

Essential normality — a unified approach in terms of local decompositions

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Dedicated to the memory of Ronald G. Douglas

ABSTRACT

In this paper, we define the asymptotic stable division property for submodules of $L_a^2(\mathbb{B}_n)$. We show that under a mild condition, a submodule with the asymptotic stable division property is p -essentially normal for all $p > n$. A new technique is developed to show that certain submodules have the asymptotic stable division property. This leads to a unified proof of most known results on essential normality of submodules as well as new results. In particular, we show that an ideal defines a p -essentially normal submodule of $L_a^2(\mathbb{B}_n)$, $\forall p > n$, if its associated primary ideals are powers of prime ideals whose zero loci satisfy standard regularity conditions near the sphere.

1. Introduction

Let \mathbb{B}_n be the open unit ball in \mathbb{C}^n . The Bergman space $L_a^2(\mathbb{B}_n)$ consists of all holomorphic functions f on \mathbb{B}_n such that

$$\|f\|^2 = \int_{\mathbb{B}_n} |f|^2 dv < \infty.$$

Here v denotes the normalized Lebesgue measure, that is, $v(\mathbb{B}_n) = 1$. For $i = 1, \dots, n$, the coordinate functions z_i acts on $L_a^2(\mathbb{B}_n)$ by multiplication:

$$M_{z_i} f = z_i f, \quad f \in L_a^2(\mathbb{B}_n).$$

The n -tuple of operators $(M_{z_1}, \dots, M_{z_n})$ are commuting and thus induces a Hilbert $\mathbb{C}[z_1, \dots, z_n]$ -module structure on $L_a^2(\mathbb{B}_n)$:

$$\mathbb{C}[z_1, \dots, z_n] \times L_a^2(\mathbb{B}_n) \rightarrow L_a^2(\mathbb{B}_n), \quad (p, f) \mapsto p(M_{z_1}, \dots, M_{z_n})f = pf.$$

For any $i, j = 1, \dots, n$, it is well known that the commutator $[M_{z_i}, M_{z_j}^*]$ belongs to the Schatten class \mathcal{C}_p , $\forall p > n$.

A closed subspace $\mathcal{P} \subset L_a^2(\mathbb{B}_n)$ that is invariant under M_{z_i} , $i = 1, \dots, n$, is called a (Hilbert) submodule of $L_a^2(\mathbb{B}_n)$. The commuting tuple (R_1, \dots, R_n) , where $R_i = M_{z_i}|_{\mathcal{P}}$, defines the module action on \mathcal{P} . Its orthogonal complement $\mathcal{Q} := \mathcal{P}^\perp$ is called a quotient module of $L_a^2(\mathbb{B}_n)$. The module action on \mathcal{Q} is defined by the tuple (S_1, \dots, S_n) , where $S_i = QM_{z_i}|_{\mathcal{Q}}$. Here, Q denotes the projection operator onto \mathcal{Q} . For $p \geq 1$, we say \mathcal{P} (\mathcal{Q}) is p -essentially normal if $[R_i, R_j^*] \in \mathcal{C}_p$ ($[S_i, S_j^*] \in \mathcal{C}_p$).

For an ideal I in $\mathbb{C}[z_1, \dots, z_n]$, let \mathcal{P}_I denote its closure in $L_a^2(\mathbb{B}_n)$. Then it is easy to see that \mathcal{P}_I is a submodule of $L_a^2(\mathbb{B}_n)$. Therefore, $\mathcal{Q}_I := \mathcal{P}_I^\perp$ is a quotient module.

Arveson–Douglas Conjecture: Suppose that I is a homogeneous ideal in $\mathbb{C}[z_1, \dots, z_n]$. Then the quotient module \mathcal{Q}_I is p -essentially normal for all $p > \dim_{\mathbb{C}} Z(I)$.

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REMARK 1.1. The Arveson–Douglas Conjecture was originally stated on the Drury–Arveson space H_n^2 . Later, it was shown that, for a homogeneous ideal I and any $p > n$, the p -essential normalities for the closures of I in the Drury–Arveson space, the Bergman space, and the Hardy space, are equivalent. Closures of non-homogeneous ideals and non-polynomial generated submodules are also considered. Submodules on other domains were also considered.

In this paper, we consider submodules in the Bergman module. We will consider p -essential normality for $p > n$. For $p > n$, the p -essential normality of a submodule \mathcal{P} is equivalent to the p -essential normality of its quotient module \mathcal{Q} .

The Arveson–Douglas Conjecture arises from Arveson’s study of row contractions in multivariable operator theory [1]–[5]. Later, Douglas [7] showed that, given an essentially normal quotient module \mathcal{Q}_I , the short exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow C^*(\{S_i\}, I) + \mathcal{K} \rightarrow C(X_I) \rightarrow 0$$

defines an element $[\mathcal{Q}_I]$ in the odd K-homology group $K_1(X_I)$ of a topological space X_I . One can show that $\mathbb{B}_n \cap Z(I) \cap \partial\mathbb{B}_n \subset X_I \subset \partial\mathbb{B}_n \cap Z(I)$. In the case I is homogeneous, $X_I = Z(I) \cap \partial\mathbb{B}_n$. The element $[\mathcal{Q}_I]$ carries geometric information of $Z(I)$. This gives a new kind of index theorem. Moreover, a positive result of the Arveson–Douglas Conjecture will lead to an analytic Grothendieck–Riemann–Roch theorem allowing singularities [9].

The existing results of the Arveson–Douglas Conjecture can be roughly categorized into three types. The first type contains results concerning varieties of dimension 1, or codimension 1. In [15], Kuo and Wang proved the cases of homogeneous ideals I when $n \leq 3$, or $\dim Z(I) \leq 1$, or I is principal. Douglas and Wang [10] showed that for a principal ideal I , not necessarily homogeneous, $\mathcal{P}_I \subset L_a^2(\mathbb{B}_n)$ is p -essentially normal for all $p > n$. Fang and Xia [13, 14] extended the results to polynomial-generated principal submodules of the Hardy space $H^2(\mathbb{B}_n)$, and, under additional assumptions, the Drury–Arveson space H_n^2 . Douglas, Guo, and Wang [8] showed that a principal submodule of $L_a^2(\Omega)$ generated by a function $h \in \text{Hol}(\overline{\Omega})$ is p -essentially normal for all $p > n$. Here, $\Omega \subset \mathbb{C}^n$ is any bounded strongly pseudoconvex domain with smooth boundary. The result was extended to the Hardy spaces $H^2(\Omega)$ by Wang and Xia [25].

The second type of results concern a geometric version of the Arveson–Douglas Conjecture. The results involve varieties with geometric conditions such as smoothness and transversality on $\partial\mathbb{B}_n$. Engliš and Eschmeier [12] showed that if a variety V is homogeneous and its only possible singular point is the origin, then the radical ideal I of all polynomials vanishing on V defines a p -essentially normal quotient module for any $p > \dim V$. Douglas, Tang, and Yu [9] showed that if I is radical and $Z(I)$ is a complete intersection space that is smooth on $\partial\mathbb{B}_n$, intersects transversely with $\partial\mathbb{B}_n$, then \mathcal{Q}_I is essentially normal. Douglas and Wang [11] showed that if I is a radical ideal and $Z(I)$ is smooth on $\partial\mathbb{B}_n$ and intersects transversely with $\partial\mathbb{B}_n$, then \mathcal{Q}_I is p -essentially normal for all $p > 2\dim Z(I)$. The result was then refined to all $p > \dim Z(I)$ by Wang and Xia [24].

The third type of results involves conditions that ensure decompositions of the submodules, or quotient modules into nice parts [17, 18, 21–23]. In particular, in [21], Shalit considered the stable division property of a submodule in H_n^2 and showed that a graded submodule with the stable division property is p -essentially normal for all $p > n$. An approximate stable division property was defined and studied in [22].

The aim of this paper is to provide a unified proof of most of the known Bergman-space results above. We define the asymptotic stable division property (Definition 3.1) and show that the asymptotic stable division property leads to essential normality. Our first main result is the following.

THEOREM 1.2 (Theorem 3.2). *Suppose that \mathcal{P} is a submodule of $L_a^2(\mathbb{B}_n)$ with the asymptotic stable division property. If the generating functions h_i are all defined in a neighborhood of $\overline{\mathbb{B}_n}$,*

and the controlling constants C_i, N_i , determined by h_i (as in Theorem 2.16), are bounded for all $i \in \Lambda$, then the submodule \mathcal{P} is p -essentially normal for all $p > n$. In particular, if the generating functions h_i are polynomials of bounded degrees, then \mathcal{P} is p -essentially normal for all $p > n$.

The proof of Theorem 3.2 involves an inequality of a new type (Theorem 2.16) that first appeared in [8]. Since principal submodules and graded submodules with the stable division property have the asymptotic stable division property trivially, Theorem 3.2 provides a unified proof for the two types of results immediately.

We will also introduce technical hypotheses (Hypothesis 1) that lead to the asymptotic stable division property (Theorem 3.4). Then we will prove our second main result.

THEOREM 1.3 (Theorem 4.1). *Suppose that I is an ideal in $\mathbb{C}[z_1, \dots, z_n]$ with primary decomposition $I = \cap_{j=1}^k I_j^{m_j}$, where I_j are prime ideals. Assume the following.*

- (1) *For each $j = 1, \dots, k$, $Z(I_j)$ has no singular points on $\partial\mathbb{B}_n$ and intersects $\partial\mathbb{B}_n$ transversely.*
- (2) *Any pair of the varieties $\{Z(I_j)\}$ does not intersect on $\partial\mathbb{B}_n$.*

Then the submodule \mathcal{P}_I has the asymptotic stable division property with generating elements $\{h_i\}$ being polynomials of uniformly bounded degrees. As a consequence, \mathcal{P}_I is p -essentially normal for all $p > n$.

In [9, 5.2], the authors mentioned a plan of studying non-radical ideals. Theorem 1.3 partially accomplishes this goal, with a different approach.

Theorem 4.1 shows that results of the second type also fit into this framework. The proof of Theorem 4.1 combines several techniques. First, we construct a covering that satisfies the bounded overlap condition for Bergman neighborhoods with large radius. The construction involves a radial-spherical decomposition method in [26]. Then we construct a decomposition formula for each covering set. The generating functions are modified from local canonical defining functions of the variety. Combining these techniques, we show that the ideals in Theorem 4.1 satisfy Hypothesis 1, and therefore are p -essentially normal for all $p > n$.

In Section 2, we provide some tools that will be used in this paper. In Section 3, we introduce the asymptotic stable division property, and give a proof of Theorem 3.2. In Sections 4–6, we prove Theorem 4.1. In the concluding remarks, we describe our future plans. In the Appendix, we prove some results involving algebraic sets. These results will be used mainly in Section 6.

2. Preliminaries

This section contains some basic tools that we are going to use in this paper. Besides the classic tools in the study of operators on the Bergman space, we will also use the theory of complex analytic sets substantively.

2.1. Arveson's lemma

The following lemma provides an approach to the Arveson–Douglas Conjecture.

LEMMA 2.1 [4]. *Suppose $\mathcal{P} \subseteq L_a^2(\mathbb{B}_n)$ is a submodule and \mathcal{Q} is the corresponding quotient module. Then for any $p > n$, the following are equivalent.*

- (1) \mathcal{P} is p -essentially normal.
- (2) \mathcal{Q} is p -essentially normal.

- (3) $[M_{z_i}, P] \in \mathcal{C}_{2p}$, $\forall i = 1, \dots, n$.
- (4) $[M_{z_i}, Q] \in \mathcal{C}_{2p}$, $\forall i = 1, \dots, n$.

Here P, Q are the projections onto \mathcal{P} and \mathcal{Q} , respectively.

Notice that

$$[M_{z_i}, Q]^* = QM_{z_i}^* - M_{z_i}^*Q = QM_{z_i}^* - QM_{z_i}^*Q = QM_{z_i}^*P.$$

Compared with the cross commutators $[M_{z_i}, M_{z_j}^*]$, the operators $QM_{z_i}^*P$ are easier to work with. We will use Lemma 2.1 in the proofs in Section 3.

2.2. Complex analytic sets

The definitions and results come from [6].

DEFINITION 2.2. Let Ω be a complex manifold. A set $A \subset \Omega$ is called a (complex) analytic subset of Ω if for each point $a \in \Omega$, there are a neighborhood U of a and functions f_1, \dots, f_N holomorphic in U such that

$$A \cap U = \{z \in U : f_1(z) = \dots = f_N(z) = 0\}.$$

A point $a \in A$ is called *regular* if there is a neighborhood V of a in Ω such that $A \cap V$ is a complex submanifold of Ω . The (complex) dimension $\dim_a A$ at a is naturally defined to be the (complex) dimension of the manifold $A \cap V$.

A point $a \in A$ is called a *singular point* of A if it is not regular. One can show that the set of regular points is dense in A . The dimension of A at a singular point is defined as

$$\dim_a A = \limsup_{\substack{z \rightarrow a \\ z \in \text{reg } A}} \dim_z A.$$

A is said to be of *pure dimension* p if $\dim_a A = p$, $\forall a \in A$.

In this paper, our main objects of study are algebraic sets in \mathbb{C}^n , that is, zero loci of polynomials in n -variables. By Hilbert's Nullstellensatz, there is a one-to-one correspondence between algebraic sets and radical polynomial ideals, and irreducible algebraic sets correspond to prime ideals. We will also consider powers of radical ideals. Thus it is convenient to use the language of holomorphic chains.

DEFINITION 2.3. A *holomorphic chain* on a complex manifold Ω is a formal, locally finite sum $\sum k_j A_j$, where A_j are pairwise distinct irreducible analytic subsets in Ω and $k_j \neq 0$ are integers.

For $f \in \text{Hol}(\Omega)$, we use the notation $f|_{\sum k_j A_j} = 0$ to indicate that f is, locally, a linear combination of functions of the form $\prod_j \prod_{i=1}^{k_j} f_{ij}$, where f_{ij} are holomorphic functions vanishing on A_j .

DEFINITION 2.4. A continuous map $f : X \rightarrow Y$ of topological spaces is called *proper* if the pre-image of every compact set $K \subseteq Y$ is a compact set in X . The spaces X and Y are assumed to be Hausdorff and locally compact.

Proper maps are important tools in the study of analytic sets. The following results will be used in the proofs.

THEOREM 2.5. *Let A be an analytic set in \mathbb{C}^n , $a \in A$, $\dim_a A = p$, $0 < p < n$, U a neighborhood of a , and $\pi : A \cap U \rightarrow U' \subset \mathbb{C}^p$, $z \mapsto z' := (z_1, \dots, z_p)$ a proper projection. Then there is an analytic subset $\sigma \subset U'$ of dimension less than p and a natural number k such that*

- (1) $\pi : A \cap U \setminus \pi^{-1}(\sigma) \rightarrow U' \setminus \sigma$ is a locally biholomorphic k -sheeted cover, in particular, $\#\pi^{-1}(z') \cap A \cap U = k$ for all $z' \in U' \setminus \sigma$;
- (2) $\pi^{-1}(\sigma)$ is nowhere dense in $A_{(p)} \cap U$. Here $A_{(p)} = \{z \in A : \dim_z A = p\}$.

In particular, if A is pure of dimension p , then $\pi^{-1}(\sigma)$ is nowhere dense in A . We say that π defines a k -sheeted analytic cover. The set σ is called the critical set of π .

A proper projection on a complex analytic set A gives rise to a set of canonical defining functions [6].

DEFINITION 2.6. (1) Let $a_1, \dots, a_k \subset \mathbb{C}^m$, not necessarily distinct. We compose the polynomial

$$P(z, w) = \langle z - a_1, w \rangle \cdots \langle z - a_k, w \rangle$$

in the variable $(z, w) \in \mathbb{C}^{2m}$. One can show that $P(z, w) \equiv^w 0$ if and only if z is one of the points a_1, \dots, a_k . Suppose

$$P(z, w) = \sum_{|\alpha|=k} P_\alpha(z) \bar{w}^\alpha,$$

where α denotes a multi-index $(\alpha_1, \dots, \alpha_m)$. Then the condition $P(z, w) \equiv^w 0$ is equivalent to that $P_\alpha(z) = 0$, $\forall \alpha$, $|\alpha| = k$. The polynomials $P_\alpha(z)$ are called the *canonical defining functions* for the system $a = \{a_1, \dots, a_k\}$.

(2) More generally, suppose that A is an analytic subset of $U = U' \times U'' \subset \mathbb{C}^n$, where $U' \subset \mathbb{C}^p$, $U'' \subset \mathbb{C}^m$ and let $\pi : (z', z'') \mapsto z' \in \mathbb{C}^p$. Suppose $\pi|_A : A \rightarrow U'$ is a k -sheeted analytic cover. Let $\sigma \subset \mathbb{C}^p$ be the critical set of $\pi|_A$. For each $z' \in U' \setminus \sigma$,

$$\pi^{-1}(z') \cap A \cap U = \{(z', a_1(z')), \dots, (z', a_k(z'))\},$$

where $a_i(z')$ are holomorphic functions defined in a small neighborhood of z' . Define

$$P_{\pi|_A}(z, w) = \langle z'' - a_1(z'), w \rangle \cdots \langle z'' - a_k(z'), w \rangle, \quad z \in U \setminus \pi^{-1}(\sigma), w \in \mathbb{C}^m$$

and

$$P_{\pi|_A}(z, w) = \sum_{|\alpha|=k} P_{\pi|_A, \alpha}(z) \bar{w}^\alpha.$$

Here, we write $\alpha = (\alpha_{p+1}, \dots, \alpha_n)$ to be consistent with the coordinates in \mathbb{C}^n . The coefficients of powers of z'' in the functions $P_{\pi|_A, \alpha}$ are locally bounded holomorphic functions on $U' \setminus \sigma$. Since $\dim \sigma < p$, they can be uniquely extended to holomorphic functions on U' . Therefore, $P_{\pi|_A, \alpha}$ extend to holomorphic functions on U (in fact, on $U' \times \mathbb{C}^m$). They are called the *canonical defining functions* for the projection π .

(3) One can also define canonical defining functions for holomorphic chains. We will only use the canonical defining functions for mA , where m is a positive integer. Then we set $P_{\pi|_{mA}}(z, w) = P_\pi(z, w)^m$ and $P_{\pi|_{mA}}(z, w) = \sum_{|\alpha|=mk} P_{\pi|_{mA}, \alpha}(z) \bar{w}^\alpha$. The functions $P_{\pi|_{mA}, \alpha}(z)$ will be the canonical defining functions for the holomorphic chain mA .

REMARK 2.7. We remark that in Definition 2.6, the functions $P_{\pi|_A}$ and $P_{\pi|_A, \alpha}$ are constructed under a specific choice of basis. Our estimates in this paper involve change of basis. It is convenient to generalize Definition 2.6 to the following ‘coordinate-free’ form. Suppose

$E = \{e_1, \dots, e_n\}$ is an orthonormal basis of \mathbb{C}^n and π is the orthonormal projection onto $\text{span}\{e_1, \dots, e_p\}$. Suppose that $\pi|_A$ is proper. Define

$$P_{\pi|_A, E}(z, w) = \Pi_{a \in \pi^{-1}\pi(z) \cap A} \langle z - a, \sum_{i=p+1}^n w_i e_i \rangle, \quad z \in \mathbb{C}^n, w \in \mathbb{C}^{n-p}.$$

Suppose that l is a unitary transformation on \mathbb{C}^n and $\pi|_{l(A)}$ is also proper. Let $E_l = \{l^{-1}e_1, \dots, l^{-1}e_n\}$. Then

$$\begin{aligned} P_{\pi|_l(A), E}(l(z), w) &= \Pi_{a \in \pi^{-1}\pi l(z) \cap l(A)} \left\langle l(z) - a, \sum_{i=p+1}^n w_i e_i \right\rangle \\ &= \Pi_{b \in l^{-1}\pi^{-1}\pi l(z) \cap A} \left\langle z - b, \sum_{i=p+1}^n w_i l^{-1}(e_i) \right\rangle. \end{aligned}$$

In other words,

$$P_{\pi|_l(A), E}(l(z), w) = P_{l^{-1}\pi l|_A, E_l}(z, w).$$

We will use this fact in the proof of Lemma 6.1. In the subsequent discussions, we will omit the subscript E where no confusion is caused.

2.3. Möbius transform and Bergman metric

For $z \in \mathbb{B}_n$, $z \neq 0$, let P_z and Q_z be the orthogonal projections from \mathbb{C}^n to $\mathbb{C}z$ and z^\perp , respectively.

DEFINITION 2.8. The Möbius transform φ_z is defined by the formula

$$\varphi_z(w) = \frac{z - P_z(w) - (1 - |z|^2)^{1/2} Q_z(w)}{1 - \langle w, z \rangle}, \quad w \in \mathbb{B}_n.$$

The following lemma contains some basic properties of the Möbius transform φ_z . One can find a proof in [20, Chapter 2].

