Feedback Strategies for a Reach-Avoid Game with a Single Evader and Multiple Pursuers

Jhanani Selvakumar, Efstathios Bakolas

***Abstract*—We address a planar multi-agent pursuit-evasion game with a terminal constraint (reach-avoid game). Specifically, we consider the problem of steering a single evader to a target location, while avoiding capture by multiple pursuers. We propose a feasible control strategy for the evader, against a group of pursuers that adopts a semi-cooperative strategy. First, we characterize a partition of the game's state-space, that allows us to determine the existence of a solution to the game based on the initial conditions of the players. Next, based on the time-derivative of an appropriately defined risk metric, we develop a nonlinear state feedback strategy for the evader which provides a feasible solution to the game. This control strategy involves switching between different control laws in different parts of the state-space. We demonstrate the efficacy of our proposed feedback control in terms of the evader's performance, through numerical simulations. We also show that for the special case of the reach-avoid game with only one pursuer, the proposed control law is successful in guiding the evader to the target location from almost all initial conditions, and ensures that the evader will remain uncaptured.**

***Index Terms*—Quickest descent control, multi-agent reach-avoid games, state space partition, feedback strategy.**

I. Introduction

Strategic interaction between multiple agents, cooperative or adversarial, is the underlying phenomenon in various situations, such as dynamic collision avoidance, surveillance evasion, and pursuit of prey, to name a few. The characterization of the motive and actions of each agent in these interactions is important in developing a successful strategy geared towards a desired outcome. Such multi-agent interactions can be put under the umbrella of dynamic non-zero-sum multi-player games. In this paper, we deal with a class of planar multi-agent pursuit-evasion games in which a single evader must reach a target location while avoiding capture by multiple pursuers. The pursuers engage in semi-cooperative relay pursuit [1] and the location of the target is known only to the evader.

The state space of a PEG is usually the multi-dimensional space of positions and velocities of all the players involved. In some cases, there may be additional states as well as state or input constraints. The exact solution to a multi-agent pursuit-evasion game (PEG) is in general onerous to obtain, due to the high dimensionality of the problem. An approximate solution can be obtained by reducing the PEG to a lower dimensional problem. To this aim, we introduce a scalar risk metric which is a function of the full state of the PEG. The risk metric allows us to characterize a partition of the state-space of the game, which encodes information about the outcome of the game, based on the initial states of the players. Then, we develop an evasion strategy induced by the risk metric, which determines a collection of nonlinear state feedback laws that guide the evader to achieve its two-fold objective (avoid capture and reach its target), such that the evader employs different control laws in different parts of the state-space.

Literature review and previous work: Adversarial or competitive interactions between two or more agents are often modeled as games, in, for instance, economics, biology, and defense [2]–[5]. These games are typically non-zero-sum, because the loss incurred by one particular player is not necessarily equal to the gain obtained by another player in the game [4]. Capture and evasion in multi-player pursuit-evasion games, where multiple agents play against a single agent, have been addressed in [1], [6]–[15]. In [1] and [10], the pursuit algorithm uses dynamic Voronoi partitions to sequentially select one of the pursuers to engage with the evader. In [8], multiple pursuers within a convex domain use a switching strategy to capture the evader by shrinking its Voronoi cell. In [12], the pursuers use a greedy pursuit policy to locate and capture the evader on a grid.

A particular class of multi-agent games called reach-avoid games consists of games with terminal constraints, state constraints (e.g., obstacles or forbidden areas) and adversarial players. This class of games is a preferred tool to model situations where a team of agents must reach a target location while avoiding obstacles or defend a target from an offensive team of agents [16]–[20]. For instance, in [18], multiple agents use a priority-based evasion system to avoid each other and reach their respective targets. In [21], a numerical solution to the multi-agent reach-avoid game is obtained by solving the associated full-dimensional Hamilton-Jacobi-Isaacs (HJI) equation. The authors of [16] and [17] present an approximate solution to the reach-avoid game, as an alternative to solving the high-dimensional HJI equation. A Nash equilibrium solution is obtained for a quadratic game involving multiple agents in [19]. Yan et al. in [22] provide an analytical solution to a three-player reach-avoid game, and characterize a partition of the state-space as well as the optimal strategies for the players in each set of the partition.

In our previous work, we have addressed the problem of guiding a single evader to a target location or a target set in the presence of multiple pursuers who engage in semi-cooperative relay pursuit, by means of semi-analytical approaches including a dynamic roadmap and static-game approximation of the dynamic game [23], [24].

Contributions: In contrast to the previously mentioned reach-avoid problems, in this work, we present an analytical solution to the problem of reaching a fixed target point. Further, our partition of the state-space reflects the fact that the evader must frequently follow curved paths to successfully reach the target.

Department of Aerospace Engineering and Engineering Mechanics, The University of Texas at Austin, Austin, TX 78712-1221, USA

We first develop a greedy analytical solution to the two-player problem in which there is only one pursuer and the evader has the two-fold goal of reaching a target location while avoiding capture. Then, we postulate that the multi-player game can be viewed as an aggregation of the individual two-player games, and we numerically solve for feasible strategies in that case. Finally, we study the efficacy of our proposed approach by extensive numerical simulations. The main contributions of this work are as follows: (i) a novel scalar risk metric for the two-player game which is used to compute a feasible feedback strategy for the evader, (ii) a partition of the state-space of the PEG, such that the outcome of the PEG can be determined by considering the membership of the initial conditions in the sets of the partition, (iii) an analytical framework to determine the evading strategy based on quickest-descent in the two-player game, and (iv) an extension of the evader's strategy to the multiple-pursuer case, wherein the evader's control is obtained using convex optimization techniques at each time step. It is also to be noted that our method makes use of a performance index that is based on time-related metrics, rather than purely geometric considerations. The use of time-related metrics can be the basis for an evasion framework which is as independent of the specific nature of the players as possible.

Structure of the paper: Section II presents the formulation of the target-seeking evasion problem. Section III presents the evader's region-based control strategy in a two-player game. The extension of this solution strategy to the case of multiple pursuers is presented in Section IV, followed by illustrative simulations and results in Section V. In Section VI, we present concluding remarks.

II. Formulation of target-seeking evasion problem

Let us consider a pursuit evasion game that takes place in $\mathbb{R}^2$, with N pursuers and one evader. The analysis in this work extends naturally to $\mathbb{R}^3$. At any given time $t \in [0, \infty)$, let the equations of motion of the players be given as follows:

$$\dot{\boldsymbol{x}}_i(t) = v_{p_i}\boldsymbol{u}_i(t), \qquad \boldsymbol{x}_i(0) = \bar{\boldsymbol{x}}_i, \tag{1a}$$

$$\dot{\boldsymbol{x}}_e(t) = v_e\boldsymbol{u}_e(t), \qquad \boldsymbol{x}_e(0) = \bar{\boldsymbol{x}}_e, \tag{1b}$$

where the state (position) of the i^{th} pursuer, P_i, for $i \in \mathcal{I} := \{1, 2, ..., N\}$, is denoted by $\boldsymbol{x}_i(t) \in \mathbb{R}^2$, and the state (position) of the single evader E, is denoted by $\boldsymbol{x}_e(t) \in \mathbb{R}^2$. The maximum speeds of the i^{th} pursuer and the evader are denoted by $v_{p_i} > 0$ and $v_e > 0$, respectively. Further, $\boldsymbol{u}_i(t) \in \mathbb{R}^2$ and $\boldsymbol{u}_e(t) \in \mathbb{R}^2$ denote the inputs of the i^{th} pursuer and the evader at time t, respectively, and are both assumed to take values in the convex and compact set $\mathcal{U} := \{\boldsymbol{u} \in \mathbb{R}^2 : \|\boldsymbol{u}\| \leq 1\}$. Note that the zero control input will be used by the players only when the game terminates (either by capture of the evader by at least one pursuer, or by the evader reaching its target location).

In our problem set-up, capture is defined as positional proximity of at least one of the pursuers to the evader within a prespecified tolerance $\ell > 0$, which is the radius of capture. The evader aims to reach a target location, denoted by $\boldsymbol{x}_T \in \mathbb{R}^2$. In practice, it is sufficient for the evader to attain positional proximity to the target with a tolerance $\epsilon > 0$. The game will terminate at time $t_f > 0$, (a) if at time $t = t_f$, the evader reaches the target while remaining uncaptured for $t \in [0, t_f)$ (in which case the evader is successful) or (b) if at time $t = t_f$, the evader is captured by a pursuer before reaching the target (in which case the pursuers are successful).

A. Formulation of the target-seeking evasion problem

Now, we give the precise formulation of the target-seeking evasion problem with N pursuers and one evader.

Problem 1: Given the initial states in the plane for all the pursuers, that is, $\bar{\boldsymbol{x}}_i \in \mathbb{R}^2$, for $i \in \mathcal{I}$, the initial state for the evader $\bar{\boldsymbol{x}}_e \in \mathbb{R}^2$, and in addition, the target location $\boldsymbol{x}_T \in \mathbb{R}^2$, along with two positive constants ℓ and ϵ, find the control input signal $\boldsymbol{u}_e(\cdot) : [0, t_f] \to \mathcal{U}$, that will steer the evader to the target location $\boldsymbol{x}_T$ within a desired tolerance ϵ at some time $t_f \in [0, \infty)$, that is, $\|\boldsymbol{x}_e(t_f) - \boldsymbol{x}_T\| \leq \epsilon$, while avoiding capture by any of the pursuers, that is, $\|\boldsymbol{x}_e(t) - \boldsymbol{x}_i(t)\| > \ell,\ \forall i \in \mathcal{I},\ \forall t \in [0, t_f)$.

Note that the game terminates at the first time instant $t_f > 0$ at which $\|\boldsymbol{x}_e(t_f) - \boldsymbol{x}_T\| \leq \epsilon$ or $\|\boldsymbol{x}_e(t_f) - \boldsymbol{x}_i(t_f)\| \leq \ell$ for some $i \in \mathcal{I}$. Typically, $\ell > \epsilon$. It is assumed that the target location, $\boldsymbol{x}_T$, is known only to the evader. Note that the evader has the two-fold goal of reaching the target location while avoiding capture, whereas the group of pursuers only aims to capture the evader as soon as possible. If the evader reaches the target at the same instant as it is captured, we consider that the evader has accomplished its goal. We assume perfect information about the states of all the players in the game at all times.

1) Relay pursuit strategy: We assume that the group of pursuers adopts the semi-cooperative strategy of relay pursuit [16]. According to this strategy, at each instant of time, the pursuer who can capture the evader in the least amount of time engages in active pursuit of the latter, while all the other pursuers remain stationary, without participating in the pursuit of the evader. We will refer to the former player as the *active pursuer*. The minimum time required for capture of the evader is called the *relay metric* in this case. If the active pursuer is designated by the index i_c, then, for all $t \in [0, t_f]$, the feedback strategy of the j^{th} pursuer is

$$\boldsymbol{u}_j^*(\boldsymbol{x}_e, \boldsymbol{x}_j) = \begin{cases} \boldsymbol{r}_j/\|\boldsymbol{r}_j\|, & \text{if } j = i_c, \\ \boldsymbol{0} & \text{otherwise,} \end{cases} \tag{2}$$

where $\boldsymbol{r}_j := \boldsymbol{x}_e - \boldsymbol{x}_j$ is the relative position vector of the evader with respect to the j^{th} pursuer. The assumption of relay pursuit for a group of pursuers is particularly relevant when the latter is distributed over the game's domain and each pursuer wishes to stay close to its initial position (e.g. agents in patrol or surveillance). In this context, it would not be prudent for all pursuers to independently and simultaneously engage in pursuit of the evader, thereby making relay pursuit an effective alternative. A schematic representation of the target-seeking pursuit-evasion problem in the presence of multiple pursuers employing the relay pursuit strategy is shown in Fig. 1.

