

# Lifted Multiplicity Codes and the Disjoint Repair Group Property

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## Abstract

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Lifted Reed Solomon Codes (Guo, Kopparty, Sudan 2013) were introduced in the context of locally correctable and testable codes. They are multivariate polynomials whose restriction to any line is a codeword of a Reed-Solomon code. We consider a generalization of their construction, which we call *lifted multiplicity codes*. These are multivariate polynomial codes whose restriction to any line is a codeword of a multiplicity code (Kopparty, Saraf, Yekhanin 2014). We show that lifted multiplicity codes have a better trade-off between redundancy and a notion of locality called the  $t$ -disjoint-repair-group property than previously known constructions. More precisely, we show that, for  $t \leq \sqrt{N}$ , lifted multiplicity codes with length  $N$  and redundancy  $O(t^{0.585}\sqrt{N})$  have the property that any symbol of a codeword can be reconstructed in  $t$  different ways, each using a disjoint subset of the other coordinates. This gives the best known trade-off for this problem for any super-constant  $t < \sqrt{N}$ . We also give an alternative analysis of lifted Reed Solomon codes using dual codes, which may be of independent interest.

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## 1 Introduction

In this work we study *lifted multiplicity codes*, and show how they provide improved constructions of codes with the  $t$ -*disjoint repair group property* ( $t$ -DRGP), a notion of locality in error correcting codes.

An *error correcting code* of length  $N$  over an alphabet  $\Sigma$  is a set  $\mathcal{C} \subseteq \Sigma^N$ . There are several desirable properties in error correcting codes, and in this paper we study the trade-off between two of them. The first is the size of  $\mathcal{C}$ , which we would like to be as big as possible

given  $N$ . The second desirable property is *locality*. Informally, a code  $\mathcal{C}$  exhibits locality if, given (noisy) access to  $c \in \mathcal{C}$ , one can learn the  $i$ 'th symbol  $c_i$  of  $c$  in sublinear time. As we discuss more below, locality arises in a number of areas, from distributed storage to complexity theory.

Two constructions of codes with locality are *lifted codes* [6] and *multiplicity codes* [15]; in fact, both of these constructions were among the first known high-rate Locally Correctable Codes. In this work, consider a combination of the two ideas in *lifted multiplicity codes*, and we show that these codes exhibit locality beyond what's known for either lifted codes or for multiplicity codes.

More precisely, we study a particular notion of locality called the  *$t$ -disjoint-repair-group property* ( $t$ -DRGP). Informally, we say that  $\mathcal{C}$  has the  $t$ -DRGP if any symbol  $c_i$  of  $c \in \mathcal{C}$  can be obtained in  $t$  different ways, each of which involves a disjoint set of coordinates of  $c$ . Formally, we have the following definition.

► **Definition 1.** A code  $\mathcal{C} \subseteq \Sigma^N$  has the  $t$ -disjoint repair property if for every  $i \in [N]$ , there is a collection of  $t$  disjoint subsets  $S_1, \dots, S_t \subseteq [N] \setminus \{i\}$ , and functions  $f_1, \dots, f_t$  so that for all  $c \in \mathcal{C}$  and for all  $j \in [t]$ ,  $f_j(c|_{S_j}) = c_i$ . The sets  $S_1, \dots, S_t$  are called repair groups.

As discussed more in Section 1.1 below, the  $t$ -DRGP naturally interpolates between many different notions of locality. The  $t$ -DRGP is well-studied both when  $t = O(1)$  is small (where it is related to Locally Repairable Codes and nearly equivalently to Private Information Retrieval Codes) and  $t = \Omega(N)$  is large (where it is equivalent to Locally Correctable Codes). For this reason, it is natural to study the  $t$ -DRGP when  $t$  is intermediate; for example, when  $t = N^a$  for  $a \in (0, 1)$ . In this case, it is possible for the size of the code  $|\mathcal{C}|$  to be quite large: more precisely, it is possible for the *rate*  $R = \frac{\log_{|\Sigma|} |\mathcal{C}|}{N}$  to approach 1 (notice that we always have  $|\mathcal{C}| \leq |\Sigma|^N$ , hence we always have  $R \leq 1$ ). Thus, the goal is to understand exactly how quickly the rate can approach 1. That is, given  $t$ , how small can the *redundancy*  $N - RN$  be?

Several works have tackled this question, and we illustrate previous results in Figure 1. Our main result is that lifted multiplicity codes improve on the best-known trade-offs for all super-constant  $t \leq \sqrt{N}$ .

## Contributions

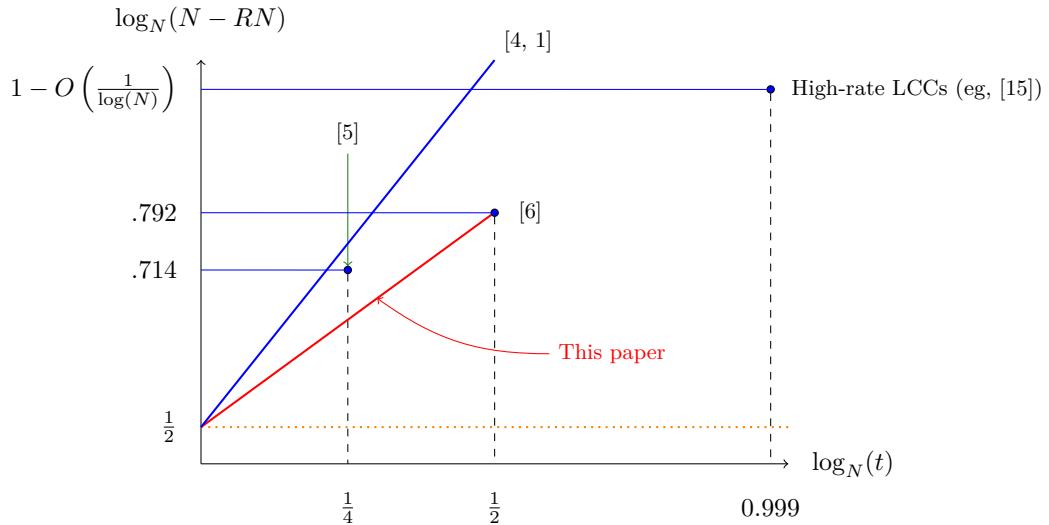
We summarize the main contributions of this work below.

1. For  $t \leq \sqrt{N}$ , we construct codes with the  $t$ -DRGP and redundancy at most

$$O\left(t^{\log_2(3)-1}\sqrt{N}\right) \approx O\left(t^{0.585}\sqrt{N}\right).$$

This gives the best known construction for all  $t$  so that  $t = \omega(1)$  and  $t \leq \sqrt{N}$ ; the only previous result that held non-trivially for a range of  $t$  was redundancy  $O(t\sqrt{N})$  [4, 2, 1] and our result also surpasses the specialized bound for  $t = N^{1/4}$  of [5]. Moreover, both our argument and our construction are quite clean.

2. We give a new analysis of bivariate lifts of multiplicity codes. Both multiplicity codes and lifted codes have been studied before (even in the context of the  $t$ -DRGP), but to the best of our knowledge the only work to consider lifted multiplicity codes is [26]. That work studies  $m$ -variate lifts of multiplicity codes, where  $m$  is large; its goal is to obtain new constructions of high-rate locally correctable codes. In the context of our discussion, this corresponds to the  $t$ -DRGP when  $t = N^{0.99}$ . In contrast, for bivariate lifts, we are able to obtain more refined bounds which lead to improved results for the  $t$ -DRGP when  $t \leq \sqrt{N}$ .



**Figure 1** The best trade-offs known between the number  $t$  of disjoint repair groups and the redundancy  $N - RN$ . Blue points and lines indicate upper bounds (possibility results), and the red line indicates our upper bound. The best lower bound (impossibility result) available is that we must have  $\log_N((1 - R)N) \geq 1/2$  for any  $t \geq 2$ , and this is shown as the dotted orange line.

## Organization

In the remainder of the introduction, we survey related work and give an overview of our approach. In Section 2, we give the formal definitions about polynomials and derivatives that we need. In Section 3, we formally define lifted multiplicity codes. In Section 4, we prove that lifted multiplicity codes have high rate, and in Section 5, we prove that they have the  $t$ -DRGP, which gives rise to our main theorem, Theorem 2.

### 1.1 Background and Related Work

#### 1.1.1 Disjoint Repair Groups

The  $t$ -DRGP and related notions have been studied both implicitly and explicitly across several communities. When  $t = O(1)$  is small, several notions related to the  $t$ -DRGP have been studied, motivated primarily by distributed storage. These include codes for Private Information Retrieval (PIR) [4, 2, 1], Locally Repairable Codes (LRCs) with availability [23, 18, 21, 22], and batch codes [10, 19, 1]. In more detail, PIR codes are basically equivalent to codes with the DRGP, with the slight difference that PIR codes generally require that every message symbol should be recoverable by many disjoint repair groups, rather than every codeword symbol. LRCs with availability are a slightly stronger notion where the disjoint repair groups should additionally be small. Batch codes are also a slightly stronger notion, where one should be able to access any  $t$ -tuple of symbols (possibly with repetition) in  $t$  disjoint ways. We refer the reader to [20] for a survey of these notions.