LEMMA 2.9. If $a, z, w \in \mathbb{B}_n$, then

(1)

$$1 - \langle \varphi_a(z), \varphi_a(w) \rangle = \frac{(1 - \langle a, a \rangle)(1 - \langle z, w \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)}.$$

(2) As a consequence of (1),

$$1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a \rangle|^2}.$$

(3) The Jacobian of the automorphism φ_z is

$$(J\varphi_z(w)) = \frac{(1 - |z|^2)^{n+1}}{|1 - \langle w, z \rangle|^{2(n+1)}}.$$

DEFINITION 2.10. The pseudo-hyperbolic metric ρ is defined by

$$\rho(z, w) = |\varphi_z(w)|, \quad z, w \in \mathbb{B}_n.$$

The *hyperbolic metric* β is defined by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}, \quad z, w \in \mathbb{B}_n.$$

β is also called the *Bergman metric* on \mathbb{B}_n . For $r > 0$ and $z \in \mathbb{B}_n$, denote

$$D(z, r) = \{w : \beta(z, w) < r\}.$$

The two metrics ρ and β define the same topology on \mathbb{B}_n . In the estimations, we will use whichever is more convenient. The following lemmas are straightforward to check. We omit the proofs.

LEMMA 2.11. *For $z, w \in \mathbb{B}_n$, we have*

- (1) $\beta(z, w) \in [-\frac{1}{2} \log(1 - \rho^2(z, w)), \log 2 - \frac{1}{2} \log(1 - \rho^2(z, w))]$;
- (2) $1 - \rho^2(z, w) \in [e^{-2\beta(z, w)}, 4e^{-2\beta(z, w)}]$.

LEMMA 2.12. *For $z, w \in \mathbb{B}_n$, the following hold.*

(1)

$$|1 - \langle z, w \rangle| > \frac{1}{2}(1 - |z|^2).$$

(2)

$$1 - |\varphi_z(w)|^2 \in \left(\frac{1}{4}(1 - |z|^2)(1 - |w|^2), 4 \frac{1 - |z|^2}{1 - |w|^2} \right).$$

2.4. Spherical distance

The following definitions and lemmas will be used in Section 5.

DEFINITION 2.13. Let $S = \partial \mathbb{B}_n$ be the unit sphere in \mathbb{C}^n . For $\zeta, \xi \in S$, the *spherical distance* $d(\zeta, \xi)$ is defined by

$$d(\zeta, \xi) = |1 - \langle \zeta, \xi \rangle|^{1/2}.$$

Then d defines a metric on S (cf. [20]). For $\delta > 0$, denote

$$Q(\zeta, \delta) = \{\xi \in S : d(\zeta, \xi) < \delta\}.$$

Let σ denote the normalized surface measure on S , that is, $\sigma(S) = 1$. For $z, w \in \mathbb{B}_n$, we will also write $d(z, w) = |1 - \langle z, w \rangle|^{1/2}$. Then d also satisfies the triangle inequality [20, Proposition 5.1.2].

LEMMA 2.14 [20, Proposition 5.1.4]. *When $n > 1$, the ratio $\sigma(Q_\delta)/\delta^{2n}$ increases from 2^{-n} to a finite limit A_0 as δ decreases from $\sqrt{2}$ to 0.*

On the punctured unit ball $\mathbb{B}_n \setminus \{0\}$, consider the projection

$$\pi_S : \mathbb{B}_n \setminus \{0\} \rightarrow S, \quad z \mapsto \frac{z}{|z|}.$$

For $z, w \in \mathbb{B}_n$, we will consider the spherical distance between their projections on S . Let us denote

$$d_S(z, w) = d(\pi_S(z), \pi_S(w)) = \left| 1 - \left\langle \frac{z}{|z|}, \frac{w}{|w|} \right\rangle \right|^{1/2}.$$

LEMMA 2.15. For $z, w \in \mathbb{B}_n$, we have

- (1) $d^2(z, w) < d_S^2(z, w) + (1 - |z|^2) + (1 - |w|^2);$
- (2) $d^2(z, w) > \frac{1}{2}d_S^2(z, w).$

Proof. By definition,

$$d^2(z, w) = |1 - \langle z, w \rangle|, \quad d_S^2(z, w) = \left| 1 - \left\langle \frac{z}{|z|}, \frac{w}{|w|} \right\rangle \right|.$$

Therefore,

$$d^2(z, w) \leq (1 - |z||w|) + |z||w||1 - \left\langle \frac{z}{|z|}, \frac{w}{|w|} \right\rangle| < (1 - |z|^2) + (1 - |w|^2) + d_S^2(z, w).$$

This proves (1).

The proof of (2) relies on the fact that $2|1 - rc| > |1 - c|$ for any $r \in (0, 1)$ and $c \in \mathbb{C}$, $|c| < 1$. So

$$d^2(z, w) = |1 - \langle z, w \rangle| > \frac{1}{2} \left| 1 - \left\langle \frac{z}{|z|}, \frac{w}{|w|} \right\rangle \right| = \frac{1}{2} d_S^2(z, w).$$

This completes the proof. \square

2.5. An inequality

In [8], the following theorem was proved, and then used to obtain p -essential normality of principal submodules.

THEOREM 2.16. Suppose that h is a holomorphic function defined in a neighborhood of $\overline{\mathbb{B}_n}$. Then there exist a constant $C > 0$ and a positive integer N , such that for any $z, w \in \mathbb{B}_n$ and any $f \in \text{Hol}(\mathbb{B}_n)$, we have

$$|h(z)f(w)| \leq C \frac{|1 - \langle z, w \rangle|^N}{(1 - |w|^2)^{n+1+N}} \int_{D(w,1)} |h(\lambda)f(\lambda)| d\nu(\lambda). \quad (2.1)$$

The constants C and N depend on the function h . In the case when h is a polynomial, the constants depend only on the degree of h . We provide a direct proof here.

THEOREM 2.17. Suppose that p is a polynomial and $N = \deg p$. Then for any $f \in \text{Hol}(\mathbb{B}_n)$ and $z, w \in \mathbb{B}_n$,

$$|p(z)f(w)| \leq C \frac{|1 - \langle z, w \rangle|^N}{(1 - |w|^2)^{n+1+N}} \int_{D(w,1)} |p(\lambda)f(\lambda)| d\nu(\lambda). \quad (2.2)$$

The constant C depends only on N .

Proof. For $w \in \mathbb{B}_n$, $w \neq 0$ and $a, b > 0$, denote

$$Q_w(a, b) = \{z \in \mathbb{B}_n : |P_w(z) - w| < a(1 - |w|^2), |Q_w(z)| < b(1 - |w|^2)^{1/2}\}.$$

From [20, 2.2.7], there exist a, b such that $D(w, 1)$ contains $Q_w(a, b)$ for any $w \in \mathbb{B}_n$, $w \neq 0$.

For a polynomial p with $\deg p = N$ and for $w \in \mathbb{B}_n$, choose an orthonormal basis $\{e_1, \dots, e_n\}$ such that $e_1 = \frac{w}{|w|}$. Then $w = (|w|, 0, \dots, 0)$. In the case $w = 0$, choose any orthonormal basis. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $|\alpha| \leq N$, applying [10, Lemma 3.2] to the one variable polynomial $\partial^{\alpha_2} \cdots \partial^{\alpha_n} p(\cdot, w_2, \dots, w_n)$, we get

$$|\partial^\alpha p(|w|, 0, \dots, 0) f(|w|, 0, \dots, 0)|$$

$$\lesssim \frac{1}{(1 - |w|^2)^{\alpha_1 + 2}} \int_{|\lambda_1 - |w|| < a(1 - |w|^2)} |\partial^{\alpha_2} \cdots \partial^{\alpha_n} p(\lambda_1, 0, \dots, 0) f(\lambda_1, 0, \dots, 0)| dv(\lambda_1).$$

Applying [10, Lemma 3.2] again to $\partial_3^{\alpha_3} \cdots \partial_n^{\alpha_n} p(\lambda_1, \cdot, w_3, \dots, w_n)$, we get

$$|\partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} p(\lambda_1, 0, \dots, 0) f(\lambda_1, 0, \dots, 0)| \lesssim \frac{1}{(1 - |w|^2)^{\alpha_2/2 + 1}} \int_{|\lambda_2| < \frac{b}{\sqrt{n-1}}(1 - |w|^2)^{1/2}} |\partial_3^{\alpha_3} \cdots \partial_n^{\alpha_n} p(\lambda_1, \lambda_2, 0, \dots, 0) f(\lambda_1, \lambda_2, 0, \dots, 0)| dv(\lambda_2).$$

Inductively, for any $k = 1, \dots, n-1$,

$$|\partial_{k+1}^{\alpha_{k+1}} \cdots \partial_n^{\alpha_n} p(\lambda_1, \dots, \lambda_k, 0, \dots, 0) f(\lambda_1, \dots, \lambda_k, 0, \dots, 0)| \lesssim \frac{1}{(1 - |w|^2)^{\alpha_k/2 + 1}} \int_{|\lambda_{k+1}| < \frac{b}{\sqrt{n-1}}(1 - |w|^2)^{1/2}} |\partial_{k+2}^{\alpha_{k+2}} \cdots \partial_n^{\alpha_n} p(\lambda_1, \dots, \lambda_{k+1}, 0, \dots, 0) f(\lambda_1, \dots, \lambda_{k+1}, 0, \dots, 0)| dv(\lambda_{k+1}).$$

Combining the inequalities above, we get

$$\begin{aligned} |\partial^\alpha p(w) f(w)| &\lesssim \frac{1}{(1 - |w|^2)^{\alpha_1 + |\alpha'|/2 + n + 1}} \int_{\substack{|\lambda_1 - w_1| < a(1 - |w|^2) \\ |\lambda'| < b(1 - |w|^2)^{1/2}}} |p(\lambda) f(\lambda)| dv(\lambda) \\ &\leqslant \frac{1}{(1 - |w|^2)^{\alpha_1 + |\alpha'|/2 + n + 1}} \int_{D(w, 1)} |p(\lambda) f(\lambda)| dv(\lambda). \end{aligned}$$

Since $p(z) = \sum_{|\alpha| \leq N} c_\alpha \partial^\alpha p(w)(z - w)^\alpha$, where c_α are the Taylor coefficients, we have

$$|p(z) f(w)| \lesssim \sum_{|\alpha| \leq N} \frac{|z_1 - |w||^{\alpha_1} |z_2|^{\alpha_2} \cdots |z_n|^{\alpha_n}}{(1 - |w|^2)^{\alpha_1 + |\alpha'|/2 + n + 1}} \int_{D(w, 1)} |p(\lambda) f(\lambda)| dv(\lambda).$$

Notice that

$$\frac{|z_1 - |w||^2}{|1 - \langle z, w \rangle|^2} + \sum_{j=2}^n (1 - |w|^2) \frac{|z_j|^2}{|1 - \langle z, w \rangle|^2} = |\varphi_w(z)|^2 < 1.$$

We have

$$|z_1 - |w|| < |1 - \langle z, w \rangle|, \quad |z_j| < \frac{|1 - \langle z, w \rangle|}{(1 - |w|^2)^{1/2}}, \quad j = 2, \dots, n.$$

Therefore,

$$\begin{aligned} |p(z) f(w)| &\lesssim \sum_{|\alpha| \leq N} \frac{|1 - \langle z, w \rangle|^{\alpha_1 + |\alpha'|/2}}{(1 - |w|^2)^{\alpha_1 + |\alpha'|/2 + n + 1}} \int_{D(w, 1)} |p(\lambda) f(\lambda)| dv(\lambda) \\ &\lesssim \frac{|1 - \langle z, w \rangle|^N}{(1 - |w|^2)^{N + n + 1}} \int_{D(w, 1)} |p(\lambda) f(\lambda)| dv(\lambda). \end{aligned}$$

From the previous argument, we know that the controlling constant depends only on N . This completes the proof. \square

2.6. Some useful computations

The following inequality will be useful in subsequent estimates. Its proof is a direct application of the Hölder's inequality.

LEMMA 2.18. For a positive integer M and $a_1, \dots, a_M > 0$,

$$(a_1 + a_2 + \dots + a_M)^2 \leq M(a_1^2 + a_2^2 + \dots + a_M^2).$$

We will use the following version of Schur's test.

LEMMA 2.19. Let $(X, d\mu)$ and $(X, d\nu)$ be measure spaces and T be an integral operator with non-negative integral kernel $K(x, y)$,

$$Tf(x) = \int_X f(y)K(x, y)d\mu(y), \quad x, y \in X.$$

Suppose that there exist a μ -measurable positive function h and a ν -measurable positive function g on X such that

$$\int_X h(y)K(x, y)d\mu(y) \leq Ag(x), \quad a.e. [\nu],$$

and

$$\int_X g(x)K(x, y)d\nu(x) \leq Bh(y), \quad a.e. [\mu].$$

then T defines a bounded operator from $L^2(X, d\mu)$ to $L^2(X, d\nu)$ and $\|T\| \leq A^{1/2}B^{1/2}$.

We want to apply Schur's test to operators determined by the following integral kernels. For any $r > 0$ and non-negative integers $l > 0$, $0 < d < n$, define

$$\begin{aligned} T_l f(z) &= \int_{\mathbb{B}_n} f(w) \frac{(1 - |w|^2)^l}{|1 - \langle z, w \rangle|^{n+1+l}} dv(w), \\ T_{d,l} f(z) &= \int_{\mathbb{B}_n} f(w) \frac{(1 - |w'|^2)^l}{|1 - \langle z', w' \rangle|^{n+1+l}} dv(w), \\ \tilde{T}_l f(z) &= \int_{\mathbb{B}_n} f(w) \frac{(1 - |w|^2)^l}{|1 - \langle z, w \rangle|^{n+1/2+l}} dv(w), \\ T_l^r f(z) &= \int_{D(z, r)} f(w) \frac{(1 - |w|^2)^l}{(1 - \langle z, w \rangle)^{n+1+l}} dv(w), \\ R_l^r f(z) &= \int_{D(z, r)^c} f(w) \frac{(1 - |w|^2)^l}{|1 - \langle z, w \rangle|^{n+1+l}} dv(w) \\ R_{l,d}^r f(z) &= \int_{D((z', 0), r)^c} f(w) \frac{(1 - |w'|^2)^l}{|1 - \langle z', w' \rangle|^{n+1+l}} dv(w). \end{aligned}$$

Here $w' = (w_1, \dots, w_d)$.

LEMMA 2.20. For the operators defined above, the following hold.

- (1) For any positive integer l and $0 < d < n$, $T_l, T_{d,l}$ define bounded operators on $L^2(\mathbb{B}_n)$. If $l \geq 1$, then T_l also defines a bounded operator on $L^2(\mathbb{B}_n, (1 - |z|^2)dv(z))$.
- (2) For $l \geq 0$, \tilde{T}_l defines a bounded operator from $L^2(\mathbb{B}_n, (1 - |z|^2)dv(z))$ to $L^2(\mathbb{B}_n)$.
- (3) For any positive integer l and for any $f \in \text{Hol}(\mathbb{B}_n)$,

$$T_l^r f(z) = c_{r,l} f(z),$$

where $c_{r,l} = \int_{D(0, r)} (1 - |z|^2)^l dv(z)$.

(4) For any $0 < d < n$, any positive integer l and any $r > 0$, the operators R_l^r and $R_{l,d}^r$ define bounded operators on $L_a^2(\mathbb{B}_n)$. Moreover,

$$\max\{\|R_l^r\|, \|R_{l,d}^r\|\} \leq \epsilon_{r,l},$$

where we have $\epsilon_{r,l} \rightarrow 0$ as $r \rightarrow \infty$, for fixed l .

Proof. We will only prove the statements for $T_{d,l}$, \tilde{T}_l , T_l^r , and $R_{l,d}^r$. We will use the Rudin–Forelli estimates [20, Proposition 1.4.10].

Let $h(z) = (1 - |z'|^2)^{-1/2}$. Then

$$\begin{aligned} & \int_{\mathbb{B}_n} \frac{(1 - |w'|^2)^l}{|1 - \langle z', w' \rangle|^{n+1+l}} h(w) dv(w) \\ &= \int_{w' \in \mathbb{B}_d} \frac{(1 - |w'|^2)^{l-1/2}}{|1 - \langle z', w' \rangle|^{n+1+l}} \int_{|w''|^2 < 1 - |w'|^2} 1 dv_{n-d}(w'') dv_d(w') \\ &\approx \int_{\mathbb{B}_d} \frac{(1 - |w'|^2)^{l+n-d-1/2}}{|1 - \langle z', w' \rangle|^{n+1+l}} dv_d(w') \\ &\lesssim (1 - |z'|^2)^{-1/2} = h(z). \end{aligned}$$

Here, v_k denotes the Lebesgue measure on \mathbb{C}^k . Similarly, we have

$$\int_{\mathbb{B}_n} \frac{(1 - |w'|^2)^l}{|1 - \langle z', w' \rangle|^{n+1+l}} h(z) dv(z) \lesssim h(w).$$

This proves (1).

To prove (2), take $h(z) = (1 - |z|^2)^{-1/2}$ and $g(w) = (1 - |w|^2)^{-1}$. We omit the calculations.

For any $f \in \text{Hol}(\mathbb{B}_n)$ and any $z \in \mathbb{B}_n$,

$$\begin{aligned} T_l^r f(z) &= \int_{D(z,r)} f(w) \frac{(1 - |w|^2)^l}{(1 - \langle z, w \rangle)^{n+1+l}} dv(w) \\ &= \int_{D(0,r)} f \circ \varphi_z(\lambda) \frac{(1 - |\varphi_z(\lambda)|^2)^l}{(1 - \langle z, \varphi_z(\lambda) \rangle)^{n+1+l}} \frac{(1 - |z|^2)^{n+1}}{|1 - \langle z, \lambda \rangle|^{2(n+1)}} dv(\lambda) \\ &= \int_{D(0,r)} f \circ \varphi_z(\lambda) \frac{(1 - |\lambda|^2)^l}{(1 - \langle \lambda, z \rangle)^{n+1+l}} dv(\lambda) \\ &= c_{r,l} f(z). \end{aligned}$$

This proves (3).

Let $E = \{(z, w) : \beta(w, (z', 0)) > r\}$. Denote β_d the Bergman metric on \mathbb{B}_d . If $(z, w) \in E$, then $1 - |\varphi_{(z', 0)}(w)|^2 < 4e^{-2r}$. Since

$$\begin{aligned} 1 - |\varphi_{(z', 0)}(w)|^2 &= \frac{(1 - |z'|^2)(1 - |w|^2)}{|1 - \langle z', w' \rangle|^2} \\ &= (1 - |\varphi_{z'}(w')|^2) \frac{1 - |w|^2}{1 - |w'|^2}, \end{aligned}$$

either $1 - |\varphi_{z'}(w')|^2 < 2e^{-r}$ or $\frac{1 - |w|^2}{1 - |w'|^2} < 2e^{-r}$. Let $E_1 = \{(z, w) : \beta_d(z', w') > \frac{1}{2}r\}$ and $E_2 = \{(z, w) : \frac{1 - |w|^2}{1 - |w'|^2} < 2e^{-r}\}$. Then $E \subset E_1 \cup E_2$. It is easy to show that the integral kernels

$\chi_{E_1} \frac{(1-|w'|^2)^l}{|1-\langle z', w' \rangle|^{n+1+l}}$ and $\chi_{E_2} \frac{(1-|w'|^2)^l}{|1-\langle z', w' \rangle|^{n+1+l}}$ define bounded operators with norms tending to 0 as $r \rightarrow \infty$. This proves (4). \square

We will also use the weighted Bergman norm. For l a positive integer and $f \in \text{Hol}(\mathbb{B}_n)$,

$$\|f\|_{L_{a,l}^2}^2 = \int_{\mathbb{B}_n} |f(z)|^2 (1-|z|^2)^l dv(z).$$

The following lemma is well known (cf. [13]).

LEMMA 2.21. *Let T be a bounded linear operator on $L_a^2(\mathbb{B}_n)$. If there exists a constant $C > 0$ such that*

$$\|Tf\|^2 \leq C \|f\|_{L_{a,1}^2}^2, \quad \forall f \in L_a^2(\mathbb{B}_n),$$

then $T \in \mathcal{C}_p$ for all $p > 2n$.