2) Times of capture and intercept: Next, we introduce a few quantities that will facilitate the discussion in this paper. At any time t, when the evader's current state is $\boldsymbol{x}_e$, and that of the single pursuer (say the i^{th} pursuer) is $\boldsymbol{x}_i$, let the min-max time-to-capture of the evader by the pursuer, ignoring all

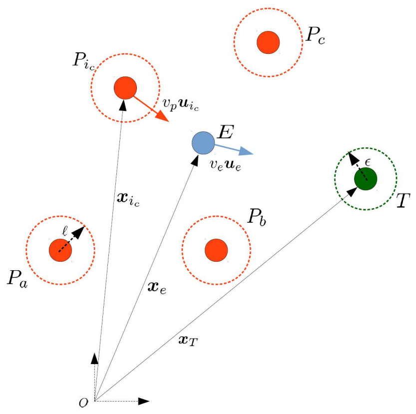


Fig. 1: Schematic representation of the target-seeking evasion problem in the presence of multiple pursuers. The pursuer P_{i_c} is the active pursuer in this figure. The target location and its radius of proximity is shown in green. The red circles illustrate the radius of capture around each pursuer.

others, be denoted by $\phi_c(\boldsymbol{x}_e, \boldsymbol{x}_i)$. The qualifier "min-max" for the time-to-capture indicates that it is the earliest time from the present that capture can occur in the game, while the pursuer tries to minimize the time-to-capture and the evader tries to maximize it. In fact, the min-max time-to-capture is the value of the min-max two-player zero-sum game. As the relative position of the evader with respect to the i^{th} pursuer at time t is given by $\boldsymbol{r}_i = \boldsymbol{x}_e - \boldsymbol{x}_i$, in view of Eq. (1), we have that

$$\dot{\boldsymbol{r}}_i = v_e \boldsymbol{u}_e - v_{p_i} \boldsymbol{u}_i, \qquad \boldsymbol{r}_i(0) = \bar{\boldsymbol{r}}_i = \bar{\boldsymbol{x}}_e - \bar{\boldsymbol{x}}_i$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{r}_i\| = \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{r}}\|\boldsymbol{r}_i\|\dot{\boldsymbol{r}}_i = \frac{\boldsymbol{r}_i^{\mathrm{T}}(v_e \boldsymbol{u}_e - v_{p_i}\boldsymbol{u}_i)}{\|\boldsymbol{r}_i\|}. \tag{3}$$

The pursuit-evasion game will end in capture, if there exists $t_f > 0$, such that $\|\boldsymbol{r}_i(t_f)\| = \ell$. If the pursuer employs the optimal pursuit action, that is, $\boldsymbol{u}_i(t) := \boldsymbol{r}_i(t)/\|\boldsymbol{r}_i(t)\|$, the pursuit-evasion game will be a free terminal time problem with two boundary conditions given by the initial and final value of $\|\boldsymbol{r}_i\|$. With these boundary conditions and Eq. (3), it is straightforward to show that the min-max time-to-capture $\phi_c(\boldsymbol{x}_e, \boldsymbol{x}_i)$ is equal to the minimum positive real solution to the following equation:

$$(v_e^2 - v_{p_i}^2)\phi^2 + 2(v_e \boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{u}_e - \ell v_{p_i})\phi + \|\boldsymbol{r}_i\|^2 - \ell^2 = 0, \tag{4}$$

where $\boldsymbol{u}_e = \boldsymbol{r}_i/\|\boldsymbol{r}_i\|$ is the direction of optimal evasion [2]. In this case, the evader acts in an antagonistic manner to the i^{th} pursuer.

Similarly, let $\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i, \boldsymbol{x}_T)$ denote the minimum time to intercept of the evader by the i^{th} pursuer, when the evader is directly headed towards the target location $\boldsymbol{x}_T$ with constant speed v_e. Note that $\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i, \boldsymbol{x}_T)$ is the value function of the minimum-time problem in which the pursuer is to intercept an evader whose direction of motion is known and constant. By its definition, $\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i, \boldsymbol{x}_T)$ is equal to the minimum real positive solution to Eq. (4) with $\boldsymbol{u}_e = (\boldsymbol{x}_T - \boldsymbol{x}_e)/\|\boldsymbol{x}_T - \boldsymbol{x}_e\|$. Note that when the minimum time to intercept is calculated, we assume that the evader does not maneuver to avoid the pursuer, and only the pursuer is maneuvering to minimize the capture time. For the evader, we can determine $\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i, \boldsymbol{x}_T)$ by solving a single quadratic equation [25].

Finally, we wish to highlight that when the evader engages in non-optimal play, $\phi_c(\boldsymbol{x}_e, \boldsymbol{x}_i)$ cannot be computed in closed-form. Then, $\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i)$ can serve as a lower bound for $\phi_c(\boldsymbol{x}_e, \boldsymbol{x}_i)$ in this case. Finally, let $\phi_T(\boldsymbol{x}_e, \boldsymbol{x}_T)$ denote the minimum time required for the evader located at $\boldsymbol{x}_e$ at time t, to reach the target $\boldsymbol{x}_T$, when there are no pursuers present. Then, in particular, $\phi_T(\boldsymbol{x}_e, \boldsymbol{x}_T) = \|\boldsymbol{x}_T - \boldsymbol{x}_e\|/v_e$.

3) Performance index for the evader: For a given initial condition, the evader's performance index is a scalar denoted by $\mathcal{P}(\boldsymbol{u}_e(\cdot))$, and is calculated at the terminal time t_f. The evader aims to minimize the value of the performance index, which is, in particular,

$$\mathcal{P}(\boldsymbol{u}_e(\cdot)) := \left.\frac{\phi_T(\boldsymbol{x}_e, \boldsymbol{x}_T) - \epsilon/v_e}{\min_i \phi_s(\boldsymbol{x}_i, \boldsymbol{x}_e, \boldsymbol{x}_T)}\right|_{t=t_f}. \tag{5}$$

Clearly, the performance index also depends on the other parameters of the game such as $\bar{\boldsymbol{x}}_e, \bar{\boldsymbol{x}}_1, ..., \bar{\boldsymbol{x}}_N, \boldsymbol{x}_T$, the control input history of the other players $\boldsymbol{u}_1(\cdot), ..., \boldsymbol{u}_N(\cdot)$, and t_f. The value of $\mathcal{P}$ is a measure of the success of the evader's strategy in reaching the target without getting captured. If the evader reaches the target within the desired tolerance and without being captured at all previous time instants, the value of the performance index is zero. However, achieving this value is not possible for all initial conditions of the game. In the event that the game will inevitably terminate in capture before the evader reaches the target, the performance index for the evader, $\mathcal{P}$, will grow unbounded. For instance, when the evader is slower than the pursuers, we can show that there is a non-empty set of initial conditions from which the evader will get captured before reaching the target. Our goal is to minimize the performance index or to make it zero, if possible.

III. Region-based control for target-seeking evader in a two-player game

In this section, we present a region-based control strategy for the evader that will minimize its performance index $\mathcal{P}$, which is given by Eq. (5). Note that the control strategy for all the pursuers is defined in Eq. (2). Since the pursuers engage in relay pursuit, in this work, we propose to view the multi-player game as an aggregation of individual two-player games. This underlines the importance of characterizing the evader's control strategy in the two-player game that involves only the closest pursuer and the evader, which is presented in the following section. Hence, let us first characterize the region-based control strategy for the evader when $N = 1$. Without loss of generality, in the subsequent discussion, we assume that the i^{th} pursuer is the closest one to the evader, that is, $i = i_c$.

A. Capturability of the evader in a two-player game

For a zero-sum game between the i^{th} pursuer and evader, under some conditions on the parameters and initial states, the evader cannot escape capture. Let us examine Eq. (4) when $\boldsymbol{u}_e(t)$ is taken to be the unit vector in the direction of optimal evasion, that is, $\boldsymbol{u}_e(t) := \boldsymbol{r}_i(t)/\|\boldsymbol{r}_i(t)\|$. Then, as stated in

Section II, $\phi_c(\boldsymbol{x}_e, \boldsymbol{x}_i)$ is equal to the minimum positive real solution to the following equation:

$$(v_e^2 - v_{p_i}^2)\phi^2 + 2(v_e\|\boldsymbol{r}_i\| - \ell v_{p_i})\phi + \|\boldsymbol{r}_i\|^2 - \ell^2 = 0. \quad (6)$$

If the evader is slower, that is, $v_e < v_{p_i}$, there is always a positive real number that satisfies Eq. (6), which is a quadratic equation in ϕ. Recall here that $\phi_c(\boldsymbol{x}_e, \boldsymbol{x}_i)$ is equal to the *value* of the two-player game.

We know that any action by the evader other than optimal evasion will result in a lower value for the game, that is, the evader will be captured sooner than the min-max time-to-capture. The ensuing equation follows from Eq. (3):

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{r}_i\| = v_e \cos\theta - v_{p_i}, \qquad \|\boldsymbol{r}_i(0)\| = \|\bar{\boldsymbol{r}}_i\|,$$

where θ is the smaller angle between the vectors $\boldsymbol{r}_i$ and $\boldsymbol{u}_e$ at time t. Then, given that $\cos\theta = 1$ only in the case of optimal evasion, we immediately conclude that, for any other maneuver by the evader, $\|\boldsymbol{r}_i\|$ will decrease faster than the optimal evasion case, which in turn implies that $\phi_c(\boldsymbol{x}_e, \boldsymbol{x}_i) \geq \phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i, \boldsymbol{x}_T)$. If $v_e \geq v_p$, and $\|\bar{\boldsymbol{r}}_i\| > \ell$, then the evader can never be captured by the pursuer provided that the evader plays optimally. Consequently, $\phi_c(\boldsymbol{x}_e, \boldsymbol{x}_i)$ is not well-defined. However, $\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i, \boldsymbol{x}_T)$ might have a finite value, as illustrated in Fig. 2 (refer also to [26]).

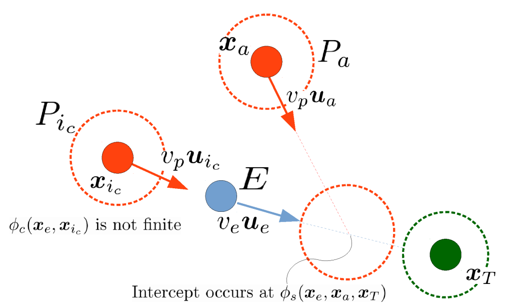


Fig. 2: When $v_e \geq v_{p_i}$, the time-to-capture for optimal evasion is not defined, as illustrated by the game between P_{i_c} and the evader E. The minimum time to intercept, on the other hand, could be finite for a certain set of initial conditions, as illustrated by the game between P_a and the evader E. Note that the two pursuers have equal speeds.

B. Risk-based partition of state space

We now propose a partition of the state-space of the two-player game, into a finite collection of pair-wise disjoint sets, whose union is equal to the state-space. For a fixed target location $\boldsymbol{x}_T$, let the augmented state-space of the game be denoted by $\mathcal{X}$, so that $\mathcal{X} := \{\boldsymbol{z} \in \mathbb{R}^4 : \boldsymbol{z} = \left[\boldsymbol{x}_e^{\mathrm{T}}\ \boldsymbol{x}_i^{\mathrm{T}}\right]^{\mathrm{T}}\}$.

