.2 — with $\xi_\star \in \mathbb{L}^*$.

(H2) The no-fold condition — Definition 1.3 — holds.

(H3) The nondegeneracy assumption $\vartheta_\star \neq 0$ holds — see (1-6).

Theorem 1.4. *Assume (H1)–(H3) hold and fix* $\epsilon > 0$. *There exists* $\delta_0 > 0$ *such that if*

$$\delta \in (0, \delta_0), \quad z \in \mathbb{D}(0, \vartheta_F - \epsilon), \quad \operatorname{dist}(\Sigma_{L^2}(\not{D}_\star), z) \geq \epsilon, \quad \lambda = E_\star + \delta z$$

then $\mathscr{P}_\delta[\zeta_\star] - \lambda$ *is invertible and*

$$(\mathscr{P}_\delta[\zeta_\star] - \lambda)^{-1} = \frac{1}{\delta} \cdot \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}^\top \cdot \Pi^* \mathcal{U}_\delta \cdot (\not{D}_\star - z)^{-1} \cdot \mathcal{U}_\delta^{-1} \Pi \cdot \overline{\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}} + \mathscr{O}_{L^2[\zeta_\star]}(\delta^{-1/3}). \tag{1-12}$$

The leading-order term in (1-12) comes with a coefficient $1/\delta$: the remainder term $\mathscr{O}_{L^2[\zeta_\star]}(\delta^{-1/3})$ is subleading when $z \in \mathbb{D}(0, \vartheta_F - \epsilon)$. Hence, Theorem 1.4 shows that the resolvents of $\mathscr{P}_\delta[\zeta_\star]$ and of $\not{D}_\star$ behave similarly, after suitable conjugations.

Theorem 1.4 applies to a spectral range that spans — modulo ϵ — the entire spectral gap of $\mathscr{P}_\delta[\zeta_\star]$ about $E_\star$. The next result describes the spectrum of $\mathscr{P}_\delta[\zeta_\star]$ in the essential spectral gap in terms of the eigenvalues

$$\vartheta_{-N} < \cdots < \vartheta_{-1} < \vartheta_0 = 0 < \vartheta_1 < \cdots < \vartheta_N$$

of the Dirac operator $\not{D}_\star$. Let X be the function space equal to

$$\{f \in C^\infty(\mathbb{R}^2 \times \mathbb{R}, \mathbb{C}) : \forall t \in \mathbb{R},\ f(\cdot, t) \in L^2_{\xi_\star} \text{ and } \exists a > 0,\ \sup e^{a|t|} |f(x,t)| < \infty\}. \tag{1-13}$$

Corollary 1.5. *Assume (H1)–(H3) hold and fix* $\vartheta_\sharp \in (\vartheta_N, \vartheta_F)$. *There exists* $\delta_0 > 0$ *such that for* $\delta \in (0, \delta_0)$ *the operator* $\mathscr{P}_\delta[\zeta_\star]$ *has exactly* $2N+1$ *eigenvalues* $\{E_{\delta,j}\}_{j \in [-N,N]}$ *in* $[E_\star - \vartheta_\sharp \delta, E_\star + \vartheta_\sharp \delta]$ *that are all simple.*

The associated eigenpairs $(E_{\delta,j}, u_{\delta,j})$ *admit full two-scale expansions in powers of* δ:

$$\begin{aligned} E_{\delta,j} &= E_\star + \vartheta_j \cdot \delta + a_2 \cdot \delta^2 + \cdots + a_M \cdot \delta^M + O(\delta^{M+1}), \\ u_{\delta,j}(x) &= f_0(x, \delta\langle k', x\rangle) + \delta \cdot f_1(x, \delta\langle k', x\rangle) + \cdots + \delta^M \cdot f_M(x, \delta\langle k', x\rangle) + o_{H^k}(\delta^M). \end{aligned}$$

In the above:

- M *and* k *are any integers*; H^k *is the* k*-th order Sobolev space.*
- *The terms* $a_m \in \mathbb{R}$, $f_m \in X$ *are recursively constructed via multiscale analysis.*
- *The leading-order term* f_0 *satisfies*

$$f_0(x,t) = \alpha_1(t) \cdot \phi_1(x) + \alpha_2(t) \cdot \phi_2(x), \quad (\not{D}_\star - \vartheta_j) \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = 0.$$

This corollary (a) mathematically validates the multiscale procedure of [Fefferman et al. 2016b] at all orders in δ, and (b) shows that all eigenvectors of $\mathscr{P}_\delta[\zeta_\star]$ are induced by the modes of $\not{D}_\star$. See Figures 4 and 5. In particular, (a) improves the result of Fefferman, Lee-Thorp, and Weinstein [loc. cit.] to arbitrary order in δ. From a general point of view, (b) represents the most important advance. It characterizes edge states topologically. It opens the way for mathematical proofs of the bulk-edge correspondence in *continuous* honeycomb structures. See Section 1G and [Drouot 2019] for further details.

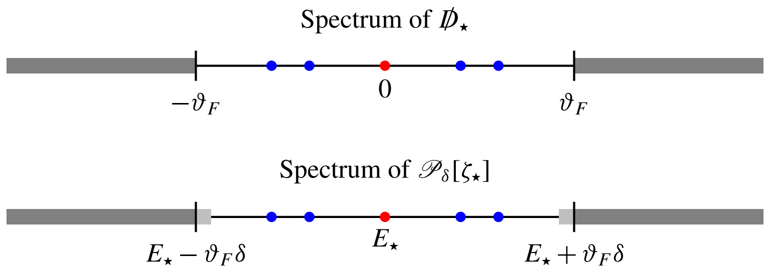

Figure 4. Eigenvalues of $\not{D}_\star$ in $(-\vartheta_F, \vartheta_F)$ (top) and eigenvalues of $\mathscr{P}_\delta$ in the spectral gap containing $E_\star$ (bottom). An approximate rescaling equal to $z \mapsto E_\star + \delta z + O(\delta^2)$ maps the top to the bottom. The red dots represent the zero eigenvalue of $\not{D}_\star$ and the corresponding one for $\mathscr{P}_\delta$. Theorem 1.4 and Corollary 1.5 do not apply in the lighter gray area near the essential spectrum.

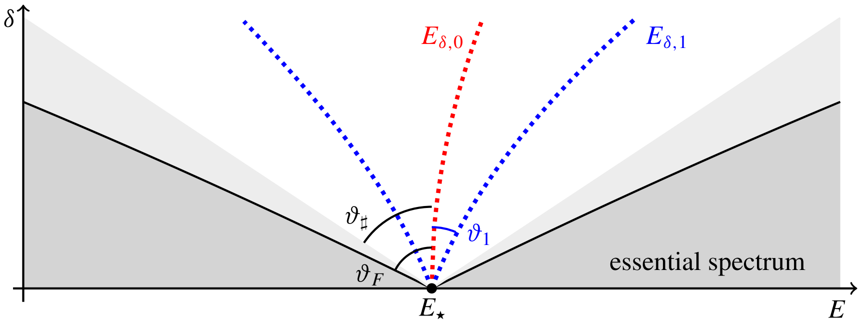

Figure 5. Discrete eigenvalues of $\not{D}_\star$ seed the bifurcation of eigenvalues of $\mathscr{P}_\delta$ (red dotted curves) from the Dirac point energy $E_\star$ (at $\delta = 0$) of P_0 as δ increases away from zero. The slopes of these curves at $\delta = 0$ (blue lines) are given by the eigenvalues of $\not{D}_\star$.

1F. ***Extension to quasimomenta near $\zeta_\star$.*** Corollary 1.5 predicts that for $\delta \in (0, \delta_0)$, $\mathscr{P}_\delta[\zeta_\star]$ has precisely $2N+1$ eigenvalues near $E_\star$. A general perturbation argument shows that $\mathscr{P}_\delta[\zeta]$ also has $2N+1$ eigenvalues for ζ close enough to $\zeta_\star$. However this argument does not specify quantitatively how close ζ needs to be to $\zeta_\star$.

We prove generalizations of Theorem 1.4 and Corollary 1.5 that hold for ζ at distance $O(\delta)$ from $\zeta_\star$; see Section 3C for statements. We show that the eigenvalues of $\mathscr{P}_\delta[\zeta_\star + \mu\delta]$ lying near $E_\star$ and of the Dirac operator

$$\not{D}(\mu) \overset{\text{def}}{=} \begin{bmatrix} 0 & \nu_\star k' \\ \overline{\nu_\star k'} & 0 \end{bmatrix} D_t + \mu \begin{bmatrix} 0 & \nu_\star \ell \\ \overline{\nu_\star \ell} & 0 \end{bmatrix} + \vartheta_\star \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \kappa, \qquad \ell \overset{\text{def}}{=} k - \frac{\langle k, k' \rangle}{|k'|^2} k',$$

are $O(\delta^2)$-away after the rescaling $z \mapsto E_\star + \delta z$.

Interestingly enough, the spectrum of $\not{D}(\mu)$ can be derived from that of $\not{D}_\star = \not{D}(0)$; see Section 3B and Figure 6. We observe that $\not{D}(\mu)$ has a topologically protected mode that bifurcates linearly from the zero

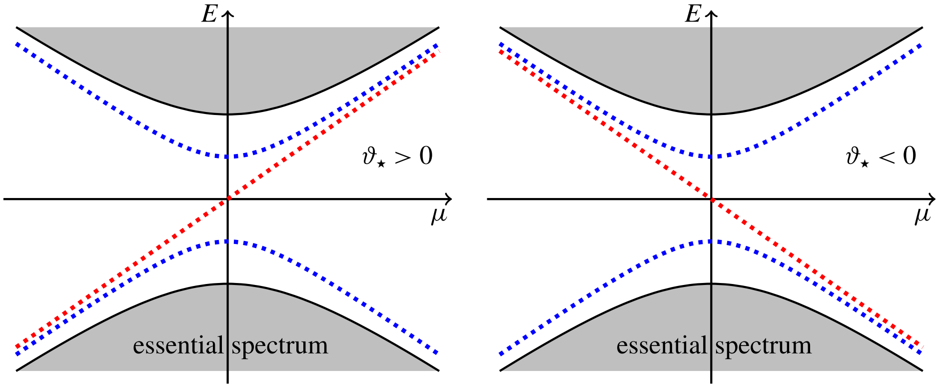

Figure 6. The spectrum of $\not{D}(\mu)$ as a function of μ. The topologically protected eigenvalue (in red) bifurcate linearly, while the nontopologically protected eigenvalues (in blue) bifurcate quadratically.

mode of $\not{D}_\star$. This suggests that under the $\mathscr{P}_\delta$ time-dependent evolution, L^2-wave packets formed from the topologically protected mode of $\not{D}(\mu)$ propagate dispersionless along the edge for a very long time.

All other modes of $\not{D}(\mu)$ are nontopologically protected and bifurcate quadratically from the modes of $\not{D}_\star$. L^2-wave packets formed from such modes should have a shorter lifetime. This suggests that topologically protected modes are more robust even in the time-dependent situation.

1G. ***A topological perspective.*** Recall that $k' \in \Lambda^*$ is the dual direction transverse to an edge $\mathbb{R}v$ and that $\lambda_{0,j}(\xi)$ are the dispersion surfaces of a honeycomb Schrödinger operator P_0. Let $(\xi_\star, E_\star) = (\xi_\star, \lambda_{0,j_\star}(\xi_\star))$ denote a Dirac point of P_0. We introduce an assumption (H4) that extends (H3) to values $\zeta \neq \zeta_\star$. It asks for the $j_\star$-th $L^2[\zeta]$-gap of $P_0[\zeta]$ to be open when $\zeta \notin \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\} \bmod 2\pi\mathbb{Z}$.

(H4) For every $\zeta \notin \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\} \bmod 2\pi\mathbb{Z}$, for every $\tau, \tau' \in \mathbb{R}$,

$$\lambda_{0,j_\star}(\zeta k + \tau k') < \lambda_{0,j_\star+1}(\zeta k + \tau' k').$$

Assumption (H4) holds for nonarmchair-type edges ($a_1 \neq a_2 \bmod 3$) and high-contrast potentials: see [Fefferman et al. 2018, Theorem 6.1 and Remark 6.5]. This follows from two general phenomena:

- Schrödinger operators with multiple-well potentials approach their tight binding limits as the depth of the wells increases [Harrell 1979; Helffer and Sjöstrand 1984; 1985; 1987; Simon 1984; Martinez 1987; 1988; Outassourt 1987; Carlsson 1990; Fefferman et al. 2018; Fefferman and Weinstein 2018];
- Wallace's tight binding model of honeycomb lattices [1947] satisfies a suitable version of (H4).

When (H1)–(H4) hold and δ is sufficiently small, the $j_\star$-th $L^2[\zeta]$-gap of $P_\delta[\zeta]$ is open. This allows us to define the spectral flow of the family

$$\zeta \in [0, 2\pi] \mapsto \mathscr{P}_\delta[\zeta]$$

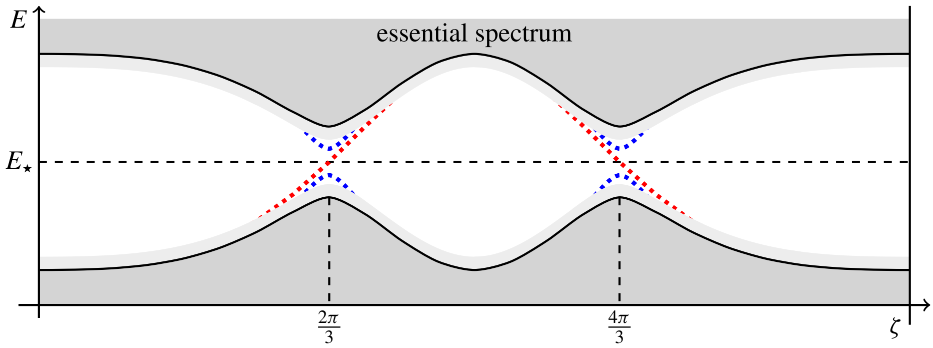

Figure 7. The spectrum of $\mathscr{P}_\delta[\zeta]$ as a function of ζ. The dark gray region represents the essential spectrum. The dotted curves are the eigenvalues of $\mathscr{P}_\delta[\zeta]$ (the edge state energies). Zooming about δ^{-1} times near $\left(\frac{2\pi}{3}, E_\star\right)$ or $\left(\frac{4\pi}{3}, E_\star\right)$ produces Figure 6. Because of complex conjugation, $\vartheta_\star^A = -\vartheta_\star^B$: near $\frac{2\pi}{3}$ $\left(\text{resp. } \frac{4\pi}{3}\right)$, the red curves move upwards (resp. downwards). This results in a spectral flow cancellation.

in the $j_\star$-th $L^2[\zeta]$-gap. It is the signed number of $L^2[\zeta]$-eigenvalues of $\mathscr{P}_\delta[\zeta]$ crossing the $j_\star$-th gap downwards as ζ runs from 0 to 2π; see, e.g., [Waterstraat 2017, §4]. Corollary 3.3 in Section 3C allows one to count precisely these eigenvalues. It leads to:

Corollary 1.6. *Assume that (H1)–(H4) hold for both Dirac points $(\xi_\star^A, E_\star)$ and $(\xi_\star^B, E_\star)$. There exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$, the spectral flow of $\mathscr{P}_\delta$ in the $j_\star$-th $L^2[\zeta]$-gap vanishes.*

This is because $\vartheta_\star^A$ and $\vartheta_\star^B$ are opposite — where $\vartheta_\star^J$ corresponds to $\vartheta_\star$ for the Dirac point $(\xi_\star^J, E_\star)$. See Figure 7. The spectral flow is a topological invariant: it does not change if a 2π-periodic family of compact operators $H^2[\zeta] \to L^2[\zeta]$ is added to $\mathscr{P}_\delta[\zeta]$. Hence Corollary 1.6 is very robust. However, it is a disappointing result: it suggests that the edge states of Corollary 1.5 shall not be topologically stable. We conjecture:

Conjecture. *Assume that (H1)–(H4) hold for both Dirac points $(\xi_\star^A, E_\star)$ and $(\xi_\star^B, E_\star)$. There exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ there exists a family $\zeta \in \mathbb{R} \mapsto B_\delta(\zeta)$ such that:*

- *$B_\delta(\zeta)$ is a compact operator $H^2[\zeta] \to L^2[\zeta]$.*
- *$B_\delta(\zeta)$ depends continuously on ζ (with respect to the operator norm on $H^2[\zeta] \to L^2[\zeta]$) and $B_\delta(\zeta + 2\pi) = B_\delta(\zeta)$ for every $\zeta \in \mathbb{R}$.*
- *$\mathscr{P}_\delta[\zeta] + B_\delta(\zeta) : H^2[\zeta] \to L^2[\zeta]$ has no eigenvalues in the essential spectral gap containing $E_\star$.*

On a positive note, our approach also applies to magnetic Schrödinger operators

$$\begin{gathered} \mathbb{P}_\delta = -(\nabla_{\mathbb{R}^2} + i\delta \cdot \kappa_\delta \cdot \mathbb{A})^2 + V, \\ \mathbb{A} \in C^\infty(\mathbb{R}^2, \mathbb{R}^2), \qquad \mathbb{A}(x + w) = \mathbb{A}(x), \quad w \in \Lambda, \qquad \mathbb{A}(-x) = -\mathbb{A}(x). \end{gathered} \tag{1-14}$$

The asymptotic operators for $\langle k', x \rangle$ near $\pm\infty$ are equal to

$$-(\nabla_{\mathbb{R}^2} + i\delta\mathbb{A})^2 + V. \tag{1-15}$$

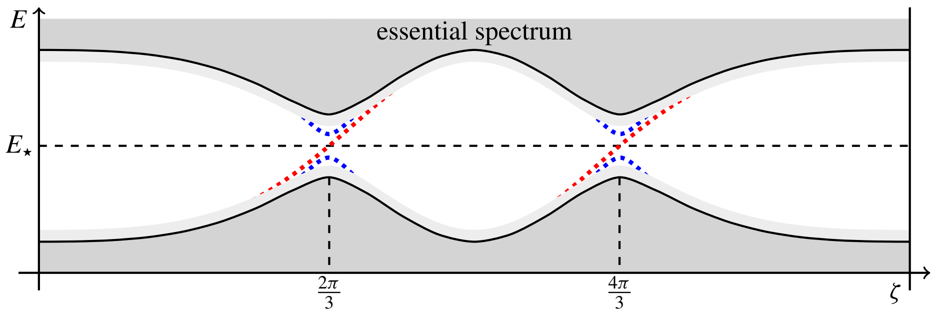

Figure 8. The spectrum of the magnetic-like perturbation $\mathbb{P}_\delta$ of P_0 for positive $\theta_\star$. The topologically protected mode of the Dirac operator induces precisely two edge-state energy curves. In contrast with Figure 7, $\theta_\star^A = \theta_\star^B$: both red curves move upwards. The resulting spectral flow is -2, indicating topologically protected states.

From a physical point of view, (1-15) models quantum particles in a magnetic field $\delta B = \delta(\partial_1 \mathbb{A}_2 - \partial_2 \mathbb{A}_1)$ (oriented in the direction $e_3 \in \mathbb{R}^3$ orthogonal to $\mathbb{R}^2$) and an electric field ∇V. Therefore $\mathbb{P}_\delta$ represents particles evolving in a near-periodic electromagnetic background, with magnetic field varying adiabatically along $\mathbb{R}v'$ from $-\delta B$ to δB. Note that the magnetic flux of B vanishes because B is periodic.

We can see (1-14) as a perturbation of $-\Delta + V$ by

$$\delta \cdot \kappa_\delta \cdot \mathbb{W}, \qquad \mathbb{W} \stackrel{\text{def}}{=} \mathbb{A} \cdot D_x + D_x \cdot \mathbb{A},$$

modulo a term of order δ^2. The perturbation $\mathbb{W}$ no longer breaks spatial inversion; instead it breaks time-reversal symmetry (complex conjugation). See [Raghu and Haldane 2008; Haldane and Raghu 2008; Lee-Thorp et al. 2019] for related models. We replace (H3) with:

(H3′) The nondegeneracy condition $\theta_\star \stackrel{\text{def}}{=} \langle \phi_1, \mathbb{W}\phi_1 \rangle_{L^2_{\xi_\star}} \neq 0$ holds.

When (H1), (H2) and (H3′) hold, the operator $\mathbb{P}_\delta[\zeta_\star]$ has an essential spectral gap centered at $E_\star$, of width of order δ — similarly to $\mathscr{P}_\delta[\zeta_\star]$. If moreover (H4) holds, then we can define the spectral flow of the family $\zeta \mapsto \mathbb{P}_\delta[\zeta]$.

Corollary 1.7. *Assume that* (H1), (H2), (H3′) *and* (H4) *hold for both Dirac points* $(\xi_\star^A, E_\star)$ *and* $(\xi_\star^B, E_\star)$. *There exists* $\delta_0 > 0$ *such that for all* $\delta \in (0, \delta_0)$, *the spectral flow of* $\mathbb{P}_\delta$ *equals* $-2 \cdot \operatorname{sgn}(\theta_\star)$.

Corollary 1.7 shows that $\mathbb{P}_\delta$ admits two topologically protected edge states; see Figure 8. This corroborates results of [Haldane and Raghu 2008; Raghu and Haldane 2008], where two quasimodes are produced via a multiscale approach. They were not proved to be topologically protected there: a statement in the spirit of Corollary 3.3 is missing. The authors perform a formal computation of the bulk index: they show that it should equal 2 or -2. We studied rigorously the bulk aspects of our problem in the recent work [Drouot 2019].

1H. *Strategy.* Our proof has three essential components:

• The simplest step consists in deriving Corollary 1.5 from Theorem 1.4; see Section 3C. Theorem 1.4 is used to count the exact number $2N+1$ of eigenvalues in the essential spectral gap (slightly away from the edges). We derive the full expansion of edge states in powers of δ using (a) the formal multiscale procedure of [Fefferman et al. 2016b] to produce $2N+1$, almost orthogonal, arbitrarily accurate quasimodes, and (b) a general selfadjoint principle that implies that these quasimodes must all be near genuine eigenvectors.

• We derive resolvent estimates for the bulk operators $P_{\pm\delta}[\zeta_\star]$. We first obtain resolvent estimates for the operators $P_{\pm\delta}(\xi) : H^2_\xi \to L^2_\xi$ in Section 4. We prove that near $(\xi_\star, E_\star)$, these operators essentially behave like Pauli matrices. In Section 5 we integrate these estimates along the dual edge $\zeta_\star k + \mathbb{R}k'$ and derive the expansion

$$(P_{\pm\delta}[\zeta_\star] - \lambda)^{-1} = \frac{1}{\delta} \cdot \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}^\top \Pi^* \cdot \mathcal{U}_\delta (\not{D}_{\star,\pm} - z)^{-1} \mathcal{U}_\delta^{-1} \cdot \Pi \overline{\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}} + \mathscr{O}_{L^2[\zeta]}(\delta^{-1/3}).$$

Above, Π and $\mathcal{U}_\delta$ are the operators introduced in Section 1E, and $\not{D}_{\star,\pm}$ are the formal limits of $\not{D}_\star$ as t goes to $\pm\infty$.

• We use a sophisticated version of the Lippmann–Schwinger principle to connect the resolvents of $\mathscr{P}_\delta[\zeta_\star]$ and of $P_{\pm\delta}[\zeta_\star]$. This requires us to construct a parametrix for $\mathscr{P}_\delta[\zeta_\star]$. After algebraic manipulations — essentially cyclicity arguments — homogenization effects take place and produce the operator $\not{D}_\star$. This leads to the resolvent estimate of Theorem 1.4.

1I. *Relation to earlier work.* The mechanism responsible for the production of edge states is the bifurcation of eigenvalues from the edge of the continuous spectrum. Such problems have a long history: see, e.g., [Tamm 1932; Schockley 1939; Simon 1976; Deift and Hempel 1986; Figotin and Klein 1997; Borisov 2007; 2011; 2015; Borisov and Gadyl'shin 2008; Parzygnat et al. 2010; Hoefer and Weinstein 2011; Zelenko 2016] for states generated by defects in periodic backgrounds; and [Golowich and Weinstein 2005; Borisov and Gadyl'shin 2006; Duchêne and Weinstein 2011; Duchêne et al. 2014; Dimassi 2016; Dimassi and Duong 2017; Drouot 2018a; 2018c; 2018d; Duchêne and Raymond 2018] for localized highly oscillatory perturbations.

Fefferman, Lee-Thorp and Weinstein [Fefferman et al. 2016a; 2016b] produced the closest results to our analysis. They were the first to prove existence of edge states for continuous honeycomb lattices in the small/adiabatic regime $\delta \to 0$. They built on their own work [Fefferman et al. 2014; 2017], where they proved existence of defect states for dislocated one-dimensional materials.

Our work improves and extends [Fefferman et al. 2016a; 2016b] in the following ways:

- It connects the resolvents of $\mathscr{P}_\delta[\zeta]$ and $\not{D}(\mu)$.
- It provides full expansions of edge states in powers of δ.
- It identifies *all* edge states with energy near Dirac point energies.

The third point allows for the topological interpretation of the results in terms of the spectral flow of $\zeta \mapsto \mathscr{P}_\delta[\zeta]$. This is a robust invariant of the system, also called the edge index. We conjecture that the modes of $\mathscr{P}_\delta[\zeta]$ should not be topologically protected: the edge index vanishes. However, for the magnetic operator $\mathbb{P}_\delta[\zeta]$ introduced in (1-14), two such states are topologically protected: they persist under large (suitable) deformations.

We refer to [Gérard et al. 1991; Panati et al. 2003; Watson et al. 2017; Watson and Weinstein 2018] for the study of similar operators with perturbations that vary adiabatically in all directions and to [De Nittis and Lein 2011; 2014; Cornean et al. 2015; 2017a; 2017b] for analysis of perturbations small with respect to the inverse scale of variation. The scaling studied here is peculiar: the perturbation varies adiabatically in one direction only.

Our strategy generalizes the one-dimensional work [Drouot et al. 2018], developed to improve the results of [Fefferman et al. 2014; 2017]. The construction of genuine edge states from quasimodes in Section 3C follows the classical approach of [Drouot et al. 2018, §3.3]. We derive the fiberwise resolvent estimates for $P_{\pm\delta}(\xi)$ in Sections 4A–4B as in [loc. cit., §4.1-4.2]. We did not prove resolvent estimates in [loc. cit.]; we used instead Fredholm determinants.

We pushed the analysis of [Drouot et al. 2018] further in [Drouot 2018b]. There, we showed that the defect states of [Fefferman et al. 2014; 2017] are topologically stable in the following sense. The model embeds naturally in a one-parameter family of *dislocated systems*, related to [Post 2003; Korotyaev 2000; Hempel and Kohlmann 2011a; 2011b; Dohnal et al. 2009; Hempel et al. 2015]. We compute the spectral flow in terms of bulk quantities. We show that it is equal to the bulk index — the Chern number of a Bloch eigenbundle for the bulk. Hence, [Drouot 2018b] provides a novel continuous setting where the bulk-edge correspondence holds — adding to [Kellendonk and Schulz-Baldes 2004a; 2004b; Taarabt 2014; Fukui et al. 2012; Bal 2017; 2018; Bourne and Rennie 2018]. A similar strategy has been developed in [Drouot 2019] to deal with magnetic honeycomb operators.

1J. *Further perspectives.* Our results stimulate future lines of research:

- Armchair-type edges are edges such that the associated dual line $\zeta_\star k + \mathbb{R}k'$ passes through both Dirac momenta $\xi_\star^A$ and $\xi_\star^B$. They correspond to the directions

$$v = a_1 v_1 + a_2 v_2, \quad v_1 \wedge v_2 = 1, \quad a_1 = a_2 \bmod 3; \tag{1-16}$$

see Section 2E. The no-fold condition barely fails for such edges: $\mathscr{P}_\delta[\zeta_\star]$ still has an essential gap in, say, the sharp-contrast regime. See [Fefferman et al. 2018, Corollary 6.3]. We expect our techniques to be robust enough to handle such edges. In particular, a 2×2 block of uncoupled Dirac operators should emerge in the resolvent estimates.

- This work may open the way to prove the no-fold conjecture of Fefferman, Lee-Thorp, and Weinstein [Fefferman et al. 2016b]. It predicts that long-lived *resonant* edge states should appear when the no-fold condition fails. This is supported by the existence of highly accurate localized quasimodes, still produced by the formal multiscale procedure of [loc. cit.]. See [Gérard and Sigal 1992; Stefanov and Vodev 1996; Tang and Zworski 1998; Stefanov 1999; 2000; Gannot 2015] for the relation between quasimodes and resonances in other settings.

- The eigenvalue curve $\zeta \mapsto E_{\delta,0}^\zeta$ of $\mathscr{P}_\delta[\zeta]$ corresponding to the topologically protected mode of $\not{D}(\mu)$ intersects $E_\star$ transversely. See the red curves in Figure 7. This contrasts with the eigenvalue curves

$\zeta \mapsto E^{\zeta}_{\delta,j}$, $j \neq 0$, which exhibit quadratic extrema near $\zeta_\star$; see the blue curves in Figure 7. This indicates that L^2-wave packets constructed from the topologically protected modes of $\not{D}(\mu)$ should have a longer lifetime. Mathematical and experimental investigations of this phenomenon would be interesting. The techniques could lead to a time-dependent analysis of quasimodes when the no-fold condition fails. See [Gérard and Sigal 1992] for a related investigation in the shape resonance context and [Carles et al. 2004; Ablowitz and Zhu 2012; 2013; Fefferman and Weinstein 2014; Arbunich and Sparber 2018] for related investigations in gapless settings.

- In [Drouot 2019], we investigate the relation between the bulk and edge indices of $\mathscr{P}_\delta[\zeta]$ or $\mathbb{P}_\delta[\zeta]$, as in [Haldane and Raghu 2008]. The bulk-edge correspondence is widely unexplored in continuous, asymptotically periodic settings: apart from [Bourne and Rennie 2018; Drouot 2018b], the only investigations concern the quantum Hall effect [Kellendonk and Schulz-Baldes 2004a; 2004b; Taarabt 2014]. The discrete setting is better understood [Kellendonk et al. 2002; Elgart et al. 2005; Graf and Porta 2013; Avila et al. 2013; Bal 2017; Shapiro 2017; Braverman 2018; Graf and Shapiro 2018; Graf and Tauber 2018; Shapiro and Tauber 2018]. It would also be nice to study it in quantum graph models of graphene — see [Kuchment and Post 2007; Becker and Zworski 2019; Becker et al. 2018; Lee 2016] for setting and spectral results.

- The recent numerical approach [Thicke et al. 2018] could be applied to $\mathbb{P}_\delta$ as δ increases away from 0. Corollary 1.7 shows that two edge states persist as long as the gap remains open. However their qualitative description (Corollary 7.4) should progressively break down as δ increases. It would be interesting to investigate numerically how their shape changes.

Notation. Here is a list of notation used in this work:

- If $z \in \mathbb{C}$, then $\bar{z}$ denotes its complex conjugate and $|z|$ its modulus. We will sometimes identify a vector $x = [x_1, x_2]^\top \in \mathbb{R}^2$ with the complex number $x_1 + ix_2$.
- $\mathbb{S}^1 \subset \mathbb{C}$ is the circle $\{z \in \mathbb{C} : |z| = 1\}$.
- $\mathbb{D}(z, r) \subset \mathbb{C}$ denotes the disk centered at $z \in \mathbb{C}$ of radius r.
- If $E, F \subset \mathbb{C}$, then $\mathrm{dist}(E, F)$ denotes the Euclidean distance between E and F.
- D_x is the operator $(1/i)[\partial_{x_1}, \partial_{x_2}]^\top = (1/i)\nabla$.
- L^2 denotes the space of square-summable functions and H^s are the classical Sobolev spaces.
- If $\mathcal{H}$ and $\mathcal{H}'$ are Hilbert spaces and $\psi \in \mathcal{H}$, we write $|\psi|_{\mathcal{H}}$ for the norm of $\mathcal{H}$; if $A : \mathcal{H} \to \mathcal{H}'$ is a bounded operator, the operator norm of A is

$$\|A\|_{\mathcal{H}\to\mathcal{H}'} \overset{\text{def}}{=} \sup_{|\psi|_{\mathcal{H}}=1} |A\psi|_{\mathcal{H}'}.$$

If $\mathcal{H} = \mathcal{H}'$, we simply write $\|A\|_{\mathcal{H}} = \|A\|_{\mathcal{H}\to\mathcal{H}}$.