To see why the  $t$ -DRGP might be relevant for distributed storage, consider a setting where some data is encoded as  $c \in \mathcal{C}$ , and then each  $c_i$  is sent to a separate server. If server  $i$  is later unavailable, we might want to reconstruct  $c_i$  without contacting too many other servers. This can be done if each symbol has one small repair group; this is the defining

property of LRCs. Now suppose that several (say,  $t - 1$ ) servers are unavailable. If  $\mathcal{C}$  has the  $t$ -DRGP then all  $t - 1$  unavailable symbols can be locally reconstructed: each node has at least  $t$  disjoint repair groups and at most  $t - 1$  of them have been compromised.

On the other hand, when  $t = \Omega(N)$  is large, the  $t$ -DRGP has been studied in the context of Locally Decodable Codes and Locally Correctable Codes (LDCs/LCCs). In fact, the  $\Omega(N)$ -DRGP is equivalent to a constant-query LCC, and the notion has been used to prove impossibility results for such codes [11, 24].

Because of these motivations, there are several constructions of  $t$ -DRGP codes for a wide range of  $t$ ; we illustrate the relevant ones in Figure 1. In the context of coded PIR, [4, 2, 1] give constructions of  $t$ -DRGP codes with redundancy  $O(t\sqrt{N})$ . This is known to be tight for  $t = 2$  [17, 25], but no better lower bound is known.<sup>1</sup> When  $t = \Omega(N)$  is very large, constructing codes with the  $t$ -DRGP is equivalent to constructing constant-query LCCs, and it is known that the rate of the code must tend to zero [24]. On the other hand, for any  $\epsilon > 0$ , when  $t = O(N^{1-\epsilon})$  is just slightly smaller, then work on high-rate LCCs [15, 6, 8, 14] (see also [1]) imply that there are codes with rate 0.99 (or any constant less than 1) with the  $t$ -DRGP.<sup>2</sup>

When  $t = \sqrt{N}$ , there are a few constructions known that beat the  $O(t\sqrt{N})$  bound mentioned above, including difference-set codes (see, e.g., [16]) and, relevant for us, lifted parity-check codes [6]. These constructions achieve redundancy  $N^{\log_4(3)} \approx N^{0.79}$  when  $t = \sqrt{N}$ . In Appendix C, we include a new proof of the fact that the lifted codes of [6] have this redundancy using a dual view of lifted codes.

When  $t < \sqrt{N}$ , there is only one construction known which beats the  $O(t\sqrt{N})$  bound, due to [5]. For the special case of  $t = N^{1/4}$ , they give a construction based on “partially lifted codes” which has redundancy  $O(N^{0.72}) = O(t^{0.88}\sqrt{N})$ .

### 1.1.2 Lifting and multiplicity codes

Lifted multiplicity codes are based on lifted codes and multiplicity codes, both of which have a long history in the study of locality in error correcting codes.

#### 1.1.2.1 Lifted Codes

Lifting was introduced by Guo, Kopparty and Sudan in [6]. The basic idea can be illustrated by Reed-Solomon (RS) codes. An RS code of degree  $d$  over  $\mathbb{F}_q$  is the code

$$\text{RS}_{d,q} = \{(f(x_1), \dots, f(x_q)) : f \in \mathbb{F}_q[X], \deg(f) < d\},$$

where  $x_1, \dots, x_q$  are the elements of  $\mathbb{F}_q$ . There is a natural multi-variate version of RS codes, known as Reed-Muller codes:

$$\text{RM}_{d,q,m} = \{(f(\mathbf{x}_1), \dots, f(\mathbf{x}_{q^m})) : f \in \mathbb{F}_q[X_1, \dots, X_m], \deg(f) < d\},$$

where  $\mathbf{x}_1, \dots, \mathbf{x}_{q^m}$  are the elements of  $\mathbb{F}_q^m$ . Reed-Muller codes have a very nice locality property, which is that the restriction of a RM codeword to a line in  $\mathbb{F}_q^m$  yields an RS codeword. This fact has been taken advantage of extensively in applications like local decoding, local list-decoding and property testing. However, RM codes have a downside,

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<sup>1</sup> When the size  $s$  of the repair groups is bounded, it is known that the redundancy must be at least  $\Omega(N \ln(t)/s)$  [22].

<sup>2</sup> In fact we may even take  $\epsilon$  slightly sub-constant using the construction of [14].

which is that if  $d < q$  (required for the above property to kick in), they have very low rate. With this inspiration, we could ask for the set  $\mathcal{C}$  which contains evaluations of *all*  $m$ -variate polynomials which restrict to low-degree univariate polynomials on every line. Surprisingly, [6] showed that this set  $\mathcal{C}$  can be much larger than the corresponding RM code! This code  $\mathcal{C}$  is called a *lifted* Reed-Solomon code, and the main structural result of [6] is that  $\mathcal{C}$  is the span of the monomials whose restrictions to lines are low-degree. This property is key when analyzing the rate of these codes. Moreover [6] showed that this is the case when we begin with *any* affine-invariant code, not just RS codes.

The original motivation for lifted codes was to construct LCCs, but [6] actually also give a code with the  $\sqrt{N}$ -DRGP, mentioned above; we give an alternate proof that this construction has the  $\sqrt{N}$ -DRGP in Appendix C. A variant of lifting was also used in [5] to construct  $N^{1/4}$ -DRGP codes; however, the analysis of this construction is quite brittle and seems difficult to extend to non-trivial constructions for  $t \neq N^{1/4}$ .

### 1.1.2.2 Multiplicity Codes

Multiplicity codes were introduced by Kopparty, Saraf and Yekhanin [15] with the goal of constructing high-rate LCCs. The basic idea of multiplicity codes is to get around the low rate of RM codes discussed above in a different way, by appending derivative information to allow for higher-degree polynomials. That is, it is not useful to have an RS code with degree  $d > q$ , since  $x^q = x$  for any  $x \in \mathbb{F}_q$ . However, if we replace the single evaluation  $f(x)$  with a vector of evaluations  $(f(x), f^{(1)}(x), \dots, f^{(r-1)}(x))$ , where  $f^{(i)}$  denotes the  $i$ 'th derivative, then it does make sense to take  $d > q$ . The  $m$ -variate multiplicity code  $\text{Mult}_{d,q,m,r}$  of degree  $d$  and order  $r$  over  $\mathbb{F}_q$  is then defined similarly to  $\text{RM}_{d,q,m}$ :

$$\text{Mult}_{d,q,m,r} = \left\{ (f^{(<r)}(\mathbf{x}_1), \dots, f^{(<r)}(\mathbf{x}_{q^m})) : f \in \mathbb{F}_q[X_1, \dots, X_m], \deg(f) < d \right\},$$

where  $f^{(<r)}(\mathbf{x}) \in \mathbb{F}_q^{\binom{m+r-1}{m}}$  is a vector containing all of the partial derivatives of  $f$  of order less than  $r$ , evaluated at  $\mathbf{x}$ . Since their introduction, multiplicity codes have found several uses beyond LCCs, including list-decoding [12, 7], and have even been used to explicitly construct codes with the  $t$ -DRGP [1].

### 1.1.2.3 Lifted Multiplicity Codes

To the best of our knowledge, the only work to study lifted multiplicity codes is the work of Wu [26]. The goal of that work is to obtain versions of multiplicity codes which are still high-rate LCCs but which require lower-order derivatives than the construction of [15]. The main result is that lifted multiplicity codes of rate  $1 - \alpha$  are LCCs with locality  $N^\epsilon$  (this corresponds roughly to having the  $t$ -DRGP with  $t = O(N^{1-\epsilon})$ ). However, since the number of variables in the lift is large, it is hard to get a very precise handle on the codimension, and in particular the codimension of the code in that work is not shown to be  $o(N)$ .

In contrast, we study bivariate lifts of multiplicity codes. By focusing only on bivariate lifts, we are able to get a more precise handle on the codimension of lifted multiplicity codes, which gives results for the  $t$ -DRGP for  $t \leq \sqrt{N}$ .

We note that the construction in [26] is similar to the construction presented here. Since this construction is somewhat non-trivial (for reasons discussed below), we include the details.

## 1.2 Our approach

We study lifted multiplicity codes to obtain improved constructions of codes with the  $t$ -DRGP. We focus on bivariate lifts in this paper in order to obtain codes with  $t$ -DRGP for  $t \leq \sqrt{N}$ . We expect that lifted multiplicity codes in more than two variables also give better codes for the  $t$ -DRGP when  $t > \sqrt{N}$ .