Proposition 1: The open sets $\mathcal{S}$, $\mathcal{F}$, and $\mathcal{R}$, where

$$\mathcal{S} := \{\boldsymbol{z} \in \mathcal{X} : \phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i, \boldsymbol{x}_T) > \phi_T(\boldsymbol{x}_e, \boldsymbol{x}_T)\}$$
$$\mathcal{F} := \{\boldsymbol{z} \in \mathcal{X} : \phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i, \boldsymbol{x}_T) < \phi_T(\boldsymbol{x}_e, \boldsymbol{x}_T) \text{ and } \phi_c(\boldsymbol{x}_e, \boldsymbol{x}_i) < \phi_T(\boldsymbol{x}_e, \boldsymbol{x}_T)\}$$
$$\mathcal{R} := \{\boldsymbol{z} \in \mathcal{X} : \phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i, \boldsymbol{x}_T) < \phi_T(\boldsymbol{x}_e, \boldsymbol{x}_T), \text{ and } \phi_c(\boldsymbol{x}_e, \boldsymbol{x}_i) > \phi_T(\boldsymbol{x}_e, \boldsymbol{x}_T)\}$$

form a partition of the state-space $\mathcal{X}$, that is,
(i) $\mathcal{X} = \bar{\mathcal{S}} \cup \bar{\mathcal{F}} \cup \bar{\mathcal{R}}$,
(ii) $\mathcal{S} \cap \mathcal{F} = \emptyset$, $\mathcal{F} \cap \mathcal{R} = \emptyset$, and $\mathcal{S} \cap \mathcal{R} = \emptyset$.

Proof 1 The closures of the sets $\mathcal{S}$, $\mathcal{F}$, and $\mathcal{R}$ include states such that $\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i, \boldsymbol{x}_T) = \phi_T(\boldsymbol{x}_e, \boldsymbol{x}_T)$, or $\phi_c(\boldsymbol{x}_e, \boldsymbol{x}_i, \boldsymbol{x}_T) = \phi_T(\boldsymbol{x}_e, \boldsymbol{x}_T)$.

Let us first prove (i). Given $\boldsymbol{x}_i$ and $\boldsymbol{x}_e$, we know that the values of ϕ_c, ϕ_s and ϕ_T are uniquely determined from Eq. (4) and from the evader-to-target distance, $\|\boldsymbol{x}_T - \boldsymbol{x}_e\|$. Consequently, each augmented state in $\mathcal{X}$ satisfies the definition of at least one of the three closed sets $\bar{\mathcal{S}}, \bar{\mathcal{F}}$, and $\bar{\mathcal{R}}$. That is, if $\boldsymbol{z} \in \mathcal{X}$, then $\boldsymbol{z} \in \bar{\mathcal{S}}$ and/or $\boldsymbol{z} \in \bar{\mathcal{F}}$ and/or $\boldsymbol{z} \in \bar{\mathcal{R}}$. Hence, $\mathcal{X} \subseteq \bar{\mathcal{S}} \cup \bar{\mathcal{F}} \cup \bar{\mathcal{R}}$. Further, by definition, $\mathcal{S} \subseteq \mathcal{X}$, where $\mathcal{X}$ is the closure of itself, and consequently, $\bar{\mathcal{S}} \subseteq \mathcal{X}$. Similarly, $\bar{\mathcal{F}} \subseteq \mathcal{X}$ and $\bar{\mathcal{R}} \subseteq \mathcal{X}$. Hence, $\bar{\mathcal{S}} \cup \bar{\mathcal{F}} \cup \bar{\mathcal{R}} \subseteq \mathcal{X}$, which implies that $\mathcal{X} = \bar{\mathcal{S}} \cup \bar{\mathcal{F}} \cup \bar{\mathcal{R}}$.

For (ii), it is enough to note that for the states which belong to the open set $\mathcal{S}$, $\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i, \boldsymbol{x}_T) > \phi_T(\boldsymbol{x}_e, \boldsymbol{x}_T)$, and thus by definition, they are excluded from $\mathcal{R} \cup \mathcal{F}$. Hence, $\mathcal{S}$ is disjoint from the other sets. Similarly, the other two sets are also disjoint from each other. ■

Remark 1 Qualitatively, whenever the augmented position vector is in the set $\mathcal{S}$, the evader can head directly to the target $\boldsymbol{x}_T$ without any danger of capture. This is because the minimum time required to reach the target location, ϕ_T, is less than the minimum time ϕ_s required to intercept the evader on its path to the target. Hence this part of the state-space is "safe" for the evader. The set $\mathcal{F}$, on the other hand, is the "unsafe" set, which consists of points in the augmented state space $\mathcal{X}$ such that the evader has no chance of reaching the target before it gets captured. This is because the maximum time available to the evader before capture, which is equal to ϕ_c, is less than ϕ_T. The "risky" set $\mathcal{R}$ includes all points in $\mathcal{X}$ for which the classification of "safe" or "unsafe" is not readily determined from the values of ϕ_s and ϕ_c alone. This is because the actual capture time for the evader might be longer than the minimum intercept time ϕ_s due to the pursuers being unaware of the evader's intent. Finally, note that the closures of $\mathcal{S}, \mathcal{F}$, and $\mathcal{R}$ may intersect.

Remark 2 For games with initial conditions in the set $\mathcal{R}$, there exist two possibilities: (i) a control signal for the evader exists such that at some time $t_s > 0$, the augmented state will be in the set $\mathcal{S}$, and (ii) there is no feasible control signal for the evader such that the augmented state at any future time $t > 0$ can end up in $\mathcal{S}$. Hence, the significance of a state being in the set $\mathcal{R}$ is that while the evader is starting outside $\mathcal{S}$, there is a possibility that the evader will eventually reach $\mathcal{S}$ without getting captured.

Knowledge of the initial partition of the state-space, in particular the sets $\mathcal{S}$ and $\mathcal{F}$, is useful to determine whether a particular configuration of the pursuer and the evader for a given target would result in the evader being captured, or the evader safely reaching the target. For subsequent analysis, we take the target location to be the origin, without loss of generality, that is, $\boldsymbol{x}_T = \mathbf{0}$, and we drop $\boldsymbol{x}_T$ as an argument to simplify notation for the times to capture and other functions.

For the two-player game, if the evader is faster than the pursuer, the set $\mathcal{F}$ will contain the augmented states corresponding to the evader lying at or within the radius of capture from the pursuer's position. This is because when $v_e \geq v_{p_i}$, the min-max time-to-capture $\phi_c(\boldsymbol{x}_e, \boldsymbol{x}_i)$ is not well-defined. For a given position $\boldsymbol{x}_i$ of the pursuer in the inertial space, let $\mathcal{S}_e(\boldsymbol{x}_i)$ and $\mathcal{F}_e(\boldsymbol{x}_i)$ denote the sets that are the projections of $\mathcal{S}$ and $\mathcal{F}$ respectively on the evader's position plane. In Fig. 3, the sets $\mathcal{S}_e(\boldsymbol{x}_i)$ and $\mathcal{F}_e(\boldsymbol{x}_i)$ for a fixed position $\boldsymbol{x}_i$ of the pursuer are shown on the evader's position plane. It is possible to analytically characterize $\mathcal{S}_e(\boldsymbol{x}_i)$ and $\mathcal{F}_e(\boldsymbol{x}_i)$, for a given position of the pursuer, and we do so in the following proposition.

Proposition 2: If the pursuer is located at time t at $\boldsymbol{x}_i$, whereas the target is at the origin ($\boldsymbol{x}_T = \boldsymbol{0}$), then, (i) the boundary $\partial\mathcal{S}_e(\boldsymbol{x}_i)$ of $\mathcal{S}_e(\boldsymbol{x}_i)$ is a circle centered at the target (origin), with radius equal to $(\|\boldsymbol{x}_i\| - \ell)v_e/v_{p_i}$, and (ii) the boundary $\partial\mathcal{F}_e(\boldsymbol{x}_i)$ of $\mathcal{F}_e(\boldsymbol{x}_i)$ is also a circle centered at $\boldsymbol{x}_i$ with radius equal to $(1 - v_e/v_{p_i})\|\boldsymbol{x}_i\| + v_e\ell/v_{p_i}$.

Proof 2 First, we prove (i). We know that $\mathcal{S}$ consists of all positions in the augmented space $\mathcal{X}$ such that $\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i) > \phi_T(\boldsymbol{x}_e)$. From Eq. (4), by re-arranging the terms, we get

$$\|\boldsymbol{r}_i + v_e\boldsymbol{u}_e\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i)\| = \ell + v_{p_i}\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i), \tag{7}$$

where $\boldsymbol{u}_e = -\boldsymbol{x}_e/\|\boldsymbol{x}_e\|$. At the boundary of $\mathcal{S}$, $\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i) = \phi_T(\boldsymbol{x}_e) = \|\boldsymbol{x}_e\|/v_e$, and so $\boldsymbol{u}_e\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i) = -\boldsymbol{x}_e/v_e$, which is substituted in Eq. (7) to give $\|\boldsymbol{r}_i - \boldsymbol{x}_e\| = \ell + v_{p_i}\|\boldsymbol{x}_e\|/v_e$, or $\|\boldsymbol{x}_e\| = (\|\boldsymbol{x}_i\| - \ell)v_e/v_{p_i}$. (ii) By definition, the set $\mathcal{F}$ consists of all states $\left[\boldsymbol{x}_e^{\mathrm{T}}, \boldsymbol{x}_i^{\mathrm{T}}\right]^{\mathrm{T}}$ in the augmented state-space $\mathcal{X}$ such that $\phi_c(\boldsymbol{x}_e, \boldsymbol{x}_i) < \phi_T(\boldsymbol{x}_e)$. In this case, in view of Eq. (4), we get $\|\boldsymbol{r}_i + v_e\boldsymbol{u}_e\phi_c(\boldsymbol{x}_e, \boldsymbol{x}_i)\| = \ell + v_{p_i}\phi_c(\boldsymbol{x}_e, \boldsymbol{x}_i)$, where $\boldsymbol{u}_e = \boldsymbol{r}_i/\|\boldsymbol{r}_i\|$, which further implies that

$$\|\boldsymbol{r}_i\| = \ell + (v_{p_i} - v_e)\phi_c(\boldsymbol{x}_e, \boldsymbol{x}_i). \tag{8}$$

In the case when $\boldsymbol{r}_i = \alpha\boldsymbol{x}_e$ for some $\alpha < 0$, the minimum time to intercept is the same as the min-max time-to-capture. Then, there exists a position of the evader $\hat{\boldsymbol{x}}_e$ such that, $\phi_c(\hat{\boldsymbol{x}}_e, \boldsymbol{x}_i) = \phi_T(\hat{\boldsymbol{x}}_e) = \phi_s(\hat{\boldsymbol{x}}_e, \boldsymbol{x}_i)$. We know that $\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i) = \phi_T(\boldsymbol{x}_e)$ at the boundary of the safe set. With this knowledge, we have from Eq. (8) and part (i) of this proof,

$$\begin{aligned}\|\boldsymbol{r}_i\| &= \ell + (v_{p_i} - v_e)\phi_T(\hat{\boldsymbol{x}}_e)\\ &= \ell + (v_{p_i}/v_e - 1)(\|\boldsymbol{x}_i\| - \ell)v_e/v_{p_i}.\end{aligned}$$

Hence, the loci of all points on the boundary $\partial\mathcal{F}_e(\boldsymbol{x}_i)$, is described by the following equation:

$$\|\boldsymbol{x}_e - \boldsymbol{x}_i\| = (1 - v_e/v_{p_i})\|\boldsymbol{x}_i\| + v_e\ell/v_{p_i},$$

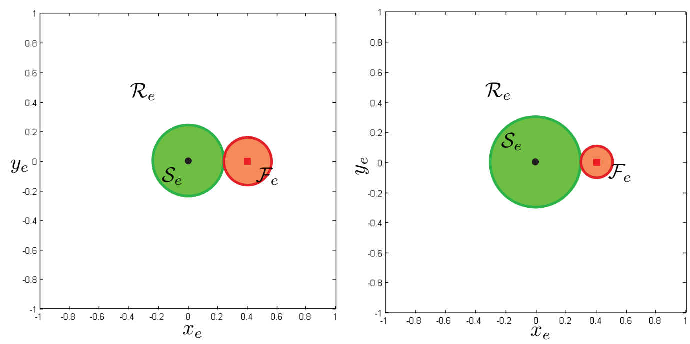
and the proof is complete. ■

Remark 3 When $v_e = v_{p_i}$, the failure set $\mathcal{F}$ is degenerate, and is simply the ball of capture of radius ℓ. Further, the set $\mathcal{S}$ shrinks as the ratio of v_e/v_{p_i} decreases, as the radius of $\partial\mathcal{S}(\boldsymbol{x}_i)$ is proportional to v_e/v_{p_i}.

