Approximating the Steady-State Periodic Solutions of Contractive Systems

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***Abstract*—We consider contractive systems whose trajectories evolve on a compact and convex state-space. It is well-known that if the time-varying vector field of the system is periodic then the system admits a unique globally asymptotically stable periodic solution. Obtaining explicit information on this periodic solution and its dependence on various parameters is important both theoretically and in numerous applications. We develop an approach for approximating such a periodic trajectory using the periodic trajectory of a simpler system (e.g. an LTI system). The approximation includes an error bound that is based on the input-to-state stability (ISS) property of contractive systems. We show that in some cases this error bound can be computed explicitly. We also use the bound to derive a new theoretical result, namely, that a contractive system with an additive periodic input behaves like a low pass filter. We demonstrate our results using several examples from systems biology.**

I. INTRODUCTION

A dynamical system is called *contractive* if any two trajectories approach each other at an exponential rate [14], [1]. This is a strong property with many important implications. For example, if the trajectories evolve on a compact and convex state-space Ω then the system admits an equilibrium point $e \in \Omega$, and since every trajectory converges to the trajectory emanating from e, e is globally asymptotically stable. Establishing this requires no explicit information on e. More generally, contractive systems with a periodic excitation *entrain*, that is, their trajectories converge to a periodic solution with the same period as the excitation [14]. In fact, contractive systems have a well-defined frequency response [21], [29]. This property is very important in applications ranging from entrainment of biological systems to periodic excitations (e.g., the 24h solar day or the periodic cell division process) to the entrainment of synchronous generators to the frequency of the electric grid. However, the proof of the entrainment property of contractive systems is based on implicit arguments (see, e.g. [31], [14]) and provides no explicit information on the periodic trajectory (except for its period).

Contraction theory has found numerous applications in systems and control theory and in systems biology (see e.g.[33], [1]). A particularly interesting line of research is based on combining contraction theory and graph theory in order to study various networks of multi-agent systems [32], [2], [30], [5].

As already noted by Desoer and Haneda [7], contractive systems satisfy a special case of the ISS property (see the survey paper [36]). Under additional conditions, contractive systems are also input-to-output stable [11]. Desoer and Haneda used the ISS property to derive bounds on the error between trajectories of a continuous-time contractive system and its time-discretized model. This is important when computing solutions of contractive systems using numerical integration schemes [15]. Sontag [37] has shown that contractive systems satisfy a "converging-input converging-output" property. A recent paper [3] used the ISS property to derive a bound on the error between trajectories of a continuous-time contractive system and those of some "simpler" continuous-time system (e.g. an LTI system). This bound is particularly useful when the simpler model can be solved explicitly.

Here, we derive new bounds on the distance between the periodic trajectory of a contractive system and the periodic trajectory of a "simpler" system, e.g. an LTI system with a periodic forcing. We show several cases where the periodic trajectory of the simpler system is explicitly known and the bound is also explicit, so this provides considerable information on the unknown periodic trajectory of the contractive system. We use one of the explicit bounds to prove that any contractive system with an additive sinusoidal forcing behaves like a low-pass filter, i.e. as the frequency of the sinusoidal signal goes to infinity the solution of the contractive system converges to an equilibrium state. This generalizes the well-known behavior of asymptotically stable LTI systems.

II. PRELIMINARIES

We briefly review some relevant properties of contractive systems. For more details, see e.g. [35], [12]. Consider the time-varying dynamical system

$$\dot{x} = f(t, x), \tag{1}$$

with the state x evolving on a positively invariant set $\Omega \subseteq \mathbb{R}^n$. Let $x(t, t_0, x_0)$ denote the solution of (1) at time $t \geq t_0$ for the initial condition $x(t_0) = x_0$.

We assume from here on that the following properties hold: (1) the state space Ω is compact and convex; (2) both $f(t, x)$ and its Jacobian $J(t, x) := \frac{\partial f}{\partial x}(t, x)$ are continuous in (t, x);

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and (3) $x(t, t_0, x_0)$ exists and is unique for all $t \geq t_0 \geq 0$ and all $x_0 \in \Omega$.

The system (1) is said to be *contractive* on Ω with respect to a vector norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ if there exists $\eta > 0$ such that

$$|x(t, t_0, a) - x(t, t_0, b)| \leq e^{-(t-t_0)\eta}|a - b| \tag{2}$$

for all $t \geq t_0 \geq 0$ and all $a, b \in \Omega$, i.e. any two trajectories approach one another at an exponential rate η. Note that this implies in particular that (1) is incrementally stable, and under our assumptions on Ω this also means that (1) is a convergent system [29]. Contraction can be defined in a more general way, for example with respect to a time- and space-varying norm [14] (see also [9]).