- If $\psi_\epsilon \in \mathcal{H}$ and $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$, we write $\psi_\epsilon = O_{\mathcal{H}}(f(\epsilon))$ when there exists $C > 0$ such that $|\psi_\epsilon|_{\mathcal{H}} \leq Cf(\epsilon)$ for $\epsilon \in (0, 1]$. If $A_\epsilon : \mathcal{H} \to \mathcal{H}$ is a linear operator and $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, we write $A_\epsilon = \mathscr{O}_{\mathcal{H}\to\mathcal{H}'}(f(\epsilon))$ when there exists $C > 0$ such that $\|A_\epsilon\|_{\mathcal{H}\to\mathcal{H}'} \leq Cf(\epsilon)$ for $\epsilon \in (0, 1]$. If $\mathcal{H} = \mathcal{H}'$, we simply write $A_\epsilon = \mathscr{O}_{\mathcal{H}}(f(\epsilon))$.

- We denote the spectrum of a (possibly unbounded) operator A on $\mathcal{H}$ by $\Sigma_{\mathcal{H}}(A)$. It splits into an essential part $\Sigma_{\mathcal{H},\mathrm{ess}}(A)$ and a discrete part $\Sigma_{\mathcal{H},\mathrm{d}}(A)$.
- Λ is the lattice $\mathbb{Z}v_1 \oplus \mathbb{Z}v_2$ — see Section 1A. An edge is a line $\mathbb{R}v \subset \mathbb{R}^2$, with $v = a_1 v_1 + a_2 v_2 \in \Lambda$, a_1, a_2 relatively prime integers. We associate to v vectors v', k and k' via (1-7).
- The space L^2_ξ consists of ξ-quasiperiodic functions with respect to Λ:
$$L^2_\xi \stackrel{\text{def}}{=} \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{C}) : u(x+w) = e^{i\langle \xi, w\rangle} u(x),\ w \in \Lambda\}.$$
- $\ell \in (\mathbb{R}^2)^*$ is the projection of k orthogonally to k':
$$\ell \stackrel{\text{def}}{=} k - \frac{\langle k, k'\rangle}{|k'|^2} k'.$$
- $L^2[\zeta]$ is the space
$$L^2[\zeta] \stackrel{\text{def}}{=} \left\{u \in L^2_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{C}) : u(x+v) = e^{i\zeta} u(x),\ \int_{\mathbb{R}^2/\mathbb{Z}v} |u(x)|^2\, dx < \infty\right\}.$$
- $V \in C^\infty(\mathbb{R}^2, \mathbb{R})$ is a honeycomb potential — see Definition 1.1.
- $W \in C^\infty(\mathbb{R}^2, \mathbb{R})$ is Λ-periodic and odd — see (1-5).
- P_δ is the operator $-\Delta + V + \delta W$ on L^2; for $\xi \in \mathbb{R}^2$, $P_\delta(\xi)$ is the operator formally equal to P_δ but acting on L^2_ξ. For $\zeta \in \mathbb{R}$, $P_\delta[\zeta]$ is the operator formally equal to P_δ but acting on $L^2[\zeta]$.
- $\mathscr{P}_\delta$ is the operator $-\Delta + V + \delta \cdot \kappa_\delta \cdot W$ on L^2, where $\kappa_\delta(x) = \kappa(\delta\langle k', x\rangle)$ and κ is a domain-wall function — see (1-8). $\mathscr{P}_\delta[\zeta]$ is the operator formally equal to $\mathscr{P}_\delta$ but acting on $L^2[\zeta]$.
- $(\xi_\star, E_\star)$ denotes a Dirac point of $P_0 = -\Delta + V$, associated to a Dirac eigenbasis (ϕ_1, ϕ_2) — see Definition 1.2.
- $\zeta_\star$ is the real number $\langle \xi_\star, v\rangle$.
- $\xi_\star^A, \xi_\star^B, \zeta_\star^A, \zeta_\star^B$ are defined in (1-4) and (1-10), respectively.
- $\nu_\star$ is a complex number associated to $(\xi_\star, E_\star)$ and to the Dirac eigenbasis (ϕ_1, ϕ_2) such that $|\nu_\star| = \nu_F$ — see Section 2C.
- $\vartheta_\star = \langle \phi_1, W\phi_1\rangle_{L^2_{\xi_\star}}$ is always assumed to be nonzero; we also define $|\vartheta_\star| = \vartheta_F$.
- The Pauli matrices are
$$\boldsymbol{\sigma_1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma_2} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \boldsymbol{\sigma_3} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

These matrices satisfy $\boldsymbol{\sigma_j}^2 = \mathrm{Id}$ and $\boldsymbol{\sigma_i}\boldsymbol{\sigma_j} = -\boldsymbol{\sigma_j}\boldsymbol{\sigma_i}$ for $i \neq j$.

2. Honeycomb potentials, Dirac points and edges

2A. ***Equilateral lattice.*** We review briefly the definitions of Section 1A. The equilateral lattice Λ is $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$ given in canonical coordinates by
$$v_1 = a\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}, \quad v_2 = a\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix},$$

where $a > 0$ is a constant such that $\mathrm{Det}[v_1, v_2] = 1$. Let $k_1, k_2 \in (\mathbb{R}^2)^*$ be dual vectors: $\langle k_i, v_j\rangle = \delta_{ij}$. Identifying $(\mathbb{R}^2)^*$ with $\mathbb{R}^2$ via the scalar product,

$$[k_1, k_2]\cdot[v_1, v_2] = \mathrm{Id} \quad\Longrightarrow\quad [k_1, k_2] = [v_1, v_2]^{-1} = \frac{1}{6a}\begin{bmatrix}\sqrt{3} & \sqrt{3}\\ 3 & -3\end{bmatrix}.$$

Our definition does not involve a factor 2π — in contrast with some other conventions. The fundamental cell $\mathbb{L}$ and dual cell $\mathbb{L}^*$ are

$$\mathbb{L} \stackrel{\text{def}}{=} \{t_1 v_1 + t_2 v_2 : t_1, t_2 \in [0, 1)\}, \qquad \mathbb{L}^* \stackrel{\text{def}}{=} \{\tau_1 k_1 + \tau_2 k_2 : \tau_1, \tau_2 \in [0, 2\pi)\}.$$

2B. ***Symmetries.*** Recall that the space of ξ-quasiperiodic functions is

$$L^2_\xi \stackrel{\text{def}}{=} \{u \in L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{C}) : u(x + w) = e^{i\langle\xi, w\rangle}u(x),\ w \in \Lambda\}.$$

We introduce three operators: $\mathcal{R}$ (rotation); $\mathcal{I}$ (spatial inversion); and $\mathcal{C}$ (complex conjugation). These are given by

$$\mathcal{R}u(x) = u(Rx), \quad R \stackrel{\text{def}}{=} \frac{1}{2}\begin{bmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{bmatrix}, \qquad \mathcal{I}u(x) = u(-x), \qquad \mathcal{C}u(x) = \overline{u(x)}.$$

We study the action of these operators on the spaces L^2_ξ. Note that $Rv_1 = -v_2$ and $Rv_2 = v_1 - v_2$. Hence, R leaves Λ invariant. If $u \in L^2_\xi$ then

$$\begin{aligned}
(\mathcal{R}u)(x + v) &= u(Rx + Rv) = e^{i\langle\xi, Rv\rangle}(\mathcal{R}u)(x) = e^{i\langle R^*\xi, v\rangle}(\mathcal{R}u)(x),\\
(\mathcal{I}u)(x + v) &= u(-x - v) = e^{-i\langle\xi, v\rangle}(\mathcal{I}u)(x),\\
(\mathcal{C}u)(x + v) &= \overline{u(x + v)} = e^{-i\langle\xi, v\rangle}(\mathcal{C}u)(x).
\end{aligned}$$

It follows that

$$\mathcal{R}L^2_\xi = L^2_{R^{-1}\xi}, \quad \mathcal{I}L^2_\xi = L^2_{-\xi}, \quad \mathcal{C}L^2_\xi = L^2_{-\xi}. \tag{2-1}$$

Let $\xi_\star^A$ and $\xi_\star^B$ be given by (1-4):

$$\xi_\star^A = \tfrac{2\pi}{3}(2k_1 + k_2), \quad \xi_\star^B = \tfrac{2\pi}{3}(k_1 + 2k_2).$$

We observe that

$$R^{-1}\xi_\star^A = \xi_\star^A + 2\pi(k_1 + k_2), \quad R^{-1}\xi_\star^B = \xi_\star^B + 2\pi k_1.$$

In particular, $R^{-1}\xi_\star = \xi_\star \mod 2\pi\Lambda^*$ when $\xi_\star \in \{\xi_\star^A, \xi_\star^B\}$. Thanks to (2-1), we see that the space $L^2_{\xi_\star}$ is $\mathcal{R}$-invariant. Since $\mathcal{R}^3 = \mathrm{Id}$, we deduce that $\mathcal{R} : L^2_{\xi_\star} \to L^2_{\xi_\star}$ has three eigenvalues: $1, \tau, \bar\tau$ with $\tau = e^{2i\pi/3}$. Since $\mathcal{R}$ is a unitary operator, $L^2_{\xi_\star}$ admits an orthogonal decomposition

$$L^2_{\xi_\star} = L^2_{\xi_\star,1} \oplus L^2_{\xi_\star,\tau} \oplus L^2_{\xi_\star,\bar\tau}, \quad L^2_{\xi_\star,z} \stackrel{\text{def}}{=} \ker_{L^2_{\xi_\star}}(\mathcal{R} - z).$$

The operator $\mathcal{CI}$ maps $L^2_{\xi_\star}$ to itself. If $u \in L^2_{\xi_\star,\tau}$ then

$$\mathcal{R}(\mathcal{CI}u)(x) = \overline{u(-Rx)} = \overline{\tau\cdot u(-x)} = \bar\tau\cdot(\mathcal{CI}u)(x).$$

Therefore $\mathcal{CI}L^2_{\xi_\star,\tau} = L^2_{\xi_\star,\bar\tau}$.

2C. *Dirac points.* We recall that $P_0 = -\Delta + V$, where V is a honeycomb potential — see Definition 1.1. We denote by

$$\lambda_{0,1}(\xi) \le \lambda_{0,2}(\xi) \le \cdots \le \lambda_{0,j}(\xi) \le \cdots \tag{2-2}$$

the dispersion surfaces of P_0, i.e., the L^2_ξ-eigenvalues of $P_0(\xi)$. Conical intersections in the band spectrum (2-2) are called Dirac points — see Definition 1.2. Fefferman and Weinstein [2012] — see also [Colin de Verdière 1991; Grushin 2009; Berkolaiko and Comech 2018; Lee 2016; Keller et al. 2018; Ammari et al. 2018] for related perspectives — showed the following result:

Theorem 2.1 [Fefferman and Weinstein 2012, Theorem 5.1]. *Let $V_0 \in C^\infty(\mathbb{R}^2, \mathbb{R})$ be a honeycomb potential such that*

$$\int_{\mathbb{L}} e^{-i(k_1+k_2)x} V_0(x)\, dx \neq 0. \tag{2-3}$$

There exists a closed countable set $\mathcal{S} \subset \mathbb{R}$ such that for every $t \in \mathbb{R} \setminus \mathcal{S}$, the operator $-\Delta_{\mathbb{R}^2} + tV_0$ admits Dirac points

$$(\xi_\star, E_\star) \in \{\xi_\star^A, \xi_\star^B\} \times \mathbb{R}, \qquad \xi_\star^A \stackrel{\text{def}}{=} \tfrac{2\pi}{3}(2k_1 + k_2), \qquad \xi_\star^B \stackrel{\text{def}}{=} \tfrac{2\pi}{3}(k_1 + 2k_2).$$

This result shows that P_0 generically admits Dirac points: the condition (2-3) excludes a hyperplane in the space of honeycomb potentials; the "bad" set $\mathcal{S}$ is countable and accounts for extraordinary cases, e.g., higher multiplicity of $E_\star$ or quadratic intersections of dispersion surfaces. When P_0 admits Dirac points, the eigenspace $\ker_{L^2_{\xi_\star}}(P_0(\xi_\star) - E_\star)$ is spanned by an orthonormal basis $\{\phi_1, \phi_2\}$, with

$$\phi_1 \in L^2_{\xi_\star,\tau}, \qquad \phi_2 = \overline{\mathcal{I}\phi_1} \in L^2_{\xi_\star,\bar{\tau}}.$$

We call (ϕ_1, ϕ_2) a Dirac eigenbasis. It is unique modulo the $\mathbb{S}^1$-action $(\phi_1, \phi_2) \mapsto (\omega\phi_1, \bar{\omega}\phi_2)$, $\omega \in \mathbb{S}^1$.

Lemma 2.2. *Let $(\xi_\star, E_\star)$ be a Dirac point of P_0 with Dirac eigenbasis (ϕ_1, ϕ_2). Then*

$$\langle \phi_1, D_x\phi_1\rangle_{L^2_{\xi_\star}} = \langle \phi_2, D_x\phi_2\rangle_{L^2_{\xi_\star}} = 0.$$

In addition, there exists $\nu_\star \in \mathbb{C}$ with $|\nu_\star| = \nu_F$ such that for all $\eta \in \mathbb{R}^2$ (canonically identified with a complex number),

$$2\langle \phi_1, (\eta \cdot D_x)\phi_2\rangle_{L^2_{\xi_\star}} = \nu_\star \eta, \qquad 2\langle \phi_2, (\eta \cdot D_x)\phi_1\rangle_{L^2_{\xi_\star}} = \overline{\nu_\star \eta}.$$

This lemma can be deduced from [Fefferman et al. 2016b, Proposition 4.5]. We include a proof in Appendix A.1. It relies on some algebraic relations relating P_0, $\mathcal{R}$ and $\mathcal{I}$, and on perturbation theory of eigenvalues.

2D. *Breaking the symmetry.* We will consider Schrödinger operators $P_\delta = -\Delta + V + \delta W$, where

$$W \in C^\infty(\mathbb{R}^2, \mathbb{R}), \qquad W(x+w) = W(x), \quad w \in \Lambda, \qquad W(x) = -W(-x).$$

Lemma 2.3. *Let $(\xi_\star, E_\star)$ be a Dirac point of P_0 with Dirac eigenbasis (ϕ_1, ϕ_2) — see Definition 1.2. Then $\langle \phi_1, W\phi_2\rangle_{L^2_{\xi_\star}} = \langle \phi_2, W\phi_1\rangle_{L^2_{\xi_\star}} = 0$. Furthermore,*

$$\vartheta_\star \stackrel{\text{def}}{=} \langle \phi_1, W\phi_1\rangle_{L^2_{\xi_\star}} = -\langle \phi_2, W\phi_2\rangle_{L^2_{\xi_\star}}.$$

See the proofs of [Fefferman et al. 2016b, (6.19), (6.20)] or Appendix A.1. These identities rely on $\mathcal{I}$ being an isometry. If $\omega \in \mathbb{S}^1$, the change $(\phi_1, \phi_2) \mapsto (\omega\phi_1, \bar{\omega}\phi_2)$ of Dirac eigenbasis leaves $\vartheta_\star$ invariant.

2E. ***Edges.*** Let a_1 and a_2 be two relatively prime integers and $v = a_1 v_1 + a_2 v_2$. Introduce $v' = b_1 v_1 + b_2 v_2$, where $a_1 b_2 - a_2 b_1 = 1$. The vectors v and v' span Λ:

$$\begin{aligned} b_1 v - a_1 v' &= (b_1 a_2 - a_1 b_2) v_2 = -v_2, \\ b_2 v - a_2 v &= (b_2 a_1 - a_2 b_1) v_1 = v_1. \end{aligned} \tag{2-4}$$

Let k and k' be dual vectors. We claim that $k = b_2 k_1 - b_1 k_2$ and $k' = -a_2 k_1 + a_1 k_2$:

$$\begin{aligned} \langle k, v\rangle &= b_2 a_1 - b_1 a_2 = 1, & \langle k, v'\rangle &= -a_2 a_1 + a_1 a_2 = 0, \\ \langle k', v\rangle &= b_2 b_1 - b_1 b_2 = 0, & \langle k', v'\rangle &= -a_2 b_1 + a_1 b_2 = 1. \end{aligned}$$

Let $(\xi_\star^A, E_\star)$ be a Dirac point in the sense of Definition 1.2 and $\mathbb{R}v$ be an edge. Assume that $\xi_\star^B$ belongs to the dual edge $\zeta_\star^A k + \mathbb{R}k' \mod 2\pi\Lambda^*$. In this case we can write $\xi_\star^B = \zeta_\star^A k + \tau k'$, with $\tau \neq \langle \xi_\star^A, v'\rangle \mod 2\pi\mathbb{Z}$. Since $\lambda_{0,j_\star}(\xi_\star^B) = E_\star$, the no-fold condition fails when $\xi_\star^B \in \zeta_\star^A k + \mathbb{R}k' \mod 2\pi\Lambda^*$ (see Definition 1.3). Given the expressions (1-4) of $\xi_\star^A$ and $\xi_\star^B$ and (1-7) of v', this arises precisely when

$$\frac{2a_1 + a_2}{3} - \frac{a_1 + 2a_2}{3} \in \mathbb{Z} \quad \Longleftrightarrow \quad a_2 - a_1 \in 3\mathbb{Z}.$$

In particular, if the no-fold condition holds then $a_1 - a_2 \neq 0 \mod 3$. This implies that $\{\zeta_\star^A, \zeta_\star^B\} = \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\} \mod 2\pi\mathbb{Z}$ because of (1-10).

3. The characterization of edge states

This work studies the eigenvalues of the operator

$$\mathscr{P}_\delta[\zeta] = -\Delta + V + \delta \cdot \kappa_\delta \cdot W : L^2[\zeta] \to L^2[\zeta].$$

Above, κ_δ is a domain-wall function — see (1-8) — and $L^2[\zeta]$ is the space (1-9). The operator $\mathscr{P}_\delta[\zeta]$ is a Schrödinger operator that interpolates between $P_\delta[\zeta]$ at $-\infty$ and $P_\delta[\zeta]$ at $+\infty$. See Figure 9. In this section we review the multiscale approach of [Fefferman et al. 2016a; 2016b] and we derive Corollary 1.5 assuming Theorem 1.4, in a slightly more general setting.

3A. ***The formal multiscale approach.*** The eigenvalue problem for $\mathscr{P}_\delta[\zeta]$ is

$$\begin{cases} (-\Delta + V(x) + \delta\kappa_\delta(x) W(x) - E_\delta) u_\delta = 0, \\ u_\delta(x+v) = e^{i\zeta} u_\delta(x), \end{cases} \qquad \int_{\mathbb{R}^2/\mathbb{Z}v} |u_\delta(x)|^2\, dx < \infty. \tag{3-1}$$

The multiscale procedure of Fefferman, Lee-Thorp, and Weinstein [Fefferman et al. 2016b, §6] produces approximate solutions of (3-1). We review it below.

We first observe that if we write a function $u_\delta \in C^\infty(\mathbb{R}^2, \mathbb{C})$ as

$$u_\delta(x) = U_\delta(x, \delta\langle k', x\rangle), \quad U_\delta \in C^\infty(\mathbb{R}^2 \times \mathbb{R}, \mathbb{C}), \tag{3-2}$$

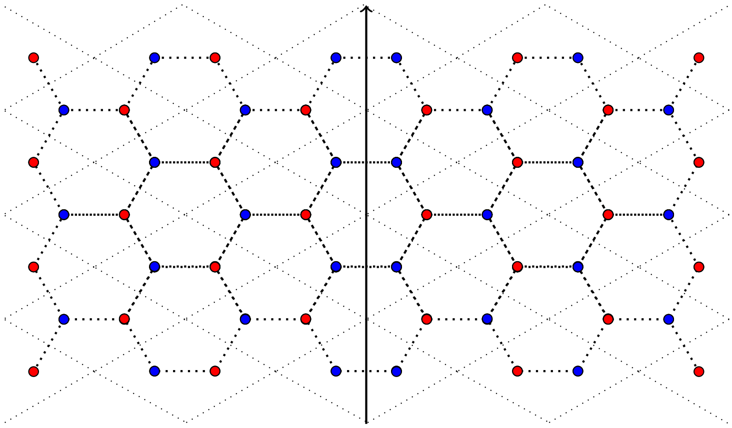

Figure 9. $\mathscr{P}_\delta[\zeta]$ is a Schrödinger operator with a typical potential represented above, with the zigzag edge $v_1 - v_2$. Each red (resp. blue) circle supports an atomic (e.g., radial) potential. The resulting potential is not periodic with respect to Λ; rather it is periodic with respect to $\mathbb{Z}v$.

then u_δ solves (3-1) if and only if U_δ solves

$$\begin{cases} ((D_x + \delta k' D_t)^2 + V(x) + \delta\kappa(t)W(x) - E_\delta)U_\delta = 0, \\ U_\delta(x+v,t) = e^{i\zeta}U_\delta(x,t), \end{cases} \qquad \int_{\mathbb{R}^2/\mathbb{Z}v} |U_\delta(x, \delta\langle k', x\rangle)|^2\,dx < \infty. \tag{3-3}$$

We now produce approximate solutions to the system (3-3) when ζ is near $\zeta_\star = \langle \xi_\star, v\rangle$. We fix $(\xi_\star, E_\star)$ a Dirac point of P_0 and we write $\zeta = \zeta_\star + \mu\delta$, $\zeta_\star = \langle \xi_\star, v\rangle$. We make an ansatz for U_δ and E_δ:

$$U_\delta(x,t) = e^{i\mu\delta\langle \ell, x\rangle} \cdot \left(\sum_{j=1,2} \alpha_j(t)\cdot\phi_j(x) + \delta\cdot V_\delta(x,t)\right), \qquad E_\delta = E_\star + \vartheta\delta + O(\delta^2), \tag{3-4}$$

where

- (ϕ_1, ϕ_2) is a Dirac eigenbasis for $(\xi_\star, E_\star)$ — see Definition 1.2;
- α_1, α_2 are smooth, exponentially decaying functions on $\mathbb{R}$, to be specified below;
- $V_\delta \in X$ — the space defined in (1-13).
- $\ell = k - (\langle k', k\rangle/|k'|^2)k'$ is the projection of k to the orthogonal of $\mathbb{R}k'$;
- $\vartheta \in \mathbb{R}$ is a real number that will be specified below.

Since $\phi_1, \phi_2 \in L^2_{\xi_\star}$, $V_\delta \in X$ and $\alpha_1, \alpha_2 \in L^2(\mathbb{R})$, the ansatz (3-4) implies

$$U_\delta(x+v,t) = e^{i\zeta}U_\delta(x,t), \qquad \int_{\mathbb{R}^2/\mathbb{Z}v} |U_\delta(x, \delta\langle k', x\rangle)|^2\,dx < \infty.$$

In particular the boundary and decay conditions of (3-3) hold under (3-4).

The eigenvalue problem (3-3) becomes a hierarchy of equations, obtained by identifying terms of orders $1, \delta, \delta^2, \dots$. Since $(P_0 - E_\star)\phi_j = 0$, the equation for the terms of order 1 is automatically satisfied.

The equation for the terms of order δ is

$$e^{i\mu\delta\langle\ell,x\rangle}(P_0-E_\star)V_\delta(x,t)+e^{i\mu\delta\langle\ell,x\rangle}(2(k'\cdot D_x)D_t+\kappa(t)W(x)-\vartheta)\sum_{j=1,2}\alpha_j(t)\phi_j(x)$$
$$+2\mu e^{i\mu\delta\langle\ell,x\rangle}(\ell\cdot D_x)\sum_{j=1,2}\alpha_j(t)\phi_j(x)=0. \quad (3\text{-}5)$$

Note that for every $t\in\mathbb{R}$, $(P_0-E_\star)V_\delta(\cdot,t)$ is orthogonal to ϕ_1 and ϕ_2. Therefore, for this system to have a solution, we must have for every $t\in\mathbb{R}$ and $k=1,2$,

$$\left\langle\phi_k,\,\big(2(k'\cdot D_x)D_t+2\mu(\ell\cdot D_x)+\kappa(t)W-\vartheta\big)\sum_{j=1,2}\alpha_j(t)\cdot\phi_j\right\rangle_{L^2_{\xi_\star}}=0. \quad (3\text{-}6)$$

The scalar products $\langle\phi_j,(k'\cdot D_x)\phi_k\rangle_{L^2_{\xi_\star}}$, $\langle\phi_j,(\ell\cdot D_x)\phi_k\rangle_{L^2_{\xi_\star}}$ and $\langle\phi_j,W\phi_k\rangle_{L^2_{\xi_\star}}$ appear in the solvability condition (3-6). They were computed in Lemmas 2.2 and 2.3. Using these formulas, (3-6) simplifies to

$$(\not{D}(\mu)-\vartheta)\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}=0,\qquad \not{D}(\mu)\stackrel{\text{def}}{=}\begin{bmatrix}0 & \nu_\star k'\\ \overline{\nu_\star k'} & 0\end{bmatrix}D_t+\mu\begin{bmatrix}0 & \nu_\star\ell\\ \overline{\nu_\star\ell} & 0\end{bmatrix}+\vartheta_\star\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\kappa.$$

This system has exponentially decaying solutions $[\alpha_1,\alpha_2]^\top$ if and only if ϑ is an eigenvalue of $\not{D}(\mu)$. Under this condition, (3-5) has a solution V_δ. In other words, this constructs a function U_δ such that (3-3) is satisfied modulo $O_X(\delta^2)$, meaning that for some $a,C>0$ and all $\delta\in(0,1)$

$$\sup_{\mathbb{R}\times\mathbb{R}^2} e^{a|t|}\cdot\left|\big((D_x+\delta k'D_t)^2+V(x)+\delta\kappa(t)W(x)-E_\delta\big)U_\delta\right|\le C\delta^2.$$

We can iterate this procedure to arbitrarily high orders in δ. It produces a function U_δ such that (3-3) is satisfied modulo $O_X(\delta^M)$ for any M. Identifying U_δ with u_δ according to (3-2), this procedure produces for any M and any eigenvalue ϑ of $\not{D}(\mu)$ a function $u_{\delta,M}$ that solves

$$(\mathscr{P}_\delta[\zeta]-E_\delta)u_{\delta,M}=O_X(\delta^M),\qquad E_\delta=E_\star+\delta\vartheta+O(\delta^2).$$

This is an approximate solution to the eigenvalue problem (3-1).

It is natural to ask whether these approximate solutions are close to eigenvectors. The work [Fefferman et al. 2016b] shows that this holds at first order in δ. Below we state results that imply that this holds at *any* order in δ. This dramatically refines the main result of [loc. cit.]. Our approach relies on resolvent estimates rather than by-hand construction of eigenvectors. It comes with further improvements of [loc. cit.]:

- the precise counting of eigenvalues of $\mathscr{P}_\delta[\zeta]$;
- an estimate that connects the resolvents of $\mathscr{P}_\delta[\zeta]$ and $\not{D}(\mu)$.

These results are stated in Section 3C and first require a spectral analysis of $\not{D}(\mu)$.

3B. ***The Dirac operator $\not{D}(\mu)$.*** The Dirac operator

$$\not{D}(\mu)=\begin{bmatrix}0 & \nu_\star k'\\ \overline{\nu_\star k'} & 0\end{bmatrix}D_t+\mu\begin{bmatrix}0 & \nu_\star\ell\\ \overline{\nu_\star\ell} & 0\end{bmatrix}+\vartheta_\star\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\kappa$$

emerges in the multiscale analysis of [Fefferman et al. 2016b]. We saw that its eigenvalues are particularly relevant in the construction of approximate eigenvectors of $\mathscr{P}_\delta[\zeta]$, $\zeta = \zeta_\star + \delta\mu$. In this section we relate the spectra of $\not{D}(\mu)$ and $\not{D}_\star = \not{D}(0)$.

Lemma 3.1. *The essential and discrete spectra of $\not{D}_\star$ and $\not{D}(\mu)$ are related through*

$$\Sigma_{L^2,\mathrm{ess}}(\not{D}(\mu)) = \mathbb{R} \setminus \big(-\sqrt{\vartheta_F^2 + \mu^2 \cdot \nu_F^2 |\ell|^2}, \sqrt{\vartheta_F^2 + \mu^2 \cdot \nu_F^2 |\ell|^2}\big),$$
$$\Sigma_{L^2,\mathrm{d}}(\not{D}(\mu)) = \big\{\mu \cdot \nu_F |\ell| \cdot \mathrm{sgn}(\vartheta_\star) : \pm\sqrt{\vartheta_j^2 + \mu^2 \cdot \nu_F^2 |\ell|^2} \text{ with } 0 \neq \vartheta_j \in \Sigma_{L^2,\mathrm{d}}(\not{D}_\star)\big\}.$$

All the eigenvalues of $\not{D}_\star$ and $\not{D}(\mu)$ are simple.

The proof of Lemma 3.1 relies on a supersymmetry: there exists a 2×2 matrix $\boldsymbol{m_2}$ such that $\boldsymbol{m_2}^2 = \mathrm{Id}$ and $\boldsymbol{m_2}\not{D}_\star = -\boldsymbol{m_2}\not{D}_\star$. We postpone it to Appendix A.2. We also mention that $\not{D}_\star$ may have more than one eigenvalue — see [Lu et al. 2018]. For a general perspective for applications of supersymmetries in spectral theory, see [Cycon et al. 1987, §6–12].

3C. *Parallel quasimomentum near $\zeta_\star$.* We are now ready to state the main result of our work. Recall that the assumptions (H1)–(H3) were introduced in Section 1E and that Π, Π^* and $\mathcal{U}_\delta$ are defined by

$$\Pi : L^2(\mathbb{R}^2/\mathbb{Z}v, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2), \qquad (\Pi f)(t) \stackrel{\mathrm{def}}{=} \int_0^1 f(sv + tv')\, ds,$$
$$\Pi^* : L^2(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}^2/\mathbb{Z}v, \mathbb{C}^2), \qquad (\Pi^* g)(x) \stackrel{\mathrm{def}}{=} g(\langle k', x\rangle),$$
$$\mathcal{U}_\delta : L^2(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2), \qquad (\mathcal{U}_\delta f)(t) \stackrel{\mathrm{def}}{=} f(\delta t).$$

Theorem 3.2. *Assume that the assumptions (H1)–(H3) hold. Fix $\mu_\sharp > 0$ and $\epsilon > 0$. There exists $\delta_0 > 0$ such that if*

$$\mu \in (-\mu_\sharp, \mu_\sharp), \quad \delta \in (0, \delta_0), \quad z \in \mathbb{D}\big(0, \sqrt{\vartheta_F^2 + \mu^2 \cdot \nu_F^2 |\ell|^2} - \epsilon\big), \quad \mathrm{dist}(\Sigma_{L^2}(\not{D}(\mu)), z) \geq \epsilon,$$
$$\zeta = \zeta_\star + \delta\mu, \quad \lambda = E_\star + \delta z$$

then $\mathscr{P}_\delta[\zeta] - \lambda$ is invertible and its resolvent $(\mathscr{P}_\delta[\zeta] - \lambda)^{-1}$ equals

$$\frac{1}{\delta} \cdot \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}^\top e^{-i\mu\delta\langle \ell, x\rangle} \cdot \Pi^* \mathcal{U}_\delta \cdot (\not{D}(\mu) - z)^{-1} \cdot \mathcal{U}_\delta^{-1} \Pi \cdot e^{i\mu\delta\langle \ell, x\rangle} \overline{\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}} + \mathscr{O}_{L^2[\zeta]}(\delta^{-1/3}).$$

It suffices to take $\mu = 0$ in Theorem 3.2 to derive Theorem 1.4.