At any time $t \in [0, t_f]$, the *log-risk* for the two-player game between the i^{th} pursuer and the evader is denoted by $\beta_i(\boldsymbol{x}_e, \boldsymbol{x}_i)$, and is defined as:

$$\beta_i(\boldsymbol{x}_e, \boldsymbol{x}_i) := \log\left(\phi_T(\boldsymbol{x}_e)/\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i)\right). \tag{9}$$

The log-risk is a measure of the capturability of the evader, whose mission is to reach the target safely. It is easy to infer that the safe set $\mathcal{S}$ comprises those augmented states which are such that the log-risk is negative, that is, $\beta_i(\boldsymbol{x}_e, \boldsymbol{x}_i) < 0$. Similarly, for states in $\mathcal{R}$ and $\mathcal{F}$, $\beta_i(\boldsymbol{x}_e, \boldsymbol{x}_i) > 0$. As the relative distance between the players approaches the radius of capture ℓ, the value of $\beta_i(\boldsymbol{x}_e, \boldsymbol{x}_i)$ tends to infinity.



(a) $v_e < v_p$ (b) $v_e = v_p$

Fig. 3: The three sets $\mathcal{S}_e$, $\mathcal{R}_e$, and $\mathcal{F}_e$, for different speeds of the evader, as projected on the evader's position space. The target location is shown as a black circle while the pursuer's fixed location is at the red square.

C. Control laws for different parts of the state-space

Now, let us characterize the individual control laws for the evader in different parts of the augmented state-space. For the following discussion, we note that the evader's position at time t is a function of its initial position and its control history in the interval $[0, t)$, that is, $\boldsymbol{x}_e = \boldsymbol{x}_e(t; \boldsymbol{u}_e(\cdot), \bar{\boldsymbol{x}}_e)$. Similarly, the position of pursuer P_i at time t is also a function of its initial position and control history, that is, $\boldsymbol{x}_i = \boldsymbol{x}_i(t; \boldsymbol{u}_i(\cdot), \bar{\boldsymbol{x}}_i)$.

Proposition 3: If at time $t \geq 0$, the augmented state $\left[\boldsymbol{x}_e^{\mathrm{T}}\ \boldsymbol{x}_i^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathcal{S}$, then the optimal control input for the evader at time t with state $\boldsymbol{x}_e$, which minimizes the performance index in Eq. (5), will be given by $\boldsymbol{u}_e^*(t) = \boldsymbol{u}_{e\mathcal{S}}(\boldsymbol{x}_e(t))$, with

$$\boldsymbol{u}_{e\mathcal{S}}(\boldsymbol{x}_e) := \begin{cases} -\boldsymbol{x}_e/\|\boldsymbol{x}_e\|, & \text{if } \|\boldsymbol{x}_e\| > \epsilon \\ \boldsymbol{0}, & \text{otherwise.} \end{cases} \tag{10}$$

Proof 3 If the initial augmented state belongs to the set $\mathcal{S}$, then $\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i) > \phi_T(\boldsymbol{x}_e)$. Consequently, if the evader moves with constant speed v_e in the direction of the target, it cannot be captured before it reaches the target. Then, the resulting value of the performance index $\mathcal{P}$ is zero. Thus, the optimal strategy for the evader is given by Eq. (10). ■

Remark 4 Note that if the evader deviates from the control input $\boldsymbol{u}_{e\mathcal{S}}(\boldsymbol{x}_e)$, it might be captured before reaching the target, despite starting from the safe set.

Corollary: If the (augmented) state of the game at time $t \geq 0$ is in $\mathcal{S}$, under optimal play by the evader as dictated by Eq. (10), the subsequent state trajectory will remain in the safe set $\mathcal{S}$ for every time instant until the end of the game, that is, $[\boldsymbol{x}_e^{\mathrm{T}}(\tau)\ \boldsymbol{x}_i^{\mathrm{T}}(\tau)]^{\mathrm{T}} \in \mathcal{S}$ for all $\tau \in [t, t_f]$.

Proof: The proof follows naturally from the definition of the safe set $\mathcal{S}$ and Proposition 3. If the evader uses the control $\boldsymbol{u}_{e\mathcal{S}}(\cdot)$ when $[\boldsymbol{x}_e^{\mathrm{T}}\ \boldsymbol{x}_i^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{S}$, then along the optimal trajectory, $\frac{\mathrm{d}}{\mathrm{d}t}\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i) = \frac{\mathrm{d}}{\mathrm{d}t}\phi_T(\boldsymbol{x}_e) = -1$. Since at time t, $\phi_s(\boldsymbol{x}_e(t), \boldsymbol{x}_i(t)) > \phi_T(\boldsymbol{x}_e(t))$, we conclude that $\phi_s(\boldsymbol{x}_e(\tau), \boldsymbol{x}_i(\tau)) > \phi_T(\boldsymbol{x}_e(\tau))$, for all $\tau \in [t, t_f]$. ■

Remark 5 For augmented states that belong to the boundary $\partial\mathcal{S}$ of set $\mathcal{S}$, the control law in Eq. (10) is still applicable, however, the evader will reach the target at the same time it will be captured. If the pursuer and the evader have equal speeds, that is, $v_e = v_{p_i}$, there exists a particular state in $\partial\mathcal{S} \cap \partial\mathcal{F}$, such that $\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i) = \phi_T(\boldsymbol{x}_e) = \phi_c(\boldsymbol{x}_e, \boldsymbol{x}_i)$, which implies that the evader has already been captured, even at time $t = 0$. We mention the existence of this initial condition for completeness, since there is no ensuing pursuit-evasion game in this case.

If the augmented state at initial time is in the set $\mathcal{R}$, the evader can maneuver to reach the target despite the fact that the latter may be initially out of safe reach.

Proposition 4: If the initial (augmented) state belongs to $\mathcal{R}$, and there exists a control signal for the evader $\boldsymbol{u}_e(\cdot)$ that drives the log-risk $\beta_i(\boldsymbol{x}_e, \boldsymbol{x}_i)$ to zero at some $t_s > 0$, then the evader can reach the target (origin) safely without being captured, that is, $\exists t_f > t_s : \|\boldsymbol{x}_e(t_f)\| < \epsilon$ and $\|\boldsymbol{r}_i(t)\| > \ell$ for all $t \in [t_s, t_f]$.

Proof 4 If the initial augmented state belongs to $\mathcal{R}$, then $\phi_T(\boldsymbol{x}_e)/\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i) > 1$ and $\phi_T(\boldsymbol{x}_e) < \phi_c(\boldsymbol{x}_e, \boldsymbol{x}_i)$. Hence, although the time required to reach the target is less than the maximum time available, by moving directly towards the target location, the evader would get captured before reaching the target. So, the evader must leave the set $\mathcal{R}$ and safely reach (without being captured during the transition) the set $\mathcal{S}$, from where it can reach the goal without being captured, by following the strategy in Eq. (10). Hence, if, by hypothesis, there exists a control input for the evader such that the ratio $\phi_T(\boldsymbol{x}_e)/\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i) = 1$ at some $t_s > 0$, we can thereafter invoke Proposition 3 for the interval $[t_s, t_f]$. ■

Remark 6 For the states in $\mathcal{R}$ for which no control history exists to achieve $\phi_T(\boldsymbol{x}_e)/\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i) = 1$ for some $t > 0$, and for the states in $\mathcal{F}$, the game will end in capture.

For all initial conditions of the game, irrespective of starting from $\mathcal{S}$, $\mathcal{R}$ or $\mathcal{F}$, we wish to minimize the performance index given in Eq. (5). To satisfy the safety constraint of no capture while heading to the target, at a given time t, we require that $\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i) \geq \phi_T(\boldsymbol{x}_e)$ for all $t \in [0, t_f]$. This is a conservative requirement, as in reality, the evader can safely reach the target if its time-to-capture by the i^{th} pursuer is greater than the time to reach the target. But when the players are engaged in non-optimal play, we cannot always compute $\phi_c(\boldsymbol{x}_e, \boldsymbol{x}_i)$ in closed form, and so, in its place, we use its conservative lower bound $\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i)$. Further, note that the safety requirement is automatically satisfied for states in $\mathcal{S}$, and we can determine that $t_f = \phi_T(\bar{\boldsymbol{x}}_e)$. For states that belong to $\mathcal{R}$ or $\mathcal{F}$ at $t = 0$, we have that $\phi_s(\bar{\boldsymbol{x}}_e, \bar{\boldsymbol{x}}_i) < \phi_T(\bar{\boldsymbol{x}}_e)$, and we cannot calculate t_f in closed form. At this point, we would like to solve for the evader's control signal $\boldsymbol{u}_e(t)$ for $t \in [0, t_s]$, such that $t_s > 0$ is the first time at which the augmented state of the game reaches the boundary of the set $\mathcal{S}$. Next, we present a quickest descent control law that will guide the evader to the boundary of the safe set, in the case that the game begins from a state in $\mathcal{R} \cup \mathcal{F}$.

1) Quickest descent control law for evader from $\mathcal{R}$ or $\mathcal{F}$: The problem of guiding the evader to the boundary of the safe set while avoiding capture is a path-constrained optimal control problem with free terminal time and control constraints. The optimal solution to this problem is analytically intractable, while a numerical solution is computationally expensive to obtain. For these reasons, we choose a quickest-descent controller, which can be formulated analytically and is inexpensive in terms of numerical computation. The quickest descent control law minimizes the time-derivative of the log-risk at every point along the trajectory of the game to achieve the desired transition from state $\mathcal{R}$ to $\mathcal{S}$. For states that are in $\mathcal{F}$ at $t = 0$, there is no feasible control history for the evader whose application would result in transition to the set $\mathcal{S}$, so we propose to use the quickest descent type control to minimize the log-risk, point wise in time, until capture occurs. The problem of finding the quickest descent control is stated as follows.

Problem 2: For the two-player target-seeking game between the evader and the i^{th} pursuer subject to the dynamics in Eq. (1), find the evader's control at each time $t \in [0, t_s]$ such that the log-risk is driven to zero at $t = t_s$, and such that $\boldsymbol{u}_e^*(t) = \hat{\boldsymbol{u}}$, where $\hat{\boldsymbol{u}}$ is the minimizer of the following cost function:

$$\mathcal{J}_t(\boldsymbol{u}) := \dot{\beta}_i(\boldsymbol{u}; \boldsymbol{x}_e(t), \boldsymbol{x}_i(t)), \tag{11}$$

which is the time-derivative of the log-risk along the trajectories $\boldsymbol{x}_e(\cdot)$ and $\boldsymbol{x}_i(\cdot)$.

The solution to Problem 2 is given by the following proposition.