We focus here on exponential contraction with respect to a *fixed* vector norm because there exist easy to check sufficient conditions, based on matrix measures, guaranteeing that (2) holds. A vector norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ induces a *matrix measure* $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$ defined by

$$\mu(A) := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(||I + \varepsilon A|| - 1),$$

where $||\cdot|| : \mathbb{R}^{n\times n} \to \mathbb{R}_+$ is the matrix norm induced by the vector norm $|\cdot|$ [40]. For example, for the ℓ_1 vector norm, denoted $|\cdot|_1$, the induced matrix measure is $\mu_1(A) = \max\{c_1(A), \dots, c_n(A)\}$, where $c_j(A) := A_{jj} + \sum_{i\neq j} |A_{ij}|$, i.e., the sum of the entries in column j of A, with non-diagonal elements replaced by their absolute values.

If the Jacobian of f satisfies

$$\mu(J(t, x)) \leq -\eta, \text{ for all } x \in \Omega \text{ and all } t \geq 0, \tag{3}$$

then (2) holds for all $t_0 \geq 0$ (see [31] for a self-contained proof).

Often it is useful to work with scaled vector norms [34], [4]. Let $|\cdot|_* : \mathbb{R}^n \to \mathbb{R}_+$ be some vector norm, and let $\mu_* : \mathbb{R}^{n\times n} \to \mathbb{R}$ denote its induced matrix measure. If $D \in \mathbb{R}^{n\times n}$ is an invertible matrix, and $|\cdot|_{*,D} : \mathbb{R}^n \to \mathbb{R}_+$ is the vector norm defined by $|z|_{*,D} := |Dz|_*$, then the induced matrix measure is $\mu_{*,D}(A) = \mu_*(DAD^{-1})$. For example, the matrix measure induced by the Euclidean norm $|\cdot|_2$ is $\mu_2(A) = \max\{\lambda : \lambda \in \Lambda\{(A + A')/2\}\}$, where $\Lambda\{A\}$ denotes the set of eigenvalues of A. Hence,

$$\begin{aligned}\mu_{2,D}(A) &= \mu_2(DAD^{-1}) \\ &= \max\{\lambda : \lambda \in \Lambda\{(DAD^{-1} + (DAD^{-1})')/2\}\}.\end{aligned}$$

The next result describes an ISS property of contractive systems with an additive input.

Theorem 1 ([7], Thm. *A) Consider the system*

$$\dot{x} = f(t, x) + u(t), \tag{4}$$

where $y \to f(t, y)$ is C^1 and $f(t, 0) = 0$ for all $t \geq t_0$ and $u(t)$ is piecewise continuous. Fix some vector norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ and suppose that (3) *holds for the induced matrix measure $\mu(\cdot)$. Then*

$$|x(t, t_0, x_0)| \leq e^{-\eta(t-t_0)}|x_0| + \int_{t_0}^{t} e^{-\eta(t-s)}|u(s)|\, ds$$

for all $t \geq t_0$.

Ref. [3] has applied this property to derive a bound on the error between trajectories of a contractive system (1) and those of a "simpler" system. Here, we consider the specific case where the vector field $f(t, x)$ is time-varying and T*-periodic* for some $T > 0$, that is, $f(t, z) = f(t + T, z)$ for all $t \geq t_0$ and all $z \in \Omega$. In this case every trajectory of (1) converges to a unique periodic solution $\gamma(t)$ of (1) with period T [31]. This entrainment property is very important in applications (see, e.g. [17], [31]). However, the proof of entrainment is based on implicit arguments and provides no information on the properties of the period trajectory (except for its period). Here, we derive a bound for the difference between $\gamma(t)$ and the periodic solution $\kappa(t)$ of some "simpler" approximating system. We also suggest suitable approximating systems.

III. BOUNDING THE DIFFERENCE BETWEEN TWO PERIODIC TRAJECTORIES

The next result provides a bound on the distance between a T-periodic trajectory of a contractive systems and a T-periodic trajectory of some *approximating system (AS).*

Theorem 2 *Consider the system* (1)*. Suppose that $f(t, x)$ and $J(t, x)$ are continuous in (t, x) and that $f(t, x)$ is T-periodic. Let $|\cdot|$ be some vector norm on $\mathbb{R}^n$ and $\mu(\cdot)$ its induced matrix measure, and suppose that* (3) *holds. Let $\gamma(t)$ be the unique periodic trajectory of* (1) *with period T. Consider another time-varying system*

$$\dot{y} = g(t, y) \tag{5}$$

and suppose that $g(t, y)$ is also T-periodic and that $\kappa(t)$ is a T-periodic trajectory of (5) *with $\kappa(t) \in \Omega$ for all $t \in [0, T]$. Define $c : \mathbb{R}_+ \to \mathbb{R}_+$ by $c(\alpha) := \int_0^\alpha e^{-\eta(\alpha-s)}|f(s, \kappa(s)) - g(s, \kappa(s))|\, \mathrm{d}s$. Then*

$$|\gamma(t) - \kappa(t)| \leq \frac{e^{-\eta t}c(T)}{1 - e^{-\eta T}} + c(t), \textit{ for all } t \in [0, T]. \tag{6}$$

Note that the bound here depends on the difference between the vector fields f and g evaluated along the periodic trajectory $\kappa(s)$ of the "simpler" y system. This is useful for example when the y system is an asymptotically stable LTI system with a sinusoidal forcing term, as then $\kappa(t)$ is known explicitly.