### 1.2.1 Definition of lifted multiplicity codes

It is not immediately obvious how to apply lifting (and in particular, the nice characterization of it developed in [6] as the span of “good” monomials) to univariate multiplicity codes. We first note that the univariate multiplicity code  $\text{Mult}_{d,q,1,r} \subseteq (\mathbb{F}_q^r)^q$  does not fit the affine-invariant framework of [6], so their results do not immediately apply. Instead, we might try to define the bivariate lift of  $\text{Mult}_{d,q,1,r}$  as the set of vectors  $(f^{(<r)}(\mathbf{x}_1), \dots, f^{(<r)}(\mathbf{x}_{q^2}))$  for all polynomials  $f$  so that every restriction of  $f$  to a line agrees with some polynomial of degree less than  $d$  on its first  $r-1$  derivatives; that is, the restriction of  $f$  is *equivalent up to order  $r$*  to a polynomial of degree less than  $d$ . This almost works, but there are two non-trivial things to deal with.

1. First, in order to get a handle on the rate of the code, as in [6] we need to characterize the polynomials  $f$  as above as the span of “good” monomials. We show in Proposition 18 that this is possible. Thus, we can alternatively define a lifted multiplicity code as the span of evaluations of monomials  $X^a Y^b$  whose restrictions to lines are equivalent up to order  $r$  to some low-degree polynomial.
2. Second, we need to take some care about what monomials we allow. With lifted RS codes, one only allows monomials  $X^a Y^b$  with individual degrees  $a, b < q$ ; otherwise, we could have multiple monomials which correspond to the same codeword which leads to problems if we are counting monomials in order to understand the dimension of the code. As we show in Lemma 14, it turns out that with multiplicity codes, we should only allow monomials  $X^a Y^b$  with  $\lfloor a/q \rfloor + \lfloor b/q \rfloor < r$ ; otherwise, we would have multiple monomials that correspond to the same codeword and this would create similar problems.

Dealing with these issues leads us to the final definition, given formally in Definition 17: a lifted multiplicity code of order  $r$  and degree  $d$  is the set of vectors  $(f^{(<r)}(\mathbf{x}_1), \dots, f^{(<r)}(\mathbf{x}_{q^m}))$  so that  $f$  is in the span of “good” monomials, where  $X^a Y^b$  is  $(q, r, d)$ -good if  $\lfloor a/q \rfloor + \lfloor b/q \rfloor < r$ , and if for every restriction of  $X^a Y^b$  to a line is equivalent up to order  $r$  to some univariate polynomial of degree less than  $d$ . We note that the work [26] considers a similar construction.

### 1.2.2 Lifted multiplicity codes have the $t$ -DRGP

In Corollary 21 we give a lower bound on the number of  $(q, r, d)$ -good monomials, and this leads to a lower bound on the dimension of the lifted multiplicity code; crucially, this can be quite a bit bigger than the dimension of the corresponding multivariate multiplicity code.

Finally, we observe that lifted multiplicity codes have the  $t$ -DRGP for a range of values of  $t$ . Similarly to previous constructions based on multivariate polynomial codes, the disjoint repair groups to recover the symbol  $f^{(<r)}(\mathbf{x})$  are given by disjoint collections of lines through  $\mathbf{x}$ . More precisely, the values  $f^{(<r)}(\mathbf{y})$  for the set of  $\mathbf{y}$  that lie on  $r$  distinct lines through  $\mathbf{x}$  can be used to recover  $f^{(<r)}(\mathbf{x})$ . Thus, the number of disjoint repair groups is  $q/r = \sqrt{N}/r$ . By adjusting  $r$ , we obtain the trade-off shown in Figure 1. Our main theorem is as follows.

► **Theorem 2.** For  $q = 2^\ell$  and  $r = 2^{\ell'}$  with  $1 \leq \ell' \leq \ell$ , there exists a code  $\mathcal{C}$  over  $\mathbb{F}_q^{(r+1)/2}$  with the following properties.

- The length of the code is  $q^2$ .
- The rate of the code is at least

$$1 - \frac{3r^{\log_2(8/3)}q^{\log_2(3)}}{\binom{r+1}{2}q^2},$$

so that the redundancy is at most

$$\frac{3r^{\log_2(8/3)}q^{\log_2(3)}}{\binom{r+1}{2}}.$$

- The code has the  $q/r$ -disjoint repair group property.

As a remark, our techniques can also recover any symbol from any one of its repair groups in polynomial time. For any  $\gamma \in [0, 1]$ , choosing  $q = 2^\ell$  and  $r = 2^{\ell'}$  with  $\gamma \approx \ell'/\ell$  gives a code with length  $N = q^2$  and redundancy at most

$$6N^{\log_4(3)-\gamma(1-\log_4(8/3))}$$

with the  $N^{(1-\gamma)/2}$ -DRGP. This is made formal in the following corollary.

► **Corollary 3.** For any  $\epsilon > 0$ , there are infinitely many  $N$  so that, for  $t = \lfloor N^\epsilon \rfloor$ , there exists a code of length  $N$  which has the  $t$ -DRGP and redundancy at most  $6t^{\log_2(3)-1}\sqrt{N}$ .

We note that Theorem 2 also yields results for constant  $t$ , not just for  $t = N^\epsilon$  as presented in Corollary 3. For example, by setting  $r = q/2$  we obtain a code with the 2-DRGP and redundancy at most  $9\sqrt{N}$ . The constant 9 is not optimal here (the optimal constant for  $t = 2$  is known to be  $\sqrt{2}$  [17]), but to the best of our knowledge, Theorem 2 does yield the best known bounds for any super-constant  $t$ .

## 2 Preliminaries

In this section, we introduce the background we need on polynomials and derivatives over finite fields. Throughout this paper, we assume that  $q$  is a power of 2. Let  $\mathbb{F}_q$  denote the finite field of order  $q$ , and let  $\mathbb{F}_q^*$  denote its multiplicative subgroup.

If  $a$  and  $b$  are nonnegative integers with binary representations  $a = \overline{a_{\ell-1} \cdots a_0}$  and  $b = \overline{b_{\ell-1} \cdots b_0}$ , then we write  $a \leq_2 b$  if  $a_i \leq b_i$  for  $i = 0, \dots, \ell - 1$ . If  $a$  is an integer, let  $(a \bmod c)$  denote the element of  $\{0, \dots, c - 1\}$  congruent to  $a \bmod c$ . We write  $a \leq_2^\ell b$  if  $(a \bmod 2^\ell) \leq_2 (b \bmod 2^\ell)$ .

As in [6], we use Lucas's theorem.

► **Proposition 4** (Lucas's theorem). Let  $p$  be a prime and  $a = \overline{a_{\ell-1} \cdots a_0}, b = \overline{b_{\ell-1} \cdots b_0}$  be written in base  $p$ . Then

$$\binom{a}{b} \equiv \prod_{i=0}^{\ell-1} \binom{a_i}{b_i} \pmod{p} \quad (1)$$

In particular, if  $p = 2$ , then  $\binom{a}{b} \equiv 1 \pmod{p}$  if and only if  $a \leq_2 b$ .

## 2.1 Polynomials and derivatives

For a vector  $\mathbf{i} = (i_1, \dots, i_m)$  of nonnegative integers, its *weight*, denoted  $\text{wt}(\mathbf{i})$ , equals  $\sum_{k=1}^m i_k$ . For a field  $\mathbb{F}$ , let  $\mathbb{F}[X_1, \dots, X_m] = \mathbb{F}[\mathbf{X}]$  be the ring of polynomials in the variables  $X_1, \dots, X_m$  with coefficients in  $\mathbb{F}$ . For a vector of nonnegative integers  $\mathbf{i} = (i_1, \dots, i_m)$  and a vector  $\mathbf{X} = (X_1, \dots, X_m)$  of variables, let  $\mathbf{X}^{\mathbf{i}}$  denote the monomial  $\prod_{j=1}^m X_j^{i_j} \in \mathbb{F}[\mathbf{X}]$ , and for a vector  $\mathbf{a} = (\alpha_1, \dots, \alpha_m) \in \mathbb{F}^m$ , let  $\mathbf{a}^{\mathbf{i}}$  denote the value  $\prod_{j=1}^m \alpha_j^{i_j}$ , where  $0^0 \stackrel{\text{def}}{=} 1$ . For nonnegative vectors  $\mathbf{i} = (i_1, \dots, i_m)$  and  $\mathbf{j} = (j_1, \dots, j_m)$ , we write  $\mathbf{i} \leq \mathbf{j}$  if  $i_k \leq j_k$  for all  $k$ . We also write  $\binom{\mathbf{i}+\mathbf{j}}{\mathbf{i}}$  to denote  $\prod_{k=1}^m \binom{i_k+j_k}{i_k}$ . For nonnegative vector  $\mathbf{i}$ , we let  $[\mathbf{X}^{\mathbf{i}}]P(\mathbf{X})$  denote the coefficient of  $\mathbf{X}^{\mathbf{i}}$  in the polynomial  $P(\mathbf{X})$ .