3. Asymptotic stable division property and essential normality

By Lemma 2.1, in order to show that a submodule \mathcal{P} is p -essentially normal, $\forall p > n$, one needs to show that $QM_{z_i}^*P$ is in \mathcal{C}_p for any $p > 2n$. That means, for $f \in \mathcal{P}$, one needs to find an element in \mathcal{P} that is close enough to $M_{z_i}^*f$. In the case when \mathcal{P} is principal with generator h , the set of functions $\{hf : f \in \mathbb{C}[z_1, \dots, z_n]\}$ is dense in \mathcal{P} . For a function hf , a reasonable approximation of $M_{z_i}^*(hf)$ will be $hM_{z_i}^*f$ (cf. [8, 10, 13–15]). In general, suppose \mathcal{P} is generated by $\{h_1, \dots, h_k\}$, it may happen that $\sum_{j=1}^k h_j f_j$ equals 0, while $\sum_{j=1}^k h_j M_{z_i}^* f_j$ does not. Thus, the distance between $M_{z_i}^*(\sum_{j=1}^k h_j f_j)$ and $\sum_{j=1}^k h_j M_{z_i}^* f_j$ may not be small (compared to $\|\sum_{j=1}^k h_j f_j\|_{L_{a,1}^2}$).

One can avoid such problems by putting restrictions on the decomposition of f . In [21], Shalit considered submodules of the Drury–Arveson module, with the stable division property. For a submodule with the stable division property, one can always find a decomposition $f = \sum_{j=1}^k h_j f_j$ with

$$\sum_{j=1}^k \|h_j f_j\| \leq C \|f\|, \quad (3.1)$$

where C is a constant depending only on \mathcal{P} . Shalit showed that graded submodules with stable division property are essentially normal.

We propose the following definition of asymptotic stable division property.

DEFINITION 3.1. Suppose that \mathcal{P} is a submodule of the Bergman module $L_a^2(\mathbb{B}_n)$. \mathcal{P} is said to have the *asymptotic stable division property* if there exists an invertible operator T on \mathcal{P} , a subset $\{h_i\}_{i \in \Lambda} \subset \mathcal{P}$, finite or countably infinite, and constants C_1, C_2 , such that for any $f \in \mathcal{P}$, there exists $\{g_i\}_{i \in \Lambda} \subset \text{Hol}(\mathbb{B}_n)$ with the following properties.

- (1) $Tf = \sum_{i \in \Lambda} h_i g_i$, where the convergence is pointwise if Λ is countably infinite.
- (2)

$$\int_{\mathbb{B}_n} \left(\sum_{i \in \Lambda} |h_i(z) g_i(z)| \right)^2 dv(z) \leq C_1 \|f\|_{L_a^2}^2.$$

(3)

$$\int_{\mathbb{B}_n} \left(\sum_{i \in \Lambda} |h_i(z)g_i(z)| \right)^2 (1 - |z|^2) dv(z) \leq C_2 \|f\|_{L^2_{a,1}}^2.$$

Similar to the case of stable division property, we have the following theorem.

THEOREM 3.2. *Suppose \mathcal{P} is a submodule of $L^2_a(\mathbb{B}_n)$ with the asymptotic stable division property. If the generating functions h_i are all defined in neighborhoods of $\overline{\mathbb{B}_n}$ and the sets of constants $\{C_i\}_{i \in \Lambda}$, $\{N_i\}_{i \in \Lambda}$, determined by h_i (as in Theorem 2.16), are bounded, then the submodule \mathcal{P} is p -essentially normal for all $p > n$. In particular, if the generating functions h_i are polynomials of bounded degrees, then \mathcal{P} is p -essentially normal for all $p > n$.*

Proof. Denote P the projection operator onto \mathcal{P} and Q the projection operator onto \mathcal{P}^\perp . By Lemma 2.1, it suffices to show that $[M_{z_k}^*, P] = QM_{z_k}^*P$ is in \mathcal{C}_p , $\forall p > 2n$. Let $N = \max\{N_i : i \in \Lambda\}$. Since C_i are uniformly bounded, by Lemma 2.12(1), there is a constant C such that inequality (2.1) holds for all h_i with constants C and N . Choose a positive integer $l > N$. Define

$$M_{z_k}^{(l)*}f(z) = c_l^{-1} \int_{\mathbb{B}_n} \bar{w}_k f(w) K_w^{(l)}(z) (1 - |w|^2)^l dv(w),$$

where $K_w^{(l)}(z) = \frac{1}{(1 - \langle z, w \rangle)^{n+l+1}}$ is the weighted reproducing kernel, and $c_l = \int_{\mathbb{B}_n} (1 - |w|^2)^l dv(w)$. For $f \in L^2_a(\mathbb{B}_n)$,

$$\begin{aligned} & |M_{z_k}^* f(z) - M_{z_k}^{(l)*} f(z)| \\ &= \left| \int_{\mathbb{B}_n} (\bar{w}_k - \bar{z}_k) f(w) K_w(z) dv(w) - c_l^{-1} \int_{\mathbb{B}_n} (\bar{w}_k - \bar{z}_k) f(w) K_w^{(l)}(z) (1 - |w|^2)^l dv(w) \right| \\ &\lesssim \int_{\mathbb{B}_n} |f(w)| \frac{|w - z|}{|1 - \langle z, w \rangle|^{n+1}} dv(w) \\ &\lesssim \int_{\mathbb{B}_n} |f(w)| \frac{1}{|1 - \langle z, w \rangle|^{n+1/2}} dv(w). \end{aligned}$$

By Lemmas 2.20 and 2.21, $M_{z_k}^* - M_{z_k}^{(l)*}$ is in \mathcal{C}_p for any $p > 2n$.

For $f \in \mathcal{P}$, by assumption, $Tf = \sum_{i \in \Lambda} h_i g_i$. Define

$$S_k f(z) = \sum_{i \in \Lambda} h_i(z) G_i(z),$$

where

$$G_i(z) = c_l^{-1} \int \bar{w}_k g_i(w) K_w^{(l)}(z) (1 - |w|^2)^l dv(w), \quad i \in \Lambda.$$

For each $i \in \Lambda$, applying Theorem 2.16, we get

$$\begin{aligned} |h_i(z) G_i(z)| &= c_l^{-1} \left| h_i(z) \int \bar{w}_k g_i(w) K_w^{(l)}(z) (1 - |w|^2)^l dv(w) \right| \\ &\leq C c_l^{-1} \int_{\mathbb{B}_n} \frac{|1 - \langle z, w \rangle|^N}{(1 - |w|^2)^{n+N+1}} \int_{D(w,1)} |h_i(\lambda) g_i(\lambda)| dv(\lambda) \frac{(1 - |w|^2)^l}{|1 - \langle z, w \rangle|^{n+l+1}} dv(w) \end{aligned}$$

$$\begin{aligned}
&= C c_l^{-1} \int_{\mathbb{B}_n} |h_i(\lambda)g_i(\lambda)| \int_{D(\lambda, 1)} \frac{1}{(1 - |w|^2)^{n+1+N-l} |1 - \langle z, w \rangle|^{n+1+l-N}} dv(w) dv(\lambda) \\
&\lesssim C \int_{\mathbb{B}_n} |h_i(\lambda)g_i(\lambda)| \frac{(1 - |\lambda|^2)^{l-N}}{|1 - \langle z, \lambda \rangle|^{n+1+l-N}} dv(\lambda).
\end{aligned}$$

By Lemma 2.20 and our assumption, $h_i G_i$ belongs to $L_a^2(\mathbb{B}_n)$. By [8, Proposition 5.5], $h_i G_i$ is in the principal submodule generated by h_i , which is contained in \mathcal{P} . Moreover,

$$\sum_{i \in \Lambda} |h_i(z)G_i(z)| \lesssim C \int_{\mathbb{B}_n} \left(\sum_{i \in \Lambda} |h_i(\lambda)g_i(\lambda)| \right) \frac{(1 - |\lambda|^2)^{l-N}}{|1 - \langle z, \lambda \rangle|^{n+1+l-N}} dv(\lambda).$$

By condition (2) in Definition 3.1, the series $\sum_{i \in \Lambda} h_i G_i$ converges weakly. Therefore, $S_k f = \sum_{i \in \Lambda} h_i G_i \in \mathcal{P}$.

Next, we show that $QM_{z_k}^{(l)*} P$ is in \mathcal{C}_p for any $p > 2n$. Once this is done, we will have $QM_{z_k}^* P = Q(M_{z_k}^* - M_{z_k}^{(l)*})P + QM_{z_k}^{(l)*} P \in \mathcal{C}_p$, which is exactly what we need.

For any $f \in \mathcal{P}$,

$$\begin{aligned}
&|M_{z_k}^{(l)*} T f(z) - S_k f(z)| \\
&\leq c_l^{-1} \sum_{i \in \Lambda} \left| \int \bar{w}_k h_i(w) g_i(w) K_w^{(l)}(z) (1 - |w|^2)^l dv(w) \right. \\
&\quad \left. - \int \bar{w}_k h_i(z) g_i(w) K_w^{(l)}(z) (1 - |w|^2)^l dv(w) \right| \\
&= c_l^{-1} \sum_{i \in \Lambda} \left| \int (\bar{w}_k - \bar{z}_k) h_i(w) g_i(w) K_w^{(l)}(z) (1 - |w|^2)^l dv(w) \right. \\
&\quad \left. - \int (\bar{w}_k - \bar{z}_k) h_i(z) g_i(w) K_w^{(l)}(z) (1 - |w|^2)^l dv(w) \right| \\
&\lesssim \sum_{i \in \Lambda} \int |h_i(w) g_i(w)| \frac{(1 - |w|^2)^l}{|1 - \langle z, w \rangle|^{n+1/2+l}} dv(w) \\
&\quad + \sum_{i \in \Lambda} \int |h_i(z) g_i(w)| \frac{(1 - |w|^2)^l}{|1 - \langle z, w \rangle|^{n+1/2+l}} dv(w).
\end{aligned}$$

Again,

$$\begin{aligned}
&\int |h_i(z) g_i(w)| \frac{(1 - |w|^2)^l}{|1 - \langle z, w \rangle|^{n+1/2+l}} dv(w) \\
&\lesssim \int \frac{|1 - \langle z, w \rangle|^N}{(1 - |w|^2)^{n+1+N}} \int_{D(w, 1)} |h_i(\lambda) g_i(\lambda)| dv(\lambda) \frac{(1 - |w|^2)^l}{|1 - \langle z, w \rangle|^{n+1/2+l}} dv(w) \\
&\lesssim \int |h_i(\lambda) g_i(\lambda)| \frac{(1 - |\lambda|^2)^{l-N}}{|1 - \langle z, \lambda \rangle|^{n+1/2+l-N}} dv(\lambda).
\end{aligned}$$

Thus,

$$|M_{z_k}^{(l)*} T f(z) - S_k f(z)| \lesssim \int \left(\sum_{i \in \Lambda} |h_i(w) g_i(w)| \right) \frac{(1 - |w|^2)^{l-N}}{|1 - \langle z, w \rangle|^{n+1/2+l-N}} dv(w).$$

Applying Lemma 2.20(2) on the right-hand side, we obtain

$$\|QM_{z_k}^{(l)*}Tf\|^2 \leq \|M_{z_k}^{(l)*}Tf - S_kf\|^2 \lesssim \int \left(\sum_{i \in \Lambda} |h_i(w)g_i(w)| \right)^2 (1 - |w|^2) dv(w) \lesssim \|f\|_{L_{a,1}^2}^2.$$

By Lemma 2.21, $QM_{z_k}^{(l)*}TP$ is in \mathcal{C}_p for any $p > 2n$. Since T is invertible on \mathcal{P} , we have $QM_{z_k}^{(l)*}P = QM_{z_k}^{(l)*}TPT^{-1}P \in \mathcal{C}_p$, $\forall p > 2n$. Thus, $QM_{z_k}^{(l)*}P$ is in \mathcal{C}_p for any $p > 2n$. By Lemma 2.1, \mathcal{P} is p -essentially normal for all $p > n$. This completes the proof. \square

REMARK 3.3. As indicated in the title, the aim of this paper is to find a unified proof that works for most known results of the Arveson–Douglas Conjecture. First, suppose that h is a holomorphic function defined in a neighborhood of $\overline{\mathbb{B}_n}$ and \mathcal{P}_h is the principal submodule generated by h . Then \mathcal{P}_h has the asymptotic stable division property trivially. Second, Theorem 3.2 generalizes Shalit’s result in that we do not require the generators to be polynomials and we do not require the submodule to be graded. In fact, by Theorems 2.16 and 3.2, any finite set of generators $\{h_i\}_{i=1}^k$, defined in a neighborhood of $\overline{\mathbb{B}_n}$, satisfying inequality (3.1) for relevant norms, generates an essentially normal submodule. Finally, we will show in Theorem 4.1 that most of the submodules in [9, 11, 12] have the asymptotic stable division property.

Before proving Theorem 4.1, let us discuss the matter with some generality. Consider the following technical hypotheses.

HYPOTHESIS 1. Suppose $I \subset \mathbb{C}[z_1, \dots, z_n]$ is an ideal, N, M are positive integers, $C > 0$. For any $\epsilon > 0$ sufficiently small, there exists $R_0 > 0$ with the following property. For $R > R_0$, there exist constants $0 < \delta < 1$, $C' > 0$, open covers $\{E_i\}_{i \in \Lambda}$, $\{F_i\}_{i \in \Lambda}$, finite or countably infinite, of $\mathbb{B}_\delta := \{z \in \mathbb{B}_n : |z| > \delta\}$, and for each $i \in \Lambda$, a subset $\{p_{ij}\}_{j \in \Gamma_i} \subseteq I$, finite or countably infinite, such that the following hold.

- (1) $E_i \subset F_i \subset \mathbb{B}_n$.
- (2) For $k \in \mathbb{N}$, denote

$$E_{ik} = \{w \in \mathbb{B}_n : \beta(w, E_i) < kR\}.$$

Then $E_{i3} \subset F_i$. Moreover, any $z \in \mathbb{B}_n$ belongs to at most M of the sets $\{F_i\}_{i \in \Lambda}$.

- (3) For any $i \in \Lambda$, $j \in \Gamma_i$, $\deg p_{ij} < N$.
- (4) For any $i \in \Lambda$ and any $f \in \mathcal{P}_I$, there exist $\{f_{ij}\}_{j \in \Gamma_i} \subseteq \text{Hol}(E_{i3})$ such that

(i)

$$\int_{E_{i3}} \left| \sum_{j \in \Gamma_i} p_{ij}(\lambda) f_{ij}(\lambda) - f(\lambda) \right|^2 dv(\lambda) \leq \epsilon \int_{F_i} |f(\lambda)|^2 dv(\lambda).$$

(ii)

$$\int_{E_{i3}} \left(\sum_{j \in \Gamma_i} |p_{ij}(\lambda) f_{ij}(\lambda)| \right)^2 dv(\lambda) \leq C \int_{F_i} |f(\lambda)|^2 dv(\lambda).$$

(iii)

$$\int_{E_{i3}} \left(\sum_{j \in \Gamma_i} |p_{ij}(\lambda) f_{ij}(\lambda)| \right)^2 (1 - |\lambda|^2) dv(\lambda)$$

$$\leq C' \int_{F_i} |f(\lambda)|^2 (1 - |\lambda|^2) dv(\lambda).$$

(iv) The maps $f \mapsto f_{ij}$ are linear.

THEOREM 3.4. *Under Hypothesis 1, \mathcal{P}_I has the asymptotic stable division property with generating functions consisting of polynomials of bounded degrees.*

Combining Theorems 3.2 and 3.4, we have the following corollary.

COROLLARY 3.5. *Assume Hypothesis 1. Then the submodule \mathcal{P}_I is p -essentially normal for all $p > n$.*

Proof of Theorem 3.4. Let I, N, M, C be as in Hypothesis 1. Let $\epsilon > 0$, $R > r > 0$ be determined later. We will always assume that $R > R_0$, where R_0 is determined by ϵ , as in Hypothesis 1. For $i \in \Lambda$, define

$$\varphi_i(z) = \begin{cases} 1 - \frac{\beta(z, E_i)}{R}, & \beta(z, E_i) \leq R \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that

- (1) $0 \leq \varphi_i \leq 1$,
- (2) $\{z : \varphi_i(z) \neq 0\} = E_{i1}$, and
- (3) $|\varphi_i(z) - \varphi_i(w)| \leq \frac{\beta(z, w)}{R}$.

Fix a positive integer $l > N$. Define the linear operator

$$\hat{T} : I \rightarrow \mathcal{P}_I, \quad f \mapsto \hat{T}f(z) = \sum_{i \in \Lambda, j \in \Gamma_i} p_{ij}(z) g_{ij}(z),$$

where

$$g_{ij}(z) = \int_{\mathbb{B}_n} \varphi_i(w) f_{ij}(w) K_w^{(l)}(z) (1 - |w|^2)^l dv(w).$$

First, we show that \hat{T} is well defined and extends to a bounded operator on \mathcal{P}_I . For each pair i, j and any $z \in \mathbb{B}_n$,

$$\begin{aligned} & |p_{ij}(z) g_{ij}(z)| \\ & \leq |p_{ij}(z)| \int \varphi_i(w) |f_{ij}(w)| \frac{(1 - |w|^2)^l}{|1 - \langle z, w \rangle|^{n+1+l}} dv(w) \\ & \lesssim \int_{E_{i1}} \frac{|1 - \langle z, w \rangle|^N}{(1 - |w|^2)^{n+1+N}} \int_{D(w, 1)} |p_{ij}(\lambda) f_{ij}(\lambda)| dv(\lambda) \frac{(1 - |w|^2)^l}{|1 - \langle z, w \rangle|^{n+1+l}} dv(w) \\ & \lesssim \int_{E_{i2}} |p_{ij}(\lambda) f_{ij}(\lambda)| \frac{(1 - |\lambda|^2)^{l-N}}{|1 - \langle z, \lambda \rangle|^{n+1+l-N}} dv(\lambda). \end{aligned} \tag{3.2}$$

Here, the second inequality comes from Theorem 2.17. Hence,

$$\|p_{ij} g_{ij}\|^2 \lesssim \int_{E_{i2}} |p_{ij} f_{ij}|^2 dv \leq C \|f\|^2.$$

Therefore, the map $f \rightarrow p_{ij} g_{ij}$ extends to a bounded linear operator from \mathcal{P}_I to the principal submodule generated by p_{ij} . By [8, Proposition 5.5], the image for any $f \in \mathcal{P}_I$ equals $p_{ij} g_{ij}$ for some $g_{ij} \in \text{Hol}(\mathbb{B}_n)$.

Next, we show that \hat{T} itself extends to a bounded linear operator. From inequality (3.2), we see that

$$\begin{aligned} \sum_{i,j} |p_{ij}(z)g_{ij}(z)| &\lesssim \sum_{i,j} \int_{E_{i2}} |p_{ij}(\lambda)f_{ij}(\lambda)| \frac{(1-|\lambda|^2)^{l-N}}{|1-\langle z, \lambda \rangle|^{n+1+l-N}} dv(\lambda) \\ &= \int_{\mathbb{B}_n} \left(\sum_i \chi_{E_{i2}}(\lambda) \sum_j |p_{ij}(\lambda)f_{ij}(\lambda)| \right) \frac{(1-|\lambda|^2)^{l-N}}{|1-\langle z, \lambda \rangle|^{n+1+l-N}} dv(\lambda). \end{aligned}$$

By Lemma 2.20 (1),

$$\begin{aligned} \|\hat{T}f\|^2 &\leq \left\| \sum_{i,j} |p_{ij}g_{ij}| \right\|^2 \\ &\lesssim \int_{\mathbb{B}_n} \left(\sum_i \chi_{E_{i2}}(\lambda) \sum_j |p_{ij}(\lambda)f_{ij}(\lambda)| \right)^2 dv(\lambda). \end{aligned}$$

By our hypotheses, for each λ , there are at most M functions $\chi_{E_{i2}}$ with $\chi_{E_{i2}}(\lambda) \neq 0$. By Lemma 2.18,

$$\left(\sum_i \chi_{E_{i2}}(\lambda) \sum_j |p_{ij}(\lambda)f_{ij}(\lambda)| \right)^2 \leq M \sum_i \chi_{E_{i2}}(\lambda) \left(\sum_j |p_{ij}(\lambda)f_{ij}(\lambda)| \right)^2.$$

Therefore,

$$\begin{aligned} \|\hat{T}f\|^2 &\leq \left\| \sum_{i,j} |p_{ij}g_{ij}| \right\|^2 \lesssim M \sum_i \int_{E_{i2}} \left(\sum_j |p_{ij}(\lambda)f_{ij}(\lambda)| \right)^2 dv(\lambda) \\ &\leq CM \sum_i \int_{F_i} |f(\lambda)|^2 dv(\lambda) \\ &\leq CM^2 \|f\|^2. \end{aligned}$$

Hence, \hat{T} defines a bounded operator.

Arguments similar to those in the proof of Theorem 3.2 show that $\hat{T}f \in \mathcal{P}_I$ for all $f \in I$.