Corollary 3.3. *Assume (H1)–(H3) hold and fix $\vartheta_\sharp \in (\vartheta_N, \vartheta_F)$ and $\mu_\sharp > 0$. There exists $\delta_0 > 0$ such that for*

$$\delta \in (0, \delta_0), \quad \mu \in (-\mu_\sharp, \mu_\sharp), \quad \zeta = \zeta_\star + \delta\mu,$$

the operator $\mathscr{P}_\delta[\zeta]$ has exactly $2N + 1$ eigenvalues $\{E_{\delta,j}^\zeta\}_{j \in [-N,N]}$ in

$$\big[E_\star - \delta\sqrt{\vartheta_\sharp^2 + \mu^2 \cdot \nu_F^2 |\ell|^2},\ E_\star + \delta\sqrt{\vartheta_\sharp^2 + \mu^2 \cdot \nu_F^2 |\ell|^2}\,\big].$$

These eigenvalues are simple. Furthermore, for each $j \in [-N, N]$, the eigenpairs $(E^\zeta_{\delta,j}, u^\zeta_{\delta,j})$ admit full expansions in powers of δ:

$$\begin{aligned} E^\zeta_{\delta,j} &= E_\star + \vartheta^\mu_j \cdot \delta + a^\mu_2 \cdot \delta^2 + \cdots + a^\mu_M \cdot \delta^M + O(\delta^{M+1}), \\ u^\zeta_{\delta,j}(x) &= e^{i(\zeta-\zeta_\star)\langle \ell, x\rangle}\big(f^\mu_0(x, \delta\langle k', x\rangle) + \cdots + \delta^M \cdot f_M(x, \delta\langle k', x\rangle)\big) + o_{H^k}(\delta^M). \end{aligned}$$

In the above expansions:

- *M and k are any integers*; *H^k is the k-th order Sobolev space.*
- *ϑ^μ_j is the j-th eigenvalue of $\not{D}(\mu)$, described in Lemma 3.1.*
- *The terms $a^\mu_m \in \mathbb{R}$ and $f^\mu_m \in X$ are recursively constructed via the multiscale analysis of [Fefferman et al. 2016b] — see Section 3A.*
- *The leading-order term f^μ_0 satisfies*

$$f^\mu_0(x, t) = \alpha^\mu_1(t)\phi_1(x) + \alpha^\mu_2(t)\phi_2(x), \quad (\not{D}(\mu) - \vartheta^\mu_j)\begin{bmatrix}\alpha^\mu_1\\ \alpha^\mu_2\end{bmatrix} = 0.$$

Proof of Corollary 3.3 assuming Theorem 3.2. In order to locate eigenvalues of $\mathscr{P}_\delta[\zeta]$, it suffices to integrate the resolvent on contours enclosing regions where Theorem 3.2 does not apply.

Let ϑ_j be an eigenvalue of $\not{D}(\mu)$ and $\epsilon > 0$ so that $\not{D}(\mu)$ has no other eigenvalues in $\mathbb{D}(\vartheta_j, \epsilon)$. We compute the residue

$$\frac{1}{2\pi i}\oint_{\partial\mathbb{D}(E_\star+\delta\vartheta_j, \epsilon\delta)} (\lambda - \mathscr{P}_\delta(\zeta))^{-1}\, d\lambda. \tag{3-7}$$

This is the projector on the spectrum of $\mathscr{P}_\delta(\zeta)$ that is enclosed by $\partial\mathbb{D}(E_\star + \delta\vartheta_j, \epsilon\delta)$. Because of Theorem 3.2 and the relation $\lambda = E_\star + \delta z$, $d\lambda = \delta\, dz$, (3-7) equals

$$\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}^\top e^{-i\mu\delta\langle \ell, x\rangle} \cdot \Pi^*\mathcal{U}_\delta \cdot \frac{1}{2\pi i}\oint_{\partial\mathbb{D}(\vartheta_j, \epsilon)} (z - \not{D}(\mu))^{-1}\, dz \cdot \mathcal{U}_\delta^{-1}\Pi \cdot e^{i\mu\delta\langle \ell, x\rangle}\overline{\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}} + \mathscr{O}_{L^2[\zeta]}(\delta^{2/3}).$$

The residue

$$\frac{1}{2\pi i}\oint_{\partial\mathbb{D}(\vartheta_j, \epsilon)} (z - \not{D}(\mu))^{-1}\, dz$$

is a rank-1 projector on $\ker_{L^2}(\not{D} - \vartheta_j)$. We write it as $\alpha^\zeta \otimes \alpha^\zeta$, where $|\alpha^\zeta|_{L^2} = 1$. We deduce that the residue (3-7) equals

$$\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}^\top e^{-i\mu\delta\langle \ell, x\rangle} \cdot \Pi^*\mathcal{U}_\delta \cdot \alpha \otimes \alpha \cdot \mathcal{U}_\delta^{-1}\Pi \cdot e^{i\mu\delta\langle \ell, x\rangle}\overline{\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}} + \mathscr{O}_{L^2[\zeta]}(\delta^{2/3}) = v^\zeta_0 \otimes v^\zeta_0 + \mathscr{O}_{L^2[\zeta]}(\delta^{2/3}),$$

where

$$v^\zeta_0 \overset{\text{def}}{=} \delta^{1/2}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}^\top e^{-i\mu\delta\langle \ell, x\rangle} \cdot \Pi^*\mathcal{U}_\delta \cdot \alpha.$$

Above we used that $(\mathcal{U}_\delta^{-1})^* = \delta \cdot \mathcal{U}_\delta$.

We deduce that (3-7) is a projector that takes the form $v_0^\zeta \otimes v_0^\zeta + \mathscr{O}_{L^2[\zeta]}(\delta^{2/3})$, where $|v_0^\zeta|_{L^2[\zeta]} = 1$. In particular, it is nonzero. Moreover, it has rank at most 1. Indeed, normalized vectors in its range must be of the form $v_0^\zeta + O_{L^2[\zeta]}(\delta^{2/3})$; therefore two of them cannot be orthogonal for δ sufficiently small. We deduce that (3-7) has rank exactly 1: $\mathscr{P}_\delta[\zeta]$ has exactly one eigenvalue in $\mathbb{D}(E_\star + \delta\vartheta_j, \epsilon\delta)$.

The rest of the proof is identical to [Drouot et al. 2018, Proof of Corollary 1]. It relies on

- the fact that $\mathscr{P}_\delta[\zeta]$ has exactly one eigenvalue in the disk enclosed by $\partial\mathbb{D}(E_\star + \delta\vartheta_j, \epsilon\delta)$ — proved just above;
- a general variational argument that shows that an approximate eigenpair (ψ, E) for a selfadjoint problem that has only one eigenvalue near E must be close to a genuine eigenpair — see [Drouot et al. 2018, Lemma 3.1];
- the construction of arbitrarily accurate approximate eigenpairs thanks to the multiscale procedure of [Fefferman et al. 2016b] — see Section 3A.

We refer to [Drouot et al. 2018, Proof of Corollary 1] for details. □

Most of the rest of the paper is devoted to the proof of Theorem 3.2.

4. The Bloch resolvent

Recall that V is a honeycomb potential — see Definition 1.1 — and that $W \in C^\infty(\mathbb{R}^2, \mathbb{R})$ is odd and Λ-periodic. In this section we study the resolvent of $P_\delta(\xi)$, the operator formally equal to $P_\delta = -\Delta + V + \delta W$ but acting on quasiperiodic spaces L^2_ξ.

Under the no-fold condition, we prove in Lemma 4.1 that $(P_\delta(\xi) - z)^{-1}$ is subdominant away from the Dirac quasimomenta $\xi_\star$. The situation is more subtle near $\xi_\star$. In Lemma 4.2 we show that when the nondegeneracy assumption (1-6) holds and (ξ, λ) is near a Dirac point $(\xi_\star, E_\star)$, $(P_\delta(\xi) - \lambda)^{-1}$ behaves like the resolvent of a rank-2 operator.

4A. *Resolvent away from Dirac momenta.* We recall that $\mathbb{L}$ is the fundamental cell associated to the generators v_1 and v_2; see (1-2). Given $\xi \in \mathbb{R}^2$, we define $\rho(\xi)$ as

$$\rho(\xi) \stackrel{\text{def}}{=} \operatorname{dist}(\xi + 2\pi\Lambda^*, \zeta_\star k + \mathbb{R}k').$$

Lemma 4.1. *Assume that the assumptions (H1) and (H2) hold. Let $c > 0$. There exist $\delta_0, \epsilon_0 > 0$ such that if*

$$\delta \in (0, \delta_0), \quad \xi \in \mathbb{L}^*, \quad \rho(\xi) \le \epsilon_0, \quad |\xi - \xi_\star| \ge \delta^{1/3}, \quad \lambda \in \mathbb{D}(E_\star, c\delta) \tag{4-1}$$

then $P_\delta(\xi) - \lambda$ is invertible and

$$\|(P_\delta(\xi) - \lambda)^{-1}\|_{L^2_\xi \to H^2_\xi} = O(\delta^{-1/3}).$$

Proof. 1. We first show that there exists $\epsilon_0 > 0$ such that

$$\xi \in \mathbb{L}^* \setminus \{\xi_\star\}, \ \ \rho(\xi) \le \epsilon_0, \quad \Longrightarrow \quad \lambda_{0,j_\star}(\xi) < E_\star - 2\epsilon_0 \cdot |\xi - \xi_\star|. \tag{4-2}$$

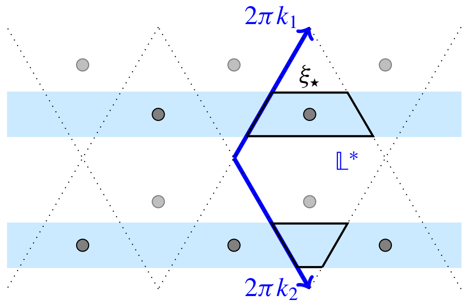

Figure 10. If $v = v_1 - v_2$ is the zigzag edge then $k' = k_1 + k_2$. An ϵ_0-neighborhood of the dual line $\zeta_\star k + \mathbb{R}k'$ is represented above as the blue strip. Lemma 4.1 applies to quasimomenta in the area enclosed in black. This domain of validity extends by periodicity to the blue strips away from $\xi_\star$ mod $2\pi\Lambda^*$.

Indeed, if this does not hold then we can find ξ_n such that

$$\xi_n \in \mathbb{L}^* \setminus \{\xi_\star\}, \quad \rho(\xi_n) \le \frac{1}{n}, \quad \lambda_{0,j_\star}(\xi_n) \ge E_\star - \frac{2}{n} \cdot |\xi - \xi_\star|.$$

Since $\xi_n \in \mathbb{L}^*$, we know ξ_n is bounded. There exists a subsequence $\xi_{\varphi(n)}$ of ξ_n that converges to an element ξ_∞ in the closure of $\mathbb{L}^*$, with $\rho(\xi_\infty) = 0$. Because $\lambda_{0,j_\star}$ is continuous, we have $\lambda_{0,j_\star}(\xi_\infty) \ge E_\star$. Since $\rho(\xi_\infty) = 0$, there exist $\eta \in \Lambda^*$ and $\tau_0 \in \mathbb{R}$ such that

$$\xi_\infty + 2\pi\eta = \zeta_\star k + \tau_0 k'.$$

We look at the function $\varphi(\tau) \stackrel{\text{def}}{=} \lambda_{0,j_\star}(\zeta_\star k + \tau k')$. It is 2π-periodic and it equals $E_\star$ precisely when $\tau = \langle \xi_\star, v' \rangle$ mod 2π because of (H2). Moreover,

$$\varphi(\langle \xi_\star, v' \rangle + \epsilon) = E_\star - \nu_F |\epsilon k'| + O(\epsilon^2).$$

Therefore, the intermediate value theorem shows that $\varphi(\tau) < E_\star$ unless $\tau = \langle \xi_\star, v' \rangle$ mod 2π. We deduce that $\tau_0 = \langle \xi_\star, v' \rangle$ mod 2π. Hence $\xi_\infty = \xi_\star$ mod $2\pi\Lambda^*$. Since ξ_∞ is in the closure of $\mathbb{L}^*$, we know $\xi_\infty = \xi_\star$. Since it also belongs to $\zeta_\star k + \mathbb{R}k'$, we have $\xi_\infty = \xi_\star$. Since $\xi_\star$ is a Dirac point, we deduce

$$E_\star - \frac{2}{\varphi(n)} \cdot |\xi_{\varphi(n)} - \xi_\star| \le \lambda_{0,j_\star}(\xi_{\varphi(n)}) \le E_\star - \nu_F \cdot |\xi_{\varphi(n)} - \xi_\star| + O(\xi_{\varphi(n)} - \xi_\star)^2.$$

This cannot hold for large n, unless $\xi_{\varphi(n)} = \xi_\star$, which is excluded. We deduce that (4-2) holds. A similar argument implies that

$$\xi \in \mathbb{L}^* \setminus \{\xi_\star\}, \quad \rho(\xi) \le \epsilon_0 \quad \Longrightarrow \quad \lambda_{0,j_\star+1}(\xi) > E_\star + 2\epsilon_0 \cdot |\xi - \xi_\star|. \tag{4-3}$$

2. From (4-2) and (4-3), we deduce that for $\delta > 0$,

$$\xi \in \mathbb{L}^*, \quad \rho(\xi) \le \epsilon_0, \quad |\xi - \xi_\star| \ge \delta^{1/3} \quad \Longrightarrow \quad \begin{cases} \lambda_{0,j_\star}(\xi) < E_\star - 2\epsilon_0\delta^{1/3}, \\ \lambda_{0,j_\star+1}(\xi) > E_\star + 2\epsilon_0\delta^{1/3}. \end{cases}$$

In particular, if $c > 0$ is given and $\lambda \in \mathbb{D}(E_\star, c\delta)$ then

$$\xi \in \mathbb{L}^*, \quad \rho(\xi) \le \epsilon_0, \quad |\xi - \xi_\star| \ge \delta^{1/3} \quad \Longrightarrow \quad \begin{cases} \mathrm{Re}(\lambda_{0,j_\star}(\xi) - \lambda) < c\delta - 2\epsilon_0\delta^{1/3}, \\ \mathrm{Re}(\lambda_{0,j_\star+1}(\xi) - \lambda) > 2\epsilon_0\delta^{1/3} - c\delta. \end{cases}$$

In particular, when δ_0 is sufficiently small, $\delta \in (0, \delta_0)$ and $\lambda \in \mathbb{D}(E_\star, c\delta)$,

$$\xi \in \mathbb{L}^*, \quad \rho(\xi) \le \epsilon_0, \quad |\xi - \xi_\star| \ge \delta^{1/3} \quad \Longrightarrow \quad \begin{cases} \mathrm{Re}(\lambda_{0,j_\star}(\xi) - \lambda) < -\epsilon_0\delta^{1/3}, \\ \mathrm{Re}(\lambda_{0,j_\star+1}(\xi) - \lambda) > \epsilon_0\delta^{1/3}. \end{cases}$$

Since the dispersion surfaces are labeled in increasing order, we deduce that if (4-1) is satisfied then

$$\mathrm{dist}(\Sigma_{L^2_\xi}(P_0(\xi)), \lambda) \ge \epsilon_0\delta^{1/3}, \quad (P_0(\xi) - \lambda)^{-1} = \mathscr{O}_{L^2_\xi}(\delta^{-1/3}).$$

We derived the estimate on $(P_0(\xi) - \lambda)^{-1}$ using the spectral theorem.

3. Assume that (4-1) holds. Thanks to step 1, $P_0(\xi) - \lambda$ is invertible and

$$P_\delta(\xi) - \lambda = P_0(\xi) - \lambda + \delta W = (P_0(\xi) - \lambda) \cdot (\mathrm{Id} + (P_0(\xi) - \lambda)^{-1}\delta W).$$

The second term equals $\mathrm{Id} + \mathscr{O}_{L^2_\xi}(\delta^{2/3})$. In particular it is invertible by a Neumann series for δ sufficiently small, with uniformly bounded inverse. We deduce that $P_\delta(\xi) - \lambda$ is invertible with inverse $\mathscr{O}_{L^2_\xi}(\delta^{-1/3})$.

4. To conclude we must show that the inverse of $P_\delta(\xi) - \lambda$ is $\mathscr{O}_{L^2_\xi \to H^2_\xi}(\delta^{-1/3})$. This is a standard consequence of the elliptic estimate: using $\delta = O(1)$, $\lambda = O(1)$, we see that for any $f \in H^2_\xi$,

$$|f|_{H^2_\xi} \le |f|_{L^2_\xi} + |\Delta f|_{H^2_\xi} \le C|f|_{L^2_\xi} + |(P_\delta(\xi) - \lambda) f|_{H^2_\xi}.$$

We apply this inequality to $f = (P_\delta(\xi) - \lambda)^{-1}u$ to deduce that

$$\|(P_\delta(\xi) - \lambda)^{-1}\|_{L^2_\xi \to H^2_\xi} \le C\|(P_\delta(\xi) - \lambda)^{-1}\|_{L^2_\xi} + 1.$$

In particular, the estimate $\mathscr{O}_{L^2_\xi}(\delta^{-1/3})$ proved in step 3 improves automatically to a bound $\mathscr{O}_{L^2_\xi \to H^2_\xi}(\delta^{-1/3})$. This completes the proof. ☐

4B. *Resolvent near Dirac momenta.* Fix a Dirac point $(\xi_\star, E_\star)$ of $P_0(\xi)$ and assume that $\vartheta_\star$ — defined in (1-6) — is nonzero. Identify $\xi - \xi_\star \in \mathbb{R}^2$ with the corresponding complex number and introduce the 2×2 matrix $M_\delta(\xi)$,

$$M_\delta(\xi) \overset{\text{def}}{=} \begin{bmatrix} E_\star + \delta\vartheta_\star & \nu_\star \cdot (\xi - \xi_\star) \\ \overline{\nu_\star \cdot (\xi - \xi_\star)} & E_\star - \delta\vartheta_\star \end{bmatrix}.$$

Lemma 4.2. *Let $\theta \in (0, 1)$. If*

$$\delta > 0, \quad \xi \in \mathbb{R}^2, \quad \vartheta_F \overset{\text{def}}{=} |\vartheta_\star| \ne 0, \quad \lambda \in \mathbb{D}\Big(E_\star, \theta\sqrt{\vartheta_F^2 \cdot \delta^2 + \nu_F^2 \cdot |\xi - \xi_\star|^2}\Big) \tag{4-4}$$

then the matrix $M_\delta(\xi) - \lambda$ is invertible and

$$\|(M_\delta(\xi) - \lambda)^{-1}\|_{\mathbb{C}^2} = O((\delta + |\xi - \xi_\star|)^{-1}).$$

Proof. The matrix $M_\delta(\xi)$ is Hermitian. It has eigenvalues

$$\mu_\delta^\pm(\xi) \stackrel{\text{def}}{=} E_\star \pm \sqrt{\vartheta_F^2 \cdot \delta^2 + \nu_F^2 \cdot |\xi - \xi_\star|^2}.$$

If (4-4) holds then the eigenvalues $\mu_\delta^\pm(\xi) - \lambda$ of $M_\delta(\xi) - \lambda$ satisfy

$$|\mu_\delta^\pm(\xi) - \lambda| \geq (1-\theta)\sqrt{\vartheta_F^2 \cdot \delta^2 + \nu_F^2 \cdot |\xi - \xi_\star|^2} \geq \frac{1-\theta}{\sqrt{2}} \cdot (\nu_F \cdot |\xi - \xi_\star| + \vartheta_F \cdot \delta).$$

By the spectral theorem, we deduce that $(M_\delta(\xi) - \lambda)^{-1}$ exists and has operator-norm bounded by $O((|\xi - \xi_\star| + \delta)^{-1})$. □

Introduce the operator

$$\Pi_0(\xi) : L^2_\xi \to \mathbb{C}^2, \qquad \Pi_0(\xi)u \stackrel{\text{def}}{=} \begin{bmatrix} \langle e^{i\langle \xi - \xi_\star, x\rangle}\phi_1, u\rangle_{L^2_\xi} \\ \langle e^{i\langle \xi - \xi_\star, x\rangle}\phi_2, u\rangle_{L^2_\xi} \end{bmatrix}. \tag{4-5}$$

Lemma 4.3. *Assume that the assumptions* (H1) *and* (H3) *hold. Let* $\theta \in (0, 1)$. *There exists* $\delta_0 > 0$ *such that if*

$$\delta \in (0, \delta_0), \quad |\xi - \xi_\star| \leq \delta^{1/3}, \quad \lambda \in \mathbb{D}\big(E_\star, \theta\sqrt{\vartheta_F^2 \cdot \delta^2 + \nu_F^2 \cdot |\xi - \xi_\star|^2}\big) \tag{4-6}$$

then $P_\delta(\xi) - \lambda$ *is invertible and*

$$(P_\delta(\xi) - \lambda)^{-1} = \Pi_0(\xi)^* \cdot (M_\delta(\xi) - \lambda)^{-1} \cdot \Pi_0(\xi) + \mathscr{O}_{L^2_\xi \to H^2_\xi}(1).$$

Proof. 1. Introduce the ξ-dependent family of vector spaces

$$\mathscr{V}(\xi) = \mathbb{C} \cdot e^{i\langle \xi - \xi_\star, x\rangle}\phi_1 \oplus \mathbb{C} \cdot e^{i\langle \xi - \xi_\star, x\rangle}\phi_2.$$

We split L^2_ξ as $\mathscr{V}(\xi) \oplus \mathscr{V}(\xi)^\perp$. With respect to this decomposition, we write $P_\delta(\xi)$ as a block-by-block operator:

$$P_\delta(\xi) - \lambda = \begin{bmatrix} A_\delta(\xi) - \lambda & B_\delta(\xi) \\ C_\delta(\xi) & D_\delta(\xi) - \lambda \end{bmatrix}. \tag{4-7}$$

We use below $\langle \cdot, \cdot \rangle$ to denote the L^2_ξ-scalar product.

2. We show that

$$B_\delta(\xi) = \mathscr{O}_{\mathscr{V}(\xi)^\perp \to \mathscr{V}(\xi)}(\delta + |\xi - \xi_\star|), \qquad C_\delta(\xi) = \mathscr{O}_{\mathscr{V}(\xi) \to \mathscr{V}(\xi)^\perp}(\delta + |\xi - \xi_\star|). \tag{4-8}$$

Note that $C_\delta(\xi) = B_\delta(\xi)^*$; hence we just have to estimate $B_\delta(\xi)$, i.e., show that

$$u \in \mathscr{V}(\xi)^\perp, \;\; |u|_{L^2_\xi} = 1 \quad \Longrightarrow \quad \langle e^{i\langle \xi - \xi_\star, x\rangle}\phi_j, P_\delta(\xi)u\rangle = O(\delta + |\xi - \xi_\star|), \tag{4-9}$$

where the implicit constant does not depend on u. We have

$$\begin{aligned} &\langle e^{i\langle \xi - \xi_\star, x\rangle}\phi_j, P_\delta(\xi)u\rangle \\ &\quad = \langle P_\delta(\xi) \cdot e^{i\langle \xi - \xi_\star, x\rangle}\phi_j, u\rangle = \langle (-\Delta + V + \delta W) \cdot e^{i\langle \xi - \xi_\star, x\rangle}\phi_j, u\rangle \\ &\quad = \langle e^{i\langle \xi - \xi_\star, x\rangle}(-\Delta + V)\phi_j, u\rangle + \langle [-\Delta, e^{i\langle \xi - \xi_\star, x\rangle}]\phi_j, u\rangle + \delta\langle W e^{i\langle \xi - \xi_\star, x\rangle}\phi_j, u\rangle \\ &\quad = (E_\star + |\xi - \xi_\star|^2)\langle e^{i\langle \xi - \xi_\star, x\rangle}\phi_j, u\rangle + 2\langle e^{i\langle \xi - \xi_\star, x\rangle}(\xi - \xi_\star) \cdot D_x\phi_j, u\rangle + \delta\langle W e^{i\langle \xi - \xi_\star, x\rangle}\phi_j, u\rangle. \end{aligned}$$

The first bracket vanishes because $u \in \mathscr{V}(\xi)^\perp$. The second and third brackets are $O(\xi - \xi_\star)$ and $O(\delta)$, respectively — and this holds uniformly in u with $|u|_{L^2_\xi} = 1$. This gives (4-9), itself implying (4-8).

3. Here we prove that if (4-6) is satisfied then

$$D_\delta(\xi) - \lambda : \mathscr{V}(\xi)^\perp \cap H^2_\xi \to \mathscr{V}(\xi)^\perp \cap L^2_\xi \text{ is invertible} \quad \text{and} \quad (D_\delta(\xi) - \lambda)^{-1} = \mathscr{O}_{\mathscr{V}(\xi)^\perp}(1).$$

It suffices to construct an operator $E_\delta(\xi, \lambda) : \mathscr{V}(\xi)^\perp \to \mathscr{V}(\xi)^\perp$ such that

$$E_\delta(\xi, \lambda) = \mathscr{O}_{\mathscr{V}(\xi)^\perp}(1), \quad E_\delta(\xi, \lambda) \cdot (D_\delta(\xi) - \lambda) = \mathrm{Id}_{\mathscr{V}(\xi)^\perp} + \mathscr{O}_{\mathscr{V}(\xi)^\perp}(\delta + |\xi - \xi_\star|). \tag{4-10}$$

The space $\mathscr{V}(\xi_\star) = \ker_{L^2_{\xi_\star}}(P_0(\xi_\star) - E_\star)$ has dimension 2; $P_0(\xi)$ depends smoothly on ξ in the sense that $e^{-i\xi x} \cdot P_0(\xi) \cdot e^{i\xi x}$ forms a smooth family of operators $H^2_0 \to L^2_0$. Therefore, there exist $\eta > 0$ and $\epsilon > 0$ such that

$$|\xi - \xi_\star| \le \epsilon \quad \Longrightarrow \quad P_0(\xi) \text{ has precisely two eigenvalues in } [E_\star - \eta, E_\star + \eta]. \tag{4-11}$$

See [Kato 1980, §VII.1.3, Theorem 1.8]. Let $\mathscr{W}(\xi)$ be the vector space spanned by the two eigenvectors of $P_0(\xi)$ with energy in $[E_\star - \eta, E_\star + \eta]$. Let $Q_0(\xi)$ be the operator formally equal to $P_0(\xi)$ but acting on $\mathscr{W}(\xi)^\perp$. From (4-11), for $|\xi - \xi_\star| \le \epsilon$, the spectrum of $Q_0(\xi)$ consists of the eigenvalues of $P_0(\xi)$ outside $[E_\star - \eta, E_\star + \eta]$. The spectral theorem implies that if δ_0 is small enough, under (4-6),

$$Q_0(\xi) - \lambda : \mathcal{W}(\xi)^\perp \cap H^2_\xi \to \mathcal{W}(\xi)^\perp \cap L^2_\xi \text{ is invertible} \quad \text{and} \quad (Q_0(\xi) - \lambda)^{-1} = \mathscr{O}_{\mathscr{W}(\xi)^\perp}(1). \tag{4-12}$$

Let $J(\xi) : \mathscr{V}(\xi)^\perp \to \mathscr{W}(\xi)^\perp$ be obtained by orthogonally projecting an element $u \in \mathscr{V}(\xi)^\perp \subset L^2_\xi$ to $\mathscr{W}(\xi)^\perp$. We set

$$E_\delta(\xi, \lambda) \stackrel{\text{def}}{=} J(\xi)^* \cdot (Q_0(\xi) - \lambda)^{-1} \cdot J(\xi) : \mathscr{V}(\xi)^\perp \to \mathscr{V}(\xi)^\perp.$$

The first estimate of (4-10) is satisfied because of (4-12). We want to check the second estimate. Observe that

$$\begin{aligned} E_\delta(\xi, \lambda) \cdot (D_\delta(\xi) - \lambda) &= E_\delta(\xi, \lambda) \cdot \pi_{\mathscr{V}(\xi)^\perp}(P_0(\xi) - \lambda + \delta W) \\ &= J(\xi)^* \cdot (Q_0(\xi) - \lambda)^{-1} \cdot J(\xi) \cdot \pi_{\mathscr{V}(\xi)^\perp}(P_0(\xi) - \lambda) + \mathscr{O}_{\mathscr{V}(\xi)}(\delta). \end{aligned} \tag{4-13}$$

Above, $\pi_{\mathscr{V}(\xi)^\perp} : L^2_\xi \to L^2_\xi$ is the orthogonal projection from L^2_ξ to $\mathscr{V}(\xi)^\perp$, also seen as an operator $L^2_\xi \mapsto \mathscr{V}(\xi)^\perp$. We introduce similarly $\pi_{\mathcal{W}(\xi)^\perp}$. Then

$$\begin{aligned} J(\xi) \cdot \pi_{\mathscr{V}(\xi)^\perp} &= \pi_{\mathscr{W}(\xi)^\perp} \cdot (\mathrm{Id} - \pi_{\mathscr{V}(\xi)}) = \pi_{\mathscr{W}(\xi)^\perp} - (\mathrm{Id} - \pi_{\mathscr{W}(\xi)}) \cdot \pi_{\mathscr{V}(\xi)} \\ &= \pi_{\mathscr{W}(\xi)^\perp} - (\pi_{\mathscr{V}(\xi)} - \pi_{\mathscr{W}(\xi)}) \cdot \pi_{\mathscr{V}(\xi)}. \end{aligned} \tag{4-14}$$

The individual eigenvectors associated to the eigenvalues of $P_0(\xi)$ in $[E_\star - \eta, E_\star + \eta]$ do not depend smoothly on ξ but the projector $\pi_{\mathscr{W}(\xi)}$ depends smoothly on ξ — see [Kato 1980, §VII1.3, Theorem 1.7]. Since $\mathscr{V}(\xi_\star) = \mathscr{W}(\xi_\star)$, this implies $\pi_{\mathscr{V}(\xi)} - \pi_{\mathscr{W}(\xi)} = \mathscr{O}_{L^2_\xi}(\xi - \xi_\star)$. We deduce that

$$J(\xi) \cdot \pi_{\mathscr{V}(\xi)^\perp} = \pi_{\mathscr{W}(\xi)^\perp} + \mathscr{O}_{\mathscr{W}(\xi)^\perp}(\xi - \xi_\star). \tag{4-15}$$

We combine (4-13) and (4-15) to obtain

$$\begin{aligned} E_\delta(\xi,\lambda)\cdot(D_\delta(\xi)-\lambda) &= J(\xi)^*\cdot(Q_0(\xi)-\lambda)^{-1}\cdot\pi_{\mathscr{W}(\xi)^\perp}(P_0(\xi)-\lambda)+\mathscr{O}_{L^2_\xi}(\delta)\\ &= J(\xi)^*\pi_{\mathscr{W}(\xi)^\perp}+\mathscr{O}_{\mathscr{V}(\xi)^\perp}(\delta+|\xi-\xi_\star|). \end{aligned}$$

The operator $J(\xi)^*$ takes an element in $\mathscr{W}(\xi)^\perp$ and projects it to $\mathscr{V}(\xi)^\perp$. By the same argument as (4-14) and (4-15) (inverting $\mathscr{V}(\xi)$ and $\mathscr{W}(\xi)$),

$$J(\xi)^*\pi_{\mathscr{W}(\xi)^\perp}=\pi_{\mathscr{V}(\xi)^\perp}+O_{\mathscr{V}(\xi)^\perp}(\xi-\xi_\star).$$

We conclude that the second estimate of (4-10) is satisfied. It follows that $D_\delta(\xi)-\lambda:\mathscr{V}(\xi)^\perp\to\mathscr{V}(\xi)^\perp$ is invertible under (4-6).