Proof. Let $z(t) := \gamma(t) - \kappa(t)$. Then

$$\dot{z}(t) = f(t, \gamma(t)) - f(t, \kappa(t)) + u(t),$$

where $u(t) := f(t, \kappa(t)) - g(t, \kappa(t))$. Note that

$$\begin{aligned}f(t, \gamma(t)) - f(t, \kappa(t)) &= M(t)(\gamma(t) - \kappa(t)) \\ &= M(t)z(t),\end{aligned}$$

where $M(t) := \int_0^1 J(t, s\gamma(t) + (1-s)\kappa(t))\, \mathrm{d}s$. Using the sub-additivity of the matrix measure to conclude that $\mu(M(t)) \leq$

$-\eta$, and using Thm. 1 gives

$$|z(t)| \le e^{-\eta t}|z(0)| + c(t). \tag{7}$$

Substituting $t = T$ yields $|z(T)| \le e^{-\eta T}|z(0)| + c(T)$. By periodicity, $z(T) = z(0)$, so $|z(0)| \le c(T)/(1 - e^{-\eta T})$. Substituting this in (7) yields (6). ■

The next example demonstrates a case where the bound (6) is tight for $t \to \infty$.

Example 1 Consider the scalar system $\dot{x} = f(t,x) := -x + 1 + \sin(2\pi t/T)$, with $T > 0$. Note that $\Omega := [0,2]$ is an invariant set of this dynamics, and that f is T-periodic. The attracting T-periodic trajectory is

$$\gamma(t) = 1 + \frac{T^2\sin(2\pi t/T) - 2\pi T\cos(2\pi t/T)}{4\pi^2 + T^2}.$$

The Jacobian of f is $J(x) = -1$, so for any vector norm the induced matrix measure satisfies $\mu(J(x)) = -1$. Consider the AS $\dot{y} = -y$, which is (vacuously) T-periodic, and admits the T-periodic solution $\kappa(t) \equiv 0$, that belongs to Ω for all t. It is not difficult to show that for $t = T - \varepsilon$, with $\varepsilon > 0$ and very small, the left- and right-hand sides of (6) become equal as $T \to \infty$. □

By the definition of $c(\cdot)$,

$$\eta c(\alpha) \le (1 - e^{-\eta\alpha}) \max_{t\in[0,\alpha]} |f(t,\kappa(t)) - g(t,\kappa(t))|$$

for all $\alpha \ge 0$, and combining this with (6) yields a simpler (and less tight) bound.

Corollary 1 *For all $\tau \ge 0$,*

$$|\gamma(\tau) - \kappa(\tau)| \le (1/\eta) \max_{t\in[0,T]} |f(t,\kappa(t)) - g(t,\kappa(t))|. \tag{8}$$

This is useful when one can establish a bound on the difference between the vector fields f and g along the periodic trajectory κ of the AS. For the special case of the L_2-norm, Corollary 1 can be obtained from the results in [22, Ch. 2].

The bound (8) demonstrates a tradeoff: if g is "close" to f then the error $f - g$ will be small, yet κ may be an unknown complicated trajectory (as we assume that f is a nonlinear vector filed). On the other hand, if g is relatively simple (e.g., the vector field of an LTI system) then κ may be known explicitly yet that difference $|f - g|$ may be large.

Thm. 2 and Corollary 1 provide a bound on the distance of the unique T-periodic trajectory of a contractive system and some T-periodic trajectory of an AS. The next step is to determine a suitable AS.

From hereon, we consider a special case of a contractive system with the form

$$\dot{x} = f(t,x) = F(x,u(t)) \tag{9}$$

where $u(t)$ is a given m-dimensional, T-periodic excitation. We also assume from here on that all the conditions in Thm. 2 hold.

The first case to consider is the AS $\dot{y} \equiv 0$ for which any $z \in \Omega$ is an equilibrium and thus a periodic solution. This yields the following.

Corollary 2 *Let $c(z) := \max_{t\in[0,T]} |F(z,u(t))|$. For all $\tau \in [0,T]$ and all $z \in \Omega$,*

$$\begin{aligned}|\gamma(\tau) - z| &\le \frac{e^{-\eta\tau}}{1 - e^{-\eta T}} \int_0^T e^{-\eta(T-s)}|F(z,u(s))|\,\mathrm{d}s \\ &\quad + \int_0^\tau e^{-\eta(\tau-s)}|F(z,u(s))|\,\mathrm{d}s \qquad (10)\\ &\le c/\eta. \qquad (11)\end{aligned}$$

The bound (11) is known in the particular case of the L_2-norm [22]. The following simple example demonstrates a case where the bounds in Corollary 2 are tight.