We will use Hasse derivatives, a notion of derivatives over finite fields:

► **Definition 5** (Hasse derivatives). *For  $P(\mathbf{X}) \in \mathbb{F}[\mathbf{X}]$  and a nonnegative vector  $\mathbf{i}$ , the  $\mathbf{i}$ -th (Hasse) derivative of  $P$ , denoted  $P^{(\mathbf{i})}(\mathbf{X})$  or  $D^{(\mathbf{i})}P(\mathbf{X})$ , is the coefficient of  $\mathbf{Z}^{\mathbf{i}}$  in the polynomial  $\tilde{P}(\mathbf{X}, \mathbf{Z}) \stackrel{\text{def}}{=} P(\mathbf{X} + \mathbf{Z}) \in \mathbb{F}[\mathbf{X}, \mathbf{Z}]$ . Thus,*

$$P(\mathbf{X} + \mathbf{Z}) = \sum_{\mathbf{i}} P^{(\mathbf{i})}(\mathbf{X}) \mathbf{Z}^{\mathbf{i}}. \quad (2)$$

For  $\mathbf{x} \in \mathbb{F}_q^m$  and  $P(X) \in \mathbb{F}_q[\mathbf{X}]$ , we use the notation  $P^{(<r)}(\mathbf{x}) \in \mathbb{F}_q^{\binom{m+r-1}{m}}$  to denote the vector containing  $P^{(\mathbf{i})}(\mathbf{x})$  for all  $\mathbf{i}$  so that  $\text{wt}(\mathbf{i}) < r$ . We record a few useful (well-known) properties of Hasse derivatives below (see [9]).

► **Proposition 6** (Properties of Hasse derivatives). *Let  $P(\mathbf{X}), Q(\mathbf{X}) \in \mathbb{F}[\mathbf{X}]^m$  and let  $\mathbf{i}, \mathbf{j}$  be vectors of nonnegative integers. Then*

1.  $P^{(\mathbf{i})}(\mathbf{X}) + Q^{(\mathbf{i})}(\mathbf{X}) = (P + Q)^{(\mathbf{i})}(\mathbf{X})$ .
2.  $(P \cdot Q)^{(\mathbf{i})}(\mathbf{X}) = \sum_{0 \leq \mathbf{e} \leq \mathbf{i}} P^{(\mathbf{e})}(\mathbf{X}) \cdot Q^{(\mathbf{i}-\mathbf{e})}(\mathbf{X})$ .
3.  $(P^{(\mathbf{i})})^{(\mathbf{j})}(\mathbf{X}) = \binom{\mathbf{i}+\mathbf{j}}{\mathbf{i}} P^{(\mathbf{i}+\mathbf{j})}(\mathbf{X})$ .

Using the above, we obtain the following useful derivative computation, and we provide a proof in Appendix A for completeness.

► **Proposition 7.** *Let  $1 \leq r < q$  with  $q$  a power of 2, and let  $P(X) = (X^q - X)^r$ . Then,*

$$P^{(i)}(X) = \begin{cases} \binom{r}{i} (X^q - X)^{r-i} & 0 \leq i \leq r \\ 0 & i > r \end{cases} \quad (3)$$

## 2.2 Polynomial local recovery

A key property exploited by earlier work on multiplicity codes [15, 13] is that  $f^{(<r)}(\mathbf{x})$  can be recovered from  $f^{(<q)}(\mathbf{y})$  for  $\mathbf{y}$  that lie on a collection of lines through  $\mathbf{x}$ . More precisely, let  $\mathcal{L}_m$  be the set of lines  $L(T)$  of the form  $\mathbf{a}T + \mathbf{b}$  with  $\mathbf{a}, \mathbf{b} \in \mathbb{F}_q^m$ . Given a multivariate polynomial  $P(\mathbf{X}) \in \mathbb{F}_q[X_1, \dots, X_m]$ , if  $L$  is the line  $\mathbf{a}T + \mathbf{b}$ , let  $P_L(T) \in \mathbb{F}_q[T]$  denote the univariate polynomial  $P(\mathbf{a}T + \mathbf{b})$ . Let  $\mathcal{L}$  be the set of lines in  $\mathbb{F}_q^2$  of the form  $L(T) = (T, \alpha T + \beta)$  for  $\alpha, \beta \in \mathbb{F}_q$ .

For simplicity – and because it is enough for our application to the  $t$ -DRGP – we will consider only bivariate polynomials in this paper, although (see for example [13]) the same basic idea works for any  $m$ . We will further specialize to lines in  $\mathcal{L}$  – that is, lines of the form  $L(T) = (T, \alpha T + \beta)$  – because it will simplify some computations later in the paper. With these restrictions, we can specialize Equation (4) of [13] to obtain the following relationship between the derivatives of  $P_L(T)$  and the derivatives of  $P(X, Y)$ .

► **Lemma 8** (Follows from, e.g., [15, 13]). Suppose that  $L_1, \dots, L_r$  are  $r$  lines in  $\mathcal{L}$  all passing through a point  $(\gamma, \delta)$ , with  $L_k$  being the line  $(T, \alpha_k T + \beta_k)$ . Then, for all polynomials  $P(X, Y) \in \mathbb{F}_q[X, Y]$ , the following matrix equality holds for all  $i = 0, \dots, r - 1$ .

$$\begin{bmatrix} P_{L_1}^{(i)}(\gamma) \\ P_{L_2}^{(i)}(\gamma) \\ \vdots \\ P_{L_{i+1}}^{(i)}(\gamma) \end{bmatrix} = \begin{bmatrix} \alpha_1^0 & \alpha_1^1 & \cdots & \alpha_1^i \\ \alpha_2^0 & \alpha_2^1 & \cdots & \alpha_2^i \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{i+1}^0 & \alpha_{i+1}^1 & \cdots & \alpha_{i+1}^i \end{bmatrix} \begin{bmatrix} P^{(i,0)}(\gamma, \delta) \\ P^{(i-1,1)}(\gamma, \delta) \\ \vdots \\ P^{(0,i)}(\gamma, \delta) \end{bmatrix}. \quad (4)$$

When lines  $L_1, \dots, L_k$  are distinct, the middle matrix in (4) is a Vandermonde matrix, and Vandermonde matrices are invertible in polynomial time. Hence, we immediately have the following corollary.

► **Corollary 9.** Suppose that  $L_1, \dots, L_r$  are  $r$  distinct lines of the form  $L_k(T) = (T, \alpha_k T + \beta_k)$  all passing through a point  $(\gamma, \delta) \in \mathbb{F}_q^2$ . For a polynomial  $P(X, Y) \in \mathbb{F}_q[X, Y]$ , given the polynomials  $P_{L_1}(T), \dots, P_{L_r}(T)$ , the derivatives  $P^{(i)}(\gamma, \delta)$  are uniquely determined and computable efficiently for all  $\mathbf{i}$  such that  $\text{wt}(\mathbf{i}) < r$ .

### 3 Lifted multiplicity codes

In this section, we define lifted multiplicity codes. As noted in the introduction, we restrict our attention to bivariate codes because this is enough for our application to the  $t$ -DRGP. However, everything in this section extends to general  $m$ -variate codes. We define bivariate lifted multiplicity codes as the vectors  $(f^{(<r)}(\mathbf{x}))_{\mathbf{x} \in \mathbb{F}_q^2}$  for polynomials  $f(X)$  that live in the span of “good” monomials. In order to define these “good” monomials, we need a few more definitions.

#### 3.1 Polynomial equivalence

We first define a notion of polynomial equivalence.

► **Definition 10.** We say that two univariate polynomials  $A(X), B(X) \in \mathbb{F}_q[X]$  are equivalent up to order  $r$ , written  $A \equiv_r B$ , if  $A^{(i)}(\gamma) = B^{(i)}(\gamma)$  for all  $i = 0, \dots, r - 1$  and  $\gamma \in \mathbb{F}_q$ .

It is easy to see that the above definition does in fact give an equivalence relation. There is a simple way to characterize this equivalence.

► **Lemma 11.** For  $A(X), B(X) \in \mathbb{F}_q[X]$  we have  $A(X) \equiv_r B(X)$  if and only if  $(X^q - X)^r | A(X) - B(X)$ .

**Proof.** By considering the polynomial  $A(X) - B(X)$ , it suffices to prove  $A(X)$  is equivalent to the zero polynomial up to order  $r$  if and only if  $(X^q - X)^r | A(X)$ . If  $A(X) = (X^q - X)^r C(X)$  for some polynomial  $C(X) \in \mathbb{F}_q[X]$ , then, by part 2 of Proposition 6 and Proposition 7, for  $0 \leq i < r$ , we have  $X^q - X | A^{(i)}(X)$ , so  $A^{(i)}(\gamma) = 0$  for all  $0 \leq i < r$  and all  $\gamma \in \mathbb{F}_q$ , so  $A(X) \equiv_r 0$ .

Conversely, suppose that  $A(X) \equiv_r 0$ . By the definition of Hasse derivatives, we have  $A(X) = A(\gamma + (X - \gamma)) = \sum_i A^{(i)}(\gamma)(X - \gamma)^i$ . Since  $A^{(i)}(\gamma) = 0$  for  $i = 0, \dots, r - 1$ , we have  $(X - \gamma)^r | A(X)$ . Thus is true for all  $\gamma$ , so  $\prod_{\gamma} (X - \gamma)^r | A(X)$ , so  $(X^q - X)^r | A(X)$ . ◀

Lemma 11 gives the following corollary.