To sum it up, \hat{T} extends to a bounded linear operator on \mathcal{P}_I and for each $f \in \mathcal{P}_I$, $\hat{T}f$ has the form $\sum_{i,j} p_{ij}g_{ij}$. Moreover,

$$\left\| \sum_{i,j} |p_{ij}g_{ij}| \right\|^2 \lesssim CM^2 \|f\|^2. \quad (3.3)$$

Since $l > N$, by Lemma 2.20, the estimates above will also give us the following:

$$\int \left(\sum_{i,j} |p_{ij}(z)g_{ij}(z)| \right)^2 (1-|z|^2) dv(z) \lesssim C'M^2 \|f\|_{L^2_{a,1}}^2. \quad (3.4)$$

Next, we want to show that inequality (1) in Definition 3.1 holds. We will obtain this by showing that a finite rank perturbation of \hat{T} is invertible on \mathcal{P}_I . For any $z \in \mathbb{B}_n$ and the

chosen $r < R$,

$$\begin{aligned}
& \left| \hat{T}f(z) - \sum_{i,j} p_{ij}(z) \int_{D(z,r)} \varphi_i(w) f_{ij}(w) K_w^{(l)}(z) (1-|w|^2)^l dv(w) \right| \\
& \leq \sum_{i,j} |p_{ij}(z)| \int_{E_{i1} \setminus D(z,r)} |f_{ij}(w)| \frac{(1-|w|^2)^l}{|1-\langle z, w \rangle|^{n+1+l}} dv(w) \\
& \lesssim \sum_{i,j} \int_{E_{i1} \setminus D(z,r)} \frac{|1-\langle z, w \rangle|^N}{(1-|w|^2)^{n+1+N}} \int_{D(w,1)} |p_{ij}(\lambda) f_{ij}(\lambda)| dv(\lambda) \frac{(1-|w|^2)^l}{|1-\langle z, w \rangle|^{n+1+l}} dv(w) \\
& \lesssim \sum_{i,j} \int_{E_{i2} \setminus D(z,r-1)} |p_{ij}(\lambda) f_{ij}(\lambda)| \frac{(1-|\lambda|^2)^{l-N}}{|1-\langle z, \lambda \rangle|^{n+1+l-N}} dv(\lambda) \\
& = \int_{D(z,r-1)^c} \left(\sum_{i,j} \chi_{E_{i2}}(\lambda) |p_{ij}(\lambda) f_{ij}(\lambda)| \right) \frac{(1-|\lambda|^2)^{l-N}}{|1-\langle z, \lambda \rangle|^{n+1+l-N}} dv(\lambda).
\end{aligned}$$

Also,

$$\begin{aligned}
& \left| \sum_{i,j} p_{ij}(z) \int_{D(z,r)} \varphi_i(w) f_{ij}(w) K_w^{(l)}(z) (1-|w|^2)^l dv(w) \right. \\
& \quad \left. - \sum_{i,j} p_{ij}(z) \varphi_i(z) \int_{D(z,r)} f_{ij}(w) K_w^{(l)}(z) (1-|w|^2)^l dv(w) \right| \\
& \leq \sum_{i,j} |p_{ij}(z)| \int_{E_{i2}} \frac{r}{R} |f_{ij}(w)| \frac{(1-|w|^2)^l}{|1-\langle z, w \rangle|^{n+1+l}} dv(w) \\
& \lesssim \frac{r}{R} \sum_{i,j} \int_{E_{i2}} \frac{|1-\langle z, w \rangle|^N}{(1-|w|^2)^{n+1+N}} \int_{D(w,1)} |p_{ij}(\lambda) f_{ij}(\lambda)| dv(\lambda) \frac{(1-|w|^2)^l}{|1-\langle z, w \rangle|^{n+1+l}} dv(w) \\
& \lesssim \frac{r}{R} \sum_{i,j} \int_{E_{i3}} |p_{ij}(\lambda) f_{ij}(\lambda)| \frac{(1-|\lambda|^2)^{l-N}}{|1-\langle z, \lambda \rangle|^{n+1+l-N}} dv(\lambda) \\
& = \frac{r}{R} \int \left(\sum_{i,j} \chi_{E_{i3}}(\lambda) |p_{ij}(\lambda) f_{ij}(\lambda)| \right) \frac{(1-|\lambda|^2)^{l-N}}{|1-\langle z, \lambda \rangle|^{n+1+l-N}} dv(\lambda).
\end{aligned}$$

Finally, notice that if $\varphi_i(z) \neq 0$, then $z \in E_{i1}$. By Lemma 2.20(3),

$$\sum_{i,j} p_{ij}(z) \varphi_i(z) \int_{D(z,r)} f_{ij}(w) K_w^{(l)}(z) (1-|w|^2)^l dv(w) = c_{r,l} \sum_{i,j} \varphi_i(z) p_{ij}(z) f_{ij}(z).$$

By Lemma 2.20 and the previous arguments,

$$\begin{aligned}
& \left\| \hat{T}f - c_{r,l} \sum_i \varphi_i \sum_j p_{ij} f_{ij} \right\| \lesssim \left(\epsilon_{r-1,l-N} + \frac{r}{R} \right) \left\| \sum_{i,j} \chi_{E_{i3}} |p_{ij} f_{ij}| \right\| \\
& \lesssim \left(\epsilon_{r-1,l-N} + \frac{r}{R} \right) (CM)^{1/2} \|f\|.
\end{aligned}$$

On the other hand, by Hypothesis 1,

$$\int_{E_{i3}} \left| \sum_j p_{ij}(\lambda) f_{ij}(\lambda) - f(\lambda) \right|^2 dv(\lambda) \leq \epsilon \int_{F_i} |f(\lambda)|^2 dv(\lambda).$$

So,

$$\begin{aligned} & \int_{\mathbb{B}_n} \left| \sum_i \varphi_i(\lambda) \sum_j p_{ij}(\lambda) f_{ij}(\lambda) - \sum_i \varphi_i(\lambda) f(\lambda) \right|^2 dv(\lambda) \\ & \leq M \sum_i \int_{E_{i1}} \left| \sum_j p_{ij}(\lambda) f_{ij}(\lambda) - f(\lambda) \right|^2 dv(\lambda) \\ & \leq \epsilon M \sum_i \int_{F_i} |f(\lambda)|^2 dv(\lambda) \\ & \leq \epsilon M^2 \|f\|^2. \end{aligned}$$

Combining the above estimates, we get

$$\left\| \hat{T}f - c_{r,l} \sum_i \varphi_i f \right\| \lesssim \left[\left(\epsilon_{r-1,l-N} + \frac{r}{R} \right) (CM)^{1/2} + \epsilon^{1/2} M \right] \|f\|.$$

Let $T_{\sum_i \varphi_i}$ be the Toeplitz operator with symbol $\sum_i \varphi_i$. Then the above implies

$$\|\hat{T}f - c_{r,l} T_{\sum_i \varphi_i} f\| \lesssim \left[\left(\epsilon_{r-1,l-N} + \frac{r}{R} \right) (CM)^{1/2} + \epsilon^{1/2} M \right] \|f\|. \quad (3.5)$$

Since C, l, N , and M are fixed, we can choose ϵ, r , and then $R > R_0$ so that

$$\left(\epsilon_{r-1,l-N} + \frac{r}{R} \right) (CM)^{1/2} + \epsilon^{1/2} M < 1/2 c_{r,l}.$$

For a positive integer k , denote

$$I_k = \{q \in I, \deg q \leq k\}.$$

Since $\sum \varphi_i \geq \chi_{\mathbb{B}_\delta}$, we can choose k large enough so that the compression of $T_{\sum_i \varphi_i}$ on $\mathcal{P}_I \ominus I_k$ is invertible on $\mathcal{P}_I \ominus I_k$ and the norm of its inverse is less than 2. Then the compression of \hat{T} on $\mathcal{P}_I \ominus I_k$ is invertible. By a block matrix argument, it is easy to see that $T := \hat{T}(P - P_{I_k}) + P_{I_k}$ is invertible on \mathcal{P}_I . Here, P denotes the orthogonal projection onto \mathcal{P}_I . Choose an orthonormal basis $\{h_i\}$ for I_k under the Bergman norm. For any $f \in \mathcal{P}_I$,

$$Tf = \sum_{i,j} p_{ij} g_{ij} + P_{I_k} f = \sum_{i,j} p_{ij} g_{ij} + \sum_i c_i h_i.$$

Then by inequalities (3.3) and (3.4),

$$\int \left(\sum_{i,j} |p_{ij} g_{ij}| \right)^2 dv \lesssim \|(P - P_{I_k})f\|^2 \leq \|f\|^2.$$

Since I_k is finitely dimensional, we also have

$$\int \left(\sum_i |c_i h_i| \right)^2 dv \lesssim \left\| \sum_i c_i h_i \right\|^2 \lesssim \|f\|^2.$$

It is also easy to see that P_{I_k} extends to a bounded operator on the closure of I in $L^2_{a,1}(\mathbb{B}_n)$. Therefore,

$$\int \left(\sum_i |c_i h_i| \right)^2 (1 - |z|^2) dv \lesssim \|f\|_{L^2_{a,1}}^2.$$

and

$$\int \left(\sum_{i,j} |p_{ij} g_{ij}| \right)^2 (1 - |z|^2) dv \lesssim \|f - P_{I_k} f\|_{L^2_{a,1}}^2 \lesssim \|f\|_{L^2_{a,1}}^2.$$

This completes the proof. \square

4. A distance estimate

As promised in Remark 3.3, we are going to show that most submodules in the known results of the geometric Arveson–Douglas Conjecture have the asymptotic stable division property. In fact, we will prove the following more general result.

THEOREM 4.1. *Suppose that I is an ideal in $\mathbb{C}[z_1, \dots, z_n]$ with primary decomposition $I = \cap_{j=1}^k I_j^{m_j}$, where I_j are prime ideals. Assume the following.*

- (1) *For any $j = 1, \dots, k$, $Z(I_j)$ has no singular points on $\partial\mathbb{B}_n$ and intersects $\partial\mathbb{B}_n$ transversely.*
- (2) *Any pair of the varieties $\{Z(I_j)\}$ does not intersect on $\partial\mathbb{B}_n$.*

Then I satisfies Hypothesis 1. Consequently, the submodule \mathcal{P}_I has the asymptotic stable division property with generating functions being polynomials of uniformly bounded degrees, and \mathcal{P}_I is p -essentially normal for all $p > n$.

We will prove Theorem 4.1 in the remaining sections.

NOTATIONS. For the remainder of this paper, we reserve the notations I , m_j , and I_j for the ones mentioned in Theorem 4.1. Denote $Z = Z(I)$, $Z_j = Z(I_j)$, $\mathcal{Z} = Z \cap \mathbb{B}_n$, and $\mathcal{Z}_j = Z_j \cap \mathbb{B}_n$.

As a preparation, we will prove a distance estimate for $Z(I)$ in this section.

PROPOSITION 4.2. *For fixed $R > 0$,*

$$\sup_{w \in D(z, R) \cap Z} \beta(w, T_z Z + z) \rightarrow 0, \quad |z| \rightarrow 1, z \in \mathcal{Z}. \quad (4.1)$$

Here, $T_z Z$ is the complex tangent space of Z at z , viewed as a linear subspace of \mathbb{C}^n .

Proof. Since Z intersects $\partial\mathbb{B}_n$ transversely, there exists $C_1 > 0$ such that, for any $z \in Z$ close enough to $\partial\mathbb{B}_n$, there exists a non-zero vector $v_z \in T_z Z$ such that

$$|Q_z(v_z)| \leq C_1 |\langle v_z, z \rangle|.$$

It is also easy to see that there exists $C_2 > 0$ such that for $r > 0$ sufficiently small and any $z \in Z$ close enough to $\partial\mathbb{B}_n$,

$$\sup_{w \in Z \cap B(z, r)} \text{dist}(w, T_z Z + z) \leq C_2 r^2.$$

Here, we use ‘dist’ to denote the Euclidean distance. Also, since $R > 0$ is fixed, there exists $C_3 > 0$ such that for any $z, w \in \mathbb{B}_n$ and $\beta(z, w) < R$,

$$|P_z(w - z)| \leq C_3(1 - |z|^2), \quad |w - z|^2 \leq C_3(1 - |z|^2).$$

For any $z \in Z$ and $w \in D(z, R) \cap Z$, let $\lambda \in T_z Z$ be such that

$$\text{dist}(w, \lambda + z) = \text{dist}(w, T_z Z + z).$$

Then

$$|w - (\lambda + z)| \leq C_2 |w - z|^2 \leq C_2 C_3 (1 - |z|^2).$$

Let $\lambda' = \lambda - \frac{\langle \lambda + z - w, z \rangle}{\langle v_z, z \rangle} v_z$. Then $\lambda' \in T_z Z$. We need to estimate $\beta(\lambda' + z, w)$. Noticing that $\langle \lambda' + z, z \rangle = \langle w, z \rangle$, we have

$$|\varphi_z(\lambda' + z) - \varphi_z(w)| = \frac{(1 - |z|^2)^{1/2} |Q_z(w - \lambda')|}{|1 - \langle w, z \rangle|} \lesssim \frac{|Q_z(w - \lambda')|}{(1 - |z|^2)^{1/2}}.$$

Since

$$\begin{aligned} |Q_z(w - \lambda')| &\leq |Q_z(w - \lambda)| + \frac{|\langle \lambda + z - w, z \rangle|}{|\langle v_z, z \rangle|} |Q_z(v_z)| \\ &= |Q_z(w - (\lambda + z))| + \frac{|\langle \lambda + z - w, z \rangle|}{|\langle v_z, z \rangle|} |Q_z(v_z)| \\ &\leq (1 + C_1) |w - (\lambda + z)| \\ &\leq (1 + C_1) C_2 C_3 (1 - |z|^2), \end{aligned}$$

we have

$$|\varphi_z(\lambda' + z) - \varphi_z(w)| \lesssim (1 - |z|^2)^{1/2}.$$

Since $w \in D(z, R)$, from the above inequality, for z close enough to $\partial \mathbb{B}_n$, $\varphi_z(w), \varphi_z(\lambda' + z)$ fall into a compact subset of \mathbb{B}_n . Hence,

$$\beta(\lambda' + z, w) = \beta(\varphi_z(\lambda' + z), \varphi_z(w)) \approx |\varphi_z(\lambda' + z) - \varphi_z(w)| \lesssim (1 - |z|^2)^{1/2}.$$

This completes the proof. \square

It is convenient to consider the following modified version of tangent space, because it is invariant under the Möbius transform φ_z .

DEFINITION 4.3. For $z \in \mathcal{Z}$, let us define the *normal tangent cone* $NT_z Z$ to be

$$NT_z Z = T_z Z \cap z^\perp + \mathbb{C}z.$$

Note that $z \in NT_z Z$.

LEMMA 4.4. For any $R > 0$, we have

$$\sup_{w \in (T_z Z + z) \cap D(z, R)} \beta(w, NT_z Z) \rightarrow 0, \quad z \in \mathcal{Z}, |z| \rightarrow 1.$$

Proof. For $z \in \mathcal{Z}$ close enough to $\partial \mathbb{B}_n$, let $z_0 = P_{T_z Z} z$. Since \mathcal{Z} intersects $\partial \mathbb{B}_n$ transversely, we can assume that $|z_0| \geq \epsilon$ for some $0 < \epsilon < 1$.

Suppose $w \in T_z Z$ and $\beta(w + z, z) < R$. Let $\lambda = P_z w + w - \langle w, z \rangle \frac{z_0}{|z_0|^2}$. Then $\lambda \in NT_z Z$ and $\langle w, z \rangle = \langle \lambda, z \rangle$. So $\varphi_z(\lambda + z)$ is well defined. Therefore,

$$|\varphi_z(z + w) - \varphi_z(z + \lambda)| = \left| \frac{(1 - |z|^2)^{1/2} \frac{\langle w, z \rangle}{|z_0|^2} Q_z(z_0)}{1 - \langle z + w, z \rangle} \right| \lesssim \frac{\frac{1 - |z|^2}{|z_0|}}{(1 - |z|^2)^{1/2}} \leq \epsilon^{-1} (1 - |z|^2)^{1/2} \rightarrow 0,$$

where the first inequality follows from the fact that $|\langle w, z \rangle| = |\langle w + z - z, z \rangle| \lesssim 1 - |z|^2$. This completes the proof. \square

Combining Proposition 4.2 and Lemma 4.4, we have the following lemma.

LEMMA 4.5. *For any $R > 0$, we have*

$$\sup_{w \in Z \cap D(z, R)} \beta(w, NT_z Z) \rightarrow 0, \quad z \in \mathcal{Z}, |z| \rightarrow 1.$$

5. A covering lemma

As a first step to proving Theorem 4.1, for arbitrarily large $R > 0$, we will construct covers $\{E_i\}_{i \in \Lambda}$ and $\{F_i\}_{i \in \Lambda}$. In [26], Xia constructed covers of \mathbb{B}_n with the bounded overlap condition (2) in Hypothesis 1. Our construction will follow the general framework of [26], but with additional requirements. First, our covers need to take the variety \mathcal{Z} into account. For this reason, we will construct the covers in two steps: first on \mathcal{Z} and then on \mathbb{B}_n . Second, to construct the decompositions on each cover set, we need the sets to be rotation invariant in certain directions. This property will be used in the proof of Lemma 6.4.

For an arbitrarily large $R > 0$, choose s' so that

$$(s' - 1) \log 2 \leq 16R < s' \log 2,$$

and then take $s = 10s'$. For a positive integer k , define

$$r_k = \sqrt{1 - 2^{-2sk}}, \quad r'_k = \sqrt{1 - 2^{-2s'k}}.$$

Write $\mathcal{Z}_k = r_k S \cap \mathcal{Z}$, where $r_k S = \{r_k x : x \in S\}$. Let $\mathcal{L}_k \subset \mathcal{Z}_k$ be maximal with respect to the following property: if $u_i, u_j \in \mathcal{L}_k$ and $u_i \neq u_j$, then $d_S(u_i, u_j) > s2^{-sk-2}$.

For $u \in \mathcal{L}_k$, set

$$\begin{aligned} \tilde{E}_u &= \left\{ z \in \mathbb{B}_n : 1 - |z|^2 > 2^{-5s}, |1 - \langle z, u \rangle| > s^{-2}, \frac{1 - |z|^2}{|1 - \langle z, u \rangle|^2} < 2^{-2s} \right\}, \\ \tilde{F}_u &= \left\{ z \in \mathbb{B}_n : 1 - |z|^2 > 2^{-6s}, |1 - \langle z, u \rangle| > \frac{1}{2}s^{-2}, \frac{1 - |z|^2}{|1 - \langle z, u \rangle|^2} < 2^{-s} \right\}, \end{aligned}$$

and

$$E_u = \varphi_u(\tilde{E}_u), \quad F_u = \varphi_u(\tilde{F}_u).$$

Let

$$\mathcal{O} = \{w \in \mathbb{B}_n : \beta(w, \mathcal{Z}) > 5s' \log 2\}.$$

Take $S_k = r'_k S \cap \mathcal{O}$. Let $\mathcal{L}'_k \subseteq S_k$ be maximal with respect to the following property: if $v_i, v_j \in S_k$ and $v_i \neq v_j$, then $d_S(v_i, v_j) > s'2^{-s'k-2}$. For $v \in \mathcal{L}'_k$, let

$$\tilde{E}'_v = \left\{ z \in \mathbb{B}_n : 1 - |z|^2 > 2^{-5s'}, |1 - \langle z, v \rangle| > s'^{-2}, \frac{1 - |z|^2}{|1 - \langle z, v \rangle|^2} < 2^{-2s'} \right\},$$

and

$$E'_v = \varphi_v(\tilde{E}'_v).$$

Set

$$\tilde{F}'_v = \tilde{E}'_{v3}, \quad F'_v = E'_{v3},$$

where for any set $A \subset \mathbb{B}_n$ and $j \in \mathbb{N}$, we denote $A_j = \{w \in \mathbb{B}_n : \beta(w, A) < jR\}$. Let K be a positive integer determined later, and $\mathcal{L} = \cup_{k \geq K} \mathcal{L}_k$, $\mathcal{L}' = \cup_{k \geq 10K} \mathcal{L}'_k$.

Let us establish some basic properties of the sets.