Proposition 5: The control input for the evader that solves Problem 2 is given by $\boldsymbol{u}_e^*(t) = \boldsymbol{u}_{e\mathcal{R}}(\boldsymbol{x}_e(t), \boldsymbol{x}_i(t))$ for all $t \in [0, t_s]$, with

$$\boldsymbol{u}_{e\mathcal{R}}(\boldsymbol{x}_e, \boldsymbol{x}_i) := \begin{cases} -\boldsymbol{\chi}_i/\|\boldsymbol{\chi}_i\|, & \text{if } \|\boldsymbol{\chi}_i\| \neq 0 \\ -\boldsymbol{x}_e/\|\boldsymbol{x}_e\|, & \text{if } \|\boldsymbol{\chi}_i\| = 0 \text{ and } \|\boldsymbol{x}_e\| > \epsilon \\ \boldsymbol{0}, & \text{otherwise,} \end{cases} \tag{12}$$

where $\boldsymbol{\chi}_i = \boldsymbol{\chi}_i(\boldsymbol{x}_e, \boldsymbol{x}_i)$ is defined at time t as follows[1]:

$$\boldsymbol{\chi}_i(\boldsymbol{x}_e, \boldsymbol{x}_i) := \frac{\boldsymbol{x}_e}{\phi_T \|\boldsymbol{x}_e\|} - \frac{\boldsymbol{\nu}}{\mu}, \tag{13}$$

with $\boldsymbol{\nu}(\boldsymbol{x}_e, \boldsymbol{x}_i) := v_e \boldsymbol{r}_i - \frac{v_e^2 \phi_s \boldsymbol{r}_i}{\|\boldsymbol{x}_e\|} + v_e^2 \phi_s \frac{(\boldsymbol{x}_e^{\mathrm{T}} \boldsymbol{r}_i)\boldsymbol{x}_e}{\|\boldsymbol{x}_e\|^3} - v_e^2 \phi_s \frac{\boldsymbol{x}_e}{\|\boldsymbol{x}_e\|}$ and $\mu(\boldsymbol{x}_e, \boldsymbol{x}_i) := \|\boldsymbol{r}_i\|^2 - \ell^2 - \phi_s \left(v_p \ell + v_e \frac{\boldsymbol{r}_i^{\mathrm{T}} \boldsymbol{x}_e}{\|\boldsymbol{x}_e\|} \right)$.

Proof 5 From the definition for the log-risk β_i, we have that

$$\dot{\beta}_i = \dot{\phi}_T/\phi_T - \dot{\phi}_s/\phi_s.$$

The expressions for $\dot{\phi}_s$ and $\dot{\phi}_T$ can be determined by differentiating Eq. (4) and the expression for ϕ_T with respect to time. After grouping of terms,

$$\dot{\beta}_i = \boldsymbol{\chi}_i^{\mathrm{T}} \boldsymbol{u}_e + \gamma_i, \tag{14}$$

[1]The arguments of the time of minimum intercept, ϕ_s, have been dropped for compactness.

where $\boldsymbol{\chi}_i$ is given by Eq. (13) and γ_i is a scalar valued function of the states of the players at time t, defined as follows:

$$\gamma_i(\boldsymbol{x}_e, \boldsymbol{x}_i) := \frac{1}{\mu}\bigg(- v_e\phi_s v_p \frac{\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{x}_e}{\|\boldsymbol{r}_i\|\|\boldsymbol{x}_e\|} + v_p\|\boldsymbol{r}_i\|\bigg). \tag{15}$$

Thus, in view of Eq. (14), at time t, given $\boldsymbol{x}_e$ and $\boldsymbol{x}_i$, the derivative $\dot{\beta}_i$ is a linear function of the evader's input. Hence, the evader's input that minimizes the time-derivative of the log-risk at time t is given as a function of $\boldsymbol{\chi}_i(\boldsymbol{x}_e, \boldsymbol{x}_i)$. In particular, the evader's input $\boldsymbol{u}_e^*(t) = \hat{\boldsymbol{u}}_1(\boldsymbol{x}_e(t), \boldsymbol{x}_i(t))$, is in the form of a non-linear state feedback, as

$$\hat{\boldsymbol{u}}_1(\boldsymbol{x}_e, \boldsymbol{x}_i) := -\boldsymbol{\chi}_i/\|\boldsymbol{\chi}_i\|, \quad \text{if } \|\boldsymbol{\chi}_i\| \neq 0.$$

If there is no information available about minimizing the derivative $\dot{\beta}_i$, as when $\boldsymbol{\chi}_i = \mathbf{0}$, the evader can choose an arbitrary admissible input. In particular, for this case, the evader's input is the unit vector in the direction of the target, that is, $\boldsymbol{u}_e^*(t) = \hat{\boldsymbol{u}}_2(\boldsymbol{x}_e(t), \boldsymbol{x}_i(t))$, where

$$\hat{\boldsymbol{u}}_2(\boldsymbol{x}_e, \boldsymbol{x}_i) := \begin{cases} -\boldsymbol{x}_e/\|\boldsymbol{x}_e\|, & \text{if } \|\boldsymbol{\chi}_i\| = 0 \text{ and } \|\boldsymbol{x}_e\| > \epsilon \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Thus, in the interval $[0, t_s]$, the evader's control input which is the solution to Problem 2 is given by Eq. (12), where $\boldsymbol{u}_{e\mathcal{R}}(\boldsymbol{x}_e, \boldsymbol{x}_i)$ is equal to $\hat{\boldsymbol{u}}_1$ or $\hat{\boldsymbol{u}}_2$, depending on the vector $\boldsymbol{\chi}_i$. ■

Remark 7 If the evader lies at a position corresponding to a state in $\mathcal{R}$, at time t, it is possible that it can still reach a position at some time $t_s > t$, at which the log-risk β_i is zero, by using the quickest descent control law in Eq. (12). This implies that the augmented state at time t_s belongs to $\partial\mathcal{S}$. This ensures that the evader will successfully reach the target. The subset of states for which such a transition to $\mathcal{S}$ is possible, is determined by the relative speeds of the two players.

Next, let us determine whether the use of the quickest descent control induced by the log-risk β_i will lead to capture of the evader in the interval $[0, t_f)$. If the pursuer engages in relay pursuit, then $\boldsymbol{u}_i = \boldsymbol{r}_i/\|\boldsymbol{r}_i\|$. Then, in view of Eq. (3), $\|\boldsymbol{r}_i\|$ satisfies the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{r}_i\| = \frac{v_e}{\|\boldsymbol{r}_i\|}\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{u}_e - v_{p_i}. \tag{16}$$

Because $\max_{\boldsymbol{u}_e \in \mathcal{U}} \|\boldsymbol{u}_e\| = 1$, we conclude that $\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{r}_i\| < 0$ provided $v_e < v_{p_i}$, that is, the relative distance between the pursuer and the evader is a monotonically decreasing function of time. For the case that $v_e = v_{p_i}$, we will examine the evolution of the relative distance with time, assuming the evader uses the quickest descent control in Eq. (12).

Proposition 6: Suppose that $v_e = v_{p_i}$. If the relative distance between the evader and the pursuer at time $t = 0$ is greater than the radius of capture, that is, if $\|\boldsymbol{r}_i(0)\| > \ell$, then, for all $\hat{\ell} \in (\ell, \|\boldsymbol{r}_i(0)\|]$, with the application of the quickest descent control law as defined in Eq. (12) but with ℓ being replaced by $\hat{\ell}$, it will hold that $\|\boldsymbol{r}_i\| \geq \hat{\ell} > \ell$, for all $t \in [0, t_f]$. Consequently, the evader will avoid capture by the pursuer for all $t > 0$.

Proof 6 Without loss of generality, let $v_e = v_{p_i} = 1$, and let us replace ℓ with $\hat{\ell}$ in Eq. (13) and Eq. (15). Then, after simplification, the new log-risk defined with respect to $\hat{\ell}$ is

$$\hat{\beta}_i(\boldsymbol{x}_e, \boldsymbol{x}_i) = \log\left(\frac{2\left(\hat{\ell}\|\boldsymbol{x}_e\| + \boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{x}_e\right)}{\|\boldsymbol{r}_i\|^2 - \hat{\ell}^2}\right), \tag{17}$$

and the corresponding $\hat{\boldsymbol{\chi}}_i = \hat{\boldsymbol{\chi}}_i(\boldsymbol{x}_e, \boldsymbol{x}_i)$, in view of Eq. (13), becomes the following:

$$\hat{\boldsymbol{\chi}}_i = \frac{2}{\|\boldsymbol{r}_i\|^2 - \hat{\ell}^2}\left[-(1 - \mathrm{e}^{-\hat{\beta}_i})\boldsymbol{r}_i + \mathrm{e}^{-\hat{\beta}_i}(\|\boldsymbol{x}_e\| + \hat{\ell})\frac{\boldsymbol{x}_e}{\|\boldsymbol{x}_e\|}\right]. \tag{18}$$

In view of Eq. (17), $\hat{\beta}_i \to \infty$, and thus $\mathrm{e}^{-\hat{\beta}_i} \to 0$, as $\|\boldsymbol{r}_i\| \to \hat{\ell}$. Then, from Eq. (18), we argue that $\hat{\boldsymbol{\chi}}_i/\|\hat{\boldsymbol{\chi}}_i\| \to -\boldsymbol{r}_i/\|\boldsymbol{r}_i\|$ as $\|\boldsymbol{r}_i\| \to \hat{\ell}$. Clearly, for small values of $\|\boldsymbol{x}_e\|$, this holds true. For the case where $\|\boldsymbol{x}_e\|$ is large, let us consider $\|\boldsymbol{x}_e\| = \Delta$, where $\Delta \gg 1$ is a large number. Let $\delta = 1/\Delta$. When $\|\boldsymbol{r}_i\|^2 - \hat{\ell}^2 = \delta$, we have $\mathrm{e}^{\hat{\beta}_i} = \frac{2(\hat{\ell}\Delta + \|\boldsymbol{r}_i\|\Delta\cos\theta_{re})}{\delta}$ where θ_{re} is the smaller angle between $\boldsymbol{r}_i$ and $\boldsymbol{x}_e$. Clearly[2], $\mathrm{e}^{\hat{\beta}_i} = \mathcal{O}(\Delta^2)$, and the second term in Eq. (18) is dominated by $\mathrm{e}^{-\hat{\beta}_i}$. Then, it is true that $\hat{\boldsymbol{\chi}}_i/\|\hat{\boldsymbol{\chi}}_i\| \to -\boldsymbol{r}_i/\|\boldsymbol{r}_i\|$ as $\|\boldsymbol{r}_i\| \to \hat{\ell}$.

When the initial augmented state belongs to the set $\mathcal{R}$, (note that, actually, $\mathcal{F}_e(\boldsymbol{x}_i)$ is simply the ball of radius ℓ centered at $\boldsymbol{x}_i$, in this case), we have the evader's control input given by Eq. (12), which is a function of $\hat{\boldsymbol{\chi}}_i$. Then from Eq. (16), we have $\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{r}_i\| = -\frac{\boldsymbol{r}_i^{\mathrm{T}}\hat{\boldsymbol{\chi}}_i}{\|\boldsymbol{r}_i\|\|\hat{\boldsymbol{\chi}}_i\|} - 1$.