► **Lemma 12.** *Let  $q$  be a power of 2 and  $r \geq 1$ . For every univariate polynomial  $A(X)$ , there exists a unique degree-at-most  $rq - 1$  polynomial  $B(X)$  such that  $A(X) \equiv_r B(X)$ . Furthermore, if  $r$  is a power of 2, then for all  $a$  such that  $\deg A - (qr - r) < a < qr$ , we have  $[X^a]A(X) = [X^a]B(X)$ .*

**Proof.** For existence of  $B(X)$ , note that, by Lemma 11, we can subtract an appropriate multiple of  $(X^q - X)^r$  from  $A(X)$  to obtain the desired  $B(X)$ . For uniqueness of  $B(X)$ , suppose that  $B_1(X)$  and  $B_2(X)$  are equivalent to  $A(X)$  up to order  $r$  and are of degree at most  $rq - 1$ . By Lemma 11, we have  $(X^q - X)^r | B_1(X) - B_2(X)$ . Additionally,  $B_1(X) - B_2(X)$  has degree at most  $rq - 1$ , so  $B_1(X) - B_2(X) = 0$ .

Now suppose  $r$  is a power of 2. Then  $(X^q - X)^r = X^{rq} + X^r$ . Above, to obtain  $B(X)$  from  $A(X)$ , we need only to subtract terms of the form  $X^{qr} + X^r, X^{qr+1} + X^{r+1}, \dots, X^{\deg A} + X^{\deg A - qr + r}$ . Thus, for  $a$  such that  $\deg A - qr + r < a < qr$ , the coefficients of  $X^a$  in  $A(X)$  and  $B(X)$  are equal. ◀

### 3.2 Type- $r$ polynomials

Define the order- $r$  evaluation map  $\text{eval}_{q,r} : \mathbb{F}_q[X, Y] \rightarrow \left(\mathbb{F}_q^{\binom{r+1}{2}}\right)^{q^2}$  by

$$\text{eval}_{q,r}(P) := (P^{(<r)}(\mathbf{x}))_{\mathbf{x} \in \mathbb{F}_q^2}, \quad (5)$$

We will want to restrict our attention to a subset of monomials  $M(X, Y) = X^a Y^b$  whose order- $r$  evaluations  $\text{eval}_{q,r}(M)$  form a basis for the space  $\{\text{eval}_{q,r}(P) : P \in \mathbb{F}_q[X, Y]\}$ . To that end, we introduce the following definition.

► **Definition 13** (Type- $r$  monomials). *Call a monomial  $X^a Y^b$  type- $r$  if  $\lfloor a/q \rfloor + \lfloor b/q \rfloor \leq r - 1$ . Let  $\mathcal{F}_{q,r}$  be the family of polynomials  $P \in \mathbb{F}_q[X, Y]$  that are spanned by type- $r$  monomials.*

It is easy to see that  $\mathcal{F}_{q,r}$  is a dimension  $\binom{r+1}{2}q^2$  vector space over  $\mathbb{F}_q$ . We now show that the type- $r$  polynomials form a basis for bivariate polynomials, up to order  $r$  equivalence.

► **Lemma 14.** *The evaluation map  $\text{eval}_{q,r} : \mathcal{F}_{q,r} \rightarrow \left(\mathbb{F}_q^{\binom{r+1}{2}}\right)^{q^2}$  is a bijection.*

**Proof of Lemma 14.** Since  $\text{eval}_{q,r}$  is a linear map and  $\mathcal{F}_{q,r}$  and  $\mathbb{F}_q^{\binom{r+1}{2}q^2}$  have the same  $\mathbb{F}_q$  dimension, it suffices to prove the map has trivial kernel. We prove by induction.

**Base Case:  $r = 1$ .** Suppose  $P \in \mathcal{F}_{q,1}$  and  $\text{eval}_1(P)$  is the 0-vector. Then  $P(X, Y) = 0$  for all  $X, Y$ . For any  $\delta \in \mathbb{F}_q$ , the polynomial  $P(X, \delta) \in \mathbb{F}_q[X]$  has degree at most  $q - 1$  but has  $q$  roots, so the polynomial must be 0. Hence,  $(Y - \delta) | P(X, Y)$  for all  $\delta$ , so  $Y^q - Y | P(X, Y)$ , which implies  $P = 0$ . This proves that  $\text{eval}_1$  has trivial kernel.

**Inductive step.** Assume  $r \geq 1$  and  $\text{eval}_{q,r}$  has trivial kernel. We prove that  $\text{eval}_{q,r+1}$  has trivial kernel.

Assume  $P(X, Y)$  is a polynomial spanned by type- $(r+1)$  monomials with all  $i$ th derivatives equal to 0 for  $\text{wt}(\mathbf{i}) < r + 1$ . Let  $\delta \in \mathbb{F}_q$  and  $B_\delta(X) \stackrel{\text{def}}{=} P(X, \delta)$ . Then, for  $0 \leq i < r$ , we have  $B_\delta^{(i)}(\gamma) = B^{(i,0)}(\gamma, \delta) = 0$  for all  $\gamma \in \mathbb{F}_q$ . Hence, for all  $\gamma \in \mathbb{F}_q$ , we have  $(X - \gamma)^r | B_\delta(X)$ . Hence,  $(X^q - X)^r | B_\delta(X)$ . Since  $\deg B_\delta(X) \leq \deg_X P(X, Y) < qr$  for all  $\delta$ , we have  $B_\delta(X) = 0$ . Thus,  $P(X, \delta)$  is the 0 polynomial for all  $\delta$ , so  $Y - \delta | P(X, Y)$  for all  $\delta$ , so  $Y^q - Y | P(X, Y)$ . Hence, we may write  $P(X, Y) = (Y^q - Y)Q(X, Y)$  for some polynomial  $Q(X, Y) \in \mathbb{F}_q[X, Y]$ .

As polynomial  $P$  is type- $(r+1)$ , polynomial  $Q$  is type- $r$ : if  $Q$  had a nonzero coefficient for  $X^a Y^b$  with  $\lfloor a/q \rfloor + \lfloor b/q \rfloor > r-1$ , then the coefficient  $X^a Y^{b+q}$  is nonzero in  $P$ , which is a contradiction. For all  $i, j$  with  $i \geq 0, j \geq 1$  and  $i+j \leq r$ , we have

$$P^{(i,j)}(X, Y) = (Y^q - Y)Q^{(i,j)}(X, Y) - Q^{(i,j-1)}(X, Y). \quad (6)$$

Here we applied part 2 of Proposition 6 and the  $r=1$  case of Proposition 7. At every  $X$  and  $Y$ , the left side is 0 by assumption on  $P$  and the right side  $Q^{(i,j-1)}(X, Y)$ . We conclude that  $Q^{(i',j')}$  evaluates to 0 everywhere for every nonnegative  $i'$  and  $j'$  satisfying  $i' + j' \leq r-1$ . Since  $Q$  is type- $r$ , we have  $Q = 0$  by the induction hypothesis, so  $P = 0$ . This completes the induction, completing the proof.  $\blacktriangleleft$

### 3.3 Definition(s) of lifted multiplicity codes

Finally we are ready to define lifted multiplicity codes, which we do below in two ways. As we will see, these two definitions are equivalent in the parameter regimes that we consider in this work. The first, more natural definition is as the set of evaluations  $\text{eval}_{q,r}(P)$  of polynomials whose restrictions to lines<sup>3</sup> are equivalent, up to order  $r$ , to a low degree polynomial:

► **Definition 15** (Lifted multiplicity codes, first definition). *The  $(q, r, d)$  (bivariate) lifted multiplicity code is a code  $\mathcal{C}$  over alphabet  $\Sigma = \mathbb{F}_q^{\binom{r+1}{2}}$  of length  $q^2$  given by*

$$\mathcal{C} = \left\{ \text{eval}_{q,r}(P) : \begin{array}{l} P \in \mathbb{F}_q[X, Y] \text{ and, for any } L(T) \in \mathcal{L}, \\ P(L(T)) \equiv_r Q(T) \text{ for some } Q \in \mathbb{F}_q[T] \text{ of degree at} \\ \text{most } d. \end{array} \right\}$$

Definition 15 is natural but difficult to get a handle on directly. Following the approach of previous work [6, 5], we will alternatively define a lifted multiplicity code as the set of vectors  $\text{eval}_{q,r}(P)$  for  $P$  which lie in the span of “good” monomials, which will make it easier to bound the rate. Informally, a monomial is  $(q, r, d)$ -good if its restriction along every line is equivalent, up to order  $r$ , to a polynomial of degree at most  $d$ .

► **Definition 16**  $((q, r, d)\text{-good polynomials})$ . *Call a monomial  $M_{a,b}(X, Y) = X^a Y^b \in \mathbb{F}_q[X, Y]$   $(q, r, d)$ -good (or simply good, when  $r$  and  $d$  are understood) if it is type- $r$  and for every line  $(T, \alpha T + \beta) \in \mathcal{L}$ , the univariate polynomial  $M_{a,b}(T, \alpha T + \beta)$  is equivalent, up to order  $r$ , to polynomial of degree less than  $d$ , and call it  $(q, r, d)$ -bad otherwise.*

Let  $\mathcal{F}_{q,r,d}$  denote the subspace of  $\mathcal{F}_{q,r}$  spanned by the  $(q, r, d)$ -good monomials. We call the elements of  $\mathcal{F}_{q,r,d}$  the  $(q, r, d)$ -good polynomials.