LEMMA 5.1. *For k, s' large enough and $u \in \mathcal{L}_k$, $v \in \mathcal{L}'_k$, the following hold.*

(1)

$$\begin{aligned} E_u &= \{w \in \mathbb{B}_n : 1 - |w|^2 \in (2^{-2(k+3)s}, 2^{-2(k+1)s}), d(u, w) < s2^{-ks}, 1 - |\varphi_u(w)|^2 > 2^{-5s}\}, \\ E'_v &= \{w \in \mathbb{B}_n : 1 - |w|^2 \in (2^{-2(k+3)s'}, 2^{-2(k+1)s'}), d(v, w) < s'2^{-ks'}, 1 - |\varphi_v(w)|^2 > 2^{-5s'}\}, \\ F_u &= \{w \in \mathbb{B}_n : 1 - |w|^2 \in (2^{-2(k+4)s}, 2^{-2ks-s}), d(u, w) < \sqrt{2}s2^{-ks}. 1 - |\varphi_u(w)|^2 > 2^{-6s}\}. \end{aligned}$$

(2)

$$E_u \subseteq D(u, 3s \log 2), \quad E'_v \subseteq D(v, 3s' \log 2), \quad F_u \subset D(u, 4s \log 2).$$

(3)

$$\{w \in \mathbb{B}_n : \beta(w, E_u) < \frac{s}{4} \log 2\} \subset F_u.$$

Thus, $E_{u4} \subset F_u$.

(4)

$$\begin{aligned} E_u &\supseteq \{w \in \mathbb{B}_n : 1 - |w|^2 \in [2^{-2(k+2)s}, 2^{-2(k+1)s}), d_S(u, w) < s2^{-ks-1}\}, \\ E'_v &\supseteq \{w \in \mathbb{B}_n : 1 - |w|^2 \in [2^{-2(k+2)s'}, 2^{-2(k+1)s'}), d_S(v, w) < s'2^{-ks'-1}\}. \end{aligned}$$

(5) *There exists a constant $M > 0$, depending only on n , such that any $z \in \mathbb{B}_n$ belongs to at most M of the sets $\{F_u\}_{u \in \mathcal{L}} \cup \{F'_v\}_{v \in \mathcal{L}'}$.*

Proof. First, we prove (1). Suppose $u \in \mathcal{L}_k$. By definition, $w \in E_u$ if and only if $\varphi_u(w) \in \tilde{E}_u$, that is,

$$1 - |\varphi_u(w)|^2 > 2^{-5s}, \quad |1 - \langle \varphi_u(w), u \rangle| > s^{-2}, \quad \frac{1 - |\varphi_u(w)|^2}{|1 - \langle \varphi_u(w), u \rangle|^2} < 2^{-2s}.$$

Since $1 - |u|^2 = 2^{-2sk}$ and by Lemma 2.9 (1), the conditions above are equivalent to

$$1 - |\varphi_u(w)|^2 > 2^{-5s}, \quad |1 - \langle w, u \rangle| < s^2 2^{-2sk}, \quad 1 - |w|^2 < 2^{-2(k+1)s}.$$

Also, by Lemma 2.12(2),

$$1 - |w|^2 = 1 - |\varphi_u \varphi_u(w)|^2 > \frac{1}{4}(1 - |u|^2)(1 - |\varphi_u(w)|^2) > \frac{1}{4}2^{-2sk-5s} > 2^{-2(k+3)s}.$$

Thus, we have

$$E_u = \{w \in \mathbb{B}_n : 1 - |w|^2 \in (2^{-2(k+3)s}, 2^{-2(k+1)s}), d(u, w) < s2^{-ks}, 1 - |\varphi_u(w)|^2 > 2^{-5s}\}.$$

The proof of (1) for E'_v and F_u is similar.

The proof of (2) is a straightforward application of Lemma 2.11 to the estimate of $1 - |\varphi_u(w)|^2$ above.

To prove (3), suppose $w \in \mathbb{B}_n$ and $\beta(w, E_u) < \frac{s}{4} \log 2$. Choose $z \in E_u$ such that $\beta(w, z) < \beta(w, E_u) + \log 2 < \frac{s}{4} \log 2 + \log 2$. Then by Lemma 2.11,

$$1 - |\varphi_w(z)|^2 \geq e^{-2\beta(z,w)} > e^{-\frac{s}{2} \log 2 - 2 \log 2} = 2^{-\frac{s}{2} - 2}.$$

By Lemma 2.12(2) and part (1) proved above,

$$1 - |w|^2 > \frac{1}{4}(1 - |z|^2)(1 - |\varphi_w(z)|^2) > \frac{1}{4} \cdot 2^{-2(k+3)s} \cdot 2^{-\frac{s}{2} - 2} > 2^{-2(k+4)s} \quad (5.1)$$

and

$$1 - |w|^2 < 4 \frac{1 - |z|^2}{1 - |\varphi_w(z)|^2} < 4 \frac{2^{-2(k+1)s}}{2^{-2 - \frac{s}{2}}} < 2^{-2ks - s}. \quad (5.2)$$

Also,

$$d^4(z, w) = |1 - \langle z, w \rangle|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{1 - |\varphi_w(z)|^2} < \frac{2^{-2(k+1)s} \cdot 2^{-2ks - s}}{2^{-\frac{s}{2} - 2}} < 2^{-4ks - 2s}.$$

Therefore, $d(z, w) < 2^{-ks - \frac{s}{2}}$. Hence,

$$d(u, w) \leq d(u, z) + d(z, w) < s2^{-ks} + 2^{-ks - \frac{s}{2}} < \sqrt{2}s2^{-ks}. \quad (5.3)$$

Finally, by Lemma 2.11,

$$\beta(z, u) < \log 2 - \frac{1}{2} \log(1 - |\varphi_u(z)|^2) < \log 2 - \frac{1}{2} \log 2^{-5s} = \left(\frac{5}{2}s + 1\right) \log 2.$$

Therefore,

$$\beta(w, u) \leq \beta(w, z) + \beta(z, u) < \frac{s}{4} \log 2 + \log 2 + \left(\frac{5}{2}s + 1\right) \log 2 < 3s \log 2$$

and

$$1 - |\varphi_u(w)|^2 > e^{-2 \cdot 3s \log 2} = 2^{-6s}. \quad (5.4)$$

From inequalities (5.1)–(5.4), we have

$$\left\{ w \in \mathbb{B}_n : \beta(w, E_u) < \frac{s}{4} \log 2 \right\} \subset F_u.$$

This proves (3).

Suppose $1 - |z|^2 \in [2^{-2(k+2)s}, 2^{-2(k+1)s}]$ and $d_S(u, z) < s2^{-ks-1}$. By Lemma 2.15,

$$\begin{aligned} d^2(u, z) &< d_S^2(u, z) + (1 - |z|^2) + (1 - |u|^2) \\ &< s^2 2^{-2ks-2} + 2^{-2(k+1)s} + 2^{-2ks} \\ &< s^2 2^{-2ks-1}. \end{aligned}$$

We have

$$1 - |\varphi_u(z)|^2 = \frac{(1 - |u|^2)(1 - |z|^2)}{|1 - \langle u, z \rangle|^2} > \frac{2^{-2ks} 2^{-2(k+2)s}}{s^4 2^{-4ks-2}} = s^{-4} 2^{-4s+2} > 2^{-5s}.$$

By (1), we have $z \in E_u$. This proves (4).

Finally, we prove (5). Suppose $z \in \mathbb{B}_n$. Let k be the positive integer such that $2^{-2(k+1)s} < 1 - |z|^2 \leq 2^{-2ks}$. If $z \in F_u$ and $u \in \mathcal{L}_l$, then by (1), $2^{-2(l+4)s} < 1 - |z|^2 < 2^{-2ls-s}$. Thus, $k-3 \leq l \leq k$. Since $z \in F_u$ and by Lemma 2.15(2), $d_S(z, u) < \sqrt{2}d(z, u) < 2s2^{-ls}$. By our construction, if $u_1 \neq u_2$, $u_1, u_2 \in \mathcal{L}_l$, then $d_S(u_1, u_2) > s2^{-sl-2}$. By Lemma 2.14, the number of $u \in \mathcal{L}_l$ such that $z \in F_u$ does not exceed $C(\frac{2s2^{-ls} + \frac{s}{2}2^{-sl-2}}{\frac{s}{2}2^{-sl-2}})^{2n} < C2^{10n}$, where $C > 0$ is a constant. Thus,

the number of $u \in \mathcal{L}$ such that $z \in F_u$ does not exceed $C2^{10n+2}$. Replacing s with s' in the proof of (3) will give us that the set $\{w \in \mathbb{B}_n : \beta(w, E'_v) < \frac{s'}{4} \log 2\}$ is contained in

$$\{w \in \mathbb{B}_n : 1 - |w|^2 \in (2^{-2(k+4)s'}, 2^{-2ks' - s'}), d(v, w) < \sqrt{2}s'2^{-ks'}, 1 - |\varphi_v(w)|^2 > 2^{-6s'}\}.$$

Thus, E'_{v3} is contained in the right-hand side of the above. Thus, similar proof will show that z belongs to at most $C2^{10n}$ of the sets $F'_v = E'_{v3}$. Altogether, the total number of $u \in \mathcal{L}$ and $v \in \mathcal{L}'$ such that $z \in F_u$ or $z \in F'_v$ does not exceed $M := C2^{10n+3}$. This completes the proof. \square

LEMMA 5.2. *For k large enough and $z \in \mathcal{Z}$ with $1 - |z|^2 \in [2^{-2(k+3)s}, 2^{-2ks}]$, there exists $u \in \mathcal{L}_k$ such that*

$$d_S(u, z) < \frac{s}{3}2^{-sk}.$$

Proof. By assumption, Z intersects $\partial\mathbb{B}_n$ transversely and has no singular points on $\partial\mathbb{B}_n$. It is easy to see that for $z \in Z$ close enough to $\partial\mathbb{B}_n$, the orthogonal projection onto $NT_z Z$, when restricted to Z , is a one-sheeted analytic cover in an Euclidean neighborhood U of z . Consider such a specific z . Choose an orthonormal basis $\{e_1, \dots, e_n\}$ of \mathbb{C}^n such that $e_1 = \frac{z}{|z|}$ and $NT_z Z = \text{span}\{e_1, \dots, e_d\}$, where d is the dimension of Z at z . There is a vector-valued holomorphic function a such that $\zeta \in Z \cap U$ if and only if $\zeta = (\zeta', a(\zeta'))$ under the new basis, where $\zeta' = (\zeta_1, \dots, \zeta_d)$, under the new basis.

We have $\frac{z'}{|z|} = (1, 0, \dots, 0) \in \mathbb{C}^d$ and $a(z') = 0$ under the new basis. Consider the paths

$$\gamma : [0, 1] \rightarrow \mathbb{C}^d, t \mapsto (1 - t)z' + t \cdot r_{k-1} \frac{z'}{|z|},$$

$$\Gamma(t) = (\gamma(t), 0) \subseteq NT_z Z,$$

and

$$\Lambda(t) = (\gamma(t), a(\gamma(t))) \subseteq \mathcal{Z}.$$

Note that $\Lambda(0) = \Gamma(0) = z$, so $1 - |\Lambda(0)|^2 \leq 2^{-2sk}$. On the other hand, by the standard inverse function theorem, there is a constant $C > 0$ such that for $z \in \mathcal{Z}$ close enough to $\partial\mathbb{B}_n$ and λ', w' in a sufficiently small Euclidean neighborhood of z' , we have

$$|a(\lambda') - a(w')| \leq C|\lambda' - w'|.$$

So, when k is large enough,

$$\begin{aligned} 1 - |\Lambda(1)|^2 &= 1 - |\gamma(1)|^2 - |a(\gamma(1))|^2 \\ &\geq 1 - |\gamma(1)|^2 - C^2|\gamma(1) - \gamma(0)|^2 \\ &\geq 2^{-2s(k-1)} - C^22^{-4sk} \\ &\geq 2^{-2sk}. \end{aligned}$$

By the intermediate value theorem, there exists $t_0 \in [0, 1]$ such that

$$1 - |\Lambda(t_0)|^2 = 2^{-2sk}.$$

Denote $z_0 = \Lambda(t_0)$. Then $z_0 \in \mathcal{Z}_k$. From our construction of \mathcal{L}_k , there exists a $u \in \mathcal{L}_k$ such that

$$d_S(u, z_0) \leq s2^{-sk-2}.$$

Also, since $\frac{z}{|z|} = \frac{\Gamma(t_0)}{|\Gamma(t_0)|}$,

$$\begin{aligned}
d_S^2(z, z_0) &= d_S^2(\Gamma(t_0), \Lambda(t_0)) \\
&< 2d^2(\Gamma(t_0), \Lambda(t_0)) \\
&= 2(1 - |\gamma(t_0)|^2) \\
&\leq 2(1 - |\Lambda(t_0)|^2) + 2|a(\gamma(t_0))|^2 \\
&\leq 2 \cdot 2^{-2sk} + 2C^2 2^{-4sk} \\
&\leq 2^{-2sk+2}.
\end{aligned}$$

Thus,

$$d_S(z, z_0) < 2^{-sk+1}.$$

So,

$$d_S(z, u) \leq d_S(z, z_0) + d_S(z_0, u) < 2^{-sk+1} + s2^{-sk-2} < \frac{s}{3}2^{-sk}.$$

This completes the proof. \square

LEMMA 5.3. For s and K large enough,

$$\{w \in \mathbb{B}_n : 1 - |w|^2 < 2^{-2s(K+1)}, \beta(w, \mathcal{Z}) < (s-1) \log 2\} \subset \bigcup_{k=K}^{\infty} \bigcup_{u \in \mathcal{L}_k} E_u.$$

Proof. Suppose $w \notin \bigcup_{l=K}^{\infty} \bigcup_{u \in \mathcal{L}_l} E_u$, and $1 - |w|^2 \in [2^{-2s(k+2)}, 2^{-2s(k+1)})$ for some $k \geq K$. For any $z \in \mathcal{Z}$, we want to show that $\beta(z, w) \geq (s-1) \log 2$.

If $1 - |z|^2 \notin (2^{-2s(k+3)}, 2^{-2sk})$, then

$$1 - |\varphi_z(w)|^2 < \min \left\{ 4 \frac{1 - |z|^2}{1 - |w|^2}, 4 \frac{1 - |w|^2}{1 - |z|} \right\} < 2^{-2s+2}.$$

Therefore,

$$\beta(z, w) > -\frac{1}{2} \log(1 - |\varphi_z(w)|^2) > (s-1) \log 2.$$

If $1 - |z|^2 \in (2^{-2s(k+3)}, 2^{-2sk})$, by Lemma 5.2, there exists $u \in \mathcal{L}_k$ such that

$$d_S(u, z) < \frac{s}{3}2^{-sk}.$$

Since $w \notin E_u$, by Lemma 5.1 (4),

$$d_S(w, u) \geq s2^{-sk-1}.$$

So,

$$d_S(w, z) \geq d_S(w, u) - d_S(z, u) > s2^{-sk-1} - \frac{s}{3}2^{-sk} = \frac{s}{6}2^{-sk}.$$

Hence,

$$d^2(w, z) > \frac{1}{2}d_S^2(w, z) > \frac{s^2}{100}2^{-2sk}.$$

Therefore, we have

$$1 - |\varphi_z(w)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle z, w \rangle|^2} < 10^4 s^{-4} 2^{-2s} < 2^{-2s}.$$

So,

$$\beta(z, w) > -\frac{1}{2} \log(1 - |\varphi_z(w)|^2) > s \log 2.$$

This completes the proof. \square

Lemma 5.3 shows that, close to the boundary, the sets E_u cover a Bergman neighborhood of the variety \mathcal{Z} . In fact, we have the following lemma.

LEMMA 5.4. *For s' and K large enough, the following are true.*

(1) *For any $k \geq 10K$ and $v \in \mathcal{L}'_k$,*

$$\{w \in \mathbb{B}_n : \beta(w, E'_v) < s' \log 2\} \cap \mathcal{Z} = \emptyset.$$

(2)

$$\left(\bigcup_{k=10K}^{\infty} \bigcup_{v \in \mathcal{L}'_k} E'_v \right) \cup \left(\bigcup_{k=K}^{\infty} \bigcup_{u \in \mathcal{L}_k} E_u \right) \supset \mathbb{B}_{\delta},$$

where $1 - \delta^2 = 2^{-2s(K+1)}$.

Proof. For $k \geq 10K$ and $v \in \mathcal{L}'_k$, by Lemma 5.1(2),

$$\{w \in \mathbb{B}_n : \beta(w, E'_v) < s' \log 2\} \subset D(v, 4s' \log 2).$$

If $\{w \in \mathbb{B}_n : \beta(w, E'_v) < s' \log 2\} \cap \mathcal{Z} \neq \emptyset$, then $\beta(v, \mathcal{Z}) < 4s' \log 2$, which contradicts the definition of \mathcal{L}'_k . This proves (1).

Suppose $w \in \mathbb{B}_{\delta}$. Then $1 - |w|^2 < 2^{-2s(K+1)}$. If $w \notin \bigcup_{l=K}^{\infty} \bigcup_{u \in \mathcal{L}_l} E_u$, then by Lemma 5.3,

$$\beta(w, \mathcal{Z}) \geq (s-1) \log 2 > 9s' \log 2.$$

Suppose $1 - |w|^2 \in [2^{-2s'(k+2)}, 2^{-2s'(k+1)}]$. Then $k > 10K$. Let $w_0 = \frac{r'_k}{|w|} w$. Then

$$1 - |\varphi_w(w_0)|^2 = \frac{(1 - |w|^2)(1 - |w_0|^2)}{|1 - \langle w, w_0 \rangle|^2} > \frac{1}{4} \frac{1 - |w|^2}{1 - |w_0|^2} \geq \frac{2^{-2s'(k+2)}}{4 \cdot 2^{-2s'k}} = 2^{-4s' - 2}.$$

By Lemma 2.11, $\beta(w, w_0) < \log 2 - \frac{1}{2} \log 2^{-4s' - 2} < 3s' \log 2$. Therefore,

$$\beta(w_0, z) \geq \beta(w, z) - \beta(w, w_0) > 6s' \log 2.$$

Thus, $w_0 \in \mathcal{O} \cap r'_k S$. By construction, there exists $v \in \mathcal{L}'_k$ such that $d_S(w_0, v) < s' 2^{-s'k - 2}$. Therefore, $d_S(w, v) = d_S(w_0, v) < s' 2^{-s'k - 2}$. By Lemma 5.1(4), $w \in E'_v$. This completes the proof. \square

6. Local decomposition formulas

To show that the I in Theorem 4.1 satisfies Hypothesis 1, we need to construct decompositions $\sum p_{ij} f_{ij}$ on the sets E_{u3} (or E'_{v3}), for $f \in I$. It is more convenient to construct the decompositions on their images under a Möbius transform.

LEMMA 6.1. *There exist a positive integer N and a constant $C > 0$, depending on I , such that the following hold. For any $R > 0$ sufficiently large, let $\{E_u\}_{u \in \mathcal{L}}$, $\{F_u\}_{u \in \mathcal{L}}$, $\{E'_v\}_{v \in \mathcal{L}'}$ and $\{F'_v\}_{v \in \mathcal{L}'}$ be as in the beginning of Section 5. Then there exists a positive integer K such that the following hold.*

(1) For each $u \in \mathcal{L}_k$, $k \geq K$, there is a finite set of polynomials $\{p_l\} \subset I$, $\sup_l \deg p_l \leq N$, and linear maps $I \rightarrow \text{Hol}(E_{u3})$, $f \mapsto f_l$ satisfying the following inequalities.

$$(i) \int_{E_{u3}} |\sum_l p_l f_l - f|^2 dv \leq C \epsilon_R^2 \int_{F_u} |f|^2 dv.$$

$$(ii) \int_{E_{u3}} (\sum_l |p_l f_l|)^2 dv \leq C \int_{F_u} |f|^2 dv.$$

Here $\epsilon_R = \epsilon_{R,0}$ is defined in Lemma 2.20.

(2) For each $v \in \mathcal{L}'_k$, $k \geq 10K$, there is a $p \in I$ such that p is non-vanishing on F'_v and $\deg p \leq N$.

Some preparations are needed for the proof of Lemma 6.1.

LEMMA 6.2. For any positive integer m, k , and $x, y \in \mathbb{C}$, $x, y \neq 0$, there are constants $\{c_j\}_{j=0}^{m-1}, \{d_j\}_{j=0}^{k-1}$, such that

$$\frac{1}{x^k} = \sum_{j=0}^{m-1} c_j \frac{(x-y)^j}{y^{k+j}} + \sum_{j=0}^{k-1} d_j \frac{(x-y)^m}{x^{k-j} y^{m+j}}.$$

Proof. We prove by induction on m . For $m = 1$ and any k ,

$$\frac{1}{x^k} = \frac{1}{y^k} + \left(\frac{1}{x^k} - \frac{1}{y^k} \right) = \frac{1}{y^k} - \sum_{j=0}^{k-1} \frac{x^j y^{k-1-j}}{x^k y^k} (x-y) = \frac{1}{y^k} - \sum_{j=0}^{k-1} \frac{(x-y)}{x^{k-j} y^{1+j}}.$$

Suppose the equation holds for $m-1$ and any k . Then

$$\begin{aligned} \frac{1}{x^k} &= \sum_{j=0}^{m-2} c'_j \frac{(x-y)^j}{y^{k+j}} + \sum_{j=0}^{k-1} d'_j \frac{(x-y)^{m-1}}{x^{k-j} y^{m-1+j}} \\ &= \sum_{j=0}^{m-2} c'_j \frac{(x-y)^j}{y^{k+j}} + \sum_{j=0}^{k-1} d'_j \frac{(x-y)^{m-1}}{y^{m-1+j}} \left(\frac{1}{y^{k-j}} - \sum_{i=0}^{k-j-1} \frac{(x-y)}{x^{k-j-i} y^{1+i}} \right) \\ &= \sum_{j=0}^{m-1} c_j \frac{(x-y)^j}{y^{k+j}} + \sum_{l=0}^{k-1} d_l \frac{(x-y)^m}{x^{k-l} y^{m+l}}. \end{aligned}$$

This completes the proof. \square

The following lemma is elementary.