4. We now study $A_\delta(\xi)-\lambda$. This operator acts on the two-dimensional space $\mathscr{V}(\xi)$; its matrix in the basis $\{e^{i\langle\xi-\xi_\star,x\rangle}\phi_1, e^{i\langle\xi-\xi_\star,x\rangle}\phi_2\}$ is

$$\begin{bmatrix}\langle e^{i\langle\xi-\xi_\star,x\rangle}\phi_1,(P_\delta(\xi)-\lambda)e^{i\langle\xi-\xi_\star,x\rangle}\phi_1\rangle & \langle e^{i\langle\xi-\xi_\star,x\rangle}\phi_1,(P_\delta(\xi)-\lambda)e^{i\langle\xi-\xi_\star,x\rangle}\phi_2\rangle\\ \langle e^{i\langle\xi-\xi_\star,x\rangle}\phi_2,(P_\delta(\xi)-\lambda)e^{i\langle\xi-\xi_\star,x\rangle}\phi_1\rangle & \langle e^{i\langle\xi-\xi_\star,x\rangle}\phi_2,(P_\delta(\xi)-\lambda)e^{i\langle\xi-\xi_\star,x\rangle}\phi_2\rangle\end{bmatrix}. \tag{4-16}$$

We observe that

$$\begin{aligned} e^{-i\langle\xi-\xi_\star,x\rangle}(P_\delta(\xi)-\lambda)e^{i\langle\xi-\xi_\star,x\rangle} &= P_\delta(\xi_\star)-\lambda+[e^{-i\langle\xi-\xi_\star,x\rangle},-\Delta]e^{i\langle\xi-\xi_\star,x\rangle}\\ &= P_\delta(\xi_\star)-\lambda+[\Delta,e^{-i\langle\xi-\xi_\star,x\rangle}]e^{i\langle\xi-\xi_\star,x\rangle}\\ &= P_\delta(\xi_\star)-\lambda+2((\xi-\xi_\star)\cdot D_x)-|\xi-\xi_\star|^2. \end{aligned}$$

Therefore the matrix elements in (4-16) are given by

$$\begin{aligned} \big\langle e^{i\langle\xi-\xi_\star,x\rangle}\phi_j,(P_\delta(\xi)-\lambda)e^{i\langle\xi-\xi_\star,x\rangle}\phi_k\big\rangle &= \big\langle\phi_j,\big(P_\delta(\xi_\star)+2(\xi-\xi_\star)\cdot D_x-\lambda-|\xi-\xi_\star|^2\big)\phi_k\big\rangle\\ &= (E_\star-|\xi-\xi_\star|^2-\lambda)\delta_{jk}+\big\langle\phi_j,(\delta W+2(\xi-\xi_\star)\cdot D_x)\phi_k\big\rangle. \end{aligned}$$

We deduce from Lemmas 2.2 and 2.3 that the matrix (4-16) is equal to $M_\delta(\xi)-\lambda+O(\xi-\xi_\star)^2$. Using a Neumann series argument based on (4-16), when (4-6) holds, $A_\delta(\xi)-\lambda$ is invertible, and

$$(A_\delta(\xi)-\lambda)^{-1}=\Pi_0(\xi)^*\cdot(M_\delta(\xi)-\lambda)^{-1}\cdot\Pi_0(\xi)+\mathscr{O}_{\mathscr{V}(\xi)}\left(\frac{|\xi-\xi_\star|^2}{\delta^2+|\xi-\xi_\star|^2}\right). \tag{4-17}$$

Because of Lemma 4.2, we also observe that

$$(A_\delta(\xi)-\lambda)^{-1}=\mathscr{O}_{\mathscr{V}(\xi)}((\delta+|\xi-\xi_\star|)^{-1}). \tag{4-18}$$

5. Schur's lemma allows us to invert block-by-block operators of the form (4-7) under certain conditions on the blocks; see [Drouot et al. 2018, Lemma 4.1] for the version needed here. We need to verify that

$$\begin{aligned} A_\delta(\xi)-\lambda:\mathscr{V}(\xi)\to\mathscr{V}(\xi)\ \text{ is invertible},\\ D_\delta(\xi)-\lambda-C_\delta(\xi)\cdot(A_\delta(\xi)-\lambda)^{-1}\cdot B_\delta(\xi):\mathscr{V}(\xi)^\perp\to\mathscr{V}(\xi)^\perp\ \text{ is invertible}. \end{aligned} \tag{4-19}$$

The first statement holds because of step 4. Regarding the second statement, we observe that because of (4-8) and (4-18),

$$C_\delta(\xi)\cdot(A_\delta(\xi)-\lambda)^{-1}\cdot B_\delta(\xi)=\mathscr{O}_{\mathscr{V}(\xi)^\perp}(\delta+|\xi-\xi_\star|)=\mathscr{O}_{\mathscr{V}(\xi)^\perp}(\delta^{1/3}).$$

Because of step 3, $D_\delta(\xi)-\lambda$ is invertible and its inverse is $\mathscr{O}_{\mathscr{V}(\xi)^\perp}(1)$. Therefore a Neumann-series argument shows that the second statement in (4-19) holds. It also shows that the inverse is $\mathscr{O}_{\mathscr{V}(\xi)^\perp}(1)$.

We apply Schur's lemma — see [Drouot et al. 2018, Lemma 4.1]. From (4-7), we obtain that $P_\delta(\xi)-\lambda : H^2_\xi\to L^2_\xi$ is invertible when (4-6) holds, and moreover

$$(P_\delta(\xi)-\lambda)^{-1}=\begin{bmatrix}(A_\delta(\xi)-\lambda)^{-1} & 0\\ 0 & 0\end{bmatrix}+\mathscr{O}_{L^2_\xi}(1).$$

Using (4-17) and the projector (4-5), we deduce that

$$(P_\delta(\xi)-\lambda)^{-1}=\Pi_0(\xi)^*\cdot(M_\delta(\xi)-\lambda)^{-1}\cdot\Pi_0(\xi)+\mathscr{O}_{L^2_\xi}(1). \tag{4-20}$$

The error term in (4-20) improves automatically to $\mathscr{O}_{L^2_\xi\to H^2_\xi}(1)$ because of elliptic regularity — see the argument at the end of the proof of Lemma 4.1. □

5. The bulk resolvent along the edge

Let $v\in\Lambda$ be the direction of an edge. We define accordingly v', k, k' and ℓ — see Section 2E. For $\zeta\in\mathbb{R}$, we set

$$L^2[\zeta]\overset{\text{def}}{=}\left\{u\in L^2_{\text{loc}}(\mathbb{R}^2,\mathbb{C}) : u(x+v)=e^{i\zeta}u(x),\ \textstyle\int_{\mathbb{R}^2/\mathbb{Z}v}|u(x)|^2\,dx<\infty\right\}.$$

Let $P_\delta[\zeta]$ be the operator formally equal to P_δ but acting on $L^2[\zeta]$. We are interested in the resolvent of $P_\delta[\zeta]$ for δ small and ζ near $\zeta_\star=\langle\xi_\star,v\rangle$. We recall

$$\begin{aligned}
\Pi &: L^2(\mathbb{R}^2/\mathbb{Z}v,\mathbb{C}^2)\to L^2(\mathbb{R},\mathbb{C}^2), && (\Pi f)(t)\overset{\text{def}}{=}\int_0^1 f(sv+tv')\,ds,\\
\Pi^* &: L^2(\mathbb{R},\mathbb{C}^2)\to L^2(\mathbb{R}^2/\mathbb{Z}v,\mathbb{C}^2), && (\Pi^* g)(x)\overset{\text{def}}{=}g(\langle k',x\rangle),\\
\mathcal{U}_\delta &: L^2(\mathbb{R},\mathbb{C}^2)\to L^2(\mathbb{R},\mathbb{C}^2), && (\mathcal{U}_\delta f)(t)\overset{\text{def}}{=}f(\delta t).
\end{aligned} \tag{5-1}$$

Let $\not{D}_\pm(\mu): H^1(\mathbb{R},\mathbb{C}^2)\to L^2(\mathbb{R},\mathbb{C}^2)$ be the formal limits of $\not{D}(\mu)$ as $t\to\pm\infty$:

$$\not{D}_\pm(\mu)\overset{\text{def}}{=}\begin{bmatrix}\vartheta_\star & \nu_\star k'\\ \overline{\nu_\star k'} & -\vartheta_\star\end{bmatrix}D_t+\mu\begin{bmatrix}0 & \nu_\star\ell\\ \overline{\nu_\star\ell} & 0\end{bmatrix}\pm\begin{bmatrix}\vartheta_\star & 0\\ 0 & -\vartheta_\star\end{bmatrix}.$$

The main result of this section relates the resolvent of $P_{\pm\delta}[\zeta]$ at $E_\star+\delta z$ to that of $\not{D}_\pm(\mu)$ at z for small enough δ. The assumptions (H1)–(H3) were defined in Section 1E.

Theorem 5.1. *Assume that the assumptions (H1)–(H3) hold and fix $\mu_\sharp>0$ and $\theta\in(0,1)$. There exists $\delta_0>0$ such that if*

$$\delta\in(0,\delta_0),\qquad \mu\in(-\mu_\sharp,\mu_\sharp),\qquad z\in\mathbb{D}\left(0,\theta\sqrt{\vartheta_F^2+\mu^2\cdot\nu_F^2|\ell|^2}\right),$$
$$\zeta=\zeta_\star+\delta\mu,\qquad \lambda=E_\star+\delta z$$

then the operators $P_{\pm\delta}[\zeta]-\lambda : H^2[\zeta]\to L^2[\zeta]$ are invertible. Furthermore,

$$(P_{\pm\delta}[\zeta]-\lambda)^{-1} = S_{\pm\delta}(\mu,z)+\mathscr{O}_{L^2[\zeta]}(\delta^{-1/3}),$$
$$(k'\cdot D_x)(P_{\pm\delta}[\zeta]-\lambda)^{-1} = S^D_{\pm\delta}(\mu,z)+\mathscr{O}_{L^2[\zeta]}(\delta^{-1/3}),$$

where

$$S_{\pm\delta}(\mu,z)\overset{\text{def}}{=}\frac{1}{\delta}\cdot\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}^\top e^{i\mu\delta\langle\ell,x\rangle}\Pi^*\cdot\mathcal{U}_\delta(\not{D}_\pm(\mu)-z)^{-1}\mathcal{U}_\delta^{-1}\cdot\Pi e^{-i\mu\delta\langle\ell,x\rangle}\overline{\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}},$$
$$S^D_{\pm\delta}(\mu,z)\overset{\text{def}}{=}\frac{1}{\delta}\cdot\begin{bmatrix}(k'\cdot D_x)\phi_1\\ (k'\cdot D_x)\phi_2\end{bmatrix}^\top e^{i\mu\delta\langle\ell,x\rangle}\Pi^*\cdot\mathcal{U}_\delta(\not{D}_\pm(\mu)-z)^{-1}\mathcal{U}_\delta^{-1}\cdot\Pi e^{-i\mu\delta\langle\ell,x\rangle}\overline{\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}}.$$

5A. *Strategy.* We first observe that it suffices to prove Theorem 5.1 for $P_\delta[\zeta]$. Indeed, to go from $P_\delta[\zeta]$ to $P_{-\delta}[\zeta]$ we simply replace W with $-W$. The only parameter to change is $\vartheta_\star$, which becomes $-\vartheta_\star$. This simply transforms $\not{D}_+(\mu)$ to $\not{D}_-(\mu)$.

To prove Theorem 5.1, we decompose $P_\delta[\zeta]$ fiberwise using the operators $P_\delta(\xi)$ (formally equal to P_δ but acting on L^2_ξ). Specifically

$$P_\delta[\zeta]=\frac{1}{2\pi}\int^\oplus_{\mathbb{R}/(2\pi\mathbb{Z})}P_\delta(\zeta k+\tau k')\cdot d\tau=\frac{1}{2\pi}\int^\oplus_{[0,2\pi]}P_\delta(\zeta k+\tau k')\cdot d\tau.$$

When $P_\delta[\zeta]-\lambda$ is invertible, we are interested in the resolvent

$$(P_\delta[\zeta]-\lambda)^{-1}=\frac{1}{2\pi}\int^\oplus_{[0,2\pi]}(P_\delta(\zeta k+\tau k')-\lambda)^{-1}\,d\tau. \tag{5-2}$$

The fiber resolvents $(P_\delta(\zeta k+\tau k')-\lambda)^{-1}$ were studied in Section 4. We first show that if $\zeta k+\tau k'$ satisfies $\rho(\zeta k+\tau k')\geq\delta^{1/3}$ then this quasimomentum does not contribute significantly to the resolvent $(P_\delta[\zeta]-\lambda)^{-1}$.

Then we study the contributions from quasimomenta $\zeta k+\tau k'$ at distance at most $\delta^{1/3}$ from $\xi_\star$. The Dirac operator $\not{D}_+(\mu)$ emerges from a rescaled direct integration of the dominant rank-2 matrix exhibited in Lemma 4.3.

5B. *Reduction to $\zeta k+\tau k'$ near $\xi_\star$.* We start the proof of Theorem 5.1. Below $\theta\in(0,1)$ and $\mu_\sharp>0$ are fixed numbers. Let n be the integer such that

$$\langle\xi_\star,v'\rangle\in[2\pi n,2\pi n+2\pi).$$

Write $\xi=\zeta k+\tau k'$, $\tau\in[2\pi n,2\pi n+2\pi)$, and introduce

$$I\overset{\text{def}}{=}\{\tau\in[2\pi n,2\pi n+2\pi):|\xi-\xi_\star|\leq\delta^{1/3}\},\qquad I^c\overset{\text{def}}{=}[2\pi n,2\pi n+2\pi)\setminus I.$$

Observe that $\rho(\xi)=\delta|\mu\ell|$. In particular for δ small enough $\rho(\xi)$ is smaller than the threshold ϵ_0 given by Lemma 4.1. That lemma yields

$$\begin{aligned}(P_\delta[\zeta]-\lambda)^{-1}&=\frac{1}{2\pi}\int^\oplus_{\tau\in I}(P_\delta(\zeta k+\tau k')-\lambda)^{-1}\,d\tau+\frac{1}{2\pi}\int^\oplus_{\tau\in I^c}(P_\delta(\zeta k+\tau k')-\lambda)^{-1}\,d\tau\\ &=\frac{1}{2\pi}\int^\oplus_{\tau\in I}(P_\delta(\zeta k+\tau k')-\lambda)^{-1}\,d\tau+\mathscr{O}_{L^2[\zeta]\to H^2[\zeta]}(\delta^{-1/3}).\end{aligned}\tag{5-3}$$

We would like to apply Lemma 4.3 to the leading term of (5-3). We check the assumptions: we must verify that when λ belongs to the range allowed in Theorem 5.1, λ belongs to the range required by Lemma 4.3. This is equivalent to

$$\mathbb{D}\big(E_\star, \theta\delta\sqrt{\vartheta_F^2+\mu^2\cdot\nu_F^2|\ell|^2}\big) \subset \mathbb{D}\big(E_\star, \theta\sqrt{\vartheta_F^2\delta^2+\nu_F^2\cdot|\xi-\xi_\star|^2}\big). \tag{5-4}$$

To check that (5-4) holds, we observe that

$$\begin{gathered} |\xi-\xi_\star|^2 = |k'|^2(\tau-\tau_\star)^2+\mu^2\delta^2|\ell|^2, \\ \tau_\star \overset{\text{def}}{=} \langle \xi_\star, v'\rangle - \mu\delta\frac{\langle k, k'\rangle}{|k'|^2}, \qquad \ell \overset{\text{def}}{=} k - \frac{\langle k, k'\rangle}{|k'|^2}k'. \end{gathered} \tag{5-5}$$

This implies

$$\theta\delta\sqrt{\vartheta_F^2+\mu^2\cdot\nu_F^2|\ell|^2} = \theta\sqrt{\vartheta_F^2\delta^2+\mu^2\nu_F^2\cdot|\ell|^2\delta^2} \le \theta\sqrt{\vartheta_F^2\delta^2+\nu_F^2\cdot|\xi-\xi_\star|^2}.$$

Therefore we can apply Lemma 4.3 to the leading term of (5-3). It shows that

$$\begin{gathered} P_\delta[\zeta] = T_\delta[\zeta] + \mathscr{O}_{L^2[\zeta]\to H^2[\zeta]}(\delta^{-1/3}), \\ T_\delta[\zeta] \overset{\text{def}}{=} \frac{1}{2\pi}\int_{\tau\in I}^{\oplus} \Pi_0(\zeta k+\tau k')^*\cdot(M_\delta(\zeta k+\tau k')-\lambda)^{-1}\cdot\Pi_0(\zeta k+\tau k')\,d\tau. \end{gathered} \tag{5-6}$$

Because of (5-5), $\tau_\star = \langle\xi_\star, v'\rangle + O(\delta)$. From Section 2E and the definition of n, we have $\langle\xi_\star, v'\rangle \in \big\{2\pi n+\frac{2\pi}{3}, 2\pi n+\frac{4\pi}{3}\big\}$. Hence $\tau_\star$ is in the interior of I for δ sufficiently small. It follows that I is an interval centered at $\tau_\star$:

$$I = [\tau_\star-\delta\cdot\alpha_\delta, \tau_\star+\delta\cdot\alpha_\delta], \qquad \alpha_\delta \overset{\text{def}}{=} \frac{\sqrt{\delta^{2/3}-\mu^2\cdot\nu_F^2|\ell|^2\delta^2}}{|k'|\delta} = \frac{\delta^{-2/3}}{|k'|} + O(\delta^{2/3}). \tag{5-7}$$

We make the substitution $\tau \mapsto \tau_\star+\delta\tau$. The vector $\zeta k+\tau k'$ becomes $\zeta k+(\tau_\star+\delta\tau)k' = \xi_\star+\delta(\tau k'+\mu\ell)$, the interval I becomes $[-\alpha_\delta, \alpha_\delta]$, $d\tau$ becomes $\delta\,d\tau$ and

$$\begin{gathered} M_\delta(\zeta k+\delta(\tau k'+\mu\ell)) = E_\star + \delta\begin{bmatrix} \vartheta_\star & \nu_\star(\tau k'+\mu\ell) \\ \overline{\nu_\star(\tau k'+\mu\ell)} & -\vartheta_\star \end{bmatrix}, \\ (M_\delta(\zeta k+\delta(\tau k'+\mu\ell))-\lambda)^{-1} = \frac{1}{\delta}\begin{bmatrix} \vartheta_\star - z & \nu_\star(\tau k'+\mu\ell) \\ \overline{\nu_\star(\tau k'+\mu\ell)} & -\vartheta_\star - z \end{bmatrix}^{-1}, \qquad z \overset{\text{def}}{=} \frac{\lambda-E_\star}{\delta}. \end{gathered}$$

We deduce that $T_\delta[\zeta]$ equals

$$\frac{1}{2\pi}\int_{|\tau|<\alpha_\delta}^{\oplus} \Pi_0(\xi_\star+\delta(\tau k'+\mu\ell))^*\cdot\begin{bmatrix} \vartheta_\star - z & \nu_\star(\tau k'+\mu\ell) \\ \overline{\nu_\star(\tau k'+\mu\ell)} & -\vartheta_\star - z \end{bmatrix}^{-1}\cdot\Pi_0(\xi_\star+\delta(\tau k'+\mu\ell))\cdot d\tau.$$

Thanks to the definition (5-7) of Π_0,

$$\Pi_0(\xi_\star+\delta(\tau k'+\mu\ell))u = \begin{bmatrix} \langle e^{i\delta\langle\tau k'+\mu\ell, x\rangle}\phi_1, u\rangle \\ \langle e^{i\delta\langle\tau k'+\mu\ell, x\rangle}\phi_2, u\rangle \end{bmatrix}.$$

We conclude that the operator $T_\delta[\zeta]$ has kernel

$$\frac{1}{2\pi}\begin{bmatrix}\phi_1(x)\\ \phi_2(x)\end{bmatrix}^\top \cdot \int_{|\tau|\leq\alpha_\delta}\begin{bmatrix}\vartheta_\star - z & \nu_\star(\tau k'+\mu\ell)\\ \overline{\nu_\star(\tau k'+\mu\ell)} & -\vartheta_\star - z\end{bmatrix}^{-1} e^{i\delta\langle \tau k'+\mu\ell, x-y\rangle}\, d\tau \cdot \overline{\begin{bmatrix}\phi_1(y)\\ \phi_2(y)\end{bmatrix}}. \tag{5-8}$$

5C. *Kernel identities and proof of Theorem 5.1.* Recall that Π, Π^* and $\mathcal{U}_\delta$ are defined in (5-1).

Lemma 5.2. *There exists $C>0$ such that for every $\delta\in(0,1)$, the following holds. Let $\Psi\in L^\infty(\mathbb{R}, M_2(\mathbb{C}))$, possibly depending on δ, and A_Ψ be the operator with kernel*

$$(x,y)\mapsto \begin{bmatrix}\phi_1(x)\\ \phi_2(x)\end{bmatrix}^\top \cdot \frac{1}{2\pi}\int_{\mathbb{R}} \Psi(\tau) e^{i\delta\langle \tau k'+\mu\ell, x-y\rangle}\, d\tau \cdot \overline{\begin{bmatrix}\phi_1(y)\\ \phi_2(y)\end{bmatrix}}.$$

Then A_Ψ is bounded on $L^2[\zeta]$ with $\|A_\Psi\|_{L^2[\zeta]}\leq C\delta^{-1}|\Psi|_\infty$, and

$$A_\Psi = \frac{1}{\delta}\cdot \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}^\top e^{i\mu\delta\langle \ell, x\rangle}\Pi^*\cdot \mathcal{U}_\delta \Psi(D_t)\mathcal{U}_\delta^{-1}\cdot \Pi e^{-i\mu\delta\langle\ell,x\rangle}\overline{\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}}. \tag{5-9}$$

If in addition $\tau\cdot\Psi\in L^\infty(\mathbb{R}, M_2(\mathbb{C}))$ then $(k'\cdot D_x)A_\Psi$ is bounded on $L^2[\zeta]$ with

$$\|(k'\cdot D_x)A_\Psi\|_{L^2[\zeta]}\leq C\delta^{-1}|\Psi|_\infty + C|\tau\cdot\Psi|_\infty$$

and

$$(k'\cdot D_x)A_\Psi = \frac{1}{\delta}\cdot\begin{bmatrix}(k'\cdot D_x)\phi_1\\ (k'\cdot D_x)\phi_2\end{bmatrix}^\top e^{i\mu\delta\langle\ell,x\rangle}\Pi^*\cdot\mathcal{U}_\delta\Psi(D_t)\mathcal{U}_\delta^{-1}\cdot\Pi e^{-i\mu\delta\langle\ell,x\rangle}\overline{\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}} + \mathscr{O}_{L^2[\zeta]}(|\langle\tau\rangle\cdot\psi|_\infty).$$

Proof. 1. We first note that the operator $\delta^{-1}\cdot\mathcal{U}_\delta\Psi(D_t)\mathcal{U}_\delta^{-1}$ has kernel

$$(t,t')\in\mathbb{R}\times\mathbb{R}^2\mapsto\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\delta\tau(t-t')}\Psi(\tau)\cdot d\tau. \tag{5-10}$$

Let δ_0 denote the Dirac mass. We claim that the operator Π has kernel

$$(t',y)\in\mathbb{R}\times\mathbb{R}^2/(\mathbb{Z}v)\mapsto\delta_0(\langle k',y\rangle - t'). \tag{5-11}$$

Fix $f\in C_0^\infty(\mathbb{R}^2/\mathbb{Z}v,\mathbb{C}^2)$. The integral

$$\int_{\mathbb{R}^2/\mathbb{Z}v}\delta_0(\langle k',y\rangle - t')f(y)\,dy$$

is well-defined. We perform the substitution $y\mapsto(\langle k,y\rangle,\langle k',y\rangle)$; the inverse substitution is $(s,t)\mapsto sv+tv'$; the Jacobian determinant is $dy=\mathrm{Det}[v,v']\cdot ds\,dt$. Since v, v' are related to v_1, v_2 by (2-4) and $\mathrm{Det}[v_1,v_2]=1$ because of (1-1), $\mathrm{Det}[v,v']=1$. The above integral becomes

$$\int_{\mathbb{R}^2/\mathbb{Z}e_1}\delta_0(t-t')f(sv+tv')\,ds\,dt = \int_{\mathbb{R}/\mathbb{Z}} f(sv+t'v')\,ds.$$

We recover the formula (5-1) for Πf. From (5-11), we deduce that the kernel of Π^* is

$$(x,t)\in\mathbb{R}^2/\mathbb{Z}v\times\mathbb{R}\mapsto\delta_0(\langle k',x\rangle - t). \tag{5-12}$$

To obtain (5-10), we compose the kernels (5-11), (5-10) and (5-12). This forces t to be $\langle k', x\rangle$ and t' to be $\langle k', y\rangle$. Hence the operator $\delta^{-1}\cdot\Pi^*\cdot\mathcal{U}_\delta\Psi(D_t)\mathcal{U}_\delta^{-1}\cdot\Pi$ has kernel

$$(x, y)\mapsto \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\delta\tau\langle k', x-y\rangle}\Psi(\tau)\cdot d\tau.$$

This implies (5-9).

2. We prove the $L^2[\zeta]$-bound. The operator Π maps $L^2(\mathbb{R}^2/\mathbb{Z}v, \mathbb{C})$ to $L^2(\mathbb{R}, \mathbb{C}^2)$, independently of δ. Its adjoint maps $L^2(\mathbb{R}, \mathbb{C}^2)$ to $L^2(\mathbb{R}^2/\mathbb{Z}v, \mathbb{C})$, independently of δ. The dilations $\mathcal{U}_\delta$ and $\mathcal{U}_\delta^{-1}$ map $L^2(\mathbb{R}, \mathbb{C}^2)$ to itself, with bounds $\delta^{-1/2}$ and $\delta^{1/2}$, respectively. The operator $\Psi(D_t)$ is a Fourier multiplier; hence it maps $L^2(\mathbb{R}, \mathbb{C}^2)$ to itself, with bound $|\Psi|_\infty$. Combining all these bounds together we get

$$\|A_\Psi\|_{L^2[\zeta]}\le C\delta^{-1}|\Psi|_\infty.$$

3. We observe that the operator $(k'\cdot D_x)A_\Psi$ has kernel

$$\begin{bmatrix}(k'\cdot D_x)\phi_1(x)\\ (k'\cdot D_x)\phi_2(x)\end{bmatrix}^\top\cdot\frac{1}{2\pi}\int_{\mathbb{R}}\Psi(\tau)e^{i\delta\langle\tau k'+\mu\ell, x-y\rangle}\,d\tau\cdot\overline{\begin{bmatrix}\phi_1(y)\\ \phi_2(y)\end{bmatrix}}$$
$$+\begin{bmatrix}\phi_1(x)\\ \phi_2(x)\end{bmatrix}^\top\cdot\frac{1}{2\pi}\int_{\mathbb{R}}\Psi(\tau)\cdot\tau\delta|k'|^2e^{it\delta\langle\tau k'+\mu\ell, x-y\rangle}\,d\tau\cdot\overline{\begin{bmatrix}\phi_1(y)\\ \phi_2(y)\end{bmatrix}}.$$

Above, we used that $\ell\cdot k'=0$. These two terms are kernels of operators studied in steps 1 and 2. The first one has $L^2[\zeta]$-operator norm controlled by $C\delta^{-1}|\Psi|_\infty$ and the second one by $C|\tau\cdot\Psi|_\infty$. □

Lemma 5.3. *Let $\vartheta_\sharp\in(0, \vartheta_F)$. There exists $C>0$ such that for any $z\in\mathbb{D}(0, \vartheta_\sharp)$, the following holds. Let $\Psi_0:\mathbb{R}\to M_2(\mathbb{C})$ be given by*

$$\Psi_0(\tau)\stackrel{\text{def}}{=}\begin{bmatrix}\vartheta_\star-z & \nu_\star(\tau k'+\mu\ell)\\ \overline{\nu_\star(\tau k'+\mu\ell)} & -\vartheta_\star-z\end{bmatrix}^{-1}. \tag{5-13}$$

Then $\tau\cdot\Psi_0\in L^\infty(\mathbb{R}, M_2(\mathbb{C}))$ and for every $a\ge 0$,

$$\sup_{|\tau|\ge a}\|\Psi_0(\tau)\|_{\mathbb{C}^2}\le Ca^{-1},\qquad \sup_{|\tau|\ge a}\|\tau\Psi_0(\tau)\|_{\mathbb{C}^2}\le C. \tag{5-14}$$

To prove Lemma 5.3, it suffices to observe that

$$\Psi_0(\tau)=-\frac{1}{|\vartheta_\star-z|^2+\nu_F^2|\ell|^2\mu^2+\nu_F^2|k'|^2\tau^2}\begin{bmatrix}\vartheta_\star-z & \nu_\star(\tau k'+\mu\ell)\\ \overline{\nu_\star(\tau k'+\mu\ell)} & -\vartheta_\star-z\end{bmatrix}.$$

In particular, $\Psi_0(\tau)=\mathscr{O}_{\mathbb{C}^2}(\tau^{-1})$. This yields the bounds (5-14). Let $\not{D}_+(\mu):H^1(\mathbb{R}, \mathbb{C}^2)\to L^2(\mathbb{R}, \mathbb{C}^2)$ be the Dirac operator defined by

$$\not{D}_+(\zeta)\stackrel{\text{def}}{=}\begin{bmatrix}\vartheta_\star & \nu_\star k'\\ \overline{\nu_\star k'} & -\vartheta_\star\end{bmatrix}D_t+\mu\begin{bmatrix}0 & \nu_\star\ell\\ \overline{\nu_\star\ell} & 0\end{bmatrix}+\begin{bmatrix}\vartheta_\star & 0\\ 0 & -\vartheta_\star\end{bmatrix}.$$

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. 1. Because of (5-6), it suffices to prove Theorem 5.1 when $P_\delta[\zeta]$ is replaced by $T_\delta[\zeta]$. We first apply Lemma 5.2 with Ψ_0 given by (5-13). It shows that

$$A_{\Psi_0} \stackrel{\text{def}}{=} \frac{1}{\delta} \cdot \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}^\top e^{i\mu\delta\langle \ell, x\rangle} \Pi^* \cdot \mathcal{U}_\delta(\not{D}_+(\mu) - z)^{-1} \mathcal{U}_\delta^{-1} \cdot \Pi e^{-i\mu\delta\langle \ell, x\rangle} \overline{\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}}$$

has kernel

$$(x, y) \mapsto \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \end{bmatrix}^\top \cdot \frac{1}{2\pi} \int_{\mathbb{R}} \Psi_0(\tau) e^{i\delta\langle \tau k' + \mu\ell, x-y\rangle} \, d\tau \cdot \overline{\begin{bmatrix} \phi_1(y) \\ \phi_2(y) \end{bmatrix}}.$$

2. We now apply Lemma 5.2 with $\Psi_1(\tau) \stackrel{\text{def}}{=} \mathbb{1}_{\mathbb{R}\setminus[-\alpha_\delta, \alpha_\delta]}(\tau) \cdot \Psi_0(\tau)$ (recall that $\alpha_\delta = |k'|^{-1}\delta^{-2/3} + O(\delta^{2/3})$ was defined in (5-7)). It shows that A_{Ψ_1} has kernel

$$\begin{bmatrix} \phi_1(x) \\ \phi_2(x) \end{bmatrix}^\top \cdot \frac{1}{2\pi} \int_{|\tau| \geq \alpha_\delta} \Psi_0(\tau) e^{i\delta\langle \tau k' + \mu\ell, x-y\rangle} \, d\tau \cdot \overline{\begin{bmatrix} \phi_1(y) \\ \phi_2(y) \end{bmatrix}}.$$

Thanks to the bounds of Lemma 5.3, $A_{\Psi_1} = \mathscr{O}_{L^2[\zeta]}(\delta^{-1}\alpha_\delta^{-1}) = \mathscr{O}_{L^2[\zeta]}(\delta^{-1/3})$.