We then (re-)define lifted multiplicity codes as order- $r$  evaluations of  $(q, r, d)$ -good polynomials.

► **Definition 17** (Lifted multiplicity codes, second definition). *The  $(q, r, d)$  (bivariate) lifted multiplicity code is a code  $\mathcal{C}$  over alphabet  $\Sigma = \mathbb{F}_q^{\binom{r+1}{2}}$  of length  $q^2$  given by*

$$\mathcal{C} = \{ \text{eval}_{q,r}(P) : P \in \mathcal{F}_{q,r,d} \}$$

<sup>3</sup> To simplify calculations, we consider restrictions to lines of the form  $L(T) = (T, \alpha T + \beta)$ . That is, we do not include lines of the form  $L(T) = (\alpha, T)$ .

We believe Definitions 17 and 15 are equivalent in general, and we prove they are for the parameters relevant to this work. One direction of containment is simple, since any  $P \in \mathcal{F}_{q,r,d}$  satisfies the requirement of Definition 15. To show the other direction, we need to show that any polynomial which satisfies the requirement of Definition 15 is also contained in  $\mathcal{F}_{q,r,d}$ ; that is, it can be written as a linear combination of  $(q,r,d)$ -good monomials. We show this in Proposition 25 in the Appendix. This implies the following proposition, and for the rest of the paper we will work with Definition 17.

► **Proposition 18.** *If  $q$  and  $r$  are powers of 2 and if  $d \geq r + q - 1$ , Definition 17 and Definition 15 are equivalent.*

## 4 The rate of lifted multiplicity codes

In this section, we bound the rate (and hence, the redundancy) of lifted multiplicity codes. Our final result on the rate is Corollary 21 below, which implies that for  $r, q$  and  $d$  of an appropriate form, the lifted multiplicity code over order  $r$  and degree  $d$  over  $\mathbb{F}_q$  has rate at least

$$1 - \frac{6}{r} \left( r - \frac{d}{q} \right)^{\log_2(4/3)}.$$

In the next section, we will choose  $d = qr - r$ , which will yield a code of rate  $1 - \frac{6}{r} \left( \frac{r}{q} \right)^{\log_2(4/3)}$  and will give us Theorem 2. We begin with a lemma that will be useful.

► **Lemma 19.** *Let  $s = 2^{\ell_s}$  and  $q = 2^\ell$  with  $\ell_s \leq \ell$ . The number of  $a_1, b_1 \in \{0, 1, \dots, q-1\}$  such that at least one of the following is true*

$$\begin{aligned} q - 1 - a_1 &\leq_2^\ell b_1 \\ q - 2 - a_1 &\leq_2^\ell b_1 \\ &\vdots \\ q - s - a_1 &\leq_2^\ell b_1 \end{aligned} \tag{7}$$

is at most  $2 \cdot 3^\ell \cdot (4/3)^{\ell_s} = 2 \cdot 3^\ell \cdot s^{\log_2(4/3)}$ .

**Proof.** Suppose we write the numbers  $(q - 1 - a_1 \bmod q), (q - 2 - a_1 \bmod q), \dots, (q - s - a_1 \bmod q)$  in binary with  $\ell$  digits (possibly with leading zeros). As these numbers span  $2^{\ell_s}$  consecutive integers mod  $q$ , when written in this binary form, their most significant  $\ell - \ell_s$  coordinates take on at most 2 values. Let  $a_2 = \lfloor \frac{(q-1-a_1 \bmod q)}{2^{\ell_s}} \rfloor$  and  $b_2 = \lfloor \frac{b_1}{2^{\ell_s}} \rfloor$  so that  $a_2, b_2 \in \{0, \dots, 2^{\ell-\ell_s} - 1\}$ , and  $a_2$  and  $b_2$  are the most significant  $\ell - \ell_s$  coordinates of  $(q - 1 - a_1 \bmod q)$  and  $b_1$ , respectively, when written in  $\ell$ -digit binary. Then if one of the equations of (7) is true, then we must have either  $a_2 \leq_2 b_2$  or  $a_2 - 1 \leq_2 b_2$ . This gives at most  $2 \cdot 3^{\ell-\ell_s}$  choices for the pair  $(a_2, b_2)$ . Given  $a_2$  and  $b_2$ , there are  $2^{\ell_s}$  choices for each of  $a_1$  and  $b_1$ , for a total of at most  $2 \cdot 3^{\ell-\ell_s} \cdot 4^{\ell_s}$  solutions to (7). ◀

► **Lemma 20.** *Let  $r = 2^{\ell_r}$ ,  $s = 2^{\ell_s}$  and  $q = 2^\ell$  with  $\ell_r, \ell_s \in \{1, \dots, \ell - 1\}$ . The number of  $(q, r, rq - s)$ -good monomials is at least  $\binom{r+1}{2} 4^\ell - 3r s^{\log_2(4/3)} \cdot 3^\ell$ .*

**Proof.** The number of type- $r$  monomials is  $\binom{r+1}{2} q^2 = \binom{r+1}{2} 4^\ell$ . A monomial  $M_{a,b}$  is  $(q, r, rq - s)$ -good if, for every  $\alpha, \beta \in \mathbb{F}_q$ , we have

$$M_{a,b,\alpha,\beta}(T) \stackrel{\text{def}}{=} T^a (\alpha T + \beta)^b = \sum_{i=0}^b \alpha^i \beta^{b-i} T^{a+i} \binom{b}{i}. \tag{8}$$

can be represented as a polynomial of degree less than  $rq - s$ . Next, we apply Lemma 12, which says that there is a unique polynomial  $B(T)$  so that  $\deg(B) \leq rq - 1$  so that  $B(T) \equiv_r M_{a,b,\alpha,\beta}(T)$ , and further that all of the coefficients  $[T^c]B(T)$  for  $\deg(M_{a,b,\alpha,\beta}) - (qr - r) < c < qr$  are equal to the corresponding coefficient of  $B(T)$ . The degree of the polynomial  $M_{a,b,\alpha,\beta}$  is at most  $(r + 1)q - 2$ , and

$$((r + 1)q - 2) - (qr - r) = r + q - 2 < qr - s$$

for any allowed choice of  $q, r, s$ , so  $[T^c]B(T) = [T^c]M_{a,b,\alpha,\beta}(T)$  for all  $c$  so that

$$qr - s \leq c \leq qr.$$

Thus, to show that  $B(T)$  has degree less than  $qr - s$ , it suffices to show that the coefficients of  $T^{qr-s}, T^{qr-s+1}, \dots, T^{qr-1}$  in  $M_{a,b,\alpha,\beta}$  are all zero.

Write  $a = a_0q + a_1$  and  $b = b_0q + b_1$  where  $a_0 + b_0 \leq r - 1$  and  $0 \leq a_1, b_1 \leq q - 1$ . Note that if  $a_0 + b_0 < r - 1$ , then for  $s' = 1, \dots, s$  coefficient  $[T^{rq-s'}]M_{a,b,\alpha,\beta}$  is always zero except possibly when  $a_0 + b_0 = r - 2$  and  $a_1 + b_1 \geq 2q - s$ . This can happen for at most  $\frac{rs^2}{2}$  pairs  $(a, b)$ . Hence, for  $a_0 + b_0 < r - 1$ , there are  $\leq \frac{rs^2}{2}$  bad monomials  $(a, b)$ .

Now assume  $a_0 + b_0 = r - 1$ . For  $s' = 1, \dots, s$ , the coefficient of  $T^{rq-s'}$  in  $T^a(\alpha T + \beta)^b$  is 0 if  $rq - s' < a$  or  $a + b < rq - s'$ . Otherwise, the coefficient is

$$\alpha^{rq-s'-a} \beta^{b-rq+s'+a} \binom{b}{rq-s'-a} = \alpha^{rq-s'-a} \beta^{b-rq+s'+a} \binom{b_0q+b_1}{b_0q+q-s'-a_1}. \quad (9)$$

By Proposition 4, the binomial coefficient is nonzero  $(\bmod 2)$  if and only if  $b_0q + q - s' - a_1 \leq_2 b_0q + b_1$ , which, as  $q$  is a power of 2, happens only if  $q - s' - a_1 \leq_2 b_1$ . Hence, if  $a_0 + b_0 = r - 1$ , the monomial  $M_{a,b}$  is  $(r, rq - s)$ -bad only if some  $s' = 1, \dots, s$  satisfies  $q - s' - a_1 \leq_2 b_1$ . Hence, by Lemma 19, for a fixed  $a_0, b_0$  with  $a_0 + b_0 = r - 1$ , there are at most  $2s^{\log_2(4/3)}3^\ell$  bad monomials  $M_{a,b}$ , so there are at most  $r \cdot s^{\log_2(4/3)}3^\ell$  bad monomials  $M_{a,b}$  over all  $a_0, b_0$  with  $a_0 + b_0 = r - 1$ . As we showed, there are at most  $\frac{rs^2}{2}$  bad monomials when  $a_0 + b_0 < r - 1$ . Hence, there are at least  $\binom{r+1}{2}4^\ell - 2rs^{\log_2(4/3)}3^\ell - \frac{rs^2}{2} \geq \binom{r+1}{2}q^2 - 3rs^{\log_2(4/3)}q^{\log_2(3/4)}$  good monomials, as desired.  $\blacktriangleleft$

Lemma 20 immediately implies Corollary 21, which in turn implies the informal result stated at the beginning of the section.