Example 2 Consider the *scalar* system $\dot{x} = F(x,u) := -ax + b$ with $a > 0$. Then $\gamma(t) \equiv b/a =: e$ is a periodic trajectory. This system is contractive with rate $\eta = a$. The bound (11) gives $|e-z| \le |-az+b|/a = |e-z|$ for any $z \in \mathbb{R}$, so this bound is tight. □

The next example demonstrates an application of Corollary 2 to a nonlinear contractive system.

Example 3 The ribosome flow model (RFM) [28] is a non-linear compartmental model describing the unidirectional flow of particles along a 1D chain of n sites using n non-linear first-order differential equations. Recently, the RFM has been extensively used to model and analyze the sequential flow of ribosomes (the particles) on groups of codons (the sites) along the mRNA molecule during translation (see, e.g. [42], [19], [18], [17], [26], [27], [25], [44], [43], [45]).

Consider the RFM with $n = 2$ and a *time-varying* initiation rate $u_0(t)$, that is,

$$\begin{aligned}\dot{x}_1 &= (1 - x_1)u_0 - \lambda_1 x_1(1 - x_2),\\ \dot{x}_2 &= \lambda_1 x_1(1 - x_2) - \lambda_2 x_2,\end{aligned} \tag{12}$$

where λ_1, λ_2 are positive constants. Suppose that $u_0(t) = \lambda_0 + \sin(2\pi t/T)$, with $\lambda_0 > 1$, $T > 0$, i.e. the initiation rate is a strictly positive periodic function with (minimal) period T. The state space here is $\Omega := [0,1]^2$. The Jacobian of (12) is $J(t,x) = \begin{bmatrix}-u_0(t) - \lambda_1(1 - x_2) & \lambda_1 x_1\\ \lambda_1(1 - x_2) & -\lambda_1 x_1 - \lambda_2\end{bmatrix}$. The off-diagonal terms are non-negative for any $x \in [0,1]^2$, so $\mu_1(J(t,x)) = \max\{-u_0(t), -\lambda_2\}$ for all $t \ge 0$ and all $x \in [0,1]^2$. Thus, the system is contractive with respect to the ℓ_1 norm with contraction rate $\eta := \min\{\lambda_0 - 1, \lambda_2\} > 0$. Let $\gamma \in [0,1]^2$ denote its unique T-periodic attractive solution. Entrainment in mRNA translation is important as biological organisms are often exposed to periodic excitations, for example the periodic cell division process. Proper biological functioning requires entrainment to such excitations [17].

Let $\bar{u}_0 := \frac{1}{T}\int_0^T u_0(s)\,\mathrm{d}s = \lambda_0$ and consider the AS

$$\begin{aligned}\dot{y}_1 &= \lambda_0(1 - y_1) - \lambda_1 y_1(1 - y_2),\\ \dot{y}_2 &= \lambda_1 y_1(1 - y_2) - \lambda_2 y_2.\end{aligned} \tag{13}$$

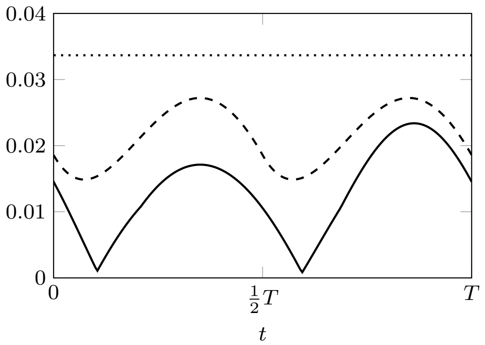

Fig. 1. The error $|\gamma(t) - e|_1$ (solid line) and the bounds (15) (dashed line) and (16) (dotted line) derived from Corollary 2.

This system admits an equilibrium point

$$e = \begin{bmatrix} \frac{\lambda_0\lambda_1 - \lambda_0\lambda_2 - \lambda_1\lambda_2 + \sqrt{d}}{2\lambda_0\lambda_1} & \frac{\lambda_0\lambda_1 + \lambda_0\lambda_2 + \lambda_1\lambda_2 - \sqrt{d}}{2\lambda_1\lambda_2} \end{bmatrix}' \in (0,1)^2, \tag{14}$$

where $d := 4\lambda_0^2\lambda_1\lambda_2 + (\lambda_0\lambda_1 - \lambda_0\lambda_2 - \lambda_1\lambda_2)^2$.