3. When we subtract the kernel of A_{Ψ_1} from the kernel of A_{Ψ_0}, we get the kernel of $T_\delta[\zeta]$; see (5-8). This shows that $T_\delta[\zeta] = A_{\Psi_0} - A_{\Psi_1}$. Hence

$$T_\delta[\zeta] = \frac{1}{\delta} \cdot \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}^\top e^{i\mu\delta\langle \ell, x\rangle} \Pi^* \cdot \mathcal{U}_\delta(\not{D}_+(\mu) - z)^{-1} \mathcal{U}_\delta^{-1} \cdot \Pi e^{-i\mu\delta\langle \ell, x\rangle} \overline{\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}} + \mathscr{O}_{L^2[\zeta]}(\delta^{-1/3}).$$

4. Lemma 5.2 and the bounds of Lemma 5.3 imply that

$$(k' \cdot D_x) A_{\Psi_0} = \frac{1}{\delta} \cdot \begin{bmatrix} (k' \cdot D_x)\phi_1 \\ (k' \cdot D_x)\phi_2 \end{bmatrix}^\top e^{i\mu\delta\langle \ell, x\rangle} \Pi^* \cdot \mathcal{U}_\delta(\not{D}_+(\mu) - z)^{-1} \mathcal{U}_\delta^{-1} \cdot \Pi e^{-i\mu\delta\langle \ell, x\rangle} \overline{\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}} + \mathscr{O}_{L^2[\zeta]}(1).$$

It also implies that $(k' \cdot D_x) A_{\Psi_1} = \mathscr{O}_{L^2[\zeta]}(\delta^{-1/3})$. We conclude that

$$(k' \cdot D_x) T_\delta[\zeta] = \frac{1}{\delta} \cdot \begin{bmatrix} (k' \cdot D_x)\phi_1 \\ (k' \cdot D_x)\phi_2 \end{bmatrix}^\top e^{i\mu\delta\langle \ell, x\rangle} \Pi^* \cdot \mathcal{U}_\delta(\not{D}_+(\mu) - z)^{-1} \mathcal{U}_\delta^{-1} \cdot \Pi e^{-i\mu\delta\langle \ell, x\rangle} \overline{\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}} + \mathscr{O}_{L^2[\zeta]}(\delta^{-1/3}). \quad \square$$

6. The resolvent of the edge operator

Recall that κ is a domain wall function — see (1-8) — and introduce the operator

$$\mathscr{P}_\delta = -\Delta + V + \delta \cdot \kappa_\delta \cdot W, \quad \kappa_\delta(x) = \kappa(\delta\langle k', x\rangle).$$

We denote by $\mathscr{P}_\delta[\zeta]$ the operator formally equal to $\mathscr{P}_\delta$ but acting on $L^2[\zeta]$. In this section we prove Theorem 3.2: we connect the resolvent of $\mathscr{P}_\delta[\zeta]$ to that of the Dirac operator $\not{D}(\mu)$ emerging from the multiscale analysis of [Fefferman et al. 2016b]:

$$\not{D}(\mu) = \begin{bmatrix} 0 & \nu_\star k' \\ \overline{\nu_\star k'} & 0 \end{bmatrix} D_t + \mu \begin{bmatrix} 0 & \nu_\star \ell \\ \overline{\nu_\star \ell} & 0 \end{bmatrix} + \vartheta_\star \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \kappa.$$

The strategy is as follows. We first prove a formula for $(\mathscr{P}_\delta[\zeta] - \lambda)^{-1}$ in terms of the asymptotic operators $(P_{\pm\delta}[\zeta] - \lambda)^{-1}$. We then apply Theorem 5.1 to exhibit the leading-order term in this formula.

We use a cyclicity argument to simplify this leading-order term. An averaging effect emerges as the driving phenomenon connecting $\mathscr{P}_\delta[\zeta]$ to $\not{D}(\mu)$. This yields Theorem 3.2.

6A. *Parametrix.* We first construct a parametrix for $\mathscr{P}_\delta[\zeta]-\lambda$. Introduce

$$\mathscr{Q}_\delta(\zeta,\lambda) \overset{\text{def}}{=} \sum_{\pm} \chi_{\pm,\delta}\cdot(P_{\pm\delta}[\zeta]-\lambda)^{-1}, \qquad \chi_\pm \overset{\text{def}}{=} \frac{1\pm\kappa}{2}. \tag{6-1}$$

This operator is well-defined — and depends holomorphically on λ — as long as $\lambda\notin\Sigma_{L^2[\zeta]}(P_\delta[\zeta])$. Formally speaking, it behaves asymptotically like $(\mathscr{P}_\delta[\zeta]-\lambda)^{-1}$.

A calculation similar to [Drouot et al. 2018, §5.2] yields

$$(\mathscr{P}_\delta[\zeta]-\lambda)\mathscr{Q}_\delta(\zeta,\lambda) = \mathrm{Id}+\mathscr{K}_\delta(\zeta,\lambda),$$

where

$$\mathscr{K}_\delta(\zeta,\lambda) = \frac{\delta}{2}\big(2(D_t\kappa)_\delta\cdot(k'\cdot D_x)+\delta|k'|^2(D_t^2\kappa)_\delta+(\kappa_\delta^2-1)W\big)\big((P_\delta[\zeta]-\lambda)^{-1}-(P_{-\delta}[\zeta]-\lambda)^{-1}\big).$$

This identity shows that if $\mathrm{Id}+\mathscr{K}_\delta(\zeta,\lambda)$ is invertible then $\mathscr{P}_\delta[\zeta]-\lambda$ is invertible. When this holds, $(\mathscr{P}_\delta[\zeta]-\lambda)^{-1}$ has an expression in terms of $\mathscr{Q}_\delta(\zeta,\lambda)$ and $\mathscr{K}_\delta(\zeta,\lambda)$:

$$(\mathscr{P}_\delta[\zeta]-\lambda)^{-1} = \mathscr{Q}_\delta(\zeta,\lambda)\cdot(\mathrm{Id}+\mathscr{K}_\delta(\zeta,\lambda))^{-1}.$$

The operators $\mathscr{Q}_\delta(\zeta,\lambda)$ and $\mathscr{K}_\delta(\zeta,\lambda)$ have expressions in terms of $(P_{\pm\delta}[\zeta]-\lambda)^{-1}$. An application of Theorem 5.1 estimates $\mathscr{Q}_\delta(\zeta,\lambda)$ and $\mathscr{K}_\delta(\zeta,\lambda)$, assuming

$$\begin{gathered}\delta\in(0,\delta_0), \quad \mu\in(-\mu_\sharp,\mu_\sharp), \quad z\in\mathbb{D}\big(0,\sqrt{\vartheta_F^2+\mu^2\cdot\nu_F^2|\ell|^2}\big), \\ \lambda = E_\star+\delta z, \quad \zeta=\zeta_\star+\delta\mu.\end{gathered} \tag{6-2}$$

We introduce the operator

$$R_0(\mu,z): L^2(\mathbb{R},\mathbb{C}^2)\to H^2(\mathbb{R},\mathbb{C}^2),$$
$$R_0(\mu,z) \overset{\text{def}}{=} (\not{D}_+(\mu)^2-z^2)^{-1} = (\nu_F^2|k'|^2D_t^2+\mu^2|\nu_\star\ell|^2+\vartheta_F^2-z^2)^{-1}.$$

It is well-defined when z is away from the spectrum of $\not{D}_\pm(\mu)$ — in particular when

$$z\in\mathbb{D}\big(0,\sqrt{\vartheta_F^2+\mu^2\cdot\nu_F^2|\ell|^2}\big).$$

Lemma 6.1. *Let $\mu_\sharp>0$, $\theta\in(0,1)$. There exists $\delta_0>0$ such that under the assumptions of Theorem 5.1, $\mathscr{K}_\delta(\zeta,\lambda)$ and $\mathscr{Q}_\delta(\zeta,\lambda)$ admit the expansions*

$$\mathscr{K}_\delta(\zeta,\lambda) = \mathcal{K}_\delta(\mu,z)+\mathscr{O}_{L^2[\zeta]}(\delta^{2/3}), \qquad \mathscr{Q}_\delta(\zeta,\lambda) = \mathcal{Q}_\delta(\mu,z)+\mathscr{O}_{L^2[\zeta]}(\delta^{-1/3}).$$

Above, $\mathcal{K}_\delta(\mu,z)$ is equal to

$$\vartheta_\star\left(2(D_t\kappa)_\delta\begin{bmatrix}k'\cdot D_x\phi_1\\ k'\cdot D_x\phi_2\end{bmatrix}^\top+(\kappa_\delta^2-1)W\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}^\top\right)e^{-i\mu\delta\langle\ell,x\rangle}\Pi^*\mathcal{U}_\delta\boldsymbol{\sigma_3}\cdot R_0(\mu,z)\cdot\mathcal{U}_\delta^{-1}\Pi e^{i\mu\delta\langle\ell,x\rangle}\overline{\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}}$$

and

$$\mathcal{Q}_\delta(\mu,z) \overset{\text{def}}{=} \frac{1}{\delta}\cdot\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}^\top e^{-i\mu\delta\langle\ell,x\rangle}\Pi^*\mathcal{U}_\delta\cdot(\not{D}(\mu)+z)\cdot R_0(\mu,z)\cdot\mathcal{U}_\delta^{-1}\Pi e^{i\mu\delta\langle\ell,x\rangle}\overline{\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}}.$$

The proof is a calculation using the relation between $\mathscr{Q}_\delta(\zeta,\lambda)$ and $\mathscr{K}_\delta(\zeta,\lambda)$ with the edge resolvents $(P_{\pm\delta}[\zeta]-\lambda)^{-1}$, and the expansions of these resolvents provided by Theorem 5.1. We defer it to Appendix A.3.

6B. *Weak convergence.* We are interested in the eigenvalues of $\mathscr{P}_\delta[\zeta]$. We previously studied eigenvalue problems in seemingly different situations [Drouot 2018a; 2018c; 2018d], as well as in a one-dimensional analog [Drouot et al. 2018]. The proofs of these results rely on a *cyclicity* principle: if A and B are two matrices then the nonzero eigenvalues of AB and BA are equal (together with their multiplicity).

Although the leading-order terms $\mathcal{K}_\delta(\mu,z)$ and $\mathcal{Q}_\delta(\mu,z)$ have complicated expressions, they exhibit a structure favorable to applying the cyclicity principle. This will provide a simple formula for the product

$$\mathcal{Q}_\delta(\mu,z)\cdot(\mathrm{Id}+\mathcal{K}_\delta(\mu,z))^{-1}$$

and complete the proof of Theorem 3.2.

A preliminary step is the computation of a weak limit that arises when permuting factors in $\mathcal{K}_\delta(\mu,z)$: the operator $L^2(\mathbb{R},\mathbb{C}^2)\to L^2(\mathbb{R},\mathbb{C}^2)$ given by

$$\begin{aligned}\vartheta_\star\mathcal{U}_\delta^{-1}\Pi e^{i\mu\delta\langle\ell,x\rangle}\overline{\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}}\cdot\left(2(D_t\kappa)_\delta\cdot\begin{bmatrix}k'\cdot D_x\phi_1\\ k'\cdot D_x\phi_2\end{bmatrix}^\top+(\kappa_\delta^2-1)W\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}^\top\right)\cdot e^{-i\mu\delta\langle\ell,x\rangle}\Pi^*\mathcal{U}_\delta\boldsymbol{\sigma_3}\\ =\vartheta_\star\mathcal{U}_\delta^{-1}\cdot\Pi\overline{\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}}\left(2(D_t\kappa)_\delta\cdot\begin{bmatrix}k'\cdot D_x\phi_1\\ k'\cdot D_x\phi_2\end{bmatrix}^\top+(\kappa_\delta^2-1)W\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}^\top\right)\boldsymbol{\sigma_3}\Pi^*\cdot\mathcal{U}_\delta.\end{aligned}\tag{6-3}$$

Lemma 6.2. *The operator (6-3) is a multiplication operator by a function $\mathscr{U}^\delta:\mathbb{R}\to M_2(\mathbb{C})$ with two-scale structure*:

$$\mathscr{U}^\delta(t)=\mathscr{U}\left(\frac{t}{\delta},t\right),\qquad \mathscr{U}\in C_0^\infty(\mathbb{R}/\mathbb{Z}\times\mathbb{R},M_2(\mathbb{C})).\tag{6-4}$$

The function $\mathscr{U}^\delta$ converges weakly to

$$\mathscr{U}^0\in C_0^\infty(\mathbb{R},M_2(\mathbb{C})),\qquad \mathscr{U}^0(t)\stackrel{\text{def}}{=}\vartheta_F^2(\kappa(t)^2-1)+\vartheta_\star\begin{bmatrix}0 & -\nu_\star k'\\ \overline{\nu_\star k'} & 0\end{bmatrix}(D_t\kappa)(t).\tag{6-5}$$

Finally, if $\mathscr{U}^\delta-\mathscr{U}^0$ is seen as a multiplication operator from H^1 to H^{-1},

$$\mathscr{U}^\delta-\mathscr{U}^0=\mathscr{O}_{H^1\to H^{-1}}(\delta).\tag{6-6}$$

Proof. 1. We set

$$\begin{aligned}F(x,t)&\stackrel{\text{def}}{=}\vartheta_\star\overline{\begin{bmatrix}\phi_1(x)\\ \phi_2(x)\end{bmatrix}}\left((D_t\kappa)(t)\cdot\begin{bmatrix}2k'\cdot D_x\phi_1(x)\\ 2k'\cdot D_x\phi_2(x)\end{bmatrix}^\top+(\kappa(t)^2-1)W(x)\begin{bmatrix}\phi_1(x)\\ \phi_2(x)\end{bmatrix}^\top\right)\boldsymbol{\sigma_3},\\ F^\delta(x)&\stackrel{\text{def}}{=}F(x,\delta\langle k',x\rangle).\end{aligned}$$

Fix $g\in C_0^\infty(\mathbb{R},\mathbb{C}^2)$. The action of the operator (6-3) on g is given by

$$\begin{aligned}(\mathcal{U}_\delta^{-1}\cdot\Pi F^\delta\Pi^*\cdot\mathcal{U}_\delta g)(t)&=\int_0^1 F^\delta\left(sv+\frac{t}{\delta}v'\right)g\left(\left\langle k',\delta\left(sv+\frac{t}{\delta}v'\right)\right\rangle\right)ds\\ &=\int_0^1 F\left(sv+\frac{t}{\delta}v',t\right)g(t)ds=\int_0^1 F\left(sv+\frac{t}{\delta}v',t\right)ds\cdot g(t).\end{aligned}$$

Therefore (6-3) is the multiplication operator by

$$\mathscr{U}^\delta(t) \stackrel{\text{def}}{=} \int_0^1 F\Big(sv + \frac{t}{\delta}v', t\Big)\, ds.$$

Note that F is Λ-periodic in x and compactly supported in t. Therefore $\mathscr{U}^\delta$ has the two-scale structure (6-4):

$$\mathscr{U}^\delta(t) = \mathscr{U}\Big(\frac{t}{\delta}, t\Big), \qquad \mathscr{U}(\tau, t) \stackrel{\text{def}}{=} \int_0^1 F(sv + \tau v', t)\, ds. \tag{6-7}$$

2. The function $\mathscr{U}$ is periodic in the first variable and compactly supported in the second one. Therefore the weak limit of $\mathscr{U}^\delta$ is

$$\mathscr{U}^0(t) \stackrel{\text{def}}{=} \int_0^1 \mathscr{U}(\tau, t)\, d\tau = \int_0^1 \int_0^1 F(sv + \tau v', t)\, d\tau\, ds = \int_{\mathbb{L}} F(x, t)\, dx. \tag{6-8}$$

In the last inequality, we changed variables: $sv + \tau v'$ became $sv_1 + \tau v_2$ (with Jacobian equal to 1); hence $[0, 1]^2$ became $\mathbb{L}$, the fundamental cell of $\mathbb{R}^2/\Lambda$ given in (1-2). Going back to the definition of F, we end up with

$$\begin{aligned}\mathscr{U}^0(t) &= \left(\begin{bmatrix} \vartheta_F^2 & 0 \\ 0 & -\vartheta_F^2 \end{bmatrix} (\kappa(t)^2 - 1) + \vartheta_\star \begin{bmatrix} 0 & \nu_\star k' \\ \overline{\nu_\star k'} & 0 \end{bmatrix} (D_t\kappa)(t) \right) \boldsymbol{\sigma_3} \\ &= \vartheta_F^2(\kappa(t)^2 - 1) + \vartheta_\star \begin{bmatrix} 0 & -\nu_\star k' \\ \overline{\nu_\star k'} & 0 \end{bmatrix} (D_t\kappa)(t).\end{aligned}$$

3. We show the quantitative estimate (6-6). Since $\mathscr{U}^\delta$ and $\mathscr{U}^0$ are functions on $\mathbb{R}$,

$$\|\mathscr{U}^\delta - \mathscr{U}^0\|_{H^1 \to H^{-1}} \le C |\mathscr{U}^\delta - \mathscr{U}^0|_{H^{-1}}.$$

See, e.g., [Drouot 2018c, Lemma 2.1]. Recall that $\mathscr{U}^\delta$ is related to $\mathscr{U}$ via (6-7). The function $\mathscr{U}$ is periodic in the first variable and compactly supported in the second variable. We write a Fourier decomposition of $\mathscr{U}$:

$$\mathscr{U}(\tau, t) = \sum_{m \in \mathbb{Z}} b_m(t) e^{2i\pi m\tau}, \qquad b_m(t) \stackrel{\text{def}}{=} \int_0^1 e^{-2i\pi m\tau'} \mathscr{U}(t, \tau')\, d\tau'.$$

Because of (6-7) and (6-8),

$$\mathscr{U}^\delta(t) - \mathscr{U}^0(t) = \sum_{m \neq 0} b_m(t) e^{2i\pi mt/\delta}.$$

In other words, $\mathscr{U}^\delta - \mathscr{U}^0$ has a highly oscillatory structure. The coefficients b_m are smooth functions of t. Their Sobolev norms decay rapidly since $\mathscr{U}$ depends smoothly on τ. We can then conclude as in the proof of [Drouot 2018a, Lemma 3.1]. □

The function $\mathscr{U}^0$ is an effective potential that arises as the homogenized limit of $\mathscr{U}^\delta$. It appears in the Dirac operator $\not{D}(\mu)$. Indeed, a computation shows that

$$\not{D}(\mu)^2 = \nu_F^2 |k'|^2 D_t^2 + \mu^2 \cdot \nu_F^2 |\ell|^2 + \vartheta_F^2 \kappa^2 + \vartheta_\star \begin{bmatrix} 0 & -\nu_\star k' \\ \overline{\nu_\star k'} & 0 \end{bmatrix} (D_t\kappa).$$

Because of (6-5), we deduce that

$$\not{D}(\mu)^2 = v_F^2|k'|^2 D_t^2 + \mu^2 \cdot v_F^2|\ell|^2 + \vartheta_F^2 + \mathscr{U}^0. \tag{6-9}$$

We will apply this identity in the next section.

6C. *A cyclicity argument.* The next result is stated abstractly. It relies on the cyclicity principle.

Lemma 6.3. *Let A, B, C, D, E be bounded operators*:

$$A : H^1(\mathbb{R}, \mathbb{C}^2) \to L^2[\zeta], \quad B : L^2[\zeta] \to L^2(\mathbb{R}, \mathbb{C}^2),$$
$$C : L^2(\mathbb{R}, \mathbb{C}^2) \to L^2[\zeta], \quad D : L^2(\mathbb{R}, \mathbb{C}^2) \to H^1(\mathbb{R}, \mathbb{C}^2),$$
$$E : L^2(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2).$$

Assume that for some $M \geq 1$:

(a) *The operator norms of A, B, C, D, E are bounded by M.*

(b) *The operator* $\mathrm{Id} + DED : L^2(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$ *is invertible and*

$$\|(\mathrm{Id} + DED)^{-1}\|_{L^2(\mathbb{R},\mathbb{C}^2)} \leq M.$$

(c) *The following estimate holds*:

$$\epsilon \stackrel{\text{def}}{=} \|D(BC - E)D\|_{L^2(\mathbb{R},\mathbb{C}^2)} \leq \frac{1}{2M}.$$

Then the operator $\mathrm{Id} + CD^2B : L^2[\zeta] \to L^2[\zeta]$ *is invertible*,

$$\begin{gathered} \|(\mathrm{Id} + CD^2B)^{-1}\|_{L^2[\zeta]} \leq 3M^5, \quad \textit{and} \\ \|AD^2B \cdot (\mathrm{Id} + CD^2B)^{-1} - AD \cdot (\mathrm{Id} + DED)^{-1} \cdot DB\|_{L^2[\zeta]} \leq 2M^6\epsilon. \end{gathered} \tag{6-10}$$

Proof. Below we use L^2 and H^1 to denote $L^2(\mathbb{R}, \mathbb{C}^2)$ and $H^1(\mathbb{R}, \mathbb{C}^2)$.

1. Recall that $\mathrm{Id} + CD^2B = \mathrm{Id} + CD \cdot DB : L^2[\zeta] \to L^2[\zeta]$ is invertible if and only if $\mathrm{Id} + DB \cdot CD : L^2 \to L^2$ is invertible. In this case, the inverses are related via

$$(\mathrm{Id} + CD^2B)^{-1} = \mathrm{Id} - CD(\mathrm{Id} + DB \cdot CD)^{-1}DB. \tag{6-11}$$

Because of (b), $\mathrm{Id} + DED$ is invertible and

$$\begin{aligned} \mathrm{Id} + DB \cdot CD &= \mathrm{Id} + DED + D(BC - E)D \\ &= (\mathrm{Id} + DED) \cdot (\mathrm{Id} + (\mathrm{Id} + DED)^{-1} \cdot D(BC - E)D). \end{aligned} \tag{6-12}$$

Because of both (b) and (c),

$$\|(\mathrm{Id} + DED)^{-1} \cdot D(BC - E)D\|_{L^2} \leq \tfrac{1}{2}.$$

This implies that $\mathrm{Id} + (\mathrm{Id} + DED)^{-1} \cdot D(BC - E)D$ is invertible by a Neumann series; the inverse has operator norm controlled by 2. Thanks to (6-12), $\mathrm{Id} + DBCD$ is invertible and the inverse has norm controlled by $2M$. Hence $\mathrm{Id}+CD^2B$ is invertible. Thanks to (6-11) and (a),

$$\|(\mathrm{Id}+CD^2B)^{-1}\|_{L^2[\zeta]} \le 1+M^2\cdot 2M\cdot M^2 \le 3M^5.$$

This proves the first estimate of (6-10).

2. Observe that

$$(\mathrm{Id}+DBCD)^{-1}-(\mathrm{Id}+DED)^{-1}=(\mathrm{Id}+DED)^{-1}\cdot D(E-BC)D\cdot(\mathrm{Id}+DBCD)^{-1}.$$

Because of the bounds proved in step 1 and of (c),

$$\|(\mathrm{Id}+DBCD)^{-1}-(\mathrm{Id}+DED)^{-1}\|_{L^2}\le 2M^2\epsilon. \tag{6-13}$$

We write

$$\begin{aligned} AD^2B\cdot(\mathrm{Id}+CD^2B)^{-1} &= AD^2B\cdot(\mathrm{Id}-CD(\mathrm{Id}+DBCD)^{-1}DB)\\ &= AD\cdot DB - AD\cdot DBCD(\mathrm{Id}+DBCD)^{-1}\cdot DB\\ &= AD\cdot(\mathrm{Id}-DBCD(\mathrm{Id}+DBCD)^{-1})\cdot DB\\ &= AD\cdot(\mathrm{Id}+DBCD)^{-1}\cdot DB.\end{aligned}$$

The operator norms of $AD : L^2\to L^2[\zeta]$ and $DB : L^2[\zeta]\to H^1$ are each bounded by M^2 because of (a). We deduce from (6-13) that

$$\|AD^2B\cdot(\mathrm{Id}+CD^2B)^{-1}-AD\cdot(\mathrm{Id}+DED)^{-1}\cdot DB\|_{L^2[\zeta]}\le 2M^6\epsilon.$$

This proves the second estimate of (6-13), hence completes the proof of the lemma. □

We would like to apply Lemma 6.3 with the choices

$$\begin{gathered} A \stackrel{\text{def}}{=} \delta^{1/2}\cdot\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}^\top e^{-i\mu\delta\langle\ell,x\rangle}\Pi^*\cdot\mathcal{U}_\delta(\not{D}(\mu)+z), \quad B \stackrel{\text{def}}{=} \frac{1}{\delta^{1/2}}\cdot\mathcal{U}_\delta^{-1}\Pi e^{i\mu\delta\langle\ell,x\rangle}\overline{\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}},\\ C \stackrel{\text{def}}{=} \delta^{1/2}\vartheta_\star\left(2(D_t\kappa)_\delta\cdot\begin{bmatrix}k'\cdot D_x\phi_1\\ k'\cdot D_x\phi_2\end{bmatrix}^\top+(\kappa_\delta^2-1)W\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}^\top\right)\cdot e^{-i\mu\delta\langle\ell,x\rangle}\Pi^*\cdot\,\mathcal{U}_\delta\boldsymbol{\sigma_3},\\ D=(\not{D}_+(\mu)^2-z^2)^{-1/2}=R_0(\mu,z)^{1/2}, \quad E=\mathscr{U}^0. \end{gathered}\tag{6-14}$$

These operators are manufactured so that

$$\mathcal{Q}_\delta(\mu,z)=\frac{1}{\delta}AD^2B, \quad \mathcal{K}_\delta(\mu,z)=CD^2B; \tag{6-15}$$

see the formula of Lemma 6.1. Recall that $\mathscr{U}^\delta$, $\mathscr{U}^0$ were defined in Lemma 6.2 and observe that $BC=\mathscr{U}^\delta\to E=\mathcal{U}^0$ (for the operator norm $H^1\to H^{-1}$). This provides the favorable setting needed for the use of the cyclicity argument (Lemma 6.3).

The definition of D requires some precision. Let $\varphi(\omega)=(\omega^2-z^2)^{-1/2}$, where the square root is holomorphic on $\mathbb{C}\setminus(-\infty,0]$. If $|z|<\sqrt{\vartheta_F^2+\mu^2\cdot\nu_F^2|\ell|^2}$ and

$$\omega\in\Sigma_{L^2}(\not{D}_+(\mu))=\mathbb{R}\setminus\big[-\sqrt{\vartheta_F^2+\mu^2\cdot\nu_F^2|\ell|^2},\sqrt{\vartheta_F^2+\mu^2\cdot\nu_F^2|\ell|^2}\big].$$

then $\mathrm{Re}(\omega^2 - z^2) > 0$. Hence φ is well-defined on the spectrum of $\not{D}_+(\mu)$. This allows to define $D = \varphi(\not{D}_+(\mu))$ using the spectral theorem.

Lemma 6.4. *Fix $\epsilon_1 > 0$, $\mu_\sharp \in \mathbb{R}$. There exists $\delta_0 > 0$ such that if*

$$\begin{gathered} \delta \in (0, \delta_0), \quad \mu \in (-\mu_\sharp, \mu_\sharp), \\ z^2 \in \mathbb{D}(0, \vartheta_F^2 + \mu^2 \cdot \nu_F^2 |\ell|^2), \quad \mathrm{dist}(\Sigma_{L^2}(\not{D}(\mu)^2), z^2) \geq \epsilon_1^2 \end{gathered} \tag{6-16}$$

then $(\mathrm{Id} + DED)^{-1}$ *and* $(\mathrm{Id} + CD^2B)^{-1}$ *are invertible on* $L^2[\zeta]$. *Moreover,*

$$\begin{aligned} AD^2B \cdot (\mathrm{Id} + CD^2B)^{-1} &= AD \cdot (\mathrm{Id} + DED)^{-1} \cdot DB + \mathscr{O}_{L^2[\zeta]}(\delta), \\ (\mathrm{Id} + CD^2B)^{-1} &= \mathscr{O}_{L^2[\zeta]}(1). \end{aligned} \tag{6-17}$$

Proof. Below we use L^2 and H^1 to denote $L^2(\mathbb{R}, \mathbb{C}^2)$ and $H^1(\mathbb{R}, \mathbb{C}^2)$. The equation (6-17) is a consequence of Lemma 6.3, assuming that the assumptions (a), (b) and (c) hold with a constant M independent of δ, μ, z satisfying (6-16).

1. We first verify (a). We observe that the only singular dependence of A, B, C and E is in δ. It arises only in the operators $\delta^{1/2}\mathcal{U}_\delta$ and $\delta^{-1/2}\mathcal{U}_\delta^{-1}$, which are both isometries on L^2. In addition,

$$\mathrm{dist}(\Sigma_{L^2}(\not{D}(\mu)^2), z^2) \geq \epsilon_1^2 \quad \Longrightarrow \quad \mathrm{dist}(\Sigma_{L^2}(\not{D}_+(\mu)^2), z^2) \geq \epsilon_1^2.$$

Therefore D is controlled by ϵ_1^{-2}, and (a) holds independently of δ, μ, z satisfying (6-16).

2. From the definition (6-14) of D, we know D is invertible. Therefore we can write

$$\mathrm{Id} + DED = D(D^{-2} + E)D.$$

Moreover, thanks to (6-9),

$$D^{-2} + E = \not{D}(\mu)^2 - z^2. \tag{6-18}$$

When z satisfies the condition of (6-16), the operator $\not{D}(\mu)^2 - z^2$ is invertible. This comes with the bound

$$\|(\not{D}(\mu)^2 - z^2)^{-1}\|_{L^2} \leq \frac{1}{\epsilon_1^2}.$$

This is independent of δ: (b) holds.

3. The operator D maps L^2 to H^1 and H^{-1} to L^2, with uniformly bounded norm in μ, z satisfying (6-16). Therefore (c) holds — possibly after shrinking δ_0 — if

$$\|BC - E\|_{H^1 \to H^{-1}} = O(\delta). \tag{6-19}$$

We observe that $BC = \mathscr{U}^\delta$ and recall that $E = \mathscr{U}^0$. Therefore (6-19) reduces to the quantitative estimate (6-6) proved in Lemma 6.2.