► **Corollary 21.** *Let  $r = 2^{\ell_r}$ ,  $s = 2^{\ell_s}$  and  $q = 2^\ell$  with  $\ell_r, \ell_s \in \{1, \dots, \ell - 1\}$ . A  $(q, r, rq - s)$  lifted multiplicity code has rate at least  $1 - 6r^{-1}s^{\log_2(4/3)}q^{\log_2(3/4)}$ .*

► **Remark 22.** We apply Corollary 21 for  $r = s \leq q$ , giving that a lifted multiplicity code of rate at least  $1 - 6r^{\log_2(2/3)}q^{\log_2(3/4)}$ . By comparison [15], a 2-variate multiplicity code of order  $r$  evaluations of degree at most  $rq - r$  polynomials over  $\mathbb{F}_q$  has rate  $\frac{\binom{rq-r+2}{2}}{\binom{r+1}{2}q^2} \leq 1 - \Omega(\frac{1}{r})$ , which is smaller than the rate of lifted multiplicity codes for  $r \ll q$ .

## 5 Disjoint repair groups of lifted multiplicity codes

Finally, we prove Theorem 2, which we repeat below.

► **Theorem (Theorem 2, restated).** *Let  $r = 2^{\ell_r}$  and  $q = 2^\ell$  with  $\ell_r < \ell$  and  $\mathcal{C}$  be the  $(q, r, rq - r)$  lifted multiplicity code, as in Definition 17.*

- *The length of the code is  $q^2$ .*
- *The rate of the code is at least  $1 - 6r^{\log_2(2/3)}q^{\log_2(3/4)}$ .*
- *The code has the  $q/r$ -disjoint repair group property.*

**Proof.** The first item follows from the definition of  $\mathcal{C}$ , and the second item is by Corollary 21. To see the third item, we show that, given a point  $(\gamma, \delta) \in \mathbb{F}_q^2$ , lines  $L_1, \dots, L_r$  passing through  $(\gamma, \delta)$ , and  $P^{(<r)}(\mathbf{y})$  at all points  $\mathbf{y}$  on the lines  $L_1, \dots, L_r$  except  $(\gamma, \delta)$  itself, we can (efficiently) recover  $P^{(<r)}(\gamma, \delta)$ . This guarantees the  $q/r$ -disjoint repair group property, because we can group the  $q$  lines of  $\mathcal{L}$  of the form  $L(T) = (T, \alpha T + \beta)$  passing through  $(\gamma, \delta)$  arbitrarily into groups of  $r$ , giving  $q/r$  disjoint repair groups. For any line  $L_k$ , the polynomial  $P_{L_k}(T)$  has degree at most  $rq-r-1$ , as  $P$  is  $(q, r, qr-r)$ -good. By taking linear combinations of directional derivatives (Lemma 8), we can efficiently compute  $P_{L_k}^{(i)}(\gamma')$  for every  $i = 0, \dots, r-1$ , every  $k = 1, \dots, r$ , and every  $\gamma' \neq \gamma$ . We can compute  $P_{L_k}(T)$  using a generalization of polynomial interpolation. This can be done in  $O(D \log D)$  time, where  $D < rq$  is the degree of the polynomial (see e.g. [3]). Hence, by Corollary 9, from  $P_{L_1}(T), \dots, P_{L_r}(T)$ , we can efficiently compute  $P^{(i,j)}(\gamma, \delta)$  for all  $i, j$  with  $0 \leq i + j \leq r - 1$ .  $\blacktriangleleft$

## 6 Conclusion

We conclude with some open questions.

1. We have shown that lifted multiplicity codes with redundancy  $O(t^{0.585}\sqrt{N})$  have the  $t$ -DRGP for a range of  $t \leq \sqrt{N}$ . However, we do not know of any general lower bounds beyond the lower bound for  $t = 2$  which implies that the redundancy must be at least  $\Omega(\sqrt{N})$  for any  $t$ . Thus, it is an open question whether or not our bound is tight or whether one can do better.
2. Lifted multiplicity codes display better locality for the  $t$ -DRGP problem for  $t \leq \sqrt{N}$ ; it is a natural question to ask whether they can be used for larger  $t$ , and in particular whether they could lead to improved constructions of locally correctable codes.

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## A Proofs of polynomial facts

**Proof of Proposition 7.** By part 2 of Proposition 6,

$$P^{(i)}(X) = \sum_{j_1+\dots+j_r=i} \prod_{k=1}^r D^{(j_k)}(X^q - X). \quad (10)$$

We have  $D^{(1)}(X^q - X) = 1$  (the field has characteristic 2). For  $2 \leq i < q$ , the  $i$ th derivative of  $X^q - X$  is  $\binom{q}{i} X^{q-i}$ , which is 0, as  $\binom{q}{i}$  is even by Proposition 4. The summand above is nonzero if and only if  $j_1, j_2, \dots, j_r \leq 1$ . When  $i \leq r$ , this happens when  $i$  of the  $j_k$ 's are 1 and  $r-i$  are 0, which happens for  $\binom{r}{i}$  choices of  $(j_1, \dots, j_r)$ . This gives  $P^{(i)}(X) = \binom{r}{i} (X^q - X)^{r-i}$  for  $0 \leq i \leq r$ . When  $i > r$ , some  $j_k$  is at least 2, in which case  $P^{(r)}(X) = 0$  for  $r < i < q$ .  $\blacktriangleleft$

**Proof of Lemma 8.** Let  $\mathbf{a}_k$  denote the vector  $(1, \alpha_k)$ , and let  $\mathbf{b}_k$  denote the vector  $(0, \beta_k)$ . By assumption, we have that  $\mathbf{a}_k \gamma + \mathbf{b}_k = (\gamma, \delta)$ . By the definition of Hasse derivatives, we have, for all  $k = 1, \dots, r$

$$\begin{aligned} P_{L_k}(T + Z) &= P(\mathbf{a}_k T + \mathbf{b}_k + \mathbf{a}_k Z) \\ &= \sum_{\mathbf{i} \in \mathbb{N}^2} P^{(\mathbf{i})}(\mathbf{a}_k T + \mathbf{b}_k) \cdot (\mathbf{a}_k Z)^{\mathbf{i}} \\ &= \sum_{\mathbf{i} \in \mathbb{N}^2} P^{(\mathbf{i})}(\mathbf{a}_k T + \mathbf{b}_k) \cdot \mathbf{a}_k^{\mathbf{i}} Z^{\text{wt}(\mathbf{i})} \\ P_{L_k}(T + Z) &= \sum_{i \geq 0} P_{L_k}^{(i)}(T) Z^i \end{aligned} \tag{11}$$

Hence, for all  $i \geq 0$  and  $k = 1, \dots, r$ , we have

$$P_{L_k}^{(i)}(T) = \sum_{\mathbf{i}: \text{wt}(\mathbf{i})=i} P^{(\mathbf{i})}(\mathbf{a}_k T + \mathbf{b}_k) \mathbf{a}_k^{\mathbf{i}} \tag{12}$$

By plugging in  $T = \gamma$ , we have for all  $i \geq 0$  and  $k = 1, \dots, r$ ,

$$P_{L_k}^{(i)}(\gamma) = \sum_{\mathbf{i}: \text{wt}(\mathbf{i})=i} P^{(\mathbf{i})}(\gamma, \delta) \mathbf{a}_k^{\mathbf{i}}. \tag{13}$$

Rewriting this in matrix form gives the desired result.  $\blacktriangleleft$

## B Proof of Proposition 18

In this appendix we prove Proposition 18 that Definitions 17 and 15 are equivalent for parameters relevant to this work.

In Definition 16, we defined an  $(q, r, d)$ -good polynomial  $P \in \mathcal{F}_{q,r,d}$  to be a polynomial spanned by the  $(q, r, d)$ -good monomials. To prove Proposition 18, it suffices to show that this definition is the same as simply looking at the restrictions of  $P$  directly. To that end, we define  $(q, r, d)$ -**good** polynomials, which end up being equivalent to  $(q, r, d)$ -good polynomials in the parameter regime of interest.

► **Definition 23.** Say that a polynomial  $P(X, Y) \in \mathcal{F}_{q,r}$  is  $(q, r, d)$ -**good** if for any line  $L(T) = (T, \alpha T + \beta)$ , there is some  $Q \in \mathbb{F}[T]$  of degree at most  $d$  so that  $P(L(T)) \equiv_r Q(T)$ .