Here,

$$F(e, u(s)) = \begin{bmatrix} (\lambda_0 + \sin(2\pi s/T))(1 - e_1) - \lambda_1 e_1(1 - e_2) \\ \lambda_1 e_1(1 - e_2) - \lambda_2 e_2 \end{bmatrix},$$

and since e is an equilibrium point of (13), $F(e, u(s)) = \begin{bmatrix}(1 - e_1)\sin(2\pi s/T) & 0\end{bmatrix}'$. Thus, (10) yields

$$\begin{aligned} |\gamma(\tau) - e|_1 \le \\ & (1 - e_1)\frac{e^{-\eta\tau}}{1 - e^{-\eta T}} \int_0^T e^{-\eta(T-s)} |\sin(2\pi s/T)| \, \mathrm{d}s \\ & + (1 - e_1) \int_0^\tau e^{-\eta(\tau - s)} |\sin(2\pi s/T)| \, \mathrm{d}s \end{aligned} \tag{15}$$

for all $\tau \in [0, T]$. Furthermore,

$$|F(e, u(t))|_1 = (1 - e_1)|\sin(2\pi t/T)| \le 1 - e_1$$

so (11) implies the simpler yet more conservative bound

$$|\gamma(\tau) - e|_1 \le (1 - e_1)/\eta, \quad \text{for all } \tau \in [0, T]. \tag{16}$$

Note that the bounds above can be computed analytically so that we obtain considerable explicit information on the periodic trajectory γ.

Fig. 1 illustrates the bounds on the periodic trajectory for the case $\lambda_0 = 4$, $\lambda_1 = 1/2$, $\lambda_2 = 4$, and $T = 2$. It may be seen that these bounds indeed provide a reasonable approximation for the ℓ_1 distance between the unknown periodic trajectory and the point e. □

The next natural AS is an LTI system that is excited by the original periodic input.

Corollary 3 *Let $e \in \Omega$ denote the globally attractive equilibrium point of the unforced dynamics of (9), i.e. $\dot{x} = F(x, 0)$. Without loss of generality, assume that $e = 0$. Let $A := \frac{\partial F}{\partial x}(0, 0)$, $B := \frac{\partial F}{\partial u}(0, 0)$, and consider the LTI AS*

$$\dot{y} = Ay + Bu := G(y, u). \tag{17}$$

Let $\kappa(t)$ be the unique T-periodic trajectory of (17) and assume that $\kappa(t) \in \Omega$ for all t. Then for all $\tau \in [0, T]$,

$$\begin{aligned} |\gamma(\tau) - \kappa(\tau)| \le & \frac{e^{-\eta\tau}}{1 - e^{-\eta T}} \int_0^T e^{-\eta(T-s)} |E(s)| \, \mathrm{d}s \\ & + \int_0^\tau e^{-\eta(\tau - s)} |E(s)| \, \mathrm{d}s \\ \le & \max_{t \in [0,T]} |E(t)|/\eta, \end{aligned} \tag{18}$$

where $E(t) := F(\kappa(t), u(t)) - G(\kappa(t), u(t))$.

We emphasize again that the advantage of the bounds here is that the integrand depends on the difference between the vector fields F and G evaluated along the solution κ of the LTI system (17). Note that our assumptions imply that A is Hurwitz and thus, for any initial condition, $y(t)$ converges to the periodic trajectory $\kappa(t)$. In some cases, this solution and the error bounds can be written explicitly. For example, if $u(t)$ is a complex exponential then $\kappa(t)$ can be easily computed using a Fourier transform and then a bound on $|F(\kappa(t), u(t)) - G(\kappa(t), u(t))|$, $t \in [0, T]$, may be straightforward to establish, as demonstrated in the example below.

Example 4 Consider again the RFM with $n = 2$ and the periodic initiation rate $u_0(t) := \lambda_0 + u(t)$, with $\lambda_0 > 1$ and $u(t) = \sin(2\pi t/T)$. Again, let e be the unique equilibrium of the system when the initiation rate is λ_0 (see (14)). Let $\delta x := x - e$. Then the linearized system is $\dot{\delta x} = A\delta x + Bu$, with $A := \begin{bmatrix} -\lambda_0 - \lambda_1(1 - e_2) & \lambda_1 e_1 \\ \lambda_1(1 - e_2) & -\lambda_1 e_1 - \lambda_2 \end{bmatrix}$, and $B := \begin{bmatrix} 1 - e_1 \\ 0 \end{bmatrix}$. Note that $\mu_1(A) = \max\{-\lambda_0, -\lambda_2\} < 0$, so, in particular, A is Hurwitz. Thus, the AS is

$$\dot{y} = A(y - e) + Bu =: G(y, u), \quad u(t) = \sin(2\pi t/T). \tag{19}$$

The difference between the vector fields evaluated along a solution of the y system is

$$\begin{aligned} & F(y, \sin(2\pi t/T)) - G(y, \sin(2\pi t/T)) = \\ & \qquad \begin{bmatrix} \lambda_1(y_1 - e_1)(y_2 - e_2) - (y_1 - e_1)\sin(2\pi t/T) \\ -\lambda_1(y_1 - e_1)(y_2 - e_2) \end{bmatrix}. \end{aligned}$$