Let us consider the set $\mathcal{A}_c := \left\{\left[\boldsymbol{x}_e^{\mathrm{T}}\ \boldsymbol{x}_i^{\mathrm{T}}\right]^{\mathrm{T}} : \|\boldsymbol{r}_i\| = \hat{\ell}\right\}$. Then, as $\left[\boldsymbol{x}_e^{\mathrm{T}}\ \boldsymbol{x}_i^{\mathrm{T}}\right]^{\mathrm{T}} \to \mathcal{A}_c$, where the convergence of the augmented state to the set $\mathcal{A}_c$ is induced by the Hausdorff metric, we can say that the time-sequence of two-dimensional relative state vectors approaches a vector that lies on the circle of radius $\hat{\ell}$ around the origin. Simply put, the norm of the relative state vector approaches $\hat{\ell}$. Then, $\frac{\mathrm{d}\|\boldsymbol{r}_i\|}{\mathrm{d}t} \to 0$ as $\|\boldsymbol{r}_i\| \to \hat{\ell}$. This means that for any given relative distance $\|\boldsymbol{r}_i\| > \ell$ at time $t = 0$, we can choose $\hat{\ell} > \ell$, such that $\|\boldsymbol{r}_i\| \geq \hat{\ell} > \ell$ over the interval $[0, t_f]$. Note that even as $\hat{\beta}_i \to \infty$, the actual log-risk $\beta_i < \infty$. In other words, when $v_e = v_{p_i}$, the evader can avoid capture for all $t \in [0, t_f]$ by applying the quickest descent control law given in Eq. (12). ■

Remark 8 From Eq. (16), we can see that when $v_e = v_{p_i}$, we can at most guarantee that the relative distance between the pursuer and the evader in the time interval $[0, t_f]$ will be strictly lower bounded by ℓ, and upper bounded by their initial relative distance. When $v_e > v_{p_i}$, however, the relative distance between the two players is not upper bounded by its initial value. In both cases, the augmented state of the game will never transition from the set $\mathcal{R}$ to the set $\mathcal{F}$.

Remark 9 In view of the expression for $\boldsymbol{\chi}_i$ in Eq. (18), which holds when $v_e = v_{p_i}$, the evader's control input is a weighted sum of the control input required for pure evasion (that is, $\boldsymbol{u}_e = \boldsymbol{r}_i/\|\boldsymbol{r}_i\|$) and the control input required for pure target

[2]Big $\mathcal{O}$ is the Landau notation, and denotes the growth rate of $\mathrm{e}^{\hat{\beta}_i}$ with respect to Δ.

seeking ($\boldsymbol{u}_e = -\boldsymbol{x}_e/\|\boldsymbol{x}_e\|$). The weights of the individual components are functions of the log-risk β_i.

The transition from set $\mathcal{R}$ to set $\mathcal{S}$ using the quickest descent control law for the evader is illustrated in Fig. 4 for a sample target-seeking evasion problem where $v_e = v_{p_i}$. In each subfigure, one can see the contours of $\dot{\beta}_i$ at specific instances in time during the game. The evader is shown as a small blue square, and moves from the region where $\dot{\beta}_i > 0$ to the region where $\dot{\beta}_i < 0$ and eventually reaches the safe set. Note that the axes of Fig. 4 are chosen such that the pursuer is always on the positive x-axis with respect to the target location (origin).

D. Characterization of the region-based control strategy for the evader

In this section, we will provide the complete characterization of the evader's strategy over the whole state-space of the game. The optimal strategy for the evader depends on the set in which the augmented state is at $t = 0$, and taking into account the transitions from one set to another, we have effectively a strategy for the evader that switches between different control laws, in particular, the ones in Eq. (10) and Eq. (12). As stated previously, if the initial augmented state vector of the game (at $t = 0$) is in $\mathcal{S}$, the evader only needs to move towards the target, thereby minimizing $\phi_T(\boldsymbol{x}_e, \boldsymbol{x}_T)$. But if the initial conditions of the game are such that the initial state vector is in $\mathcal{F} \cup \mathcal{R}$, then the evader will have to first try to render $\beta_i = 0$.

The evader's region-based control strategy for the two-player game is given as shown in Fig. 5a. For all times $t \in [0, t_f]$, $\boldsymbol{u}_e^*(t) = \boldsymbol{u}_{\mathcal{X}}(\boldsymbol{x}_e(t), \boldsymbol{x}_i(t))$, where

$$\boldsymbol{u}_{\mathcal{X}}(\boldsymbol{x}_e, \boldsymbol{x}_i) = \begin{cases} \boldsymbol{u}_{e_{\mathcal{S}}}(\boldsymbol{x}_e), & \text{if } [\boldsymbol{x}_e^{\mathrm{T}} \ \boldsymbol{x}_i^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{S} \\ \boldsymbol{u}_{e_{\mathcal{R}}}(\boldsymbol{x}_e, \boldsymbol{x}_i), & \text{otherwise.} \end{cases} \tag{19}$$

The velocity field of the evader using the region-based strategy is illustrated in Fig. 5b, for a sample case, and for a fixed position of the pursuer. The blue arrows show the evader's velocity in $\mathcal{R}$ and the black arrows show the evader's velocity in $\mathcal{S}$. The pursuer is marked as a red square, and the target, by a black dot.

Proposition 7: Let us assume that $v_e = v_{p_i}$. If the choice of $\hat{\ell}$, where $\hat{\ell} \in (\ell, \|\boldsymbol{r}_i(0)\|]$, is such that $\|\boldsymbol{x}_i\| > \hat{\ell}$, then, as $\left[\boldsymbol{x}_e^{\mathrm{T}} \ \boldsymbol{x}_i^{\mathrm{T}}\right]^{\mathrm{T}} \to \partial\mathcal{S}$, $\|\boldsymbol{u}_{e_{\mathcal{R}}}(\boldsymbol{x}_e, \boldsymbol{x}_i) - \boldsymbol{u}_{e_{\mathcal{S}}}(\boldsymbol{x}_e)\| \to 0$.

Proof 7 Without loss of generality, let $v_e = v_{p_i} = 1$. If we consider the definition of the log-risk with respect to $\hat{\ell}$ as in Eq. (17), the points in the boundary of the new safe set $\partial\hat{\mathcal{S}}$ for this case are given by $\|\boldsymbol{x}_e\| = \|\boldsymbol{x}_i\| - \hat{\ell}$. If the evader starts close to $\partial\hat{\mathcal{S}}$, then $\|\boldsymbol{x}_e\| = \|\boldsymbol{x}_i\| - \hat{\ell} + \rho$, where $\rho > 0$ is a very small number. We also know that at the boundary $\partial\hat{\mathcal{S}}$, $\hat{\beta}_i = 0$, since the minimum time to intercept and the time-to-target are equal. It is simple to show that when $\|\boldsymbol{x}_e\| = \|\boldsymbol{x}_i\| - \hat{\ell} + \rho$ in Eq. (17), $\hat{\beta}_i$ is equal to a small but positive number. Then, from Eq. (18), we have

$$\frac{\hat{\boldsymbol{\chi}}_i}{\|\hat{\boldsymbol{\chi}}_i\|} = \frac{-(1-\mathrm{e}^{-\hat{\beta}_i})\boldsymbol{r}_i + \mathrm{e}^{-\hat{\beta}_i}(\|\boldsymbol{x}_i\| + \rho)\frac{\boldsymbol{x}_e}{\|\boldsymbol{x}_i\| - \hat{\ell} + \rho}}{\|(1-\mathrm{e}^{-\hat{\beta}_i})\boldsymbol{r}_i - \mathrm{e}^{-\hat{\beta}_i}(\|\boldsymbol{x}_i\| + \rho)\frac{\boldsymbol{x}_e}{\|\boldsymbol{x}_i\| - \hat{\ell} + \rho}\|}$$

We note that $\hat{\beta}_i \to 0^+$ as $\rho \to 0^+$, and in addition, $\|\boldsymbol{x}_i\| > \hat{\ell}$. Further, let us note that in view of Eq. (17), $\hat{\beta}_i > \beta_i$, and so, we have that $\beta_i \to 0^+$ as $\hat{\beta}_i \to 0^+$. Then, as $\rho \to 0^+$, $\frac{\hat{\boldsymbol{\chi}}_i}{\|\hat{\boldsymbol{\chi}}_i\|} \to \frac{\boldsymbol{x}_e}{\|\boldsymbol{x}_e\|}$, which implies, in view of Eq. (10) and Eq. (12), we see that as, $\hat{\beta}_i \to 0^+$, which is equivalent to $\left[\boldsymbol{x}_e^{\mathrm{T}} \ \boldsymbol{x}_i^{\mathrm{T}}\right]^{\mathrm{T}} \to \partial\mathcal{S}$, we have $\|\boldsymbol{u}_{e_{\mathcal{R}}}(\boldsymbol{x}_e, \boldsymbol{x}_i) - \boldsymbol{u}_{e_{\mathcal{S}}}(\boldsymbol{x}_e)\| \to 0$. ■

Remark 10 When the augmented state is close to $\partial\mathcal{S}$, the time-derivative of the relative distance between the target and the evader satisfies the following equation: $\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{x}_e\| = \frac{\boldsymbol{x}_e^{\mathrm{T}}\boldsymbol{u}_e}{\|\boldsymbol{x}_e\|} = -1$. Hence, if the augmented state is close to the boundary of the set $\mathcal{S}$, but belongs to the set $\mathcal{R}$, the optimal action for the evader is to follow the straight line path to the target.

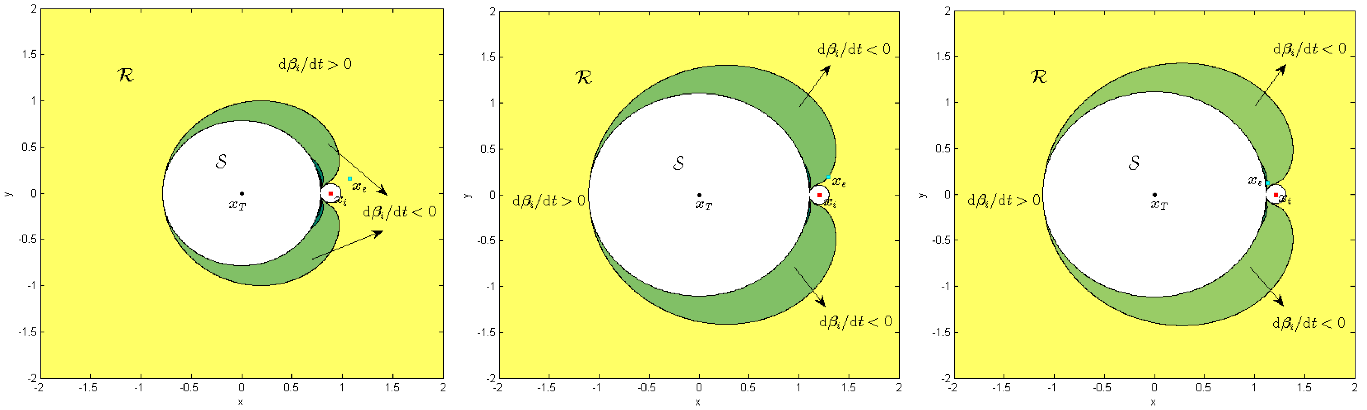
IV. Region-based control for target-seeking evader with multiple pursuers

We now extend the characterization of the evader's strategy for the case with a single pursuer to the case in which there are multiple pursuers, that is, $N > 1$. When there are more than two players, the game can be viewed as an aggregation of the individual two-player games between the evader and each pursuer. The actions of the evader with respect to one of the pursuers, however, will affect the states of the two-player games with respect to the other pursuers. Hence, the partition of the augmented state-space as well as the feedback control strategy require a distinct characterization in this case.