4. Because of steps 1, 2 and 3, we can apply Lemma 6.3. It yields Lemma 6.4. □

According to this lemma, when (6-16) holds, $\mathrm{Id} + \mathcal{K}_\delta(\mu, z)$ is invertible. Hence

$$\mathcal{Q}_\delta(\mu, z) \cdot (\mathrm{Id} + \mathcal{K}_\delta(\mu, z))^{-1}$$

is well-defined. Thanks to (6-15),

$$\begin{aligned}\mathcal{Q}_\delta(\mu,z)\cdot(\mathrm{Id}+\mathcal{K}_\delta(\mu,z))^{-1} &= \frac{1}{\delta}AD\cdot(\mathrm{Id}+DED)^{-1}\cdot DB+\mathscr{O}_{L^2[\zeta]}(1)\\ &= \frac{1}{\delta}AD\cdot D^{-1}(D^{-2}+E)^{-1}D^{-1}\cdot DB+\mathscr{O}_{L^2[\zeta]}(1)\\ &= \frac{1}{\delta}A\cdot(D^{-2}+E)^{-1}\cdot B+\mathscr{O}_{L^2[\zeta]}(1).\end{aligned}$$

We now plug in the formula (6-14) for A, B, C, D, E, and we use the relation (6-18). This yields

$$\begin{aligned}&\mathcal{Q}_\delta(\mu,z)\cdot(\mathrm{Id}+\mathcal{K}_\delta(\mu,z))^{-1}\\ &\quad=\frac{1}{\delta}\cdot\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}^\top e^{-i\mu\delta\langle\ell,x\rangle}\Pi^*\mathcal{U}_\delta\cdot(\not{D}(\mu)+z)\cdot(\not{D}(\mu)^2-z^2)^{-1}\cdot\mathcal{U}_\delta^{-1}\Pi e^{i\mu\delta\langle\ell,x\rangle}\overline{\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}}+\mathscr{O}_{L^2[\zeta]}(1)\\ &\quad=\frac{1}{\delta}\cdot\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}^\top e^{-i\mu\delta\langle\ell,x\rangle}\cdot\Pi^*\mathcal{U}_\delta\cdot(\not{D}(\mu)-z)^{-1}\cdot\mathcal{U}_\delta^{-1}\Pi\cdot e^{i\mu\delta\langle\ell,x\rangle}\overline{\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}}+\mathscr{O}_{L^2[\zeta]}(1).\end{aligned}\tag{6-20}$$

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. 1. Fix $\epsilon>0$ and $\mu_\sharp>0$. Fix $z\in\mathbb{C}$ satisfying

$$z\in\mathbb{D}\Big(0,\sqrt{\vartheta_F^2+\mu^2\cdot\nu_F^2|\ell|^2}-\frac{\epsilon}{3}\Big),\qquad \mathrm{dist}(\Sigma_{L^2}(\not{D}(\mu)^2),z^2)\geq\frac{\epsilon^2}{9}.\tag{6-21}$$

Note that this does not quite correspond to the assumptions of Theorem 3.2. Instead it is a stronger form of the assumptions of Lemma 6.4 with $\epsilon_1=\epsilon/3$. The equation (6-21) implies that $\mathrm{Id}+\mathcal{K}_\delta(\mu,z)$ is invertible. Apply Lemma 6.1 with

$$\theta=1-\frac{\epsilon}{3\sqrt{\vartheta_F^2+\mu^2\cdot\nu_F^2|\ell|^2}}.$$

It implies that

$$\mathrm{Id}+\mathscr{K}_\delta(\zeta,\lambda)=\mathrm{Id}+\mathcal{K}_\delta(\mu,z)+\mathscr{O}_{L^2[\zeta]}(\delta^{2/3}).$$

Hence — after possibly shrinking δ_0 — the operator $\mathrm{Id}+\mathscr{K}_\delta(\zeta,\lambda)$ is invertible. The inverses of $\mathrm{Id}+\mathscr{K}_\delta(\zeta,\lambda)$ and $\mathrm{Id}+\mathcal{K}_\delta(\mu,z)$ are related via

$$(\mathrm{Id}+\mathscr{K}_\delta(\zeta,\lambda))^{-1}=(\mathrm{Id}+\mathcal{K}_\delta(\mu,z))^{-1}+\mathscr{O}_{L^2[\zeta]}(\delta^{2/3}),$$

because $(\mathrm{Id}+\mathcal{K}_\delta(\mu,z))^{-1}=(\mathrm{Id}+CD^2B)^{-1}$ is uniformly bounded by Lemma 6.4. It follows that under (6-21), $\mathscr{P}_\delta[\zeta]-\lambda$ is invertible and

$$(\mathscr{P}_\delta[\zeta]-\lambda)^{-1}=\mathscr{Q}_\delta(\zeta,\lambda)\cdot(\mathrm{Id}+\mathscr{K}_\delta(\zeta,\lambda))^{-1}.$$

2. Observe that $\mathcal{Q}_\delta(\mu,z)=\mathscr{O}_{L^2[\zeta]}(\delta^{-1})$: this comes from the relation between $\mathscr{Q}_\delta(\zeta,\lambda)$ and $\mathcal{Q}_\delta(\mu,z)$ provided by Lemma 6.1. We deduce that $\mathscr{P}_\delta[\zeta]-\lambda$ is invertible and

$$(\mathscr{P}_\delta[\zeta]-\lambda)^{-1}=\mathcal{Q}_\delta(\mu,z)\cdot(\mathrm{Id}+\mathcal{K}_\delta(\mu,z))^{-1}+\mathscr{O}_{L^2[\zeta]}(\delta^{-1/3}).$$

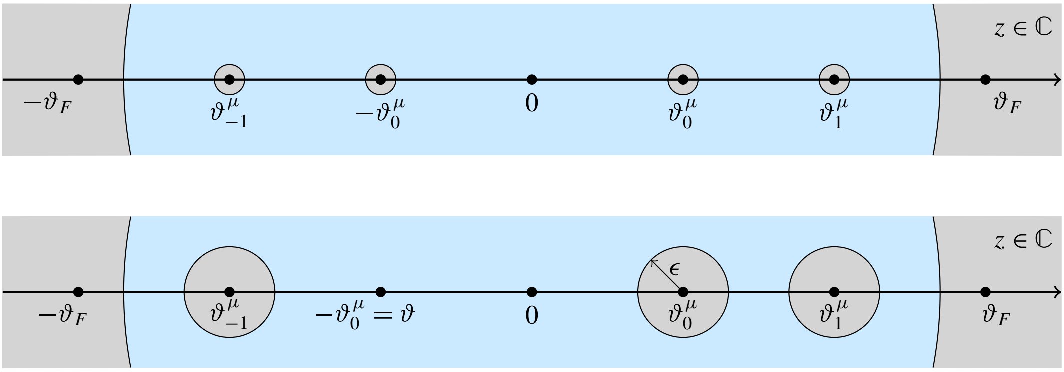

Figure 11. The top blue area represents the domain of validity of (6-22) provided by steps 1 and 2. The bottom blue area represents the domain of validity of (6-22) as specified by Theorem 3.2. In step 3 we prove that (6-22) holds near $\vartheta = -\vartheta_0^{\mu}$, at the price of increasing $\epsilon/3$ to ϵ.

Thanks to (6-20), this simplifies to

$$(\mathscr{P}_\delta[\zeta]-\lambda)^{-1} = \frac{1}{\delta}\cdot \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}^{\top} e^{-i\mu\delta\langle \ell,x\rangle}\,\Pi^*\mathcal{U}_\delta\cdot(\not{D}(\mu)-z)^{-1}\cdot\mathcal{U}_\delta^{-1}\,\Pi e^{i\mu\delta\langle\ell,x\rangle}\,\overline{\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}} + \mathscr{O}_{L^2[\zeta]}(\delta^{-1/3}). \quad (6\text{-}22)$$

See Figure 11.

3. The estimate (6-22) is valid as long as z satisfies (6-21). There is a subtlety here: (6-21) does not quite correspond to the assumption of Theorem 3.2. To conclude the proof, we must show that (6-21) is unnecessarily strong. In other words, we assume in these final steps that

$$z \in \mathbb{D}\big(0, \sqrt{\vartheta_F^2+\mu^2\cdot\nu_F^2|\ell|^2}-\epsilon\big), \qquad \operatorname{dist}(\Sigma_{L^2}(\not{D}(\mu)), z) \geq \epsilon, \qquad \operatorname{dist}(\Sigma_{L^2}(\not{D}(\mu)^2), z^2) < \frac{\epsilon^2}{9}.$$

The third condition implies

$$\operatorname{dist}(\Sigma_{L^2}(\not{D}(\mu)), z) < \frac{\epsilon}{3} \quad \text{or} \quad \operatorname{dist}(\Sigma_{L^2}(-\not{D}(\mu)), z) < \frac{\epsilon}{3}.$$

From the second condition, we deduce that $\operatorname{dist}(\Sigma_{L^2}(-\not{D}(\mu)), z) < \epsilon/3$. The spectra of $\not{D}(\mu)$ and $-\not{D}(\mu)$ differ by at most one eigenvalue:

$$\Sigma_{L^2}(-\not{D}(\mu)) \setminus \Sigma_{L^2}(\not{D}(\mu)) \subset \{\vartheta\}, \qquad \vartheta \overset{\text{def}}{=} -\mu\cdot\nu_F|\ell|\cdot\operatorname{sgn}(\vartheta_\star), \quad (6\text{-}23)$$

see Lemma 3.1. Hence, z must belong to $\mathbb{D}(\vartheta, \epsilon/3)$.

4. Because of step 3, the proof of Theorem 3.2 is complete if we can show that (6-22) holds when

$$z \in \mathbb{D}\big(0, \sqrt{\vartheta_F^2+\mu^2\cdot\nu_F^2|\ell|^2}-\epsilon\big), \quad \operatorname{dist}(\Sigma_{L^2}(\not{D}(\mu)), z) \geq \epsilon, \qquad z \in \mathbb{D}\Big(\vartheta, \frac{\epsilon}{3}\Big).$$

Fix $s \in \partial\mathbb{D}(\vartheta, \epsilon/3)$. Then, $|z-s| < 2\epsilon/3$. This implies that

$$s \in \mathbb{D}\Big(0, \sqrt{\vartheta_F^2+\mu^2\cdot\nu_F^2|\ell|^2}-\frac{\epsilon}{3}\Big), \quad \operatorname{dist}(\Sigma_{L^2}(\not{D}(\mu)), s) \geq \frac{\epsilon}{3}, \quad |\vartheta - s| = \frac{\epsilon}{3}.$$

Because of (6-23), s satisfies

$$\operatorname{dist}(\Sigma_{L^2}(\not{D}(\mu)), s) \geq \frac{\epsilon}{3}, \quad \operatorname{dist}(\Sigma_{L^2}(-\not{D}(\mu)), s) = \frac{\epsilon}{3} \quad \Longrightarrow \quad \operatorname{dist}(\Sigma_{L^2}(\not{D}(\mu)^2), s) \geq \frac{\epsilon^2}{9}.$$

In particular, s satisfies (6-21).

Therefore steps 1 and 2 apply to $s \in \partial\mathbb{D}(\vartheta, \epsilon/3)$. They yield

$$(\mathscr{P}_\delta[\zeta] - E_\star - \delta s)^{-1}$$
$$= \frac{1}{\delta} \cdot \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}^\top e^{-i\mu\delta\langle \ell, x\rangle} \Pi^* \mathcal{U}_\delta \cdot (\not{D}(\mu) - s)^{-1} \cdot \mathcal{U}_\delta^{-1} \Pi e^{i\mu\delta\langle \ell, x\rangle} \overline{\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}} + \mathscr{O}_{L^2[\zeta]}(\delta^{-1/3}). \quad (6\text{-}24)$$

Note that $(\not{D}(\mu) - s)^{-1}$ has no poles in the disk $\mathbb{D}(\vartheta, \epsilon/3)$: otherwise z could not be at distance at least ϵ from $\Sigma_{L^2}(\not{D}(\mu))$. Thus, integrating (6-24) over the circle $\partial\mathbb{D}(\vartheta, \epsilon/3)$,

$$\frac{1}{2\pi i} \oint_{\partial\mathbb{D}(\vartheta, \epsilon/3)} (\mathscr{P}_\delta[\zeta] - E_\star - \delta s)^{-1} \, ds = \mathscr{O}_{L^2[\zeta]}(\delta^{-1/3}). \quad (6\text{-}25)$$

We substitute $\lambda = E_\star + \delta s$ in (6-25) to get

$$\frac{1}{2\pi i} \oint_{\partial\mathbb{D}(E_\star + \delta\vartheta, \epsilon\delta/3)} (\mathscr{P}_\delta[\zeta] - \lambda)^{-1} \, d\lambda = \mathscr{O}_{L^2[\zeta]}(\delta^{2/3}). \quad (6\text{-}26)$$

Equation (6-26) implies that $(\mathscr{P}_\delta[\zeta] - \lambda)^{-1}$ cannot have a pole in $\mathbb{D}(E_\star + \vartheta\delta, \epsilon\delta/3)$. Indeed, since $\mathscr{P}_\delta[\zeta]$ is selfadjoint, the nonzero residues of its resolvent are nonzero projectors, and hence have $L^2[\zeta]$-operator norm at least equal to 1.

We deduce that $s \mapsto (\mathscr{P}_\delta[\zeta] - E_\star - \delta s)^{-1}$ is holomorphic in the disk $\mathbb{D}(\vartheta, \epsilon/3)$, and so is the leading term in (6-24). Their difference is bounded by $\mathscr{O}_{L^2[\zeta]}(\delta^{-1/3})$ on the boundary of the disk. By the maximum principle, this difference is $\mathscr{O}_{L^2[\zeta]}(\delta^{-1/3})$ also inside the disk. This shows that (6-24) holds when s is in the disk $\mathbb{D}(\vartheta, \epsilon/3)$. Equivalently (6-22) holds when $z \in \mathbb{D}(\vartheta, \epsilon/3)$. This completes the proof of Theorem 3.2. □

7. A topological perspective

7A. *The role of $\vartheta_\star^A$ and $\vartheta_\star^B$ in the spectral flow.* Assume that P_0 has Dirac points $(\xi_\star^A, E_\star)$ and $(\xi_\star^B, E_\star)$ — where $\xi_\star^A$ and $\xi_\star^B$ were defined in (1-4). Following Definition 1.2, these Dirac points are associated to Dirac eigenbases (ϕ_1^A, ϕ_2^A) and (ϕ_1^B, ϕ_2^B):

$$\phi_1^J \in L^2_{\xi_\star^J, \tau}, \ \phi_2^J \in L^2_{\xi_\star^J, \bar{\tau}}, \quad J = A, B, \quad \text{and} \quad \vartheta_\star^J = \langle \phi_1^J, W\phi_1^J \rangle_{L^2_{\xi_\star^J}}. \quad (7\text{-}1)$$

We recall that $\vartheta_\star^J$ does not depend on the choice of Dirac eigenbasis satisfying (7-1). The next result is a key identity — see also [Lee-Thorp et al. 2019, §7.1].

Lemma 7.1. *The identity $\vartheta_\star^A + \vartheta_\star^B = 0$ holds.*

Proof. 1. We claim that $\mathcal{I}\phi_1^A \in L^2_{\xi_\star^B, \tau}$. Thanks to (1-4),

$$-\xi_\star^A = -\tfrac{2\pi}{3}(2k_1 + k_2) = \tfrac{2\pi}{3}(k_1 + 2k_2) = \xi_\star^B \mod 2\pi\Lambda^*.$$

Because $\phi_1^A \in L^2_{\xi_\star^A,\tau}$,

$$(\mathcal{I}\phi_1^A)(x+w) = \phi_1^A(-x-w) = e^{-i\langle \xi_\star^A, w\rangle}(\mathcal{I}\phi_1^A)(x) = e^{i\langle \xi_\star^B, w\rangle}(\mathcal{I}\phi_1^A)(x),$$
$$(\mathcal{R}\mathcal{I}\phi_1^A)(x) = \phi_1^A(-Rx) = \tau\phi_1^A(-x) = \tau(\mathcal{I}\phi_1^A)(x).$$

It follows that $\mathcal{I}\phi_1^A \in L^2_{\xi_\star^B,\tau}$ — as claimed. The same calculation shows that $\mathcal{I}\phi_2^A \in L^2_{\xi_\star^B,\bar{\tau}}$.

The operator P_0 is $\mathcal{I}$-invariant. Thus, $\mathcal{I}\phi_1^A$ and $\mathcal{I}\phi_2^A$ form an orthonormal basis of $\ker_{L^2_{\xi_\star}}(P_0(\xi_\star^B) - E_\star)$, and $(\mathcal{I}\phi_1^A, \mathcal{I}\phi_2^A)$ is a Dirac eigenbasis for $(\xi_\star^B, E_\star)$.

2. Because W is odd and $\vartheta_\star^B$ does not depend on the choice of Dirac eigenbasis,

$$\vartheta_\star^B = \langle \mathcal{I}\phi_1^A, W\mathcal{I}\phi_1^A\rangle_{L^2_{\xi_\star^B}} = -\langle \phi_1^A, W\phi_1^A\rangle_{L^2_{\xi_\star^A}} = -\vartheta_\star^A. \qquad \square$$

Recall the assumption (H4): for every $\zeta \notin \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\} \bmod 2\pi\mathbb{Z}$ and $\tau, \tau' \in \mathbb{R}$,

$$\lambda_{0,j_\star}(\zeta k + \tau k') < \lambda_{0,j_\star+1}(\zeta k + \tau' k').$$

Lemma 7.2. *Assume (H1)–(H4) hold for both $\xi_\star^A$ and $\xi_\star^B$. There exists a function $E \in C^0(\mathbb{R}/(2\pi\mathbb{Z}), \mathbb{R})$ with $E(\zeta_\star^A) = E(\zeta_\star^B) = E_\star$ and such that*

$$\forall \zeta \in \mathbb{R}, \quad E(\zeta) \notin \Sigma_{L^2[\zeta],\mathrm{ess}}(\mathscr{P}_\delta[\zeta]).$$

Moreover, there exist $\mu_\flat > 0$ and $\delta_0 > 0$ such that if

$$\delta \in (0, \delta_0), \quad \zeta \in [0, 2\pi], \quad \left|\zeta - \tfrac{2\pi}{3}\right| \ge \mu_\flat\delta, \quad \left|\zeta - \tfrac{4\pi}{3}\right| \ge \mu_\flat\delta,$$

then the operator $\mathscr{P}_\delta[\zeta]$ has no spectrum in $[E(\zeta) - \delta, E(\zeta) + \delta]$.

Proof. 1. Set $r(\zeta) = \mathrm{dist}\left(\zeta, \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}\right)$. We first show that there exists $a > 0$ such that for $\zeta \in [0, 2\pi]$,

$$\inf_{\tau,\tau' \in \mathbb{R}} (\lambda_{0,j_\star+1}(\zeta k + \tau' k') - \lambda_{0,j_\star}(\zeta k + \tau k')) \ge 4a \cdot r(\zeta). \tag{7-2}$$

Otherwise, we can find $\zeta_n \in [0, 2\pi]$, $\tau_n, \tau_n' \in \mathbb{R}$, such that

$$\lambda_{0,j_\star+1}(\zeta_n k + \tau_n' k') - \lambda_{0,j_\star}(\zeta_n k + \tau_n k') \le \frac{r(\zeta_n)}{n} = \frac{1}{n} \cdot \mathrm{dist}\left(\zeta_n, \left\{\tfrac{2\pi}{3}, \tfrac{4\pi}{3}\right\}\right). \tag{7-3}$$

Using periodicity of the dispersion curves, we can assume that τ_n, τ_n' both live in $[0, 2\pi]$. In particular we can pass to converging subsequences: there exist ζ_∞, τ_∞ and τ_∞' with

$$\lambda_{0,j_\star}(\zeta_\infty k + \tau_\infty k') = \lambda_{0,j_\star+1}(\zeta_\infty k + \tau_\infty' k'). \tag{7-4}$$

Because of (H4), $\zeta_\infty \in \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\} = \{\zeta_\star^A, \zeta_\star^B\} \bmod 2\pi$. In the proof of Lemma 4.1, we showed that

$$\zeta_\star \in \left\{\tfrac{2\pi}{3}, \tfrac{4\pi}{3}\right\}, \quad \tau, \tau' \in \mathbb{R} \quad \Longrightarrow \quad \lambda_{0,j_\star}(\zeta_\star k + \tau k') \le E_\star, \quad \lambda_{0,j_\star+1}(\zeta_\infty k + \tau' k') \ge E_\star.$$

Thanks to (7-4), we deduce that $\lambda_{0,j_\star+1}(\zeta_\infty k + \tau_\infty' k') = E_\star = \lambda_{0,j_\star}(\zeta_\infty k + \tau_\infty k')$. The no-fold condition implies that $\zeta_\infty k + \tau_\infty k' = \zeta_\infty k + \tau_\infty' k' = \xi_\star$, where $\xi_\star \in \{\xi_\star^A, \xi_\star^B\}$ is a Dirac-point momentum. In particular,

$\zeta_n k + \tau_n' k'$ and $\zeta_n k + \tau_n k'$ both converge to $\xi_\star$. We deduce that for n sufficiently large,

$$\lambda_{0,j_\star+1}(\zeta_n k + \tau_n' k') - \lambda_{0,j_\star}(\zeta_n k + \tau_n k') \geq \nu_F |\zeta_n k + \tau_n' k' - \xi_\star| \geq \nu_F |k'| \cdot r(\zeta_n),$$

because $\langle \xi_\star, v\rangle \in \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$. This contradicts (7-3). We deduce that (7-2) holds for some $a > 0$. Without loss of generalities, we assume below that $a < \nu_F |\ell|$.

2. Define

$$E(\zeta) \stackrel{\text{def}}{=} 2a \cdot r(\zeta) + \sup_{\tau \in \mathbb{R}} \lambda_{0,j_\star}(\zeta k + \tau k').$$

This is a continuous, 2π-periodic function. Observe that for every $\xi \in \zeta k + \mathbb{R}k'$

$$\lambda_{0,j_\star}(\xi) \leq E(\zeta) - 2a \cdot r(\zeta) \leq E(\zeta) + 2a \cdot r(\zeta) \leq \lambda_{0,j_\star+1}(\xi). \tag{7-5}$$

Assume that $a \cdot r(\zeta) \geq \delta$ and that $\lambda \in [E(\zeta) - \delta, E(\zeta) + \delta]$. Since the dispersion surfaces are labeled in increasing order, we deduce that

$$\xi \in \zeta k + \mathbb{R}k' \quad \Longrightarrow \quad \operatorname{dist}(\Sigma_{L^2_\xi}(P_0(\xi)), \lambda) \geq a \cdot r(\zeta).$$

The reconstruction formula (5-2) and the spectral theorem yield

$$a \cdot r(\zeta) \geq \delta, \;\; \lambda \in [E(\zeta) - \delta, E(\zeta) + \delta] \quad \Longrightarrow \quad \|(P_0[\zeta] - \lambda)^{-1}\|_{L^2[\zeta]} \leq \frac{1}{a \cdot r(\zeta)}. \tag{7-6}$$

3. We now observe that under the assumptions of (7-6),

$$\mathscr{P}_\delta[\zeta] - \lambda = (P_0[\zeta] - \lambda) \cdot (\mathrm{Id} + \delta \cdot (P_0[\zeta] - \lambda)^{-1} \cdot \kappa_\delta W). \tag{7-7}$$

Because of (7-6) and since κ, W are in L^∞, there exist $\delta_0 > 0$ and $\mu_\flat > 0$ with

$$\delta \in (0, \delta_0), \;\; \zeta \in [0, 2\pi], \;\; r(\zeta) \geq \mu_\flat \delta \quad \Longrightarrow \quad \|\delta \cdot (P_0[\zeta] - \lambda)^{-1} \cdot \kappa_\delta W\|_{L^2[\zeta]} \leq \tfrac{1}{2}.$$

In particular, the second factor in the right-hand side of (7-7) is invertible via a Neumann series. We deduce that $\mathscr{P}_\delta[\zeta] - \lambda$ is invertible. This implies that $\mathscr{P}_\delta[\zeta]$ has no spectrum in $[E(\zeta) - \delta, E(\zeta) + \delta]$, as long as $r(\zeta) \geq \mu_\flat \delta$.

4. It remains to show that $E(\zeta)$ is not in the essential spectrum of $\mathscr{P}_\delta[\zeta]$, independently of ζ. Because of step 3, this holds for every ζ such that $r(\zeta) \geq \mu_\flat \delta$. Fix ζ such that $r(\zeta) < \mu_\flat \delta$. Let $\xi_\star$ be a Dirac point closest to $\zeta k + \mathbb{R}k'$: the distance between $\xi_\star$ and the line $\zeta k + \mathbb{R}k'$ is $r(\zeta)|\ell|$. Because of (7-5),

$$\lambda_{0,j_\star}(\zeta k + \tau k') + 2a \cdot r(\zeta) \leq E(\zeta) \leq \lambda_{0,j_\star+1}(\zeta k + \tau k') - 2a \cdot r(\zeta).$$

Since $\xi_\star$ is a Dirac point, we get

$$E_\star - (\nu_F|\ell| - 2a) \cdot r(\zeta) + O(r(\zeta)^2) \leq E(\zeta) \leq E_\star + (\nu_F|\ell| + 2a) \cdot r(\zeta) + O(r(\zeta)^2).$$

Hence, for δ sufficiently small,

$$E(\zeta) \in [E_\star - (\nu_F|\ell| - a) \cdot r(\zeta), E_\star + (\nu_F|\ell| - a) \cdot r(\zeta)].$$

Fix $\theta \in (0,1)$ such that $\nu_F|\ell| - a = \theta\nu_F|\ell|$; θ exists because $a \in (0, \nu_F|\ell|)$. Then

$$E(\zeta) \in \mathbb{D}\big(E_\star, \theta\sqrt{\vartheta_F^2\delta^2 + r(\zeta)^2 \cdot \nu_F^2|\ell|^2}\big).$$

Apply Theorem 5.1 with $\mu_\sharp > \mu_\flat$: for δ sufficiently small and $|\zeta - \zeta_\star| < \mu_\sharp\delta$, we have $E(\zeta) \notin \Sigma_{L^2[\zeta],\mathrm{ess}}(P_{\pm\delta}(\zeta))$. This implies that $E(\zeta)$ is not in the essential spectrum of $\mathscr{P}_\delta[\zeta]$ as long as $r(\zeta) < \mu_\flat\delta$, which concludes the proof. □

Lemma 7.2 allows us to define the spectral flow of the family $\zeta \mapsto \mathscr{P}_\delta[\zeta]$ as ζ runs from 0 to 2π: it is the signed number of eigenvalues of $\mathscr{P}_\delta[\zeta]$ that cross the curve $\zeta \mapsto E(\zeta)$ (with downwards crossings counted positively). Because $\mathscr{P}_\delta[\zeta]$ depends periodically on ζ, the spectral flow of $\mathscr{P}_\delta$ is a topological invariant. We refer to [Waterstraat 2017, §4] for an introduction to spectral flow. We are now ready to prove Corollary 1.6.

Proof of Corollary 1.6. We split $[0,2\pi]$ in three parts: $[0,2\pi] = I_A \cup I_B \cup I_0$ with

$$I_J \stackrel{\text{def}}{=} [\zeta_\star^J - \mu_\flat\delta, \zeta_\star^J + \mu_\flat\delta], \qquad J = A, B, \qquad I_0 \stackrel{\text{def}}{=} [0,2\pi] \setminus (I_A \cup I_B),$$

where we identified $\zeta_\star^J$ with their reduction modulo $2\pi\mathbb{Z}$. The spectral flow of $\zeta \in I_0 \mapsto \mathscr{P}_\delta[\zeta]$ through $E_\star$ vanishes because of Lemma 7.2.

In order to compute the spectral flow of $\zeta \in I_J \mapsto \mathscr{P}_\delta[\zeta]$ through $E_\star$, we fix $\mu_\sharp > \mu_\flat$, $\vartheta_\sharp > \vartheta_N$ and we apply Corollary 3.3. This result allows us to precisely count the number $N_\pm^J$ of eigenvalues of $\mathscr{P}_\delta[\zeta_\star^J \pm \mu_\flat\delta]$ in the set

$$\mathcal{E} \stackrel{\text{def}}{=} \big[E_\star - \delta\sqrt{\vartheta_\sharp^2 + \mu_\flat^2 \cdot \nu_F^2|\ell|^2}, E_\star\big]$$

in terms of the number of eigenvalues $2N+1$ of the Dirac operator $\not{D}(\mu)$. Thanks to Lemma 3.1, we find

$$N_-^J = N+1, \;\; N_+^J = N \quad \text{if } \vartheta_\star^J > 0, \qquad N_-^J = N, \;\; N_+^J = N+1 \quad \text{if } \vartheta_\star^J < 0.$$

In particular, the spectral flow of $\zeta \in I_J \mapsto \mathscr{P}_\delta[\zeta]$ through $E_\star$ is $N_+^J - N_-^J = -\operatorname{sgn}(\vartheta_\star^J)$ — see, e.g., [Waterstraat 2017, §4.1]. Since $\vartheta_\star^A$ and $\vartheta_\star^B$ have opposite sign, the spectral flow of the whole family $\zeta \in [0,2\pi] \mapsto \mathscr{P}_\delta[\zeta]$ vanishes. □

7B. *Magnetic perturbations of honeycomb Schrödinger operators.* Let V be a honeycomb potential and $\mathbb{A} \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ be Λ-periodic, odd and real-valued. Set

$$\mathbb{P}_\delta \stackrel{\text{def}}{=} (D_x + \delta \cdot \kappa_\delta \cdot \mathbb{A})^2 + V.$$

This operator is a nonlocal perturbation of $P_0 = -\Delta + V$, where $\delta \cdot \kappa_\delta \cdot \mathbb{A}$ plays the role of a perturbing magnetic potential. We introduce similarly to $\mathscr{P}_\delta[\zeta]$ the operator $\mathbb{P}_\delta[\zeta]$ formally equal to $\mathbb{P}_\delta$ but acting on $L^2[\zeta]$. We observe that

$$\begin{aligned}\mathbb{P}_\delta[\zeta] &= -\Delta + V + \delta \cdot \kappa_\delta \cdot (\mathbb{A}D_x + D_x\mathbb{A}) + \delta^2((k' \cdot D_x\kappa)_\delta + \kappa_\delta^2|\mathbb{A}|^2) \\ &= P_0 + \delta \cdot \kappa_\delta \cdot \mathbb{W} + \mathscr{O}_{L^2[\zeta]}(\delta^2) \quad \text{where } \mathbb{W} \stackrel{\text{def}}{=} \mathbb{A} \cdot D_x + D_x \cdot \mathbb{A}.\end{aligned} \tag{7-8}$$

We first state a simple analog of Lemma 2.3:

Lemma 7.3. *Let $(\xi_\star, E_\star)$ be a Dirac point of P_0 with Dirac eigenbasis (ϕ_1, ϕ_2) — see Definition 1.2. Then $\langle \phi_1, \mathbb{W}\phi_2\rangle_{L^2_{\xi_\star}} = \langle \phi_2, \mathbb{W}\phi_1\rangle_{L^2_{\xi_\star}} = 0$. Furthermore,*

$$\theta_\star \stackrel{\text{def}}{=} \langle \phi_1, \mathbb{W}\phi_1\rangle_{L^2_{\xi_\star}} = -\langle \phi_2, \mathbb{W}\phi_2\rangle_{L^2_{\xi_\star}}.$$

See Appendix A.1 or [Lee-Thorp et al. 2019, Proposition 5.1] for the proof. Below we state Corollary 7.4, which is the analog of Corollary 3.3 for the magnetic operator $\mathbb{P}_\delta[\zeta]$. We assume:

(H3′) The nondegeneracy condition $\theta_\star \neq 0$ holds.

When (H3′) holds and δ is small enough, the operator $\mathbb{P}_\delta[\zeta]$ has an essential spectral gap centered at $E_\star$ of width $\sim \delta$. Indeed (H3′) implies that $P_0 + \delta \cdot \kappa_\delta \cdot \mathbb{W}$ admits such a gap — as for $\mathscr{P}_\delta$ when (H3) is satisfied. This gap can only be moved by $O(\delta^2)$ under the perturbation $\mathscr{O}_{L^2[\zeta]}(\delta^2)$ of (7-8). We introduce the operator

$$\not{\mathbb{D}}(\mu) \stackrel{\text{def}}{=} \begin{bmatrix} 0 & \nu_\star k' \\ \overline{\nu_\star k'} & 0 \end{bmatrix} D_t + \mu \begin{bmatrix} 0 & \nu_\star \ell \\ \overline{\nu_\star \ell} & 0 \end{bmatrix} + \theta_\star \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \kappa.$$

We denote by $\{\theta_j^\mu\}_{j=-n}^n$ its eigenvalues. They are all simple — see Lemma 3.1 — and lie in $(-\theta_F, \theta_F)$, where $\theta_F = |\theta_\star|$.