Let $\hat{g}(s) := \begin{bmatrix} \hat{g}_1(s) \\ \hat{g}_2(s) \end{bmatrix} = (sI - A)^{-1}B$, and let $\kappa(t) : \mathbb{R} \to \mathbb{R}^2$ be the unique periodic trajectory of (19) defined for all $-\infty < t < \infty$. For a complex number v, let $\angle v$ denote the argument of v. Then

$$\kappa(t) - e = \begin{bmatrix} |\hat{g}_1(j\omega)| \sin(\omega t + \angle\hat{g}_1(j\omega)) \\ |\hat{g}_2(j\omega)| \sin(\omega t + \angle\hat{g}_2(j\omega)) \end{bmatrix},$$

with $\omega := 2\pi/T$, and Corollary 3 implies

$$\begin{aligned} |\gamma(t) - \kappa(t)|_1 & \le \max_{t \in [0,T]} |F(\kappa(t), u(t)) - G(\kappa(t), u(t))|_1/\eta \\ & = \left(2\lambda_1 |\hat{g}_1(j\omega)||\hat{g}_2(j\omega)| + |\hat{g}_1(j\omega)|\right)/\eta \end{aligned} \tag{20}$$

where $\eta := \min\{\lambda_0 - 1, \lambda_2\}$ as before. Note that the bound here depends on the frequency of the periodic excitation. The more exact bound in (18) can be computed numerically.

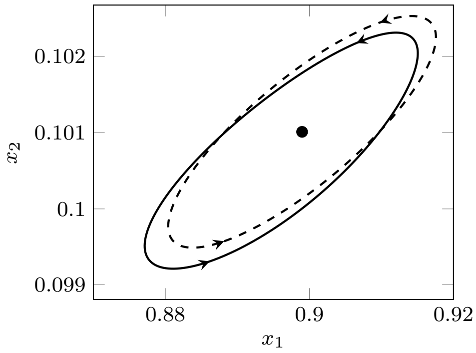

Fig. 2. Periodic trajectory $\gamma(t)$ of the RFM (solid line) and the periodic trajectory $\kappa(t)$ of the linearized system (dashed line) when $u_0(t) = 4 + \sin(2\pi t/T)$. The equilibrium e for $\lambda_0 = 4$ is marked by a dot.

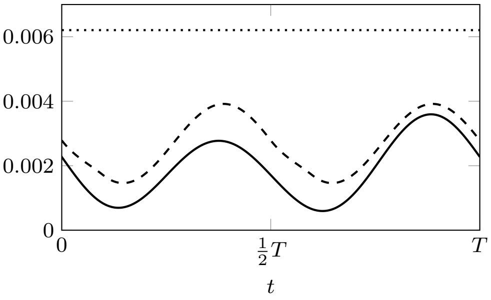

Fig. 3. RFM in Example 4. The error $|\gamma(t) - \kappa(t)|_1$ (solid line) and the bounds (18) (dashed lines) and (20) (dotted line).

For $\lambda_1 = 1/2$, $\lambda_2 = 4$, and $T = 2$, Fig. 2 shows the equilibrium point when $\lambda_0 = 4$, the periodic trajectory when the initiation rate is $u_0(t) = 4 + \sin(2\pi t/T)$, and the periodic trajectory of the linearized system. Fig. 3 illustrates the bounds from Corollary 3. It may be observed that these bounds provide a reasonable estimate of the error. $\square$

The bound (20) has some interesting implications. For example, if $\hat{g}_1(j\omega) = 0$ for some ω then (20) implies that $\gamma(t) \equiv \kappa(t)$ for a sinusoidal excitation with frequency ω. Similarly, if $\lim_{\omega\to\infty} \hat{g}_1(j\omega) = 0$ then (20) implies that for a high frequency sinusoidal forcing term, γ will approach κ. Note that the conclusions on γ here are based on *properties of the LTI system*. The next section uses this idea to derive a new theoretical result on the response of contractive systems to a sinusoidal input.

IV. CONTRACTIVE SYSTEMS ARE LOW-PASS FILTERS

We consider a contractive system with an additive input and show that for a high-frequency sinusoidal input, the periodic trajectory of the contractive system is very similar to that of a suitable asymptotically stable LTI system. For the sake of simplicity, we state this for the case of a scalar control.