LEMMA 6.3. Suppose $0 < a < b$ and d is a positive integer. Write $B_a^b = \{z \in \mathbb{C}^d : a < |z| < b\}$. Then for any $f \in \text{Hol}(b\mathbb{B}_d)$, we have

(1)

$$\int_{b\mathbb{B}_d} |f|^2 dv \leq \frac{b}{b-a} \int_{B_a^b} |f|^2 dv.$$

(2) For any multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$,

$$\int_{B_a^b} f(w) \overline{w^\alpha} dv(w) = c_{a,b,\alpha} \partial^\alpha f(0).$$

Here $c_{a,b,\alpha} = \frac{1}{\alpha!} \int_{B_a^b} |w^\alpha|^2 dv(w)$.

LEMMA 6.4. Let d, m be positive integers, $0 < d < n$. For $R > 0$ and a positive integer k that are large enough, the following hold. Let $s = 10s'$, where s' is the unique integer satisfying

$(s' - 1) \log 2 \leq 16R < s' \log 2$. Write $r = \sqrt{1 - 2^{-2ks}}$, $u = (r, 0, \dots, 0)$, and define $E := \tilde{E}_u$, $F := \tilde{F}_u$ as in the beginning of Section 5, that is,

$$E = \left\{ z \in \mathbb{B}_n : 1 - |z|^2 > 2^{-5s}, |1 - \langle z, u \rangle| > s^{-2}, \frac{1 - |z|^2}{|1 - \langle z, u \rangle|^2} < 2^{-2s} \right\},$$

and

$$F = \left\{ z \in \mathbb{B}_n : 1 - |z|^2 > 2^{-6s}, |1 - \langle z, u \rangle| > \frac{1}{2}s^{-2}, \frac{1 - |z|^2}{|1 - \langle z, u \rangle|^2} < 2^{-s} \right\}.$$

Write $U' = \pi(F)$, where π is the projection $\pi : \mathbb{C}^n \rightarrow \mathbb{C}^d$, $z \mapsto z' := (z_1, \dots, z_d)$. Then there exist $\delta > 0$, depending only on d, m, R , and $C > 0$, depending only on d, m , with the following properties.

Suppose $a : U' \rightarrow \mathbb{C}^{n-d}$ is holomorphic and $|a(z')| < \delta, \forall z' \in U'$. Write $A = \{(z', a(z')) : z' \in U'\}$. Let $\{P_\alpha = P_{\pi|_{mA}, \alpha}\}$ be the set of canonical defining functions with respect to $\pi|_{mA}$. Here $\alpha = (\alpha_{d+1}, \dots, \alpha_n)$, $|\alpha| = m$. Then there exist linear maps $f \mapsto f_\alpha$, where $f \in \text{Hol}(\mathbb{B}_n)$, $f|_{mA} = 0$, and $f_\alpha \in \text{Hol}(E_3)$, with the following properties.

(1)

$$\int_{E_3} |f - \sum_{|\alpha|=m} P_\alpha f_\alpha|^2 dv \leq C \epsilon_R^2 \int_F |f|^2 dv.$$

(2) For any $\alpha, \beta, |\alpha| = |\beta| = m$,

$$\int_{E_3} |P_\beta f_\alpha|^2 dv \leq C \int_F |f|^2 dv.$$

Here $\epsilon_R = \epsilon_{R,0}$ is the constant defined in Lemma 2.20.

Proof. To simplify notations, we write $\pi(z) = (z', 0)$ and $p(z) = (z', a(z'))$.

We will construct the functions f_α by decomposing the reproducing kernel $K_w(z)$. For $z \in F$ and $w \in \mathbb{B}_n$, applying Lemma 6.2 for $x = 1 - \langle z, w \rangle$, $y = 1 - \langle p(z), w \rangle$, we get

$$\begin{aligned} K_w(z) &= \sum_{j=0}^{m-1} c_j \frac{\langle a(z') - z'', w'' \rangle^j}{(1 - \langle p(z), w \rangle)^{n+1+j}} + \sum_{j=0}^n d_j \frac{\langle a(z') - z'', w'' \rangle^m}{(1 - \langle z, w \rangle)^{n+1-j} (1 - \langle p(z), w \rangle)^{m+j}} \\ &= I(z, w) + II(z, w). \end{aligned}$$

Notice that $\langle a(z') - z'', w'' \rangle^m = P_{\pi|_{mA}}(z, w'') = \sum_{|\alpha|=m} P_\alpha(z) \overline{w'^{\alpha}}$. Write

$$G(z, w) = \sum_{j=0}^n d_j \frac{1}{(1 - \langle z, w \rangle)^{n+1-j} (1 - \langle p(z), w \rangle)^{m+j}}.$$

Then $II(z, w) = \sum_{|\alpha|=m} G(z, w) \overline{w'^{\alpha}} P_\alpha(z)$. Let

$$f_\alpha(z) = c_R^{-1} \int_{F_1} f(w) G(z, w) \overline{w'^{\alpha}} dv(w).$$

Here $c_R = c_{R,0}$ is the constant in Lemma 2.20. Then

$$(P_\beta f_\alpha)(z) = c_R^{-1} \int_{F_1} f(w) G(z, w) \overline{w'^{\alpha}} P_\beta(z) dv(w). \quad (6.1)$$

We will show that the functions satisfy inequalities (1) and (2).

First, we prove inequality (2). Notice that by definition, $1 - |z|^2 > 2^{-6s}, \forall z \in F$. If $\delta < 2^{-6s-2}$ and $|a(\xi')| \leq \delta, \forall \xi' \in U'$, then for any $z \in E_3, w \in F$, we have

$$\begin{aligned} |1 - \langle p(z), w \rangle| &= |1 - \langle z', w' \rangle - \langle a(z'), w'' \rangle| \geq |1 - \langle z', w' \rangle| - |a(z')| > |1 - \langle z', w' \rangle| - 2^{-6s-2} \\ &\geq \frac{1}{2} |1 - \langle z', w' \rangle|. \end{aligned}$$

Here, the last inequality holds because

$$|1 - \langle z', w' \rangle| \geq 1 - |w| \geq \frac{1}{2} (1 - |w|^2) > 2^{-6s-1}.$$

Thus,

$$|G(z, w)| \lesssim \sum_{j=0}^n \frac{1}{|1 - \langle z, w \rangle|^{n+1-j} |1 - \langle z', w' \rangle|^{m+j}}.$$

For each pair $\alpha, \beta, |\alpha| = |\beta| = m$ and $z \in E_3, w \in F$, since

$$|\overline{w''^\alpha}| \leq |w|^{m/2} \leq (1 - |w'|^2)^{m/2} \lesssim |1 - \langle z', w' \rangle|^{m/2},$$

and

$$|P_\beta(z)| \leq |a(z') - z''|^{m/2} \lesssim |1 - \langle z', w' \rangle|^{m/2},$$

we have

$$|G(z, w) \overline{w''^\alpha} P_\beta(z)| \lesssim \sum_{j=0}^n \frac{1}{|1 - \langle z, w \rangle|^{n+1-j} |1 - \langle z', w' \rangle|^j} \lesssim \frac{1}{|1 - \langle z, w \rangle|^{n+1}} + \frac{1}{|1 - \langle z', w' \rangle|^{n+1}}.$$

By (6.1) and Lemma 2.20, we obtain inequality (2).

Next, we prove (1). For any $z \in E_3$, by Lemma 5.1(2) and (3), $D(z, R) \subset F$. By Lemma 2.20(3),

$$\int_F f(w) K_w(z) dv(w) - c_R f(z) = \int_{F \setminus D(z, R)} f(w) K_w(z) dv(w). \quad (6.2)$$

Thus,

$$\begin{aligned} f(z) - \sum_{\alpha} P_\alpha f_\alpha &= c_R^{-1} \left(\int_F f(w) K_w(z) dv(w) - \int_{F \setminus D(z, R)} f(w) K_w(z) dv(w) - \int_F f(w) II(z, w) dv(w) \right) \\ &= c_R^{-1} \left(\int_F f(w) I(z, w) dv(w) - \int_{F \setminus D(z, R)} f(w) K_w(z) dv(w) \right). \end{aligned} \quad (6.3)$$

Each term of I has the form $\frac{(a(z') - z'')^\alpha \overline{w''^\alpha}}{(1 - \langle p(z), w \rangle)^{n+1+j}}$, where $0 \leq j \leq m-1$ and $|\alpha| = j$. We want to prove that their integrals with f are small. First, since $E_3 \subset F$, $E_3 \subset \{z \in \mathbb{B}_n : 1 - |z|^2 > 2^{-6s}\}$, which is contained in a compact subset of \mathbb{B}_n , we can find $C_1 > 0$, depending only on m, R , such that

$$\left| \frac{(a(z') - z'')^\alpha \overline{w''^\alpha}}{(1 - \langle p(z), w \rangle)^{n+1+j}} - \frac{(-z'')^\alpha \overline{w''^\alpha}}{(1 - \langle \pi(z), w \rangle)^{n+1+j}} \right| \leq C_1 |a(z')| \leq C_1 \delta, \quad \forall z \in E_3, w \in F.$$

Therefore,

$$\left| \int_F f(w) I(z, w) dv(w) - \int_F f(w) \sum_{j=0}^{m-1} c_j \frac{\langle -z'', w'' \rangle^j}{(1 - \langle z', w' \rangle)^{n+1+j}} dv(w) \right| \lesssim C_1 \delta \int_F |f(w)| dv(w). \quad (6.4)$$

Notice that for each $w' \in U'$, we have $|1 - \langle w', u' \rangle| > \frac{1}{2}s^{-2}$, and therefore,

$$\pi^{-1}(w') \cap F = \{(w', w'') : 1 - |w'|^2 - 2^{-s}|1 - \langle w', u' \rangle|^2 < |w''|^2 < 1 - |w'|^2 - 2^{-6s}\}.$$

Each fiber is either a spherical shell or a ball in \mathbb{C}^{n-d} . From this and Lemma 6.3(2), it is easy to see that for $z' \in \pi(E_3)$,

$$\left| \int_F f(w) \sum_{j=0}^{m-1} c_j \frac{\langle -z'', w'' \rangle^j}{(1 - \langle z', w' \rangle)^{n+1+j}} dv(w) \right| \quad (6.5)$$

$$\lesssim \sum_{j=0}^{m-1} \int_{U'} \frac{1}{|1 - \langle z', w' \rangle|^{n+1+j}} dv(w') \sup_{z' \in \pi(E_3), |\alpha| \leq m-1} |\partial^\alpha f(z', 0)| \quad (6.6)$$

$$\lesssim 2^{6s(n+m)} \sup_{z' \in \pi(E_3), |\alpha| \leq m-1} |\partial^\alpha f(z', 0)|. \quad (6.7)$$

It remains to show that $\sup_{z' \in \pi(E_3), |\alpha| \leq m-1} |\partial^\alpha f(z', 0)|$ is sufficiently small. Obtain F_1 by filling the ‘holes’ in F , that is,

$$F_1 = \{(w', w'') \in \mathbb{B}_n : w' \in U', |w''|^2 < 1 - |w'|^2 - 2^{-6s}\}.$$

By Lemma 5.1, $E_3 \subset D(0, 4s \log 2)$ and $F \supset E_4$. Thus, for δ small enough, the set of points $\{\pi(z) : z \in E_3\} \cup \{p(z) : z \in E_3\}$ has a positive Euclidean distance (independent of k) to the boundary of F_1 . So, there exists a constant $C_2 > 0$ (depending only on d, m, R) such that $\forall z \in E_3, \forall g \in \text{Hol}(F_1)$ and $\forall \alpha, |\alpha| \leq m-1$,

$$|(\partial^\alpha g)(p(z)) - (\partial^\alpha g)(\pi(z))|^2 \leq C_2^2 |p(z) - \pi(z)|^2 \int_{F_1} |g(w)|^2 dv(w) \leq C_2^2 \delta^2 \int_{F_1} |g(w)|^2 dv(w).$$

Since $f|_{mA} = 0$, we have $(\partial^\alpha f)(p(z)) = 0$ for all $z \in E_3$ and $\alpha = (\alpha_{d+1}, \dots, \alpha_n)$, $|\alpha| \leq m-1$. Thus,

$$\sup_{z' \in \pi(E_3), |\alpha| \leq m-1} |(\partial^\alpha f)(z', 0)|^2 \leq C_2^2 \delta^2 \int_{F_1} |f(w)|^2 dv(w), \quad \forall z \in E_3. \quad (6.8)$$

We need to compare the L^2 -norms on F_1 and F . For any $w' \in U'$, if $1 - |w'|^2 - 2^{-s}|1 - \langle w', u' \rangle|^2 > 0$, then for s large enough, we have

$$\begin{aligned} & \frac{1 - |w'|^2 - 2^{-6s}}{(1 - |w'|^2 - 2^{-6s}) - (1 - |w'|^2 - 2^{-2s}|1 - \langle w', u' \rangle|^2)} \\ &= \frac{1 - |w'|^2 - 2^{-6s}}{2^{-2s}|1 - \langle w', u' \rangle|^2 - 2^{-6s}} \\ &\leq \frac{1}{2^{-2s-2}s^{-4} - 2^{-6s}} \leq 2^{2s+3}s^4. \end{aligned}$$

Then by Lemma 6.3 and a simple double integral argument, we have

$$\int_{F_1} |g|^2 dv \leq 2^{2s+3}s^4 \int_F |g|^2 dv, \quad \forall g \in \text{Hol}(F_1). \quad (6.9)$$

Combining inequalities (6.5), (6.8), and (6.9), we get

$$\left| \int_F f(w) \sum_{j=0}^{m-1} c_j \frac{\langle -z'', w'' \rangle^j}{(1 - \langle z', w' \rangle)^{n+1+j}} dv(w) \right| \lesssim 2^{6s(n+m)} C_2 \delta \cdot 2^{s+2} s^2 \left(\int_F |f|^2 dv \right)^{1/2}.$$

Then by inequality (6.4) and Holder's inequality, we get

$$\left| \int_F f(w) I(z, w) dv(w) \right| \lesssim (C_1 \delta + 2^{6s(n+m)} C_2 \delta \cdot 2^{s+2} s^2) \left(\int_F |f|^2 dv \right)^{1/2}.$$

If we choose δ small enough, we can make $(C_1 \delta + 2^{6s(n+m)} C_2 \delta \cdot 2^{s+2} s^2) \leq \epsilon_R$. Then inequality (1) follows from inequality (6.3), Lemma 2.20, and our estimates above. This completes the proof. \square

The following lemma is a simplified version of Lemma 6.1(1).

LEMMA 6.5. Suppose that J is a prime ideal in $\mathbb{C}[z_1, \dots, z_n]$. Write $A = Z(J)$. Suppose that A has no singular point in $\partial\mathbb{B}_n$ and intersects $\partial\mathbb{B}_n$ transversely. Then for a positive integer m , there exist a positive integer N and constant $C > 0$ with the following property. For $R > 0$ sufficiently large, there exists a positive integer K such that the following hold. Let s' be the positive integer such that $(s' - 1) \log 2 \leq 16R < s' \log 2$ and $s = 10s'$.

(1) For each $u \in A$, $1 - |u|^2 = 2^{-2sk}$, $k \geq K$, define E_u and F_u as in the beginning of Section 5. Then there is a finite set of polynomials $\{p_l\} \subset J^m$, $\sup_l \deg p_l \leq N$, and linear maps $J \rightarrow \text{Hol}(E_{u3})$, $f \mapsto f_l$ satisfying the following inequalities.

- (i) $\int_{E_{u3}} |\sum_l p_l f_l - f|^2 dv \leq C \epsilon_R^2 \int_{F_u} |f|^2 dv$.
- (ii) $\int_{E_{u3}} (\sum_l |p_l f_l|)^2 dv \leq C \int_{F_u} |f|^2 dv$.

(2) For each $v \in \mathbb{B}_n$, $\beta(v, A) > 5s' \log 2$, $1 - |v|^2 = 2^{-2s'k}$, $k \geq 10K$, define F'_v as in the beginning of Section 5. Then there exists a polynomial $p \in J^m$ such that p is non-vanishing on F'_v and $\deg p \leq N$.

Proof. We want to apply Lemma 6.4. A Möbius transform φ_u will allow us to transfer between decompositions on E_{u3} and \tilde{E}_{u3} . Also, we need to make adjustments so that the generating functions p_l are polynomials in J^m .

Denote $d = \dim A$. For any $\zeta \in \partial\mathbb{B}_n \cap A$, since A intersects $\partial\mathbb{B}_n$ transversely at ζ , A is not contained in $\zeta^\perp + \zeta$. By Theorem A.6, the set

$$\tilde{\mathcal{U}}_\zeta = \{l \in U(\zeta^\perp) : \overline{(l \oplus 1)(NT_\zeta A)}^\perp \cap \overline{A - \zeta} = \emptyset\}$$

is dense in $U(\zeta^\perp)$. Here, we denote L^\perp the orthogonal complement of a linear space L in the complex projective space \mathbb{P}_n (cf. the Appendix) and $U(\zeta^\perp)$ the space of all unitary transforms on $\zeta^\perp \cong \mathbb{C}^{n-1}$. We consider $l \oplus 1$ as acting on the ζ^\perp by l and acting on $\mathbb{C}\zeta$ as identity.

Let $\delta > 0$ be the constant in Lemma 6.4 determined by d, m, R . If $\overline{NT_\zeta A}^\perp \cap \overline{A - \zeta} = \emptyset$, then take $L_\zeta = NT_\zeta A$. Otherwise, choose $l \in \tilde{\mathcal{U}}_\zeta$ close enough to the identity I_{ζ^\perp} so that the Hausdorff distance

$$\text{dist}_H(L_\zeta \cap \mathbb{B}_n, NT_\zeta A \cap \mathbb{B}_n) \leq \delta/4,$$

where $L_\zeta = (l \oplus 1)(NT_\zeta A)$. In either case, we have $\zeta \in L_\zeta$, $\overline{L_\zeta}^\perp \cap \overline{A - \zeta} = \emptyset$, and

$$\text{dist}_H(L_\zeta \cap \mathbb{B}_n, NT_\zeta A \cap \mathbb{B}_n) \leq \delta/4.$$

Choose a new basis $\{e_1, \dots, e_n\}$ such that $L_\zeta = \text{span}\{e_1, \dots, e_d\}$. Denote π_ζ the projection from \mathbb{C}^n onto L_ζ .

By Lemma A.4, $\pi_\zeta|_{A-\zeta}$ is proper. Since ζ is a regular point of A and A intersects $\partial\mathbb{B}_n$ transversely, π_ζ defines a one-sheeted analytic cover of $A-\zeta$ in a small neighborhood of 0. Apply Theorem A.5 to $A-\zeta$ and L_ζ . Let U_1, \mathcal{U} , and \mathcal{W} be as in Theorem A.5. Thus, for any $w \in \mathcal{W}$ and $l \in \mathcal{U}$, the function

$$\frac{P_{\pi_\zeta|_{l(A-\zeta)}}(z, w)}{P_{\pi_\zeta|_{l(A-\zeta) \cap U_1}}(z, w)}$$

is non-vanishing and holomorphic in U_1 .