Corollary 7.4. *Assume that (H1), (H2) and (H3′) hold and fix $\theta_\sharp \in (\theta_N, \theta_F)$ and $\mu_\sharp > 0$. There exists $\delta_0 > 0$ such that for*

$$\delta \in (0, \delta_0), \qquad \mu \in (-\mu_\sharp, \mu_\sharp), \qquad \zeta = \zeta_\star + \delta\mu,$$

the operator $\mathbb{P}_\delta[\zeta]$ has exactly $2n+1$ eigenvalues $\{\lambda^\zeta_{\delta,j}\}_{j\in[-n,n]}$ in

$$\Big[E_\star - \delta\sqrt{\theta_\sharp^2 + \mu^2 \cdot \nu_F^2|\ell|^2},\ E_\star + \delta\sqrt{\theta_\sharp^2 + \mu^2 \cdot \nu_F^2|\ell|^2}\,\Big].$$

These eigenvalues are simple. Furthermore, for each $j \in [-N, N]$, the eigenpairs $(\lambda^\zeta_{\delta,j}, v^\zeta_{\delta,j})$ admit full expansions in powers of δ:

$$\begin{aligned} \lambda^\zeta_{\delta,j} &= E_\star + \theta_j^\mu \cdot \delta + b_2^\mu \cdot \delta^2 + \cdots + b_M^\mu \cdot \delta^M + O(\delta^{M+1}), \\ v^\zeta_{\delta,j}(x) &= e^{i(\zeta-\zeta_\star)\langle \ell, x\rangle}\big(g_0^\mu(x, \delta\langle k', x\rangle) + \cdots + \delta^M \cdot g_M^\mu(x, \delta\langle k', x\rangle)\big) + o_{H^k}(\delta^M). \end{aligned}$$

In the above expansions:

- *M and k are any integers; H^k is the k-th order Sobolev space.*
- *θ_j^μ is the j-th eigenvalue of $\not{\mathbb{D}}(\mu)$.*
- *The terms $b_m^\mu \in \mathbb{R}$, $g_m^\mu \in X$ are recursively constructed via multiscale analysis.*
- *The leading-order term g_0^μ satisfies*

$$g_0^\mu(x,t) = \beta_1^\mu(t)\phi_1(x) + \beta_2^\mu(t)\phi_2(x), \qquad (\not{\mathbb{D}}(\mu) - \theta_j^\mu)\begin{bmatrix} \beta_1^\mu \\ \beta_2^\mu \end{bmatrix} = 0.$$

The proof is identical to that of Theorem 3.2 and Corollary 3.3; we do not reproduce it here. Let $\theta_\star^J$ be the coefficient $\theta_\star$ associated to the Dirac point $(\xi_\star^J, E_\star)$. The main difference between $\mathscr{P}_\delta[\zeta]$ and $\mathbb{P}_\delta[\zeta]$ lies in the next identity — see also [Lee-Thorp et al. 2019, §7.1].

Lemma 7.5. *The identity $\theta_\star^A = \theta_\star^B$ holds.*

Proof. Because of step 1 in the proof of Lemma 7.1, $(\mathcal{I}\phi_1^A, \mathcal{I}\phi_2^A)$ is a Dirac eigenbasis for $(\xi_\star^B, E_\star)$. Since $\theta_\star^B$ does not depend on the choice of Dirac eigenbasis and $\mathbb{W}$ commutes with $\mathcal{I}$,

$$\theta_\star^B = \langle \mathcal{I}\phi_1^A, \mathbb{W}\mathcal{I}\phi_1^A \rangle_{L^2_{\xi_\star^B}} = \langle \phi_1^A, \mathbb{W}\phi_1^A \rangle_{L^2_{\xi_\star^A}} = \theta_\star^A. \qquad \square$$

Corollary 1.7 has the same proof as Corollary 1.6. We find that the spectral flow of $\mathbb{P}_\delta$ in the $j_\star$-th gap as ζ runs from 0 to 2π is equal to

$$-\operatorname{sgn}(\theta_\star^A) - \operatorname{sgn}(\theta_\star^B) = -2 \cdot \operatorname{sgn}(\theta_\star).$$

Appendix

A.1. *Proofs of some identities.* We prove the identities relating the Dirac eigenbasis and W. Similar proofs arise in [Fefferman et al. 2016b; 2017; Lee-Thorp et al. 2019].

Proof of Lemma 2.2. Below we use $\langle \cdot\,, \cdot \rangle$ instead of $\langle \cdot\,, \cdot \rangle_{L^2_{\xi_\star}}$ to simplify notations.

1. We first analyze the (2-vector) $\langle \phi_1, D_x\phi_1 \rangle$. We observe that $\langle \phi_1, D_x\phi_1 \rangle \in \mathbb{R}^2$ because D_x is selfadjoint. Since $\phi_1 \in L^2_{\xi_\star,\tau}$,

$$\langle \phi_1, D_x\phi_1 \rangle = \langle \mathcal{R}\phi_1, \mathcal{R}D_x\phi_1 \rangle = \langle \tau\phi_1, \mathcal{R}D_x\mathcal{R}^{-1} \cdot \tau\phi_1 \rangle = \langle \phi_1, (\mathcal{R}D_x\mathcal{R}^{-1}) \cdot \phi_1 \rangle.$$

As $\mathcal{R}D_x\mathcal{R}^{-1} = R^{-1}D_x$, we conclude that $\langle \phi_1, D_x\phi_1 \rangle$ is either 0 or an eigenvector of R. Since the latter cannot be real, we conclude $\langle \phi_1, D_x\phi_1 \rangle = 0$. The same argument applies to $\langle \phi_2, D_x\phi_2 \rangle$.

2. We now analyze $\langle \phi_1, D_x\phi_2 \rangle$. Since $\phi_1 \in L^2_{\xi_\star,\tau}$ and $\phi_2 \in L^2_{\xi_\star,\bar\tau}$

$$\langle \phi_1, D_x\phi_2 \rangle = \langle \mathcal{R}\phi_1, \mathcal{R}D_x\phi_2 \rangle = \langle \tau\phi_1, \mathcal{R}D_x\mathcal{R}^{-1} \cdot \bar\tau\phi_2 \rangle = \bar\tau^2 \langle \phi_1, (\mathcal{R}D_x\mathcal{R}^{-1}) \cdot \phi_2 \rangle.$$

As $\mathcal{R}D_x\mathcal{R}^{-1} = R^{-1}D_x$ and $\bar\tau^2 = \tau$, we deduce $R\langle \phi_1, D_x\phi_2 \rangle = \tau\langle \phi_1, D_x\phi_2 \rangle$. This yields $\langle \phi_1, D_x\phi_2 \rangle \in \ker_{\mathbb{C}^2}(R-\tau)$. This eigenspace is $\mathbb{C} \cdot [1, i]^\top$; thus there exists $\nu_\star \in \mathbb{C}$ with

$$2\langle \phi_1, D_x\phi_2 \rangle = \nu_\star \cdot \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

If we identify the point $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$ with $\eta_1 + i\eta_2 \in \mathbb{C}$, then

$$2\langle \phi_1, (\eta \cdot D_x)\phi_2 \rangle = 2\langle \phi_1, (\eta_1 D_{x_1} + \eta_2 D_{x_2})\phi_2 \rangle = \nu_\star\eta_1 + i\nu_\star\eta_2 = \nu_\star\eta.$$

Above $\nu_\star\eta$ denotes the multiplication of $\nu_\star$ with $\eta = \eta_1 + i\eta_2$. Taking the complex conjugate of this identity and observing that $\eta \cdot D_x$ is a selfadjoint operator, we get

$$2\langle \phi_2, (\eta \cdot D_x)\phi_1 \rangle = \overline{\nu_\star\eta}.$$

3. It remains to show that $|\nu_\star| = \nu_F$. Fix $\eta \in \mathbb{R}^2$ with $|\eta| = 1$. Because of perturbation theory of eigenvalues, the operator $P_0(\xi_\star + t\eta)$ has precisely two eigenvalues near $E_\star$ when t is sufficiently small—

see [Kato 1980, §VII1.3, Theorem 1.8]. Because $(\xi_\star, E_\star)$ is a Dirac point of P_0, they are

$$E_\star \pm \nu_F t + O(t^2). \tag{A-1}$$

Let $\xi = \xi_\star + t\eta$. We want to construct approximate eigenvectors of $P_0(\xi)$. Let $a, b \in \mathbb{C}^2$, $\mu \in \mathbb{R}$, and $v \in H^2_{\xi_\star}$, with $v = O_{H^2_{\xi_\star}}(1)$ uniformly in t. Then

$$\begin{aligned} e^{-it\langle\eta,x\rangle}(P_0 - E_\star + \mu t)e^{it\langle\eta,x\rangle} \cdot (a\phi_1 + b\phi_2 + tv) \\ &= ((D_x + t\eta)^2 + V - E_\star + \mu t)(a\phi_1 + b\phi_2 + tv) \\ &= t(P_0 - E_\star)v + t(2\eta \cdot D_x + \mu)(a\phi_1 + b\phi_2) + O_{L^2_{\xi_\star}}(t^2). \end{aligned} \tag{A-2}$$

We now construct v such that

$$(P_0 - E_\star)v + (2\eta \cdot D_x + \mu)(a\phi_1 + b\phi_2) = 0. \tag{A-3}$$

This equation admits a solution if and only if $(2\eta \cdot D_x + \mu)(a\phi_1 + b\phi_2)$ is orthogonal to ϕ_1 and ϕ_2. This solvability condition is equivalent to

$$\begin{cases} \langle\phi_1, (2\eta \cdot D_x + \mu)(a\phi_1 + b\phi_2)\rangle = 0, \\ \langle\phi_2, (2\eta \cdot D_x + \mu)(a\phi_1 + b\phi_2)\rangle = 0, \end{cases} \iff \begin{cases} \nu_\star\eta \cdot b + \mu a = 0, \\ \overline{\nu_\star\eta} \cdot b + \mu a = 0. \end{cases} \tag{A-4}$$

A nontrivial solution of (A-4) exists if and only if

$$\operatorname{Det}\begin{bmatrix} \mu & \nu_\star\eta \\ \overline{\nu_\star\eta} & \mu \end{bmatrix} = 0 \iff |\nu_\star\eta|^2 = |\nu_\star|^2 = \mu^2.$$

Therefore, when $\mu = |\nu_\star|$, we can construct $(a, b) \neq (0, 0)$ satisfying (A-4) for $\mu = \pm|\nu_\star|$. With this choice, (A-3) admits a solution v. It follows from (A-2) that

$$(P_0(\xi) - E_\star + |\nu_\star|t) \cdot e^{it\langle\eta,x\rangle}(a\phi_1 + b\phi_2 + tv) = O(t^2).$$

In other words, we constructed an $O(t^2)$-accurate quasimode for $P_0(\xi)$, with energy $E_\star + |\nu_\star|t$. A general principle — see, e.g., [Drouot et al. 2018, Lemma 3.1] — implies that $P_0(\xi)$ has an eigenvalue at $E_\star - |\nu_\star|t + O(t^2)$. Because of (A-1), this eigenvalue must be $E_\star - \nu_F t + O(t^2)$. This implies $|\nu_\star| = \nu_F$ and completes the proof. □

Proof of Lemma 2.3. Below we use $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_{L^2_{\xi_\star}}$ to simplify notation. We start by proving the first identity. Since $\mathcal{I}$ is an isometry and $\mathcal{I}\phi_2 = \bar\phi_1$,

$$\langle\phi_2, W\phi_1\rangle = \langle\mathcal{I}\phi_2, \mathcal{I}W\mathcal{I}\phi_1\rangle = -\langle\bar\phi_1, W\bar\phi_2\rangle = -\langle\phi_2, W\phi_1\rangle.$$

This implies $\langle\phi_2, W\phi_1\rangle = 0$. Using that W is real-valued, $\langle\phi_1, W\phi_2\rangle = 0$ as well. We prove now the second identity: for the same reasons as above,

$$\langle\phi_1, W\phi_1\rangle = \langle\mathcal{I}\phi_1, \mathcal{I}W\mathcal{I}\phi_1\rangle = -\langle\bar\phi_2, W\bar\phi_2\rangle = -\langle\phi_2, W\phi_2\rangle. \qquad \square$$

Proof of Lemma 7.3. Below we use $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_{L^2_{\xi_\star}}$ to simplify notations. We start by proving the first identity. Since $\mathcal{I}$ is an isometry from $L^2_{\xi^A_\star}$ to $L^2_{\xi^B_\star}$ and $\mathcal{I}\phi_2 = \bar\phi_1$, $\mathcal{I}\mathbb{W} = \mathbb{W}\mathcal{I}$,

$$\langle\phi_2, \mathbb{W}\phi_1\rangle = \langle\mathcal{I}\phi_2, \mathcal{I}\mathbb{W}\mathcal{I}\phi_1\rangle = \langle\bar\phi_1, \mathbb{W}\bar\phi_2\rangle.$$

Moreover, $\overline{\mathbb{W}} = -\mathbb{W}$ because $\mathbb{A}$ is real-valued and $D_x = (1/i)\nabla$. Therefore,

$$\langle \phi_2, \mathbb{W}\phi_1 \rangle = -\langle \bar{\phi}_1, \bar{\mathbb{W}}\phi_2 \rangle = -\langle \mathbb{W}\phi_2, \phi_1 \rangle = -\langle \phi_2, \mathbb{W}\phi_1 \rangle.$$

We used in the last equality the selfadjointness of $\mathbb{W}$. We deduce $\langle \phi_2, \mathbb{W}\phi_1 \rangle = 0$. Similarly, $\langle \phi_1, \mathbb{W}\phi_2 \rangle = 0$.

We prove now the second identity: for the same reasons as above,

$$\langle \phi_1, \mathbb{W}\phi_1 \rangle = \langle \mathcal{I}\phi_1, \mathcal{I}\mathbb{W}\phi_1 \rangle = -\langle \bar{\phi}_2, \mathbb{W}\bar{\phi}_2 \rangle = -\langle \bar{\phi}_2, \overline{\mathbb{W}\phi_2} \rangle = -\langle \phi_2, \mathbb{W}\phi_2 \rangle. \qquad \square$$

A.2. ***Spectrum of the Dirac operator.***

Proof of Lemma 3.1. 1. Introduce the matrices

$$\boldsymbol{m_1} = \frac{1}{\nu_F |k'|} \begin{bmatrix} 0 & \nu_\star k' \\ \overline{\nu_\star k'} & 0 \end{bmatrix}, \quad \boldsymbol{m_2} = \frac{1}{\nu_F |\ell|} \begin{bmatrix} 0 & \nu_\star \ell \\ \overline{\nu_\star \ell} & 0 \end{bmatrix}, \quad \boldsymbol{m_3} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Note that $\boldsymbol{m_j}^2 = \mathrm{Id}$. Moreover, the matrices $\boldsymbol{m_j}$ anticommute: $\boldsymbol{m_j m_k} + \boldsymbol{m_k m_j} = 0$ when $j \neq k$. Indeed, $\boldsymbol{m_1 m_2} + \boldsymbol{m_2 m_1}$ equals

$$\frac{1}{\nu_F |k'| \cdot \nu_F |\ell|} \begin{bmatrix} \nu_\star k' \cdot \overline{\nu_\star \ell} + \nu_\star \ell \cdot \overline{\nu_\star k'} & 0 \\ 0 & \overline{\nu_\star k'} \cdot \nu_\star \ell + \overline{\nu_\star \ell} \cdot \nu_\star k' \end{bmatrix} = \frac{2\,\mathrm{Re}(k'\bar{\ell})}{|\ell k'|} = 0,$$

because $\mathrm{Re}(k'\bar{\ell}) = \langle k', \ell \rangle = 0$. With this notation,

$$\not{D}(\mu) = \nu_F |k'| \boldsymbol{m_1} D_t + \mu \cdot \nu_F |\ell| \boldsymbol{m_2} + \vartheta_\star \boldsymbol{m_3} \kappa = \not{D}_\star + \mu \cdot \nu_F |\ell| \boldsymbol{m_2}.$$

2. The formula for the essential spectrum is derived by looking at those of the asymptotic operators:

$$\not{D}_\pm(\mu) \stackrel{\text{def}}{=} \nu_F |k'| \boldsymbol{m_1} D_t + \mu \cdot \nu_F |\ell| \boldsymbol{m_2} \pm \vartheta_\star \boldsymbol{m_3}.$$

These are Fourier multipliers. Their essential spectrum corresponds to the possible eigenvalues of their symbol as the Fourier parameter runs through $\mathbb{R}$. We find

$$\Sigma_{L^2,\mathrm{ess}}(\not{D}_\pm(\mu)) = \mathbb{R} \setminus \left(-\sqrt{\vartheta_F^2 + \mu^2 \cdot \nu_F^2 |\ell|^2}, \sqrt{\vartheta_F^2 + \mu^2 \cdot \nu_F^2 |\ell|^2}\right).$$

3. We start by studying the bifurcation of the zero mode of $\not{D}_\star = \not{D}(0)$. This mode satisfies the equation $\not{D}(0)u = 0$ or equivalently

$$(\nu_F |k'| \partial_t + \vartheta_\star i \boldsymbol{m_1 m_3} \kappa) u = 0.$$

The matrix $i\boldsymbol{m_1 m_3}$ has eigenvalues ± 1. Let u_0 be an eigenvector of $i\boldsymbol{m_1 m_3}$ associated with the eigenvalue $\mathrm{sgn}(\vartheta_\star)$ and set

$$u(t) = u_0 \cdot \exp\left(-\frac{\vartheta_F}{\nu_F |k'|} \int_0^t \kappa(s)\, ds\right).$$

A direct calculation shows that u is an eigenvector of $\not{D}(0)$.

We claim that $\boldsymbol{m_2} u_0 = \mathrm{sgn}(\vartheta_\star) u_0$. Since $i\boldsymbol{m_1 m_3} u_0 = \mathrm{sgn}(\vartheta_\star) u_0$,

$$i\boldsymbol{m_2 m_1 m_3} u_0 = \mathrm{sgn}(\vartheta_\star) \boldsymbol{m_2} u_0, \quad i\boldsymbol{m_2 m_1 m_3} = \frac{i}{|k'\ell|} \begin{bmatrix} \ell \bar{k}' & 0 \\ 0 & -\bar{\ell} k' \end{bmatrix}.$$

Recall that $\mathrm{Re}(\ell\bar{k}') = 0$ because ℓ and k' are orthogonal. Therefore $-\bar{\ell}k' = \ell\bar{k}'$, and we deduce that

$$\mathrm{sgn}(\vartheta_\star)\boldsymbol{m_2}u_0 = -\frac{i}{|k'\ell|}k'\bar{\ell}u_0 \quad \Longrightarrow \quad \boldsymbol{m_2}u_0 = \mathrm{sgn}(\mathrm{Im}(k'\bar{\ell}))\cdot\mathrm{sgn}(\vartheta_\star)u_0.$$

We recall that $k' = -a_2k_1 + a_1k_2$, $k = b_2k_1 - b_1k_2$, $a_2b_1 - b_2a_1 = 1$ — see Section 2E. Hence

$$\mathrm{Im}(k'\bar{\ell}) = \mathrm{Det}[k, k'] = (a_2b_1 - b_2a_1)\cdot\mathrm{Det}[k_1, k_2] = 1 > 0.$$

We deduce that $\boldsymbol{m_2}u_0 = \mathrm{sgn}(\vartheta_\star)u_0$ and $\boldsymbol{m_2}u = \mathrm{sgn}(\vartheta_\star)u$.

We recall that $\not{D}(\mu) = \not{D}_\star + \mu\cdot\nu_F|\ell|\boldsymbol{m_2}$, $\not{D}_\star u = 0$ and obtain

$$\not{D}(\mu)u = \mu\cdot\nu_F|\ell|\cdot\mathrm{sgn}(\vartheta_\star)u.$$

This shows that $\mu\cdot\nu_F|\ell|\cdot\mathrm{sgn}(\vartheta_\star)$ is an eigenvalue of $\not{D}(\mu)$.

4. Let $\vartheta_j > 0$ be an eigenvalue of $\not{D}_\star$. Since $\boldsymbol{m_2}\not{D}_\star = -\not{D}_\star\boldsymbol{m_2}$, we deduce that $-\vartheta_j$ is also eigenvalue of $\not{D}_\star$. The respective eigenvectors are denoted by f_+, f_- and are related via $\boldsymbol{m_2}f_+ = f_-$. We look for an eigenpair $(E, a_+f_+ + a_-f_-)$ of $\not{D}(\mu) = \not{D}_\star + \mu\cdot\nu_F|\ell|\boldsymbol{m_2}$: it suffices to solve the equation

$$\begin{aligned}(\not{D}_\star + \mu\nu_F|\ell|\boldsymbol{m_2})\sum_\pm a_\pm f_\pm = E\sum_\pm a_\pm f_\pm \quad &\Longleftrightarrow \quad \sum_\pm \pm\vartheta_j a_\pm f_\pm + \mu\cdot\nu_F|\ell|a_\pm f_\mp = E\sum_\pm a_\pm f_\pm \\ &\Longleftrightarrow \quad (\vartheta_j\boldsymbol{\sigma_3} + \mu\cdot\nu_F|\ell|\boldsymbol{\sigma_1})\begin{bmatrix}a_+\\a_-\end{bmatrix} = Ea.\end{aligned}$$

This is equivalent to (E, a) being an eigenpair of $\vartheta_j\boldsymbol{\sigma_3} + \mu\cdot\nu_F|\ell|\boldsymbol{\sigma_1}$. Thus we conclude that $E = \pm\sqrt{\vartheta_j^2 + \mu^2\cdot\nu_F^2|\ell|^2}$ are both eigenvalues of $\not{D}(\mu)$.

5. So far we only showed that the eigenvalues of $\not{D}_\star$ induce eigenvalues of $\not{D}(\mu)$. We must prove the converse statement. Without loss of generality, $\mu \neq 0$. We first deal with eigenvalues of $\not{D}(\mu)$ which *apparently* do not bifurcate from the zero mode of $\not{D}_\star$. That is, we assume first that (E, f) is an eigenpair of $\not{D}(\mu) = \not{D}_\star + \mu\cdot\nu_F|\ell|\boldsymbol{m_2}$, with $E \neq \mathrm{sgn}(\vartheta_\star)\cdot\nu_F|\ell|\mu$.

We first claim that f and $g = \boldsymbol{m_2}f$ are linearly independent. Otherwise, we would have $f = \boldsymbol{m_2}f$ or $f = -\boldsymbol{m_2}f$ because $\boldsymbol{m_2}^2 = \mathrm{Id}$. This would imply respectively in the first and second cases

$$\not{D}_\star f = (E - \mu\cdot\nu_F|\ell|)f \quad \text{or} \quad \not{D}_\star f = (E + \mu\cdot\nu_F|\ell|)f. \tag{A-5}$$

In particular, f is an eigenvector of $\not{D}_\star$ with $\boldsymbol{m_2}f$ and f colinear. Because of step 3 it must be a zero mode of $\not{D}_\star$. Because of step 2 we must have $\boldsymbol{m_2}f = \mathrm{sgn}(\vartheta_\star)f$. Going back to (A-5), $E = \mathrm{sgn}(\vartheta_\star)\cdot\nu_F|\ell|$, which contradicts our assumption.

We now look for an eigenpair of $\not{D}_\star$ in the form $(\vartheta_j, af + bg)$. We get the equation

$$\begin{aligned}\not{D}_\star(af + bg) = \vartheta_j(af + bg) \quad &\Longleftrightarrow \quad a(Ef - \mu\cdot\nu_F|\ell|g) + b(\mu\cdot\nu_F|\ell|f - Eg) = \vartheta_j(af + bg) \\ &\Longleftrightarrow \quad (E\boldsymbol{\sigma_1} + i\mu\cdot\nu_F|\ell|\boldsymbol{\sigma_2})\begin{bmatrix}a\\b\end{bmatrix} = \vartheta_j\begin{bmatrix}a\\b\end{bmatrix}.\end{aligned}$$

Hence, ϑ is an eigenvalue of $E\boldsymbol{\sigma_1} + i\mu\cdot\nu_F|\ell|\boldsymbol{\sigma_2}$; equivalently, $\vartheta_j = \pm\sqrt{E^2 - \mu^2\cdot\nu_F^2|\ell|^2}$.

6. To conclude we deal with the case of an eigenpair (E, f) of $\not{D}(\mu)$ with $E = \mu \cdot \nu_F|\ell| \cdot \operatorname{sgn}(\vartheta_\star)$ — i.e., when E seemingly bifurcates from the zero mode of $\not{D}_\star$.

We claim that f and $g = \boldsymbol{m_2} f$ are colinear. Otherwise, following the last part of step 5, we would be able to construct $[a, b]^\top$, an eigenvector of $\operatorname{sgn}(\vartheta_\star)\boldsymbol{\sigma_1} + i\boldsymbol{\sigma_2}$ such that $af + bg$ is an eigenvector of $\not{D}_\star$. The matrix $\operatorname{sgn}(\vartheta_\star)\boldsymbol{\sigma_1} + i\boldsymbol{\sigma_2}$ has only one eigenvector, which is either $[0, 1]^\top$ or $[1, 0]^\top$. Therefore either f or g — but not both — is an eigenvector of $\not{D}_\star$. This implies that f or g is a zero eigenvector of $\not{D}_\star$. In particular, f and $\boldsymbol{m_2} f$ (or g and $\boldsymbol{m_2} g$) are colinear — which is a contradiction.

It follows that $f = \boldsymbol{m_2} f$ or $f = -\boldsymbol{m_2} f$. If $\boldsymbol{m_2} f = \operatorname{sgn}(\vartheta_\star) f$, we are done. In the other case, we deduce the existence of an eigenpair $(f, 2\mu \cdot \nu_F|\ell| \cdot \operatorname{sgn}(\vartheta_\star))$ of $\not{D}_\star$. This would require f and $\boldsymbol{m_2} f$ to be colinear, which is impossible. This completes the proof of the converse statement.

7. The argument presented in steps 5 and 6 shows that the eigenvalues of $\not{D}(\mu)$ and $\not{D}_\star$ have the same multiplicity. Appendix C of [Drouot et al. 2018] shows that $\not{D}_\star$ has only simple eigenvalues. □

A.3. *A calculation.*

Proof of Lemma 6.1. 1. From Theorem 5.1, when (6-2) is satisfied,

$$(P_\delta[\zeta] - \lambda)^{-1} \pm (P_{-\delta}[\zeta] - \lambda)^{-1} = S_\delta(\mu, z) \pm S_{-\delta}(\mu, z) + \mathscr{O}_{L^2[\zeta]}(\delta^{-1/3}).$$

A calculation yields

$$S_\delta(\mu, z) \pm S_{-\delta}(\mu, z) = \frac{1}{\delta} \cdot \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}^\top e^{-i\mu\delta\langle \ell, x\rangle} \Pi^* \cdot \mathcal{U}_\delta \big((\not{D}_+(\mu) - z)^{-1} \pm (\not{D}_-(\mu) - z)^{-1} \big) \mathcal{U}_\delta^{-1} \cdot \Pi e^{i\mu\delta\langle \ell, x\rangle} \overline{\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}}.$$

We now compute the resolvent difference $(\not{D}_+(\mu) - z)^{-1} \pm (\not{D}_-(\mu) - z)^{-1}$. We have

$$(\not{D}_+(\mu) - z)^{-1} + (\not{D}_-(\mu) - z)^{-1} = 2 \begin{bmatrix} z & \nu_\star k' D_t + \mu \nu_\star \ell \\ \overline{\nu_\star k'} D_t + \mu \overline{\nu_\star \ell} & z \end{bmatrix} R_0(\mu, z),$$

$$(\not{D}_+(\mu) - z)^{-1} - (\not{D}_-(\mu) - z)^{-1} = 2\vartheta_\star \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R_0(\mu, z) = 2\vartheta_\star \boldsymbol{\sigma_3} R_0(\mu, z).$$

Above we recall that $R_0(\mu, z) = (\nu_F^2 |k'|^2 D_t^2 + \mu^2 \cdot \nu_F^2 |\ell|^2 + \vartheta_F^2 - z^2)^{-1}$. This implies that

$$\begin{aligned} &S_\delta(\mu, z) + S_{-\delta}(\mu, z) \\ &\quad = \frac{2}{\delta} \cdot \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}^\top e^{-i\mu\delta\langle \ell, x\rangle} \Pi^* \cdot \mathcal{U}_\delta \begin{bmatrix} z & \nu_\star k' D_t + \mu \nu_\star \ell \\ \overline{\nu_\star k'} D_t + \mu \overline{\nu_\star \ell} & z \end{bmatrix} R_0(\mu, z) \mathcal{U}_\delta^{-1} \cdot \Pi e^{i\mu\delta\langle \ell, x\rangle} \overline{\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}} \end{aligned}$$

and

$$S_\delta(\mu, z) - S_{-\delta}(\mu, z) = \frac{2}{\delta} \cdot \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}^\top e^{-i\mu\delta\langle \ell, x\rangle} \Pi^* \cdot \mathcal{U}_\delta \vartheta_\star \boldsymbol{\sigma_3} R_0(\mu, z) \mathcal{U}_\delta^{-1} \cdot \Pi e^{i\mu\delta\langle \ell, x\rangle} \overline{\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}}.$$

We similarly obtain

$$(k' \cdot D_x)(S_\delta(\mu, z) - S_{-\delta}(\mu, z)) = \frac{2}{\delta} \cdot \begin{bmatrix} (k' \cdot D_x)\phi_1 \\ (k' \cdot D_x)\phi_2 \end{bmatrix}^\top e^{-i\mu\delta\langle \ell, x\rangle} \Pi^* \cdot \mathcal{U}_\delta \vartheta_\star \boldsymbol{\sigma_3} R_0(\mu, z) \mathcal{U}_\delta^{-1} \cdot \Pi e^{i\mu\delta\langle \ell, x\rangle} \overline{\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}}.$$

2. From the definition of $\mathscr{K}_\delta[\zeta](z)$, we see that

$$\begin{aligned}\mathscr{K}_\delta[\zeta](z) &= \tfrac{1}{2}\big([-\Delta, \kappa_\delta] + \delta(\kappa_\delta^2 - 1)W\big)\big((P_\delta[\zeta]-\lambda)^{-1} - (P_{-\delta}[\zeta]-\lambda)^{-1}\big) \\ &= \tfrac{1}{2}\big(2(D_t\kappa)_\delta \cdot (k'\cdot D_x) + \delta(\kappa_\delta^2-1)W\big)\big(S_\delta(\mu,z) - S_{-\delta}(\mu,z)\big) + \mathscr{O}_{L^2[\zeta]}(\delta^{2/3}).\end{aligned}$$

Thanks to step 1, the leading-order term is

$$\mathcal{K}_\delta(\mu,z)$$
$$\stackrel{\text{def}}{=} \vartheta_\star\left(2(D_t\kappa)_\delta\cdot\begin{bmatrix}k'\cdot D_x\phi_1\\ k'\cdot D_x\phi_2\end{bmatrix}^\top + (\kappa_\delta^2-1)W\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}^\top\right)\cdot e^{-i\mu\delta\langle\ell,x\rangle}\Pi^*\cdot\ \mathcal{U}_\delta\boldsymbol{\sigma_3}R_0(\mu,z)\mathcal{U}_\delta^{-1}\cdot\Pi e^{i\mu\delta\langle\ell,x\rangle}\overline{\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}}.$$

3. Because of the definition (6-1) and Theorem 5.1,

$$\begin{aligned}\mathscr{Q}_\delta(\zeta,\lambda) &= \frac{1}{2}\cdot((P_\delta[\zeta]-\lambda)^{-1} + (P_{-\delta}[\zeta]-\lambda)^{-1}) + \frac{\kappa_\delta}{2}\cdot((P_\delta[\zeta]-\lambda)^{-1} - (P_{-\delta}[\zeta]-\lambda)^{-1}) \\ &= \frac{1}{2}(S_\delta(\mu,z)+S_{-\delta}(\mu,z)) + \frac{\kappa_\delta}{2}\cdot(S_\delta(\mu,z)-S_{-\delta}(\mu,z)) + \mathscr{O}_{L^2[\zeta]}(\delta^{-1/3}).\end{aligned}$$

Thanks to the first step, the leading-order term is

$$\begin{aligned}\mathcal{Q}_\delta(\mu,z) \stackrel{\text{def}}{=} \frac{1}{\delta}\cdot\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}^\top e^{-i\mu\delta\langle\ell,x\rangle}\Pi^*\cdot\mathcal{U}_\delta\cdot\begin{bmatrix} z & \nu_\star k'D_t + \mu\nu_\star\ell \\ \overline{\nu_\star k'}D_t + \mu\overline{\nu_\star\ell} & z\end{bmatrix} R_0(\mu,z)\cdot\mathcal{U}_\delta^{-1}\cdot\Pi e^{i\mu\delta\langle\ell,x\rangle}\overline{\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}} \\ + \kappa_\delta\cdot\frac{1}{\delta}\cdot\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}^\top e^{-i\mu\delta\langle\ell,x\rangle}\Pi^*\cdot\mathcal{U}_\delta\cdot\vartheta_\star\boldsymbol{\sigma_3}R_0(\mu,z)\cdot\mathcal{U}_\delta^{-1}\cdot\Pi e^{i\mu\delta\langle\ell,x\rangle}\overline{\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}}.\end{aligned}$$

A key identity is $\kappa_\delta\Pi^*\mathcal{U}_\delta = \Pi^*\mathcal{U}_\delta\kappa$. Therefore, we deduce that

$$\mathcal{Q}_\delta(\mu,z) = \frac{1}{\delta}\cdot\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}^\top e^{-i\mu\delta\langle\ell,x\rangle}\Pi^*\cdot\mathcal{U}_\delta\cdot\begin{bmatrix}\vartheta_\star\kappa + z & \nu_\star k'D_t + \mu\nu_\star\ell \\ \overline{\nu_\star k'}D_t + \mu\overline{\nu_\star\ell} & -\vartheta_\star\kappa + z\end{bmatrix} R_0(\mu,z)\cdot\mathcal{U}_\delta^{-1}\cdot\Pi e^{i\mu\delta\langle\ell,x\rangle}\overline{\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}}.$$

The operator

$$\not{D}(\mu) + z = \begin{bmatrix}\vartheta_\star\kappa + z & \nu_\star k'D_t + \mu\nu_\star\ell \\ \overline{\nu_\star k'}D_t + \mu\overline{\nu_\star\ell} & -\vartheta_\star\kappa + z\end{bmatrix}$$

emerges and we end up with

$$\mathcal{Q}_\delta(\mu,z) = \frac{1}{\delta}\cdot\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}^\top e^{-i\mu\delta\langle\ell,x\rangle}\Pi^*\cdot\mathcal{U}_\delta(\not{D}(\mu)+z)\cdot R_0(\mu,z)\mathcal{U}_\delta^{-1}\cdot\Pi e^{i\mu\delta\langle\ell,x\rangle}\overline{\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}}. \qquad \square$$

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References

[Ablowitz and Zhu 2012] M. J. Ablowitz and Y. Zhu, "Nonlinear waves in shallow honeycomb lattices", *SIAM J. Appl. Math.* **72**:1 (2012), 240–260. MR Zbl

[Ablowitz and Zhu 2013] M. J. Ablowitz and Y. Zhu, "Nonlinear wave packets in deformed honeycomb lattices", *SIAM J. Appl. Math.* **73**:6 (2013), 1959–1979. MR Zbl

[Ammari et al. 2018] H. Ammari, B. Fitzpatrick, H. Lee, E. O. Hiltunen, and S. Yu, "Honeycomb-lattice Minnaert bubbles", preprint, 2018. arXiv

[Ando et al. 1975] T. Ando, Y. Matsumoto, and Y. Uemura, "Theory of Hall effect in a two-dimensional electron system", *J. Phys. Soc. Japan* **39**:2 (1975), 279–288.