Theorem 3 *Consider the system $\dot{x} = f(x) + bu$ with scalar input $u(t) = a\cos(\omega t + \phi)$, and suppose that it admits a compact and convex invariant state space $\Omega \subset \mathbb{R}^n$. Assume that for some matrix measure induced by a norm $|\cdot|$,*

$$\mu\left(\frac{\partial f}{\partial x}(x)\right) \le -\eta < 0, \quad \textit{for all } x \in \Omega. \tag{21}$$

Let $\gamma(\cdot) : \mathbb{R} \to \Omega$ be the unique, attracting, $T := 2\pi/\omega$ periodic orbit of this system. Without loss of generality, assume that the globally asymptotically stable equilibrium point[1] *$e \in \Omega$ of the unforced system $\dot{x} = f(x)$ is $e = 0$. Let $A := \frac{\partial f}{\partial x}(0)$, and consider the AS*

$$\dot{y} = Ay + bu := G(y, u). \tag{22}$$

Let $\hat{g}(s) := (sI - A)^{-1}b$ and let $\kappa(t)$ be the unique T-periodic trajectory of (22)*, so that the ith coordinate of $\kappa(t)$ is*

$$\kappa_i(t) = a|\hat{g}_i(j\omega)|\cos(\omega t + \phi + \angle\hat{g}_i(j\omega)), \ i = 1, \dots, n. \tag{23}$$

Then

$$\max_{t\in[0,T]} |\gamma(t) - \kappa(t)| = O(1/\omega). \tag{24}$$

Proof. For any τ,

$$\begin{aligned} |\gamma(\tau) - \kappa(\tau)|\eta &\le \max_{t\in[0,T]} |f(\kappa(t)) - A\kappa(t)| \\ &\le \max_{t\in[0,T]} O(|\kappa(t)|). \end{aligned} \tag{25}$$

Since

$$\hat{g}(s) = \frac{\operatorname{adj}(sI - A)}{\det(sI - A)} b, \tag{26}$$

where adj denotes the adjugate, (23) implies that $|\kappa_i(t)| = O(1/\omega)$ for all i and all $t \in [0, T]$. Combining this with (25) completes the proof. $\blacksquare$

Note that (26) implies that the linear system behaves as a low-pass filter (LPF), in the sense that $\hat{g}(j\omega) \to 0$ as $\omega \to \infty$, and thus the bound (24) implies that the contractive system is also an LPF.

Example 5 We consider a basic model for an externally driven transcriptional module that is ubiquitous in both biology and synthetic biology (see, e.g., [6], [31]):

$$\begin{aligned} \dot{x}_1 &= u - \delta x_1 + k_1 x_2 - k_2(e_T - x_2)x_1, \\ \dot{x}_2 &= -k_1 x_2 + k_2(e_T - x_2)x_1, \end{aligned} \tag{27}$$

where $\delta, k_1, k_2, e_T > 0$. Here $x_1(t)$ is the concentration at time t of a transcriptional factor X that regulates a downstream transcriptional module by binding to a promoter with concentration $e(t)$ yielding a protein-promoter complex Y with concentration $x_2(t)$. The binding reaction is reversible with binding and dissociation rates k_2 and k_1, respectively. The linear degradation rate of X is δ, and as the promoter is not subject to decay, its total concentration, e_T, is conserved, so $e(t) = e_T - x_2(t)$ for all $t \ge 0$. The input $u(t)$ might represent for example the concentration of an enzyme or of a second messenger that activates X, so we assume that $u(t) \ge 0$ for all $t \ge 0$.

Trajectories of (27) evolve on $[0, \infty) \times [0, e_T]$. For an input satisfying $0 \le u(t) \le c$ for all $t \ge 0$, the set $\Omega := [0, (c +$

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[1] Existence of e follows from (21).

$k_1e_T)/\delta] \times [0, e_T]$ is a convex and compact invariant set.

Ref. [31] has shown that (27) is contractive with respect to a certain weighted L_1 norm. Indeed, the Jacobian of (27) is $J(x) = \begin{bmatrix} -\delta - k_2(e_T - x_2) & k_1 + k_2x_1 \\ k_2(e_T - x_2) & -k_1 - k_2x_1 \end{bmatrix}$, so for $D := \operatorname{diag}(d, 1)$, with $d > 0$,

$$DJ(x)D^{-1} = \begin{bmatrix} -\delta - k_2(e_T - x_2) & (k_1 + k_2x_1)d \\ k_2(e_T - x_2)/d & -k_1 - k_2x_1 \end{bmatrix}. \quad (28)$$

The off-diagonal terms here are non-negative, and this means that for any $d \in (\frac{k_2e_T}{k_2e_T+\delta}, 1)$,

$$\mu_{1,D}(J(x)) \leq -\eta, \text{ for all } \begin{bmatrix} x_1 & x_2 \end{bmatrix}' \in \Omega,$$

where $\eta := \min\{k_1(1-d), \delta + k_2e_T(1-d^{-1})\} > 0$. Thus, (27) is contractive with respect to the scaled norm $|\cdot|_{1,D}$ with contraction rate η.