The partition of the augmented state-space for the multiple pursuer game can be obtained, by naturally extending the definition of the sets for the single pursuer game. In particular, $\mathcal{S} := \cap_{i=1}^{N}\mathcal{S}^i$ where $\mathcal{S}^i$ denotes the "safe" set for the game involving only the i^{th} pursuer and the evader. Similarly, $\mathcal{F} := \cup_{i=1}^{N}\mathcal{F}^i$, whereas the risky set $\mathcal{R} := \mathrm{int}(\mathcal{X}\backslash(\bar{\mathcal{S}} \cup \bar{\mathcal{F}}))$. For $N = 2$, the projection of the sets $\mathcal{S}$ and $\mathcal{F}$ on the evader's position space for fixed positions of the two pursuers is shown in Fig. 6. The set in white is the "risky" set $\mathcal{R}_e(\boldsymbol{x}_1, \boldsymbol{x}_2)$, and the sets $\mathcal{S}_e(\boldsymbol{x}_i)$ and $\mathcal{F}_e(\boldsymbol{x}_i)$ for $i = 1, 2$ are in green and red respectively. The dotted green circle is the boundary of the safe set $\mathcal{S}_e^2(\boldsymbol{x}_2)$ corresponding to the second pursuer, but the actual safe set $\mathcal{S}_e(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is the intersection of the individual safe sets, and is shown in solid green.

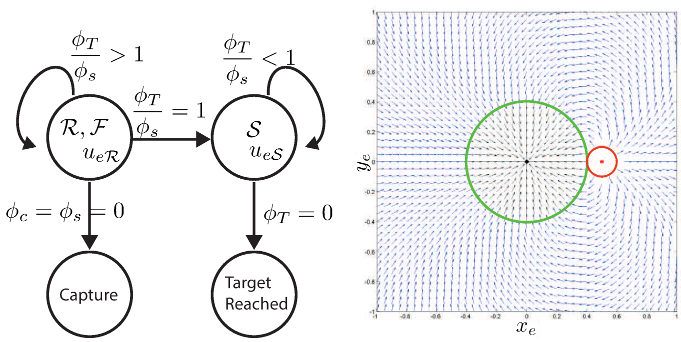
If the initial augmented state of the game belongs to the set $\mathcal{S}$, then the evader can go directly to the target at no risk of capture. If the initial state is in $\mathcal{F}$ or $\mathcal{R}$, then the proposed approach for the evader is to try to simultaneously minimize the risk ratio with respect to each pursuer. Let us assume that the maximum speed of all the players is equal to unity. Ideally, subject to the dynamic equations in Eq. (1), we would like to find a control signal for the evader, that is, $\boldsymbol{u}_e(t)$ for all $t \in [0, t_f]$, such that the time-derivative of the log-risk $\dot{\beta}_i$ is minimized simultaneously for all $i \in \mathcal{I}$, and for all $t \in [0, t_f]$. Since we have multiple criteria to minimize, however, we must perform a weighted least-squares approach to obtain a feasible control signal for the evader, point-wise in time. To this aim, we first form a vector whose elements are the individual log-risks at time t, as follows:

$$\boldsymbol{b}(\boldsymbol{x}_e, \boldsymbol{x}_1, ..., \boldsymbol{x}_N) := \begin{bmatrix} \beta_1(\boldsymbol{x}_e, \boldsymbol{x}_1) \\ \vdots \\ \beta_N(\boldsymbol{x}_e, \boldsymbol{x}_N) \end{bmatrix}. \tag{20}$$



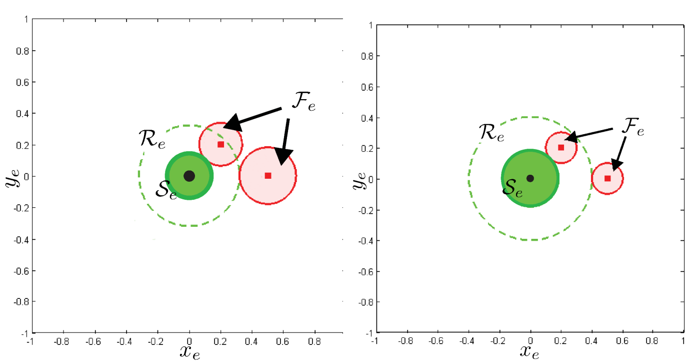
(a) $v_e = v_{p_i}, t = 0$ (b) $v_e = v_{p_i}, t = t_1$ (c) $v_e = v_{p_i}, t = t_2$

Fig. 4: The contours of the $\dot{\beta}_i$ function, projected on the evader's position space. The safe set is the white region while the evader is located at the blue square. The green regions correspond to negative values for $\dot{\beta}_i$.



(a) Region-based control strategy. (b) Evader's velocity field.

Fig. 5: For $N = 1$, a schematic representation of the evader's control law, and the evader's velocity field after the application of the control law, in the sets $\mathcal{R}$ and $\mathcal{S}$, for a fixed position of the pursuer.



(a) $v_e < v_p$ (b) $v_e = v_p$

Fig. 6: The three sets $\mathcal{S}_e$, $\mathcal{R}_e$, and $\mathcal{F}_e$, for different speeds of the evader, as projected on the evader's position space, for $N = 2$. The target location is shown as a black circle while the pursuer locations are the red squares.

Then, we have at time t the total derivative of $\boldsymbol{b}$ along the trajectories of all players satisfy:

$$\dot{\boldsymbol{b}}(\boldsymbol{u}_e; \boldsymbol{x}_e, \boldsymbol{x}_1, ..., \boldsymbol{x}_N) = \begin{bmatrix} \dot{\beta}_1(\boldsymbol{u}_e; \boldsymbol{x}_e, \boldsymbol{x}_1) \\ \vdots \\ \dot{\beta}_N(\boldsymbol{u}_e; \boldsymbol{x}_e, \boldsymbol{x}_N) \end{bmatrix} \tag{21}$$

where $\dot{\beta}_i(\boldsymbol{u}_e; \boldsymbol{x}_e, \boldsymbol{x}_i) = \nabla_{\boldsymbol{x}_e}\beta_i\dot{\boldsymbol{x}}_e + \nabla_{\boldsymbol{x}_i}\beta_i\dot{\boldsymbol{x}}_i = \boldsymbol{\chi}_i^{\mathrm{T}}\boldsymbol{u}_e + \gamma_i$, and $\boldsymbol{\chi}_i = \boldsymbol{\chi}_i(\boldsymbol{x}_e, \boldsymbol{x}_i)$ as defined in Eq. (13) and $\gamma_i = \gamma_i(\boldsymbol{x}_e, \boldsymbol{x}_i)$ as defined in Eq. (15). Given $\boldsymbol{x}_e$ and $\boldsymbol{x}_i$ for all $i \in \mathcal{I}$, let a function $\boldsymbol{\lambda}(\cdot) : \mathcal{U} \to \mathbb{R}^{N+1}$, where

$$\boldsymbol{\lambda}(\boldsymbol{u}_e) := \dot{\boldsymbol{b}}(\boldsymbol{u}_e; \boldsymbol{x}_e, \boldsymbol{x}_1, ..., \boldsymbol{x}_N). \tag{22}$$

Ideally, at time t, the choice of the control input $\boldsymbol{u}_e(t)$ for the evader should ensure that the time-derivative of each β_i function is individually minimized. That is, $\dot{\beta}_i = \min_{\boldsymbol{u}_e}(\boldsymbol{\chi}_i^{\mathrm{T}}\boldsymbol{u}_e + \gamma_i) = -\|\boldsymbol{\chi}_i\| + \gamma_i$, as follows from the controller defined in Eq. (12), for all $i \in \mathcal{I}$, and where $\boldsymbol{\chi}_i = \boldsymbol{\chi}_i(\boldsymbol{x}_e, \boldsymbol{x}_i)$. The desired vector value for the derivative $\dot{\boldsymbol{b}}$ is given by

$$\boldsymbol{\lambda}_{des}(\boldsymbol{x}_e, \boldsymbol{x}_1, ..., \boldsymbol{x}_N) := \begin{bmatrix} -\|\boldsymbol{\chi}_1\| \\ \vdots \\ -\|\boldsymbol{\chi}_N\| \end{bmatrix} + \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_N \end{bmatrix}. \tag{23}$$

The evader's objective, as stated above, is to minimize the derivative of the log-risk with respect to all the pursuers simultaneously and with equal weighting given to the minimization of the individual β_i at each instant of time t. During the course of a game, however, all the pursuers may not need to be considered with the same priority by the evader. For instance, if $N = 3$, it is possible that the location of the third pursuer is irrelevant to the evader since that pursuer can never capture the evader before the others. In that case, it is prudent to only consider the game between the evader and the first two pursuers. The notion of the "relevance" of the i^{th} pursuer to the game is introduced in the form of a weight w_i assigned to the $\dot{\beta}_i$ element in the vector $\boldsymbol{\lambda}$. The weight $w_i = w_i(\boldsymbol{x}_e, \boldsymbol{x}_i)$ is chosen as a function of the minimum time to intercept $\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i)$ as follows:

$$w_i(\boldsymbol{x}_e, \boldsymbol{x}_i) = 1 + \mathrm{e}^{-\phi_s(\boldsymbol{x}_e, \boldsymbol{x}_i)}, \quad i \in \mathcal{I}. \tag{24}$$

Note that the weight w_i is such that if the minimum time to intercept by a pursuer is large, then the evader is considered to be at less risk of being captured by that pursuer, compared to a pursuer whose minimum time to intercept is small. Hence, the evader weighs the log-risk of the latter pursuer more than the others. Now, let $\boldsymbol{W}(\boldsymbol{x}_e, \boldsymbol{x}_1, ..., \boldsymbol{x}_N) := \text{diag}\,[\bar{w}_1(\boldsymbol{x}_e, \boldsymbol{x}_1), ..., \bar{w}_N(\boldsymbol{x}_e, \boldsymbol{x}_N)]$, where

$$\bar{w}_i(\boldsymbol{x}_e, \boldsymbol{x}_i) := w_i(\boldsymbol{x}_e, \boldsymbol{x}_i)/(\sum_{i=1}^{N} w_i(\boldsymbol{x}_e, \boldsymbol{x}_i)). \tag{25}$$

Now, we introduce the following optimization problem:

Problem 3: Given $\boldsymbol{\lambda}$ *as defined in Eq.* (22)*, and the vector* $\boldsymbol{\lambda}_{des}$ *as given by Eq.* (23)*, find the evader's input* $\hat{\mathbf{u}}_e$ *which minimizes the cost function* [3]

$$\begin{aligned}\mathcal{J}_w(\boldsymbol{u}_e) &:= \|\boldsymbol{\lambda}_{des} - \boldsymbol{\lambda}(\boldsymbol{u}_e)\|^2_{\boldsymbol{W}} \\ &= (\boldsymbol{\lambda}_{des} - \boldsymbol{\lambda}(\boldsymbol{u}_e))^{\mathrm{T}}\, \boldsymbol{W}\, (\boldsymbol{\lambda}_{des} - \boldsymbol{\lambda}(\boldsymbol{u}_e))\end{aligned} \tag{26}$$

subject to the constraint $\|\boldsymbol{u}_e\| \leq 1$.

Proposition 8: The evader's control input $\boldsymbol{u}_e^*(t)$ *for the vector case at each time* $t \in [0, t_s]$*, where* t_s *is the first time at which the evader reaches the boundary of set* $\mathcal{S}$*, is given by the solution to the quadratically constrained quadratic program in Problem 3. That is,* $\boldsymbol{u}_e^*(t) = \hat{\boldsymbol{u}}_e$ *for all* $t \in [0, t_s]$.