[Arbunich and Sparber 2018] J. Arbunich and C. Sparber, "Rigorous derivation of nonlinear Dirac equations for wave propagation in honeycomb structures", *J. Math. Phys.* **59**:1 (2018), art. id. 011509. MR Zbl

[Avila et al. 2013] J. C. Avila, H. Schulz-Baldes, and C. Villegas-Blas, "Topological invariants of edge states for periodic two-dimensional models", *Math. Phys. Anal. Geom.* **16**:2 (2013), 137–170. MR Zbl

[Bal 2017] G. Bal, "Topological protection of perturbed edge states", preprint, 2017. arXiv

[Bal 2018] G. Bal, "Continuous bulk and interface description of topological insulators", preprint, 2018. arXiv

[Becker and Zworski 2019] S. Becker and M. Zworski, "Magnetic oscillations in a model of graphene", *Comm. Math. Phys.* **367**:3 (2019), 941–989. MR

[Becker et al. 2018] S. Becker, R. Han, and S. Jitomirskaya, "Cantor spectrum of graphene in magnetic fields", preprint, 2018. arXiv

[Berkolaiko and Comech 2018] G. Berkolaiko and A. Comech, "Symmetry and Dirac points in graphene spectrum", *J. Spectr. Theory* **8**:3 (2018), 1099–1147. MR Zbl

[Borisov 2007] D. I. Borisov, "Some singular perturbations of periodic operators", *Teoret. Mat. Fiz.* **151**:2 (2007), 207–218. In Russian; translated in *Theor. Math. Phys.* **151**:2 (2007), 614–624. MR Zbl

[Borisov 2011] D. I. Borisov, "On the spectrum of a two-dimensional periodic operator with a small localized perturbation", *Izv. Ross. Akad. Nauk Ser. Mat.* **75**:3 (2011), 29–64. In Russian; translated in *Izv. Math.* **75**:3 (2011), 471–505. MR Zbl

[Borisov 2015] D. I. Borisov, "On the band spectrum of a Schrödinger operator in a periodic system of domains coupled by small windows", *Russ. J. Math. Phys.* **22**:2 (2015), 153–160. MR Zbl

[Borisov and Gadyl'shin 2006] D. I. Borisov and R. R. Gadyl'shin, "The spectrum of the Schrödinger operator with a rapidly oscillating compactly supported potential", *Teoret. Mat. Fiz.* **147**:1 (2006), 58–63. In Russian; translated in *Theor. Math. Phys.* **147**:1 (2006), 496–500. MR Zbl

[Borisov and Gadyl'shin 2008] D. I. Borisov and R. R. Gadyl'shin, "On the spectrum of a periodic operator with small localized perturbation", *Izv. Ross. Akad. Nauk Ser. Mat.* **72**:4 (2008), 37–66. In Russian; translated in *Izv. Math.* **72**:4 (2008), 659–688). MR Zbl

[Bourne and Rennie 2018] C. Bourne and A. Rennie, "Chern numbers, localisation and the bulk-edge correspondence for continuous models of topological phases", *Math. Phys. Anal. Geom.* **21**:3 (2018), art. id. 16. MR Zbl

[Braverman 2018] M. Braverman, "The spectral flow of a family of Toeplitz operators", preprint, 2018. arXiv

[Brendel et al. 2017] C. Brendel, V. Peano, O. Painter, and F. Marquardt, "Snowflake topological insulator for sound waves", preprint, 2017. arXiv

[Carles et al. 2004] R. Carles, P. A. Markowich, and C. Sparber, "Semiclassical asymptotics for weakly nonlinear Bloch waves", *J. Statist. Phys.* **117**:1-2 (2004), 343–375. MR Zbl

[Carlsson 1990] U. Carlsson, "An infinite number of wells in the semi-classical limit", *Asymptotic Anal.* **3**:3 (1990), 189–214. MR Zbl

[Colin de Verdière 1991] Y. Colin de Verdière, "Sur les singularités de van Hove génériques", pp. 99–110 in *Analyse globale et physique mathématique* (Lyon, 1989), Mém. Soc. Math. France (N.S.) **46**, Soc. Math. France, Paris, 1991. MR Zbl

[Cornean et al. 2015] H. D. Cornean, V. Iftimie, and R. Purice, "Peierls substitution and magnetic pseudo-differential calculus", preprint, 2015. arXiv

[Cornean et al. 2017a] H. D. Cornean, B. Helffer, and R. Purice, "Low lying spectral gaps induced by slowly varying magnetic fields", *J. Funct. Anal.* **273**:1 (2017), 206–282. MR Zbl

[Cornean et al. 2017b] H. D. Cornean, B. Helffer, and R. Purice, "Peierls' substitution for low lying spectral energy windows", preprint, 2017. arXiv

[Cycon et al. 1987] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*, Springer, 1987. MR Zbl

[De Nittis and Lein 2011] G. De Nittis and M. Lein, "Applications of magnetic ΨDO techniques to SAPT", *Rev. Math. Phys.* **23**:3 (2011), 233–260. MR Zbl

[De Nittis and Lein 2014] G. De Nittis and M. Lein, "Effective light dynamics in perturbed photonic crystals", *Comm. Math. Phys.* **332**:1 (2014), 221–260. MR Zbl

[Deift and Hempel 1986] P. A. Deift and R. Hempel, "On the existence of eigenvalues of the Schrödinger operator $H - \lambda W$ in a gap of $\sigma(H)$", *Comm. Math. Phys.* **103**:3 (1986), 461–490. MR Zbl

[Delplace et al. 2017] P. Delplace, J. B. Marston, and A. Venaille, "Topological origin of equatorial waves", *Science* **358**:6366 (2017), 1075–1077. MR Zbl

[Dimassi 2016] M. Dimassi, "Semi-classical asymptotics for the Schrödinger operator with oscillating decaying potential", *Canad. Math. Bull.* **59**:4 (2016), 734–747. MR Zbl

[Dimassi and Duong 2017] M. Dimassi and A. T. Duong, "Scattering and semi-classical asymptotics for periodic Schrödinger operators with oscillating decaying potential", *Math. J. Okayama Univ.* **59**:1 (2017), 149–174. MR Zbl

[Dohnal et al. 2009] T. Dohnal, M. Plum, and W. Reichel, "Localized modes of the linear periodic Schrödinger operator with a nonlocal perturbation", *SIAM J. Math. Anal.* **41**:5 (2009), 1967–1993. MR Zbl

[Drouot 2018a] A. Drouot, "Bound states for rapidly oscillatory Schrödinger operators in dimension 2", *SIAM J. Math. Anal.* **50**:2 (2018), 1471–1484. MR Zbl

[Drouot 2018b] A. Drouot, "The bulk-edge correspondence for continuous dislocated systems", preprint, 2018. arXiv

[Drouot 2018c] A. Drouot, "Resonances for random highly oscillatory potentials", *J. Math. Phys.* **59**:10 (2018), art. id. 101506. MR Zbl

[Drouot 2018d] A. Drouot, "Scattering resonances for highly oscillatory potentials", *Ann. Sci. Éc. Norm. Supér.* (4) **51**:4 (2018), 865–925. MR Zbl

[Drouot 2019] A. Drouot, "The bulk-edge correspondence for continuous honeycomb lattices", preprint, 2019. arXiv

[Drouot et al. 2018] A. Drouot, C. L. Fefferman, and M. I. Weinstein, "Defect modes for dislocated periodic media", preprint, 2018. arXiv

[Duchêne and Raymond 2018] V. Duchêne and N. Raymond, "Spectral asymptotics for the Schrödinger operator on the line with spreading and oscillating potentials", *Doc. Math.* **23** (2018), 599–636. MR Zbl

[Duchêne and Weinstein 2011] V. Duchêne and M. I. Weinstein, "Scattering, homogenization, and interface effects for oscillatory potentials with strong singularities", *Multiscale Model. Simul.* **9**:3 (2011), 1017–1063. MR Zbl

[Duchêne et al. 2014] V. Duchêne, I. Vukićević, and M. I. Weinstein, "Scattering and localization properties of highly oscillatory potentials", *Comm. Pure Appl. Math.* **67**:1 (2014), 83–128. MR Zbl

[Elgart et al. 2005] A. Elgart, G. M. Graf, and J. H. Schenker, "Equality of the bulk and edge Hall conductances in a mobility gap", *Comm. Math. Phys.* **259**:1 (2005), 185–221. MR Zbl

[Fefferman and Weinstein 2012] C. L. Fefferman and M. I. Weinstein, "Honeycomb lattice potentials and Dirac points", *J. Amer. Math. Soc.* **25**:4 (2012), 1169–1220. MR Zbl

[Fefferman and Weinstein 2014] C. L. Fefferman and M. I. Weinstein, "Wave packets in honeycomb structures and two-dimensional Dirac equations", *Comm. Math. Phys.* **326**:1 (2014), 251–286. MR Zbl

[Fefferman and Weinstein 2018] C. L. Fefferman and M. I. Weinstein, "Edge states of continuum Schrödinger operators for sharply terminated honeycomb structures", preprint, 2018. arXiv

[Fefferman et al. 2014] C. L. Fefferman, J. P. Lee-Thorp, and M. I. Weinstein, "Topologically protected states in one-dimensional continuous systems and Dirac points", *Proc. Natl. Acad. Sci. USA* **111**:24 (2014), 8759–8763. MR Zbl

[Fefferman et al. 2016a] C. L. Fefferman, J. P. Lee-Thorp, and M. I. Weinstein, "Bifurcations of edge states: topologically protected and non-protected in continuous 2D honeycomb structures", *2D Materials* **3**:1 (2016), art. id. 014008.

[Fefferman et al. 2016b] C. L. Fefferman, J. P. Lee-Thorp, and M. I. Weinstein, "Edge states in honeycomb structures", *Ann. PDE* **2**:2 (2016), art. id. 12. MR Zbl

[Fefferman et al. 2017] C. L. Fefferman, J. P. Lee-Thorp, and M. I. Weinstein, *Topologically protected states in one-dimensional systems*, Mem. Amer. Math. Soc. **1173**, Amer. Math. Soc., Providence, RI, 2017. MR Zbl

[Fefferman et al. 2018] C. L. Fefferman, J. P. Lee-Thorp, and M. I. Weinstein, "Honeycomb Schrödinger operators in the strong binding regime", *Comm. Pure Appl. Math.* **71**:6 (2018), 1178–1270. MR Zbl

[Figotin and Klein 1997] A. Figotin and A. Klein, "Localized classical waves created by defects", *J. Statist. Phys.* **86**:1-2 (1997), 165–177. MR Zbl

[Fu et al. 2007] L. Fu, C. L. Kane, and E. J. Mele, "Topological insulators in three dimensions", *Phys. Rev. Lett.* **98**:10 (2007), art. id. 106803.

[Fukui et al. 2012] T. Fukui, K. Shiozaki, T. Fujiwara, and S. Fujimoto, "Bulk-edge correspondence for Chern topological phases: a viewpoint from a generalized index theorem", *J. Phys. Soc. Japan* **81**:11 (2012), art. id. 114602.

[Gannot 2015] O. Gannot, "From quasimodes to resonances: exponentially decaying perturbations", *Pacific J. Math.* **277**:1 (2015), 77–97. MR Zbl

[Gérard and Sigal 1992] C. Gérard and I. M. Sigal, "Space-time picture of semiclassical resonances", *Comm. Math. Phys.* **145**:2 (1992), 281–328. MR Zbl

[Gérard et al. 1991] C. Gérard, A. Martinez, and J. Sjöstrand, "A mathematical approach to the effective Hamiltonian in perturbed periodic problems", *Comm. Math. Phys.* **142**:2 (1991), 217–244. MR

[Golowich and Weinstein 2005] S. E. Golowich and M. I. Weinstein, "Scattering resonances of microstructures and homogenization theory", *Multiscale Model. Simul.* **3**:3 (2005), 477–521. MR Zbl

[Graf and Porta 2013] G. M. Graf and M. Porta, "Bulk-edge correspondence for two-dimensional topological insulators", *Comm. Math. Phys.* **324**:3 (2013), 851–895. MR Zbl

[Graf and Shapiro 2018] G. M. Graf and J. Shapiro, "The bulk-edge correspondence for disordered chiral chains", *Comm. Math. Phys.* **363**:3 (2018), 829–846. MR Zbl

[Graf and Tauber 2018] G. M. Graf and C. Tauber, "Bulk-edge correspondence for two-dimensional Floquet topological insulators", *Ann. Henri Poincaré* **19**:3 (2018), 709–741. MR Zbl

[Grushin 2009] V. V. Grushin, "Application of the multiparameter theory of perturbations of Fredholm operators to Bloch functions", *Mat. Zametki* **86**:6 (2009), 819–828. MR Zbl

[Haldane and Raghu 2008] F. D. M. Haldane and S. Raghu, "Possible realization of directional optical waveguides in photonic crystals with broken time-reversal symmetry", *Phys. Rev. Lett.* **100**:1 (2008), art. id. 013904.

[Halperin 1982] B. I. Halperin, "Quantized Hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential", *Phys. Rev. B* **25**:4 (1982), 2185–2190.

[Harrell 1979] E. M. Harrell, "The band-structure of a one-dimensional, periodic system in a scaling limit", *Ann. Physics* **119**:2 (1979), 351–369. MR Zbl

[Hatsugai 1993] Y. Hatsugai, "Chern number and edge states in the integer quantum Hall effect", *Phys. Rev. Lett.* **71**:22 (1993), 3697–3700. MR Zbl

[Helffer and Sjöstrand 1984] B. Helffer and J. Sjöstrand, "Multiple wells in the semiclassical limit, I", *Comm. Partial Differential Equations* **9**:4 (1984), 337–408. MR Zbl

[Helffer and Sjöstrand 1985] B. Helffer and J. Sjöstrand, "Puits multiples en limite semi-classique, II: Interaction moléculaire, symétries, perturbation", *Ann. Inst. H. Poincaré Phys. Théor.* **42**:2 (1985), 127–212. MR Zbl

[Helffer and Sjöstrand 1987] B. Helffer and J. Sjöstrand, "Effet tunnel pour l'équation de Schrödinger avec champ magnétique", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **14**:4 (1987), 625–657. MR

[Hempel and Kohlmann 2011a] R. Hempel and M. Kohlmann, "Spectral properties of grain boundaries at small angles of rotation", *J. Spectr. Theory* **1**:2 (2011), 197–219. MR Zbl

[Hempel and Kohlmann 2011b] R. Hempel and M. Kohlmann, "A variational approach to dislocation problems for periodic Schrödinger operators", *J. Math. Anal. Appl.* **381**:1 (2011), 166–178. MR Zbl

[Hempel et al. 2015] R. Hempel, M. Kohlmann, M. Stautz, and J. Voigt, "Bound states for nano-tubes with a dislocation", *J. Math. Anal. Appl.* **431**:1 (2015), 202–227. MR Zbl

[Hoefer and Weinstein 2011] M. A. Hoefer and M. I. Weinstein, "Defect modes and homogenization of periodic Schrödinger operators", *SIAM J. Math. Anal.* **43**:2 (2011), 971–996. MR Zbl

[Hsieh et al. 2008] D. Hsieh, D. Qian, A. L. Wray, Y. Y. Xia, Y. Hor, R. Cava, and M. Z. Hasan, "A topological Dirac insulator in a quantum spin Hall phase", *Nature* **452** (2008), 970–974.

[Jotzu et al. 2014] G. Jotzu, M. Messer, R. Desbuquois, M. Lebrat, T. Uehlinger, D. Greif, and T. Esslinger, "Experimental realization of the topological Haldane model with ultracold fermions", *Nature* **515** (2014), 237–240.

[Kane and Mele 2005a] C. L. Kane and E. J. Mele, "Quantum spin Hall effect in graphene", *Phys. Rev. Lett.* **95**:22 (2005), art. id. 226801.

[Kane and Mele 2005b] C. L. Kane and E. J. Mele, "Z_2 topological order and the quantum spin Hall effect", *Phys. Rev. Lett.* **95**:14 (2005), art. id. 146802.

[Kato 1980] T. Kato, *Perturbation theory for linear operators*, Grundlehren der Mathematischen Wissenschaften **132**, Springer, 1980. Zbl

[Kellendonk and Schulz-Baldes 2004a] J. Kellendonk and H. Schulz-Baldes, "Boundary maps for C^*-crossed products with $\mathbb{R}$ with an application to the quantum Hall effect", *Comm. Math. Phys.* **249**:3 (2004), 611–637. MR Zbl

[Kellendonk and Schulz-Baldes 2004b] J. Kellendonk and H. Schulz-Baldes, "Quantization of edge currents for continuous magnetic operators", *J. Funct. Anal.* **209**:2 (2004), 388–413. MR Zbl

[Kellendonk et al. 2002] J. Kellendonk, T. Richter, and H. Schulz-Baldes, "Edge current channels and Chern numbers in the integer quantum Hall effect", *Rev. Math. Phys.* **14**:1 (2002), 87–119. MR Zbl

[Keller et al. 2018] R. T. Keller, J. L. Marzuola, B. Osting, and M. I. Weinstein, "Spectral band degeneracies of $\frac{\pi}{2}$-rotationally invariant periodic Schrödinger operators", *Multiscale Model. Simul.* **16**:4 (2018), 1684–1731. MR Zbl

[Khanikaev et al. 2007] A. B. Khanikaev, S. H. Mousavi, W.-K. Tse, M. Kargarian, A. H. MacDonald, and G. Shvets, "Topological insulators with inversion symmetry", *Phys. Rev. B* **76**:4 (2007), art. id. 045302.

[Kitaev 2009] A. Kitaev, "Periodic table for topological insulators and superconductors", pp. 22–30 in *Advances in theoretical physics: Landau Memorial Conference* (Chernogolokova, Russia, 2008), edited by V. Lebedev and M. Feigel'man, AIP Conference Proceedings **1134**, AIP, 2009. Zbl

[von Klitzing et al. 1980] K. von Klitzing, G. Dorda, and M. Pepper, "New method for high-accuracy determination of the fine-structure constant based on quantized Hall resistance", *Phys. Rev. Lett.* **45**:6 (1980), 494–497.

[Korotyaev 2000] E. Korotyaev, "Lattice dislocations in a 1-dimensional model", *Comm. Math. Phys.* **213**:2 (2000), 471–489. MR Zbl

[Kuchment and Post 2007] P. Kuchment and O. Post, "On the spectra of carbon nano-structures", *Comm. Math. Phys.* **275**:3 (2007), 805–826. MR Zbl

[Lee 2016] M. Lee, "Dirac cones for point scatterers on a honeycomb lattice", *SIAM J. Math. Anal.* **48**:2 (2016), 1459–1488. MR Zbl

[Lee-Thorp et al. 2019] J. P. Lee-Thorp, M. I. Weinstein, and Y. Zhu, "Elliptic operators with honeycomb symmetry: Dirac points, edge states and applications to photonic graphene", *Arch. Ration. Mech. Anal.* **232**:1 (2019), 1–63. MR Zbl

[Lu et al. 2018] J. Lu, A. B. Watson, and M. I. Weinstein, "Dirac operators and domain walls", preprint, 2018. arXiv

[Martinez 1987] A. Martinez, "Estimations de l'effet tunnel pour le double puits, I", *J. Math. Pures Appl.* (9) **66**:2 (1987), 195–215. MR Zbl

[Martinez 1988] A. Martinez, "Estimations de l'effet tunnel pour le double puits, II: États hautement excités", *Bull. Soc. Math. France* **116**:2 (1988), 199–229. MR Zbl

[Moore and Balents 2007] J. E. Moore and L. Balents, "Topological invariants of time-reversal-invariant band structures", *Phys. Rev. B* **75**:12 (2007), art. id. 121306.

[Nash et al. 2015] L. M. Nash, D. Kleckner, A. Read, V. Vitelli, A. M. Turner, and W. T. M. Irvine, "Topological mechanics of gyroscopic metamaterials", *Proc. Natl. Acad. Sci. USA* **112**:47 (2015), 14495–14500.

[Outassourt 1987] A. Outassourt, "Comportement semi-classique pour l'opérateur de Schrödinger à potentiel périodique", *J. Funct. Anal.* **72**:1 (1987), 65–93. MR Zbl

[Ozawa et al. 2018] T. Ozawa, H. M. Price, A. Amo, N. Goldman, M. Hafezi, L. Lu, M. Rechtsman, D. Schuster, J. Simon, O. Zilberberg, and I. Carusotto, "Topological photonics", preprint, 2018. arXiv

[Panati et al. 2003] G. Panati, H. Spohn, and S. Teufel, "Space-adiabatic perturbation theory", *Adv. Theor. Math. Phys.* **7**:1 (2003), 145–204. MR

[Parzygnat et al. 2010] A. Parzygnat, K. Lee, Y. Avniel, and S. Johnson, "Sufficient conditions for two-dimensional localization by arbitrarily weak defects in periodic potentials with band gaps", *Phys. Rev. B* **81**:15 (2010), art. id. 155324.

[Perrot et al. 2018] M. Perrot, P. Delplace, and A. Venaille, "Topological transition in stratified fluids", preprint, 2018. arXiv

[Post 2003] O. Post, "Eigenvalues in spectral gaps of a perturbed periodic manifold", *Math. Nachr.* **261/262** (2003), 141–162. MR Zbl

[Raghu and Haldane 2008] S. Raghu and F. D. M. Haldane, "Analogs of quantum-Hall-effect edge states in photonic crystals", *Phys. Rev. A* **78**:3 (2008), art. id. 033834.

[Rechtsman et al. 2013] M. C. Rechtsman, J. M. Zeuner, Y. Plotnik, Y. Lumer, D. Podolsky, F. Dreisow, S. Nolte, M. Segev, and A. Szameit, "Photonic Floquet topological insulators", *Nature* **496** (2013), 196–200.

[Reed and Simon 1978] M. Reed and B. Simon, *Methods of modern mathematical physics, IV: Analysis of operators*, Academic, New York, 1978. MR Zbl

[Roy 2009] R. Roy, "Topological phases and the quantum spin Hall effect in three dimensions", *Phys. Rev. B* **79**:19 (2009), art. id. 195322.

[Ryu et al. 2010] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, "Topological insulators and superconductors: tenfold way and dimensional hierarchy", *New J. Phys.* **12** (2010), art. id. 065010.

[Schockley 1939] W. Schockley, "On the surface states associated with a periodic potential", *Phys. Rev.* **317**:4 (1939), 317–323.

[Shapiro 2017] J. Shapiro, "The bulk-edge correspondence in three simple cases", preprint, 2017. arXiv

[Shapiro and Tauber 2018] J. Shapiro and C. Tauber, "Strongly disordered Floquet topological systems", preprint, 2018. arXiv

[Simon 1976] B. Simon, "The bound state of weakly coupled Schrödinger operators in one and two dimensions", *Ann. Physics* **97**:2 (1976), 279–288. MR Zbl

[Simon 1984] B. Simon, "Semiclassical analysis of low lying eigenvalues, III: Width of the ground state band in strongly coupled solids", *Ann. Physics* **158**:2 (1984), 415–420. MR Zbl

[Singha et al. 2011] A. Singha, M. Gibertini, B. Karmakar, S. Yuan, M. Polini, G. Vignale, M. I. Kastnelson, A. Pinczuk, L. N. Pfeiffer, K. W. West, and V. Pellegrini, "Two-dimensional Mott–Hubbard electrons in an artificial honeycomb lattice", *Science* **332**:6034 (2011), 1176–1179.

[Stefanov 1999] P. Stefanov, "Quasimodes and resonances: sharp lower bounds", *Duke Math. J.* **99**:1 (1999), 75–92. MR Zbl

[Stefanov 2000] P. Stefanov, "Resonances near the real axis imply existence of quasimodes", *C. R. Acad. Sci. Paris Sér. I Math.* **330**:2 (2000), 105–108. MR Zbl

[Stefanov and Vodev 1996] P. Stefanov and G. Vodev, "Neumann resonances in linear elasticity for an arbitrary body", *Comm. Math. Phys.* **176**:3 (1996), 645–659. MR Zbl

[Taarabt 2014] A. Taarabt, "Equality of bulk and edge Hall conductances for continuous magnetic random Schrödinger operators", preprint, 2014. arXiv

[Tamm 1932] I. Tamm, "Über eine mögliche Art der Elektronenbindung an Kristalloberflächen", *Phys. Z. Sowjetunion* **1** (1932), 733–746. Zbl

[Tang and Zworski 1998] S.-H. Tang and M. Zworski, "From quasimodes to resonances", *Math. Res. Lett.* **5**:3 (1998), 261–272. MR Zbl

[Thicke et al. 2018] K. Thicke, A. Watson, and J. Lu, "Computation of bound states of semi-infinite matrix Hamiltonians with applications to edge states of two-dimensional materials", preprint, 2018. arXiv

[Thouless et al. 1982] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, "Quantized Hall conductance in a two-dimensional periodic potential", *Phys. Rev. Lett.* **49**:6 (1982), 405–408.

[Wallace 1947] P. R. Wallace, "The band theory of graphite", *Phys. Rev.* **71**:9 (1947), 622–634. Zbl

[Wang et al. 2008] Z. Wang, Y. D. Chong, J. D. Joannopoulos, and M. Soljačić, "Reflection-free one-way edge modes in a gyromagnetic photonic crystal", *Phys. Rev. Lett.* **100**:1 (2008), art. id. 013905.

[Waterstraat 2017] N. Waterstraat, "Fredholm operators and spectral flow", *Rend. Semin. Mat. Univ. Politec. Torino* **75**:1 (2017), 7–51. MR

[Watson and Weinstein 2018] A. Watson and M. I. Weinstein, "Wavepackets in inhomogeneous periodic media: propagation through a one-dimensional band crossing", *Comm. Math. Phys.* **363**:2 (2018), 655–698. MR Zbl

[Watson et al. 2017] A. B. Watson, J. Lu, and M. I. Weinstein, "Wavepackets in inhomogeneous periodic media: effective particle-field dynamics and Berry curvature", *J. Math. Phys.* **58**:2 (2017), art. id. 021503. MR Zbl

[Yu et al. 2008] Z. Yu, G. Veronis, Z. Wang, and S. Fan, "One-way electromagnetic waveguide formed at the interface between a plasmonic metal under a static magnetic field and a photonic crystal", *Phys. Rev. Lett.* **100**:2 (2008), art. id. 023902.

[Zelenko 2016] L. Zelenko, "Virtual bound levels in a gap of the essential spectrum of the weakly perturbed periodic Schrödinger operator", *Integral Equations Operator Theory* **85**:3 (2016), 307–345. MR Zbl

[Zhang et al. 2009] H. Zhang, C.-X. Liu, X.-L. Qi, X. Dai, Z. Fang, and S.-C. Zhang, "Topological insulators in Bi_2Se_3, Bi_2Te_3 and Sb_2Te_3 with a single Dirac cone on the surface", *Nature Phys.* **5** (2009), 438–442.

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Cover image: The figure shows the outgoing scattered field produced by scattering a plane wave, coming from the northwest, off of the (stylized) letters P A A. The total field satisfies the homogeneous Dirichlet condition on the boundary of the letters. It is based on a numerical computation by Mike O'Neil of the Courant Institute.

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