Start with an open neighborhood U of ζ . For $u \in A \cap U$, let $L_u = L_\zeta \cap u^\perp + \mathbb{C}u$ and denote π_u the projection from \mathbb{C}^n onto it. Since $\zeta \in L_\zeta$, the definition is consistent at ζ , and the spaces L_u vary continuously with u . Thus, for u close enough to ζ , we can find $l_u \in \mathcal{U}$ such that $l_u^{-1}(L_\zeta) = L_u$. So $\pi_u = l_u^{-1}\pi_\zeta l_u$. For any $w \in \mathcal{W}$,

$$\frac{P_{\pi_\zeta|_{l_u(A-\zeta)}}(z, w)}{P_{\pi_\zeta|_{l_u(A-\zeta) \cap U_1}}(z, w)}$$

is non-vanishing in U_1 . Equivalently, for any $w \in \mathcal{W}$,

$$\frac{P_{\pi_\zeta|_{l_u(A)}}(z, w)}{P_{\pi_\zeta|_{l_u(A) \cap (U_1 + l_u(\zeta))}}(z, w)} \tag{6.10}$$

is non-vanishing in $U_1 + l_u(\zeta)$. Note that the definition of canonical defining functions depends on the choice of a basis. Let $e_{u,i} = l_u^{-1}(e_i)$. By Remark 2.7, under the new basis $\{e_{u,1}, \dots, e_{u,n}\}$, we have

$$\psi_w(z) := \frac{P_{\pi_u|_A}(z, w)}{P_{\pi_u|_{A \cap (l_u^{-1}U_1 + \zeta)}}(z, w)}$$

is non-vanishing for $z \in l_u^{-1}U_1 + \zeta$ and $w \in \mathcal{W}$. It is easy to see that $l_u^{-1}(U_1) + \zeta \supset U_2, \forall u \in U$, for some open neighborhood U_2 of ζ . Let us use \hat{P} with appropriate subscripts to denote the locally defined canonical defining functions depending on a one-sheeted analytic cover on the piece of manifold $A \cap U$, or its image under a Möbius transform. Then we have shown that for any $u \in U$ and $w \in \mathcal{W}$,

$$\psi_w(z) = \frac{P_{\pi_u|_A}(z, w)}{\hat{P}_{\pi_u|_A}(z, w)}$$

is non-vanishing on U_2 .

On the other hand, both L_u and the normal tangent spaces $NT_u A$ vary continuously. By shrinking the neighborhood U , we can also assume that

$$\text{dist}_H(L_u \cap \mathbb{B}_n, NT_u A \cap \mathbb{B}_n) \leq \delta/2, \quad \forall u \in U \cap A.$$

By Lemma 4.5 and 5.1(2), again, by shrinking U , we also have for any $u \in U$,

$$\text{dist}(z, NT_u A) \leq \delta/2, \quad \forall z \in \varphi_u(A) \cap \tilde{F}_u.$$

Thus,

$$\text{dist}(z, L_u) \leq \delta, \quad \forall z \in \varphi_u(A) \cap \tilde{F}_u.$$

For any $f \in J^m$, define $U_u f(z) = f \circ \varphi_u(z) \cdot k_u(z)$, where $k_u(z) = \frac{(1-|u|^2)^{(n+1)/2}}{(1-\langle z, u \rangle)^{n+1}}$. Then $U_u f|_{m\varphi_u(A)} = 0$. Applying Lemma 6.4 to $F = \tilde{F}_u$ and $\varphi_u(A)$, we can find F_α such that

$$\int_{\tilde{E}_{u3}} \left| U_u f - \sum_{|\alpha|=m_j} \hat{P}_\alpha F_\alpha \right|^2 dv \leq C \epsilon_R^2 \int_{\tilde{F}_u} |U_u f|^2 dv$$

and

$$\int_{\tilde{E}_{u3}} \left(\sum_{|\alpha|=m_j} |\hat{P}_\alpha F_\alpha| \right)^2 dv \leq C \int_{\tilde{F}_u} |U_u f|^2 dv.$$

Here $\hat{P}_\alpha = \hat{P}_{\pi_u|_{m\varphi_u(A)}, \alpha}$, that is, $\sum_{|\alpha|=m} \hat{P}_\alpha(z) \overline{w^\alpha} = \hat{P}_{\pi_u|_{\varphi_u(A)}}^m(z, w)$. Therefore,

$$\begin{aligned} & \int_{E_{u3}} |f(z) - \sum_{|\alpha|=m} \hat{P}_\alpha \circ \varphi_u(z) F_\alpha \circ \varphi_u(z) k_u(z)|^2 dv(z) \\ &= \int_{\tilde{E}_{u3}} |U_u f - \sum_{|\alpha|=m} \hat{P}_\alpha F_\alpha|^2 dv(z) \\ &\leq C \epsilon_R^2 \int_{\tilde{F}_u} |U_u f|^2 dv(z) \\ &= C \epsilon_R^2 \int_{F_u} |f(z)|^2 dv(z). \end{aligned}$$

Similarly, for each pair $\alpha, \beta, |\alpha| = |\beta| = m$,

$$\int_{E_{u3}} |\hat{P}_\beta \circ \varphi_u(z) F_\alpha \circ \varphi_u(z) k_u(z)|^2 dv(z) \leq C \int_{F_u} |f(z)|^2 dv(z).$$

Using the formula for φ_u , it is easy to verify that

$$\hat{P}_{\pi_u|_{\varphi_u(A)}}(\varphi_u(z), w) = -\frac{(1-|u|^2)^{1/2}}{1-\langle z, u \rangle} \hat{P}_{\pi_u|_A}(z, w). \quad (6.11)$$

Let $\Gamma = \{\alpha = (\alpha_{d+1}, \dots, \alpha_n) : |\alpha| = m\}$ and let $K = \#\Gamma$. By Lemma A.2, we can choose K distinct points $\{w_i\}_{i=1}^K \subset \mathcal{W}$ (independent of u) such that the vectors $\{(w_i^\alpha)_{\alpha \in \Gamma}\}_{i=1}^K$ form a basis of \mathbb{C}^K . Let W be the $K \times K$ matrix $[\overline{w_i^\alpha}]_{\alpha \in \Gamma, i=1, \dots, K}$. Then W is invertible. Suppose $W^{-1} = (c_{i,\alpha})$.

Denote $p_i(z) = P_{\pi_u|_A}^m(z, w_i)$ and $\psi_i = \psi_{w_i}$, $i = 1, \dots, K$. Since J is prime, by Theorem A.3 and Hilbert's Nullstellensatz, each p_i is a polynomial in J^m . Denote N_0 the upper bound of the degrees of the canonical defining functions of A , as in Theorem A.3. Then we have $\deg p_i \leq mN_0$. Set $N = mN_0$. Let

$$f_i(z) = \sum_{\alpha} \left(-\frac{(1-|u|^2)^{1/2}}{1-\langle z, u \rangle} \right)^m c_{i,\alpha} \psi_i(z)^{-m} F_\alpha \circ \varphi_u(z) k_u(z),$$

then

$$\begin{aligned} p_i(z) f_i(z) &= \sum_{\alpha} \left(-\frac{(1-|u|^2)^{1/2}}{1-\langle z, u \rangle} \right)^m c_{i,\alpha} \hat{P}_{\pi_u|_A}(z, w_i)^m F_\alpha \circ \varphi_u(z) k_u(z) \\ &= \sum_{\alpha} c_{i,\alpha} \hat{P}_{\pi_u|_{\varphi_u(A)}}^m(\varphi_u(z), w_i) F_\alpha \circ \varphi_u(z) k_u(z) \\ &= \sum_{\alpha} \sum_{\beta} c_{i,\alpha} \overline{w_i^\beta} \hat{P}_\beta \circ \varphi_u(z) F_\alpha \circ \varphi_u(z) k_u(z), \end{aligned}$$

and

$$\begin{aligned}
\sum_i p_i(z) f_i(z) &= \sum_{\alpha} \sum_{\beta} \sum_i c_{i,\alpha} \overline{w_i^{\beta}} \hat{P}_{\beta} \circ \varphi_u(z) F_{\alpha} \circ \varphi_u(z) k_u(z) \\
&= \sum_{\alpha} \sum_{\beta} \delta_{\alpha,\beta} \hat{P}_{\beta} \circ \varphi_u(z) F_{\alpha} \circ \varphi_u(z) k_u(z) \\
&= \sum_{\alpha} \hat{P}_{\alpha} \circ \varphi_u(z) F_{\alpha} \circ \varphi_u(z) k_u(z).
\end{aligned}$$

From this, it is easy to see that the functions p_i and f_i satisfy inequalities (i) and (ii).

For each $\zeta \in A \cap \partial \mathbb{B}_n$, we have found a neighborhood $U_{\zeta} := U$ such that for any $u \in U_{\zeta}$, we have a decomposition with the stated properties. By compactness, we can cover $A \cap \partial \mathbb{B}_n$ by finitely many such neighborhoods. This proves (1).

Next, we prove (2). Suppose $v \in \mathbb{B}_n$ and $\beta(v, A) > 5s' \log 2$. By Lemma 5.1(2), $F'_v = E'_{v3} \subset D(v, 4s' \log 2)$. Let $\xi \in A$ be a point such that $\beta(v, \xi) = \beta(v, A)$. Since

$$\beta(\varphi_{\xi}(v), 0) = \beta(v, \xi) = \beta(v, A) > 5s' \log 2,$$

by Lemma 2.11(2),

$$1 - |\varphi_{\xi}(v)|^2 \leq 4e^{-2\beta(\xi, v)} < 2^{-10s' + 2}.$$

By [20, 2.2.7], the Euclidean diameter of $D(\varphi_{\xi}(v), 4s' \log 2)$ does not exceed $2^{-s'}$. Since $\varphi_{\xi}(F'_v) \subset \varphi_{\xi}(D(v, 4s' \log 2)) = D(\varphi_{\xi}(v), 4s' \log 2)$, the Euclidean diameter of $\varphi_{\xi}(F'_v)$ does not exceed $2^{-s'}$.

Also, we can assume that v is close enough to $\partial \mathbb{B}_n$ (equivalently, k is large enough) so that ξ is a regular point of A . A simple computation (cf. [23, Lemma 3.12]) shows that $\varphi_{\xi}(v)$ is perpendicular to $T_0 \varphi_{\xi}(A) = \varphi_{\xi}(T_{\xi} A + \xi)$ (cf. [20, Proposition 2.4.2]).

Let $\delta > 0$ be a sufficiently small constant to be determined later. Suppose $\zeta \in A \cap \partial \mathbb{B}_n$. In the proof of (1), we have constructed an open neighborhood U of ζ and an open subset \mathcal{W} in \mathbb{C}^{n-d} such that for any $u \in U \cap A$ and any $w \in \mathcal{W}$,

$$\psi_w(z) = \frac{P_{\pi_{u|A}}(z, w)}{\hat{P}_{\pi_{u|A}}(z, w)}$$

is non-vanishing on U_2 , where U_2 is another open neighborhood of ζ . Here, the canonical defining functions are constructed based on the basis $\{e_{u,1}, \dots, e_{u,n}\}$ as in the proof of (1). For v close enough to ζ , we can assume that $u \in U$ and $F'_v \subset U_2$. Thus, by shrinking U , we will have that $\psi_w(z)$ is non-vanishing on F'_v for any $v \in U$.

Notice that although \mathcal{W} depends on our choice of δ , from the proof of Theorem A.5 (which is used in the proof of (1)), by shrinking the set U , we can always ensure that the volume of $\Pi(\mathcal{W})$ is greater than half of the volume of \mathbb{P}_{n-d-1} . Here, Π denotes the canonical map from $\mathbb{C}^{n-d} \setminus \{0\}$ to \mathbb{P}_{n-d-1} .

On the other hand, from the previous argument, we know that the Euclidean diameter of $\varphi_{\xi}(F'_v)$ is less than $2^{-s'}$, $\varphi_{\xi}(v)$ is perpendicular to $\varphi_{\xi}(T_{\xi} A + \xi)$, and that $1 - |\varphi_{\xi}(v)|^2 \leq 4e^{-2\beta(\xi, v)} < 2^{-10s' + 2}$. By Lemmas 4.4 and 4.5 and our construction in (1), for v close enough to $\partial \mathbb{B}_n$, we will have

$$\text{dist}_H(\varphi_{\xi}(A) \cap D(0, 5s \log 2), L_{\xi} \cap D(0, 5s \log 2)) \leq 2\delta.$$

Denote $\mathcal{W}_1 = \{w \in \mathbb{C}^{n-d} : \hat{P}_{\pi_{\xi|A}}(z, w) \text{ is non-vanishing on } \varphi_{\xi}(F'_v)\}$. For s' large enough and δ small enough, we will have that the volume of $\Pi(\mathcal{W}_1)$ is greater than half of the volume of \mathbb{P}_{n-d-1} . Choose such a δ . Then $\mathcal{W} \cap \mathcal{W}_1$ is non-empty. Choose a $w_0 \in \mathcal{W} \cap \mathcal{W}_1$. Let U be the open neighborhood of ζ determined by δ . Then for any $v \in U$, by (6.11) and the fact that

$w_0 \in \mathcal{W}_1$, $\hat{P}_{\pi_\xi|_A}(z, w_0)$ is non-vanishing on F'_v . Since $w_0 \in \mathcal{W}$, $\frac{P_{\pi_\xi|_A}(z, w_0)}{\hat{P}_{\pi_\xi|_A}(z, w_0)}$ is also non-vanishing on F'_v . Thus, $P_{\pi_\xi|_A}(z, w_0)$ is non-vanishing on F'_v . By Theorem A.3 and Hilbert's Nullstellensatz, $P_{\pi_\xi|_A}(z, w_0)$ is a polynomial of degree less than N in J . Thus, $p = P_{\pi_\xi|_A}^m(z, w_0) \in J^m$ is non-vanishing on F'_v and $\deg p \leq N$. This proves (2). \square

We are ready to prove Lemma 6.1.

Proof of Lemma 6.1. Notice that the varieties Z_j are disjoint in a neighborhood of $\partial\mathbb{B}_n$. If $\zeta \in Z_j \cap \partial\mathbb{B}_n$, then for each $i \neq j$, there exists $p_i \in I_i^{m_i}$ such that $p_i(\zeta) \neq 0$. Thus, for each $\zeta \in Z_j \cap \partial\mathbb{B}_n$, we can find a neighborhood V_ζ and $k-1$ polynomials $p_i \in I_i^{m_i}$, $i \neq j$ such that p_i are non-vanishing on V_ζ . Choose finitely many open sets V_ζ that cover $Z \cap \partial\mathbb{B}_n$. Suppose $F_u \subset V_\zeta$, where $\zeta \in Z_j \cap \partial\mathbb{B}_n$. For $f \in I$, let P_l , f_l be the functions constructed in Lemma 6.5, for $I_j^{m_j}$. Then we can simply replace P_l with $p_l \prod_{i \neq j} p_i$ and f_l with $\frac{f_l}{\prod_{i \neq j} p_i}$. This proves (1). The proof for (2) is similar. \square

We are ready to prove Theorem 4.1.

Proof of Theorem 4.1. For any $\epsilon > 0$ sufficiently small, let C be the constant in Lemma 6.1. Choose $R_0 > 0$ so that $C\epsilon_{R_0-3}^2 < \epsilon$. Let $\{E_u\}_{u \in \mathcal{L}}$, $\{E'_v\}_{v \in \mathcal{L}'}$, $\{F_u\}_{u \in \mathcal{L}}$, and $\{F'_v\}_{v \in \mathcal{L}'}$ be determined by R as in Section 5. Let K , N be the positive integers in Lemma 6.1. Let $\delta = \sqrt{1 - 2^{-2m(K+1)}}$. Finally, let $\{E_i\} = \{E_u\}_{u \in \cup_{k \geq K} \mathcal{L}_k} \cup \{E'_v\}_{v \in \cup_{k \geq 10K} \mathcal{L}'_k}$. The polynomials p_{ij} will be the corresponding polynomials in Lemma 6.1. The conditions (1) and (2) in Hypothesis 1 follow from Lemma 5.1, where we take $M = 2^{10n+3}$. The conditions (3) and (4)(i)(ii)(iv) follow from Lemma 6.1. By Lemma 5.1(2), $(1 - |\lambda|^2)$ is comparable to $(1 - |u|^2)$ for $\lambda \in F_u$ and $(1 - |\lambda|^2)$ is comparable to $(1 - |v|^2)$ for $\lambda \in F'_v$. From this, condition (4)(iii) follows immediately.

Therefore I satisfies Hypothesis 1. The rest of the theorem follows from Theorem 3.4. This completes the proof. \square

7. Concluding remarks

In this paper, we have provided a unified proof for most known results of the Arveson–Douglas Conjecture. In fact, we have proved the stronger result that the submodules under our consideration have the asymptotic stable division property. We raise the following question.

QUESTION. Suppose that I is an ideal in $\mathbb{C}[z_1, \dots, z_n]$. Find sufficient conditions for I to have the asymptotic stable division property. Find sufficient conditions for I to have the asymptotic stable division property with generating elements $\{h_i\}$ being polynomials of uniformly bounded degrees.

By Theorem 3.2, a positive result on this question will lead to a positive result of the Arveson–Douglas Conjecture. We would also like to explore other applications of the asymptotic stable division property, for example, in index theory.

The techniques we have developed in this paper are aimed at getting more general results. For the next step, we plan to consider the following examples.

- (1) Arbitrary union of smooth, transversal varieties.
- (2) Varieties with certain type of singular points on $\partial\mathbb{B}_n$. For example, singular points ζ with tangent cones being linear subspaces. This will cover the classic example of singular point, $z_1^2 = z_2^3$ at 0.

Tools, for example, from [6] and [19] can be useful.

We will also use the techniques to study the Arveson–Douglas Conjecture in connection with the L^2 -extension problem [27]. The covering constructed in Section 5 can be useful in constructing a holomorphic extension.

Appendix

For an algebraic set $A \subset \mathbb{C}^n$ with pure dimension, we can show that the functions P_π and $P_{\pi,\alpha}$ in Definition 2.6 are polynomials in z and $\bar{w} := (\bar{w}_{p+1}, \dots, \bar{w}_n)$. Let us first consider a simple case.

LEMMA A.1. *Suppose that $A \subset \mathbb{C}^n$ is an algebraic set of pure dimension $n - 1$, and suppose*

$$\pi : A \rightarrow \mathbb{C}^{n-1}, z \mapsto z' := (z_1, \dots, z_{n-1})$$

is proper. Then $P_\pi(z, w)$ and $P_{\pi,\alpha}(z)$ are polynomials in z and \bar{w} .

Proof. In this case, there is only one canonical defining function. If π is k -sheeted, then $\alpha = k$, and

$$P_\pi(z, w_n) = P_{\pi,k}(z) \bar{w}_n^k.$$

It suffices to show that $P_{\pi,k}$ is a polynomial. By definition, $P_{\pi,k}$ is the Weierstrass polynomial determined by π .

The algebraic set A decomposes into finitely many irreducible algebraic sets. It is easy to see that $P_{\pi,k}$ is just the product of the canonical defining functions of the irreducible components. Without loss of generality, we can assume that A is itself irreducible.

By [16, Proposition 1.13], there is an irreducible polynomial $Q(z)$ such that $A = Z(Q)$. Clearly, $\deg_{z_n} Q \geq k$. Write $l = \deg_{z_n} Q$ and write Q as a polynomial in z_n with coefficients in $\mathbb{C}[z_1, \dots, z_{n-1}]$,

$$Q(z) = q_l(z') z_n^l + \dots$$

Consider the set

$$B = \{z \in A : q_l(z') = 0, \text{ or } \partial_n Q(z) = 0\}.$$

$\pi(B)$ is the set of points $z' \in \mathbb{C}^{n-1}$ such that $Q(z', \cdot)$ do not have l distinct simple roots. Since Q is irreducible, B is an analytic subset of dimension $\leq n - 2$. By [6, Proposition 3.3.2], $\pi(B) \subset \mathbb{C}^{n-1}$ is an analytic subset of dimension $\leq n - 2$. Thus, $\mathbb{C}^{n-p} \setminus \pi(B)$ is dense in \mathbb{C}^{n-1} . If $l > k$, for any $z' \in \mathbb{C}^{n-1} \setminus \pi(B)$, $\pi^{-1}(z')$ contains $l > k$ distinct points, a contradiction. Thus, $\deg_{z_n} Q = k$. Also, if $\deg q_k > 0$, then for $z' \notin \sigma$, comparing the two polynomials in z_n ,

$$Q(z', z_n) = q_k(z') z_n^k + \dots$$

and

$$P_{\pi,k}(z', z_n) = (z_n - a_1(z')) \cdots (z_n - a_k(z')),$$

we get $Q(z', z_n) = q_k(z') P_{\pi,k}(z', z_n)$, $z' \notin \sigma, z_n \in \mathbb{C}$. Since σ is nowhere dense, $Q \equiv q_k P_{\pi,k}$. But then if $q_k(z') = 0$ at some z' , we will have infinitely many points on the fiber $\pi^{-1}(z')$, a contradiction. Thus, q_k is a constant. Without loss of generality, we assume $q_k \equiv 1$. Then $P_{\pi,k} = Q$ is a polynomial. This completes the proof. \square

LEMMA A.2. *For any open set $U \subset \mathbb{B}_n$ and any finite collection of indexes*

$$F \subset \{\alpha = (\alpha_1, \dots, \alpha_n) : \alpha_i \in \mathbb{N}\},$$

let $K = \#F$. Then there exist K distinct points $\{w_i\}_{i=1}^k$ in U such that the vectors $\{(w_i^\alpha)_{\alpha \in F}\}_{i=1}^K$ in \mathbb{C}^K are linearly independent.