For  $a, b \geq 0$  and  $\alpha, \beta \in \mathbb{F}_q$ , let

$$M_{a,b,\alpha,\beta}(T) \stackrel{\text{def}}{=} T^a(\alpha T + \beta)^b. \tag{14}$$

By Lemma 12, for all  $\alpha, \beta \in \mathbb{F}_q$  and all  $a, b \geq 0$ , there exists a unique polynomial  $M_{a,b,\alpha,\beta}^*(T)$  with  $\deg M_{a,b,\alpha,\beta}^*(T) \leq rq - 1$  such that the polynomial  $T^a(\alpha T + \beta)^b$  is equivalent to  $M_{a,b,\alpha,\beta}^*(T)$  up to order  $r$ .

► **Lemma 24.** Let  $q$  and  $r$  be powers of 2, let  $s$  satisfy  $r + q - 1 \leq s \leq rq - 1$ , and let  $X^a Y^b$  be a type- $r$  monomial. Then  $[T^s] M_{a,b,\alpha,\beta}^*(T) = \binom{b}{s-a} \alpha^{s-a} \beta^{a+b-s}$ .

**Proof.** For all type- $r$  monomials  $X^a Y^b$ , we have  $\deg M_{a,b,\alpha,\beta} \leq a + b \leq rq + q - 2$ , so  $\deg M_{a,b,\alpha,\beta} - (rq - r) < s < rq$ . By Lemma 12, we have

$$[T^s] M_{a,b,\alpha,\beta}^*(T) = [T^s] M_{a,b,\alpha,\beta}(T) = \binom{b}{s-a} \alpha^{s-a} \beta^{a+b-s}, \tag{15}$$

as desired.  $\blacktriangleleft$

► **Proposition 25.** Let  $q$  and  $r$  be powers of 2 with  $r \leq q$  and let  $d \geq r + q - 1$ . Let  $P(X, Y) = \sum_{i=1}^k \zeta_i X^{a_i} Y^{b_i}$  be an  $(q, r, d)$ -good polynomial. Then, for each  $i$ , the monomial  $X^{a_i} Y^{b_i}$  is  $(q, r, d)$ -good.

**Proof.** For all  $\alpha, \beta \in \mathbb{F}_q$ , let

$$P_{\alpha, \beta}^*(T) \stackrel{\text{def}}{=} \sum_{i=1}^k \zeta_i M_{a_i, b_i, \alpha, \beta}^*(T). \quad (16)$$

By definition of  $P_{\alpha, \beta}$ , we have  $P_{\alpha, \beta}(T) \equiv_r P_{\alpha, \beta}^*(T)$  for all  $\alpha, \beta$ , and, by Lemma 12,  $P_{\alpha, \beta}^*(T)$  is the unique polynomial of degree at most  $rq - 1$  with this property. Since  $P(X, Y)$  is  $(q, r, d)$ -good,  $\deg P_{\alpha, \beta}^*(T) < d$  for all  $\alpha, \beta \in \mathbb{F}_q$ . Hence, for all  $s$  with  $d \leq s \leq rq - 1$  and all  $\alpha, \beta \in \mathbb{F}_q$ , we have

$$0 = [T^s] P_{\alpha, \beta}^*(T) = \sum_{i=1}^k [T^s] \zeta_i M_{a_i, b_i, \alpha, \beta}^*(T) = \sum_{i=1}^k \binom{b_i}{s - a_i} \zeta_i \alpha^{s-a_i} \beta^{a_i+b_i-s}. \quad (17)$$

where the last equality follows from Lemma 24 and that  $s \geq d \geq r + q - 1$ . For each  $s$ , the last sum can be viewed as a bivariate polynomial in  $\alpha$  and  $\beta$  (when  $s - a_i < 0$  or  $a_i + b_i - s < 0$ , the binomial  $\binom{b_i}{s - a_i}$  is 0, so this is indeed a polynomial). Furthermore, since this polynomial has degree at most  $q - 1$  in each of  $\alpha$  and  $\beta$  and evaluates to 0 everywhere, this polynomial is the zero polynomial, so  $\binom{b_i}{s - a_i} \equiv 0 \pmod{2}$  for all  $s$  with  $d \leq s \leq rq - 1$  and all  $i$ . Hence, by Lemma 24, we have  $\deg M_{a_i, b_i, \alpha, \beta}^*(T) < d$  for all  $i$  and  $\alpha, \beta \in \mathbb{F}_q$ , so the monomial  $X^{a_i} Y^{b_i}$  is  $(q, r, d)$ -good for all  $i$ . ◀

Proposition 25 shows that any  $(q, r, d)$ -good polynomial is also  $(q, r, d)$ -good, in the sense of Definition 17. It is clear that any  $(q, r, d)$ -good polynomial is also  $(q, r, d)$ -good. In the language of this appendix, Definition 15 says that

$$\mathcal{C} = \{\text{eval}_r(P) : P \in \mathbb{F}_q[X, Y], P \text{ is } (q, r, d)\text{-good}\},$$

while Definition 17 says that

$$\mathcal{C} = \{\text{eval}_r(P) : P \in \mathbb{F}_q[X, Y], P \text{ is } (q, r, d)\text{-good}\}.$$

Thus, we have proved Proposition 18, which says that the two definitions are equivalent provided that  $q$  and  $r$  are powers of 2 and that  $d \geq r + q - 1$ .

## C Lifted codes via dual codes

It was shown in [6] that bivariate lifted parity-check codes over  $\mathbb{F}_q$ , where  $q = 2^\ell$ , have co-dimension  $3^\ell$ . Here, we give an alternative proof using dual codes. The techniques in this proof are not directly related to the techniques that we used in the main body of the paper, but we found this alternative proof illuminating so we include it.

Let  $q = 2^\ell$ . Recall  $\mathcal{L}$  is the set of lines expressible as  $L(T) = (T, \alpha T + \beta)$  where  $\alpha, \beta \in \mathbb{F}_q$ . One way to think about codes with locality is by considering their dual code. If the code is a subset of  $\mathbb{F}_q^{q \times q}$ , then the dual code corresponds to lines of repair groups. Given a line  $L(T)$  in  $\mathcal{L}$ , define the corresponding dual codeword:

$$(c_L^\perp)_{ij} \stackrel{\text{def}}{=} \begin{cases} 1 & (i, j) = L(t) \text{ for some } t \in \mathbb{F}_q \\ 0 & \text{o/w} \end{cases} \quad (18)$$

Let

$$V_{\mathcal{L}} \stackrel{\text{def}}{=} \text{span} \{c_L^\perp : L \in \mathcal{L}\}. \quad (19)$$

Note that  $V_{\mathcal{L}}$  is spanned by  $4^\ell$  elements, so the trivial bound on the dimension is  $4^\ell$ . We give the following improved bound, matching the analysis of [6].

► **Lemma 26.** *The subspace  $V_{\mathcal{L}}$  has dimension at most  $3^\ell$ .*

**Proof.** A codeword  $c_L^\perp$  is the evaluation of the following polynomial on  $\mathbb{F}_q^{q \times q}$ :

$$P_L(X, Y) \stackrel{\text{def}}{=} \prod_{\beta \neq \beta_L} (\alpha_L X + \beta - Y). \quad (20)$$

If  $(X, Y) \notin L$ , then the polynomial evaluates to 0 as  $Y - \alpha_L X \neq \beta_L$ , and otherwise it evaluates to

$$\prod_{\beta \neq \beta_L} (\beta - \beta_L) = \prod_{\beta \in \mathbb{F}_q^*} \beta = 1. \quad (21)$$

For  $a + b \geq q$ , the coefficient of  $X^a Y^b$  in  $P_L(X, Y)$  is 0. For  $a + b \leq q$ , the coefficient of  $X^a Y^b$  in  $P_L(X, Y)$  is

$$\binom{a+b}{a} \alpha_L^a (-1)^b \sum_{\substack{\beta_1, \dots, \beta_{q-1-a-b} \in \mathbb{F}_q \\ \text{distinct, } \neq \beta_L}} \prod_{j=1}^{q-1-a-b} \beta_j. \quad (22)$$

This is because we first chose  $a+b$  terms that contain  $X$  or  $Y$ , then choose which terms are  $X$  and which terms are  $Y$ , and this gives us  $a$  many  $\alpha_L$ 's and  $b$  many  $-1$ 's, and we sum over the choices of the  $\beta$  terms that we choose. Hence, the only  $a, b$  such that  $[X^a Y^b] P_L(X, Y) \neq 0$  for any  $L$  are the pairs  $(a, b)$  such that  $a + b \leq q - 1$  and  $\binom{a+b}{a} \equiv 1 \pmod{2}$ . There are at most  $3^\ell$  pairs by Proposition 4. It follows that the polynomials  $P_L(X, Y)$  are spanned by  $3^\ell$  monomials  $X^a Y^b$  with  $\binom{a+b}{a} \equiv 1 \pmod{2}$ . Hence, the vector space  $V_{\mathcal{L}}$  is spanned by  $3^\ell$  dual codewords in  $\mathbb{F}_q^{q \times q}$  and thus has dimension at most  $3^\ell$ . ◀