Linearizing (27) yields $\dot{y} = G(y, u) = Ay + bu$, with $A := \begin{bmatrix} -\delta - k_2e_T & k_1 \\ k_2e_T & -k_1 \end{bmatrix}$, and $b := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so

$$\begin{aligned} \hat{g}(s) &= (sI - A)^{-1}b \\ &= \frac{1}{s^2 + (\delta + k_1 + k_2e_T)s + \delta k_1} \begin{bmatrix} s + k_1 \\ k_1 \end{bmatrix}. \end{aligned}$$

Since $f(y) - Ay = k_2y_1y_2 \begin{bmatrix} 1 & -1 \end{bmatrix}'$, the bound (25) yields

$$\begin{aligned} \eta|D(\gamma(\tau) - \kappa(\tau))|_1 &\leq k_2 \max_{t\in[0,T]} |\kappa_1(t)\kappa_2(t) \begin{bmatrix} 1 & -1 \end{bmatrix}'|_{1,D} \\ &\leq k_2(d+1) \max_{t\in[0,T]} |\kappa_1(t)\kappa_2(t)|. \end{aligned} \quad (29)$$

Note that for any input $u(t) = \sum_{i=1}^{p} a_i \cos(\omega_i t + \phi_i)$ the periodic trajectory $\kappa(t)$ is explicitly known and thus the bound (29) is explicit. Fig. 4 depicts the trajectories of both the contractive system (27) and of the LTI system for the parameters $k_1 = 1$, $k_2 = 5$, $\delta = 1$, $e_T = 2$, and the excitation $u(t) = \cos(\omega t)$ for two different values of ω.[2] It may be seen that for a larger value of ω the difference between γ and κ decreases, as anticipated by (24). □

V. DISCUSSION

Contractive systems entrain to periodic excitations. Analyzing the corresponding periodic solution of the contractive system and its dependence on various parameters is an important theoretical question with many potential applications. It is known that such contractive systems posses a well-defined frequency response leading to a nonlinear Bode plot [21], however, computing this response is generally challenging as it requires solving a partial differential equation. In this paper, we developed approximation schemes for periodic solution of a contractive system using an LTI system and, using the ISS property of contractive systems, provided bounds on the approximation error. An important advantage of these bounds is that in some cases they can be computed explicitly. This also led to a new theoretical result on the behavior of contractive systems with a sinusoidal additive input.

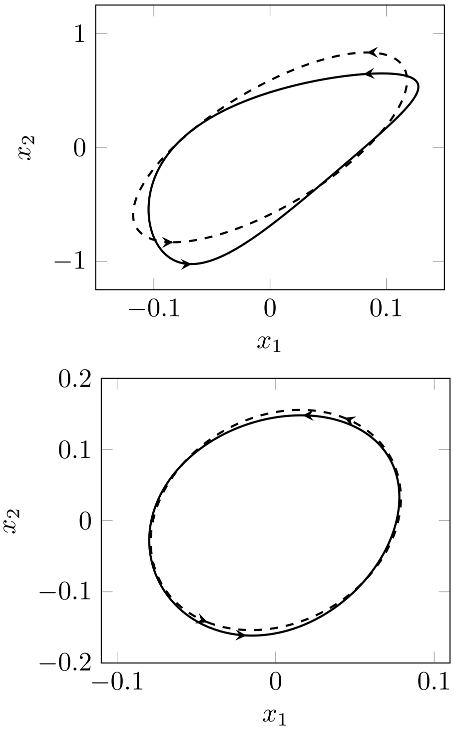

Fig. 4. Trajectories γ (solid line) and κ (dashed line) for the system in Example 5 for $\omega = 1$ (top) and $\omega = 5$ (bottom). Note the different scales in the figures.

Interesting topics for further research include: (1) deriving more theoretical results using the explicit bounds described here; (2) the design of an excitation signal that yields a *prespecified* periodic trajectory for a contractive system. This issue arises for example in synthetic biology, where an important goal is to design programmable biochemical oscillators (see e.g., [8], [10], [38], [41]), and this objective can be addressed in part by the nonlinear Bode plots introduced in [21], [22]; (3) defining other systematic approaches to determining approximating systems, for example, by employing the classical describing function method [13, Ch. 10.4]; and (4) the extension of the results presented here to more general classes of dynamical systems. For example, systems that are contractive for restricted classes of inputs [24], [23]. See also [20] for a special class of infinite-dimensional systems that admit a frequency response function, and [39] for analysis of a class of systems where solutions converge to an equilibrium if the excitation is sufficiently fast.

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[2] This control is not positive for all times, yet for the initial conditions in the simulations the trajectory remains in a convex and compact region in which the off-diagonal terms in (28) are non-negative and contraction holds.

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