Proof. Let

$$L = \text{span}\{(w^\alpha)_{\alpha \in F} : w \in U\}.$$

It suffices to show that $L = \mathbb{C}^K$. Otherwise, choose a non-zero vector $(a_\alpha)_{\alpha \in F}$ that is perpendicular with L . Define

$$f(w) = \sum_{\alpha \in F} \bar{a}_\alpha w^\alpha.$$

Then f is an analytic polynomial that vanishes on the open set U . Therefore f is identically zero. So $a_\alpha = 0, \forall \alpha \in F$, a contradiction. This completes the proof. \square

THEOREM A.3. Suppose that $A \subset \mathbb{C}^n$ is an algebraic set of pure dimension p . Suppose that the projection

$$\pi : A \rightarrow \mathbb{C}^p, z \mapsto z' := (z_1, \dots, z_p)$$

is proper. Then the functions $P_\pi(z, w)$ and $P_{\pi, \alpha}(z)$ are polynomials in z and \bar{w} . Moreover, there exists a positive integer N , depending only on A , such that for any choice of basis and any proper projection π , the degrees of $P_\pi(z, w)$ (in z) and $P_{\pi, \alpha}$ are less than N .

Proof. Let $\sigma \subset \mathbb{C}^p$ be the critical set of π . Suppose that π is k -sheeted. Fix any $z' \notin \sigma$, we have $\pi^{-1}(z') = \{(z', a_1(z')), \dots, (z', a_k(z'))\}$, where $\{a_1(z'), \dots, a_k(z')\}$ are k distinct points in \mathbb{C}^{n-p} . The set

$$\mathcal{W} := \{w \in \mathbb{C}^{n-p} : \langle a_1(z'), w \rangle, \dots, \langle a_k(z'), w \rangle \text{ are distinct at least for one } z' \in U'\},$$

is open in \mathbb{C}^{n-p} . Fix any $w \in \mathcal{W}$. Consider the projections

$$\pi_w : A \rightarrow \mathbb{C}^{p+1}, z \mapsto (z', \langle z'', w \rangle)$$

and

$$\pi'_w : \pi_w(A) \subset \mathbb{C}^{p+1} \rightarrow \mathbb{C}^p, (z', \langle z'', w \rangle) \mapsto z'.$$

Then $\pi = \pi'_w \circ \pi_w$. By [6, 3.1(2)], both π_w and π'_w are proper maps. By [6, Theorem 3.2, Proposition 3.3.2], $\pi_w(A)$ is a pure algebraic set in \mathbb{C}^{p+1} of dimension p , and π'_w is a k -sheeted analytic cover. By Lemma A.1, $P_{\pi'_w}((z', \lambda), \xi)$ is a polynomial in z' , λ , and ξ . Also, from the proofs of [6, Theorem 3.2, Proposition 3.3.2], the degree (in (z', λ)) of $P_{\pi'_w}((z', \lambda), \xi)$ has a upper bound determined by any set of generators of the ideal $\{p \in \mathbb{C}[z_1, \dots, z_n] : p|_A = 0\}$, which we denote by N . Checking by definition, we have the equation

$$P_\pi(z, w) = P_{\pi'_w}((z', \langle z'', w \rangle), 1).$$

Thus, for any $w \in \mathcal{W}$, $P_\pi(z, w)$ is a polynomial in z and $\deg_z P_\pi(z, w) \leq N$. Fix an order of the set $\Gamma = \{\alpha = (\alpha_{p+1}, \dots, \alpha_n) : |\alpha| = k\}$, and let $K = \#\Gamma$. By Lemma A.2, we can choose K distinct points $\{w_j\}_{j=1}^K \subset \mathcal{W}$ such that the $K \times K$ matrix $W = (\overline{w_j^\alpha})_{\alpha \in \Gamma, j=1, \dots, K}$ is invertible. From the equations

$$P_\pi(z, w_j) = \sum_{\alpha \in \Gamma} P_{\pi, \alpha}(z) \overline{w_j^\alpha}, \quad j = 1, \dots, K,$$

we can solve $P_{\pi, \alpha}$ as linear combinations of $\{P_\pi(z, w_j)\}_{j=1}^K$. Thus, $P_{\pi, \alpha}$ are polynomials in z , and then $P_\pi(z, w)$ is a polynomial in z and \bar{w} . Moreover, $\deg_z P_\pi(z, w) \leq N$, $\deg P_{\pi, \alpha}(z) \leq N$. This completes the proof. \square

Let \mathbb{P}_n denote the n -dimensional complex projective space. Then \mathbb{C}^n can be viewed as a subset of \mathbb{P}_n via the natural embedding

$$\mathbb{C}^n \rightarrow \mathbb{P}_n, \quad (z_1, \dots, z_n) \mapsto [1, z_1, \dots, z_n].$$

For any algebraic set $A \subset \mathbb{C}^n$, its closure \overline{A} in \mathbb{P}_n is an analytic subset of \mathbb{P}_n . For a p -dimensional linear space $L \subset \mathbb{C}^n$, \overline{L} is a p -dimensional linear space in \mathbb{P}_n . Its orthogonal complement in \mathbb{P}_n is of dimension $n - p - 1$.

$$\overline{L}^\perp = \{[z_0, \dots, z_n] : (z_0, \dots, z_n) \perp (1, w), w \in L\}.$$

We have the following lemma [6, 7.3].

LEMMA A.4. *Let A be a pure p -dimensional projective algebraic set in \mathbb{P}_n and let $L \subset \mathbb{P}_n$ be a complex $n - p - 1$ -dimensional plane not intersecting A . Then the projection $\pi : A \rightarrow L^\perp$ is proper.*

The following theorem will be used in the proof of Theorem 4.1.

THEOREM A.5. *Suppose that $A \subset \mathbb{C}^n$ is an algebraic set of pure dimension $p < n$ and $0 \in A$. Assume the following.*

(1) *Denote $L = \{(z', 0) : z' \in \mathbb{C}^p\} \subset \mathbb{C}^n$ and suppose $\overline{L}^\perp \cap \overline{A} = \emptyset$. Therefore, if we denote $\pi : \mathbb{C}^n \rightarrow \mathbb{C}^p$, $z \mapsto z' := (z_1, \dots, z_p)$, then $\pi|_A$ is proper.*

(2) *$U = U' \times U''$, where $0 \in U' \subset \mathbb{C}^p$ and $0 \in U'' \subset \mathbb{C}^{n-p}$ are open sets. $\pi|_{A \cap U}$ is also proper. Moreover, $\pi^{-1}(0) \cap A \cap U = \{0\}$.*

Then there exist open sets $0 \in U'_1 \subset U'$, $0 \in U''_1 \subset U''$, an open neighborhood \mathcal{U} of the $n \times n$ identity matrix $I_{n \times n}$ in $U(n)$, and an open set $\mathcal{W} \subset \mathbb{C}^{n-p}$ with the following properties. Denote $U_1 = U'_1 \times U''_1$. For any $l \in \mathcal{U}$, the projections $\pi|_{l(A)}$ and $\pi|_{l(A) \cap U_1}$ are proper. Moreover, for any $w \in \mathcal{W}$, the function

$$\psi_w(z) = \frac{P_{\pi|_{l(A)}}(z, w)}{P_{\pi|_{l(A) \cap U_1}}(z, w)}$$

is non-vanishing and holomorphic in U_1 .

Proof. Since $\pi|_A$ is proper, we know that $\pi^{-1}(0) \cap A$ consists of finitely many points. Suppose

$$(\pi^{-1}(0) \cap A) \setminus \{(0, 0)\} = \{(0, a_1), \dots, (0, a_k)\},$$

where $a_i \in \mathbb{C}^{n-p}$. Take an open set $\mathcal{W} \subset \mathbb{C}^{n-p}$ whose closure is contained in the open set $\{w \in \mathbb{C}^{n-p} : \langle a_i, w \rangle \neq 0\}$. We can find open neighborhoods $a_i \in V_i \subset \mathbb{C}^{n-p}$, $0 \in U''_1 \subset \mathbb{C}^{n-p}$ such that

$$\langle \lambda - z'', w \rangle \neq 0, \quad \forall i, \forall \lambda \in V_i, \forall z'' \in U''_1, \forall w \in \mathcal{W}.$$

We claim that there exist an open neighborhood \mathcal{U} of $I_{n \times n}$ in $U(n)$ and an open set $0 \in U'_1 \subset U'$ with the following property. Denote $U_1 = U'_1 \times U''_1$. Then $\pi|_{l(A)}$ and $\pi|_{l(A) \cap U_1}$ are proper. Moreover,

$$\pi^{-1}(U'_1) \cap l(A) \subset U_1 \cup \left(\bigcup_{i=1}^k U'_1 \times V_i \right).$$

By assumption, $\overline{L}^\perp \cap \overline{A} = \emptyset$. Therefore, we can find an open neighborhood \mathcal{U} of $I_{n \times n}$ in $U(n)$ such that $\forall l \in \mathcal{U}$, $\overline{l^{-1}(L)^\perp} \cap \overline{A} = \emptyset$. Thus, the projection from A onto $l^{-1}(L)$ is proper.

Equivalently, $\pi|_{l(A)}$ is proper. Also, by the proof of [6, Theorem 7.4.2], A is contained in the union of a ball $B := \{z : |z| < R\}$ and a cone $K := \{z : |z''| < C|z'|\}$.

On the other hand, by [6, Corollary 4.2], we can take U'_2 small enough so that

$$\pi^{-1}(U'_2) \cap A \subset (U'_2 \times U''_1) \cup \left(\bigcup_{i=1}^k U'_2 \times V_i \right).$$

If we shrink \mathcal{U} , we can ensure that $\forall l \in \mathcal{U}$, $l^{-1}\pi^{-1}(U'_2) \setminus B$ is outside the cone K . Then $l^{-1}\pi^{-1}(U'_2) \cap A \subset B$. Then if we shrink \mathcal{U} again, we can find $0 \in U'_1 \subset U'_2$ so that $\forall l \in \mathcal{U}$, $l^{-1}\pi^{-1}(U'_1) \cap B \subset \pi^{-1}(U'_2)$. Thus,

$$l^{-1}\pi^{-1}(U'_1) \cap A = l^{-1}\pi^{-1}(U'_1) \cap B \cap A \subset \pi^{-1}(U'_2) \cap A \subset (U'_2 \times U''_1) \cup \left(\bigcup_{i=1}^k U'_2 \times V_i \right).$$

If we replace the right-hand side with a compact neighborhood \mathcal{N} of $A \cap \pi^{-1}(U'_1)$ contained in $(U'_2 \times U''_1) \cup (\bigcup_{i=1}^k U'_2 \times V_i)$, then the same method will give us $l^{-1}\pi^{-1}(U'_1) \cap A \subset \mathcal{N}$. Then we can shrink \mathcal{U} again to ensure $l(\mathcal{N}) \subset (U'_2 \times U''_1) \cup (\bigcup_{i=1}^k U'_2 \times V_i)$. Then we have $\forall l \in \mathcal{U}$,

$$\pi^{-1}(U'_1) \cap l(A) \subset (U'_2 \times U''_1) \cup \left(\bigcup_{i=1}^k U'_2 \times V_i \right).$$

Then obviously,

$$\pi^{-1}(U'_1) \cap l(A) \subset (U'_1 \times U''_1) \cup \left(\bigcup_{i=1}^k U'_1 \times V_i \right).$$

This proves our claim.

The open sets U''_1 and V_i can be chosen to be disjoint. By [6, 3.1(3)], $\pi|_{l(A) \cap U_1}$ is also proper. Let $z' \in U'_1$ be outside the critical sets of both projections. Then

$$\pi^{-1}(z') \cap l(A) = \{(z', b_1(z')), \dots, (z', b_l(z')); (z', a_1(z')), \dots, (z', a_k(z'))\}.$$

Here $b_j(z') \in U''_1$, $a_j(z') \in \cup_i V_i$. By definition, for $z \in U_1$ and $w \in \mathcal{W}$,

$$P_{\pi|_{l(A)}}(z, w) = \prod_{j=1}^l \langle z'' - b_j(z'), w \rangle \times \prod_{j=1}^k \langle z'' - a_j(z'), w \rangle = P_{\pi|_{l(A) \cap U_1}}(z, w) \times \psi_w(z),$$

where

$$\psi_w(z) = \prod_{j=1}^k \langle z'' - a_j(z'), w \rangle.$$

From our construction, it is straightforward that ψ_w is non-vanishing on U_1 . This completes the proof. \square

THEOREM A.6. *Suppose that A is a p -dimensional irreducible affine algebraic set in \mathbb{C}^n and $0 \in A$. Assume that $A \not\subseteq e_n^\perp$. Let*

$$\mathcal{G}_n = \{L \in G(p, n) : e_n \in L\}$$

and

$$\tilde{\mathcal{G}}_n = \{L \in \mathcal{G}_n : \overline{L}^\perp \cap \overline{A} = \emptyset\}.$$

Here, $G(p, n)$ is the Grassmannian. Then $\tilde{\mathcal{G}}_n$ is a dense open set in \mathcal{G}_n . Equivalently, let

$$\tilde{\mathcal{U}}_n = \{l \in U(n-1) : \text{span}\{l(e_1), \dots, l(e_{p-1}), e_n\} \in \tilde{\mathcal{G}}_n\}.$$

Then $\tilde{\mathcal{U}}_n$ is dense in $U(n-1)$.

Proof. The two statements are clearly equivalent. Let us prove the first statement. Consider the canonical projection.

$$\Pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}_n, \quad (z_0, z_1, \dots, z_n) \mapsto [z_0, z_1, \dots, z_n].$$

For $L \in G(p, n)$, denote $\tilde{L} = \Pi^{-1}(\overline{L}^\perp) \cup \{0\}$ and $\tilde{A} = \Pi^{-1}(\overline{A}) \cup \{0\}$. Then $\overline{L}^\perp \cap \overline{A} = \emptyset$ if and only if $\tilde{L} \cap \tilde{A} = \{0\}$. Since A is irreducible and has dimension p , \tilde{A} is a homogeneous irreducible algebraic set of dimension $p+1$. \tilde{L} is a linear subspace of dimension $n-p$. The condition that $e_n \in L$ is equivalent to that $\tilde{L} \subset L_n := e_n^\perp \subset \mathbb{C}^{n+1}$. Let $\tilde{A}_n := \tilde{A} \cap L_n$. Then $L \in \tilde{\mathcal{G}}_n$ if and only if $\tilde{L} \subset L_n$ and $\tilde{L} \cap \tilde{A}_n = \{0\}$.

We claim that $\dim \tilde{A}_n \leq p$. Otherwise, $\dim \tilde{A}_n = p+1$. Since \tilde{A} is irreducible, it cannot properly contain any algebraic set of the same dimension. So $\tilde{A} = \tilde{A}_n$ and therefore $\tilde{A} \subset L_n$. However, this implies that $A = \overline{A} \cap \mathbb{C}^n \subset \Pi(\tilde{A} \setminus \{0\}) \cap \mathbb{C}^n \subset \Pi(L_n \setminus \{0\}) \cap \mathbb{C}^n = e_n^\perp$. A contradiction. Thus, $\dim \tilde{A} \leq p$.

Assume $\tilde{L} \subset L_n$. Both \tilde{L} and \tilde{A}_n are homogeneous varieties in $L_n \cong \mathbb{C}^n$. Thus, $\tilde{L} \cap \tilde{A}_n = \{0\}$ if and only if their preimages in \mathbb{P}_{n-1} do not intersect. The preimages of the two varieties have dimension $n-p-1$, $\leq p-1$, respectively. Thus, the set of $\tilde{L} \subset L_n$ not intersecting \tilde{A} forms a dense open set in $G(n-p, n)$. From this, it is easy to see that $\tilde{\mathcal{G}}_n$ is a dense open set in \mathcal{G}_n . This completes our proof. \square

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References

1. W. ARVESON, ‘Subalgebras of C^* -algebras. III. Multivariable operator theory’, *Acta Math.* 181 (1998) 159–228.
2. W. ARVESON, ‘The curvature invariant of a Hilbert module over $\mathbb{C}[z_1, \dots, z_d]$ ’, *J. reine angew. Math.* 522 (2000) 173–236.
3. W. ARVESON, ‘The Dirac operator of a commuting d -tuple’, *J. Funct. Anal.* 189 (2002) 53–79.
4. W. ARVESON, ‘ p -Summable commutators in dimension d ’, *J. Operator Theory* 54 (2005) 101–117.
5. W. ARVESON, ‘Quotients of standard Hilbert modules’, *Trans. Amer. Math. Soc.* 359 (2007) 6027–6055.
6. E. M. CHIRKA, *Complex analytic sets*, Mathematics and its Applications 46 (Soviet Series) (Kluwer Academic Publishers Group, Dordrecht, 1989).
7. R. G. DOUGLAS, ‘A new kind of index theorem’, *Analysis, geometry and topology of elliptic operators* (ed. B. Booss; World Scientific Publishing, Hackensack, NJ, 2006) 369–382.
8. R. G. DOUGLAS, K. GUO and Y. WANG, ‘On the p -essential normality of principal submodules of the Bergman module on strongly pseudoconvex domains’, arXiv:1812.03211.
9. R. G. DOUGLAS, X. TANG and G. YU, ‘An analytic Grothendieck Riemann Roch theorem’, *Adv. Math.* 294 (2016) 3–19.
10. R. G. DOUGLAS and K. WANG, ‘A harmonic analysis approach to essential normality of principal submodules’, *J. Funct. Anal.* 261 (2001) 3155–3180.
11. R. G. DOUGLAS and Y. WANG, ‘Geometric Arveson-Douglas Conjecture and holomorphic extensions’, *Indiana Univ. Math. J.* 66 (2017) 1499–1535.
12. M. ENGLIŠ and J. ESCHMEIER, ‘Geometric Arveson-Douglas conjecture’, *Adv. Math.* 274 (2015) 606–630.
13. Q. FANG and J. XIA, ‘Essential normality of polynomial-generated submodules: Hardy space and beyond’, *J. Funct. Anal.* 265 (2013) 2991–3008.
14. Q. FANG and J. XIA, ‘On the essential normality of principal submodules of the Drury-Arveson module’, *Indiana Univ. Math. J.* 67 (2018) 1439–1498.
15. K. GUO and K. WANG, ‘Essentially normal Hilbert modules and K -homology’, *Math. Ann.* 340 (2008) 907–934.

16. R. HARTSHORNE, *Algebraic geometry*, Graduate Texts in Mathematics 52 (Springer, New York-Heidelberg, 1977) xvi+496.
17. M. KENNEDY, ‘Essential normality and the decomposability of homogeneous submodules’, *Trans. Amer. Math. Soc.* 367 (2015) 293–311.
18. M. KENNEDY and O. SHALIT, ‘Essential normality and the decomposability of algebraic varieties’, *New York J. Math.* 18 (2012) 877–890.
19. J. L. LÓPEZ and N. M. TEMME, ‘Multi-point Taylor expansions of analytic functions’, *Trans. Amer. Math. Soc.* 356 (2004) 4323–4342.
20. W. RUDIN, *Function theory on the unit ball of \mathbb{C}^n* , Reprint of the 1980 edition, Classics in Mathematics (Springer, Berlin, 2008) xiv+436.
21. O. SHALIT, ‘Stable polynomial division and essential normality of graded Hilbert modules’, *J. Lond. Math. Soc.* (2) 83 (2011) 273–289.
22. O. SHALIT and S. BISWAS, ‘Stable division and essential normality: the non-homogeneous and quasi homogeneous cases’, *Indiana Univ. Math. J.* 67 (2018) 169–185.
23. Y. WANG and R. G. DOUGLAS, ‘Geometric Arveson-Douglas conjecture — decomposition of varieties’, *Complex Anal. Oper. Theory* 12 (2018) 1267–1290.
24. Y. WANG and J. XIA, ‘Essential normality for quotient modules and complex dimensions’, *J. Funct. Anal.* 276 (2019) 1061–1096.
25. Y. WANG and J. XIA, ‘Essential normality of principal submodules of the Hardy module on a strongly pseudo-convex domain’, *J. Operator Theory*, to appear.
26. J. XIA, ‘A double commutant relation in the Calkin algebra on the Bergman space’, *J. Funct. Anal.* 274 (2018) 1631–1656.
27. X. ZHOU, ‘A survey on L^2 extension problem’, *Complex geometry and dynamics*, Abel Symposium 10 (eds J. E. Fornæss, M. Irgens and E. F. Wold; Springer, Cham, 2015) 291–309.

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