Proof 8 By construction, the matrix $\boldsymbol{W}$ is always positive definite. Note that $\boldsymbol{\lambda}$ is an affine function of $\boldsymbol{u}_e$, and the quadratic form with respect to a positive definite matrix is a convex function. Therefore, the cost function $\mathcal{J}_w$, which is a composition of a convex quadratic function with an affine function of $\boldsymbol{u}_e$, is convex in $\boldsymbol{u}_e$. Similarly, the constraint on the magnitude of the evader's input, given by $\|\boldsymbol{u}_e\| \leq 1$, is also convex. Hence, we conclude that Problem 3 is a convex program, and in particular, a quadratically constrained quadratic program. By the formulation of the cost function in Eq. (26), the solution to this program is the solution to the quadratically constrained least squares problem, which, in turn, is the control input for the evader, point-wise in time. Hence, $\boldsymbol{u}_e^*(t) = \hat{\boldsymbol{u}}_e$ for all $t \in [0, t_s]$. ■

The solution $\hat{\boldsymbol{u}}_e$ to Problem 3 is implicitly dependent on the states of the players at time t. Once the evader reaches the boundary of set $\mathcal{S}$, we can use the control input in Eq. (10). Then, the region-based feedback strategy for the target-seeking evader with multiple pursuers is given as follows:

$$\boldsymbol{u}_e^*(t) = \begin{cases} \boldsymbol{u}_{e_{\mathcal{S}}}(\boldsymbol{x}_e), & \text{if } [\boldsymbol{x}_e^{\mathrm{T}}\ \boldsymbol{x}_1^{\mathrm{T}}\ ...\ \boldsymbol{x}_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{S} \\ \boldsymbol{u}_{e_{\mathcal{R}}}(\boldsymbol{x}_e, \boldsymbol{x}_1, ..., \boldsymbol{x}_N), & \text{otherwise,} \end{cases} \tag{27}$$

where $\boldsymbol{u}_{e_{\mathcal{S}}}(\boldsymbol{x}_e)$ is defined in Eq. (10) and $\boldsymbol{u}_{e_{\mathrm{R}}}(\boldsymbol{x}_e, \boldsymbol{x}_1, ..., \boldsymbol{x}_N) := \hat{\boldsymbol{u}}_e$. Note that when all players have unit speeds, the projections of set $\mathcal{F}$ are balls of radius ℓ around each pursuer.

V. Numerical simulations

In this section, we present numerical simulations that illustrate the performance of the proposed control strategy against one or more pursuers that use the relay pursuit strategy. The various parameters chosen are: $v_{p_i} = 1$ for all $i \in \mathcal{I}$, $v_e = 1$, $\ell = 0.1$, and $\epsilon = 0.001$. In all figures in this section, the target is marked as a black circle and the path of the evader is shown in green and that of the pursuers in red. In Fig. 7a, evader-pursuer trajectories in the position space are shown for a sample case when $N = 2$. The pursuers engage in relay pursuit with only the pursuer P_1 being active through the game. The game illustrated in Fig. 7b features only the active players of the game illustrated in Fig. 7a. That is, in Fig. 7b, only the evader and pursuer P_1 are present. The evader clearly follows a different trajectory in this case compared to the one in Fig. 7a, and this is because in Fig. 7a, the evader takes into account the presence of a second pursuer which has the potential to be activated. In Fig. 7c and Fig. 7a, the non-active pursuers are stationary at their initial locations. The respective augmented states in each sample case are initially in the set $\mathcal{R}$.

TABLE I: Outcome of games - $v_e = 1$

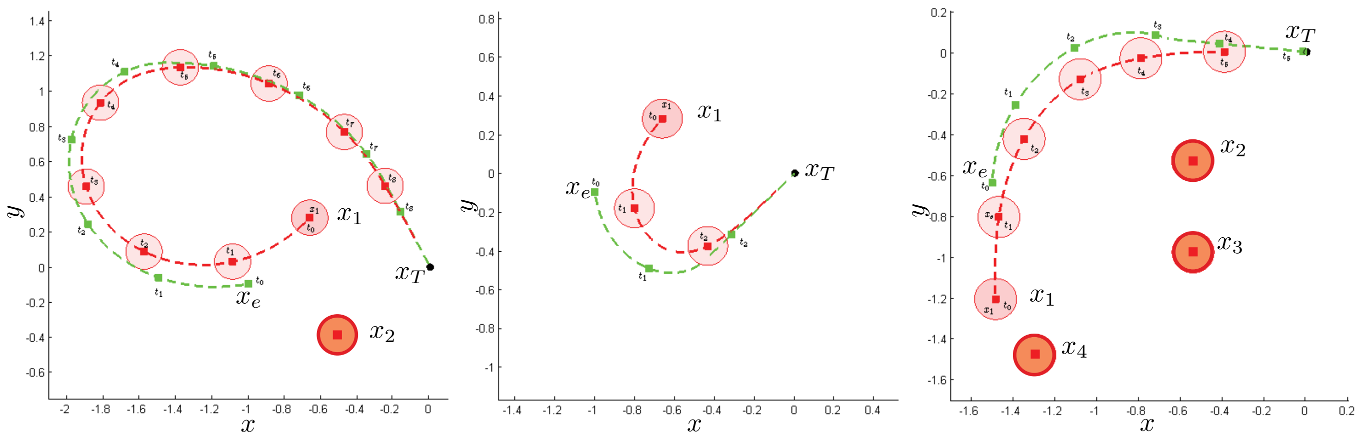
Solution Method	$N=1$	$N=2$	$N=4$	$\mathcal{P}$ at t_f
Evader Captured	0.00%	26.88%	41.66%	∞
Target Reached	97.47%	72.04%	57.89%	0

In order to test the performance of the proposed control strategy, we simulated the target-seeking evasion game for a large number of random initial conditions for the pursuer and the evader, and random target positions. In all cases, the initial augmented state was in the risky set $\mathcal{R}$ and the evader followed the region-based control strategy as given in Eq. (19) and Eq. (27). The time limit for each simulation was fixed at 100 time units, to ensure the completion of each simulation in finite time. The outcomes of the target-seeking evasion game for initial states that begin from $\mathcal{R}$ are shown for different values of N in Table I. For $N = 1$, of about 10^5 cases of uniformly random initial conditions chosen from the set $\mathcal{R}$, there were no cases of capture. The evader reached the target in all cases, except when the players were positioned such that $\boldsymbol{r}_i$ and $\boldsymbol{x}_e$ were collinear and the pursuer was between the evader and the target. In such a case, the optimal action for the evader is pure evasion, and so the evader will neither reach the target nor get captured. When $N > 1$, we notice that as N increases, the percentage of games that end in capture of the evader also increases, as one would expect. The percentage of inconclusive games in Table I correspond to those cases in which the evader was neither captured nor did it reach the target within the fixed time limit.

VI. Conclusion and Future Work

In this paper, we addressed a reach-avoid problem in which the evader must reach a fixed target location while avoiding capture by a group of pursuers. In our proposed solution, the evader avoids being captured by pursuers which adopt a semi-cooperative strategy known as relay pursuit, which is a more efficient pursuit strategy than simultaneous non-cooperative pursuit. We first characterized the solution for the corresponding two-player reach-avoid game, with only one pursuer and the evader. We have shown that the state-space of the game can be partitioned into three non-overlapping sets,

[3] where the arguments corresponding to the states have been dropped



(a) $N = 2$. (b) $N = 1$, P_2 is absent. (c) $N = 4$.

Fig. 7: Evader-pursuer trajectories, where two pursuers are present in Fig. 7a, and the second pursuer P_2 is absent in Fig. 7b, and with $N = 4$ in Fig. 7c.The pursuers' and evader's positions are marked at specific points in time during the game.

and this partition allows us to determine the outcome of the game at initial time. We also developed a quickest-descent control strategy for the evader, which depends on its location in the state-space. Subsequently, we extended these results to the game with multiple pursuers, where the feedback strategy of the evader was determined by the solution to a convex minimization problem. Finally, we illustrated the performance of the proposed control strategy for the evader using a large number of numerical simulations. We conclude that the proposed region-based control strategy performs well for a small number of pursuers, independent of other parameters of the game such as the initial relative distance between the players or the capture radius. In future work, we will improve the performance of the region-based strategy as the number of pursuers increases, by exploring modifications of the quickest descent control law applied to the multi-player case. We will explore, in particular, different weighting functions for the cost function associated with the quickest descent control law, and also, the case where the position information available to the evader is noisy.

Acknowledgments: This work was supported by the NSF under Grant 1562339.

References

[1] E. Bakolas and P. Tsiotras, Relay pursuit of a maneuvering target using dynamic Voronoi diagrams, *Automatica*, vol. 48, no. 9, pp. 2213–2220, 2012.

[2] R. Isaacs, *Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization*. New York: Dover Publication, 1999.

[3] W. Scott and N. Leonard, Pursuit, herding and evasion: A three-agent model of caribou predation, in *ACC 2013*, pp. 2978–2983, June, 2013.

[4] T. Basar and G. J. Olsder, Dynamic Noncooperative Game Theory. London: SIAM, 1995.

[5] A. Haurie, J. B. Krawczyk, G. Zaccour, *et al.*, *Games and dynamic games*. Singapore: World Scientific Publishing Co. Pte. Ltd., 2012.

[6] A. Perelman, T. Shima, and I. Rusnak, Cooperative differential games strategies for active aircraft protection from a homing missile, *J. Guid. Contr. Dynam.*, vol. 34, no. 3, pp. 761–773, 2011.

[7] Z. Fuchs, P. Khargonekar, and J. Evers, Cooperative defense within a single-pursuer, two-evader pursuit evasion differential game, in *CDC 2010*, pp. 3091–3097, Dec., 2010.

[8] S. Pan, H. Huang, J. Ding, W. Zhang, D. Stipanovic, and C. Tomlin, Pursuit, evasion and defense in the plane, in *ACC 2012*, pp. 4167–4173, June, 2012.

[9] H. Huang, W. Zhang, J. Ding, D. Stipanovic, and C. Tomlin, Guaranteed decentralized pursuit-evasion in the plane with multiple pursuers, in *CDC-ECC 2011*, pp. 4835–4840, Dec., 2011.

[10] W. Sun and P. Tsiotras, A sequential pursuer-target assignment problem under external disturbances, in *CDC 2013*, pp. 3994–3999, Dec., 2013.

[11] E. Bakolas, Evasion from a group of pursuers with double integrator kinematics, in *CDC 2013*, pp. 1472–1477, Dec., 2013.

[12] J. P. Hespanha, H. J. Kim, and S. Sastry, Multiple-agent probabilistic pursuit-evasion games, in *CDC 1999*, pp. 2432–2437, Dec 1999.

[13] J. P. Hespanha, M. Prandini, and S. Sastry, Probabilistic pursuit-evasion games: A one-step Nash approach, in *CDC 2000*, pp. 2272-2277, Dec 2000.

[14] M. Ramana and M. Kothari, Pursuit strategy to capture high-speed evaders using multiple pursuers, *Journal of Guidance, Control, and Dynamics*, vol. 40, no. 1, pp. 139-149, 2016.

[15] M. Kothari, J. G. Manathara, and I. Postlethwaite, Cooperative multiple pursuers against a single evader, *Journal of Intelligent and Robotic Systems*, vol. 86, no. 3–4, pp. 551-567, 2017.

[16] M. Chen, Z. Zhou, and C. J. Tomlin, Multiplayer reach-avoid games via pairwise outcomes, *IEEE Transactions on Automatic Control*, vol. 62, no. 3, pp. 1451-1457, 2017.

[17] Z. Zhou, R. Takei, H. Huang, and C. J. Tomlin, A general, open-loop formulation for reach-avoid games, in *CDC 2012*, pp. 6501-6506, 2012.

[18] M. Chen, J. F. Fisac, S. Sastry, and C. J. Tomlin, Safe sequential path planning of multi-vehicle systems via double-obstacle Hamilton-Jacobi-Isaacs variational inequality, in *ECC 2015*, pp. 3304-3309, 2015.

[19] I. Rusnak, The lady, the bandits and the body guards a two team dynamic game, *IFAC Proceedings Volumes*, vol. 38, no. 1, pp. 441–446, 2005.

[20] E. Garcia, D. W. Casbeer, and M. Pachter, Design and analysis of state-feedback optimal strategies for the differential game of active defense, *IEEE Transactions on Automatic Control*, vol. 64, no. 2, pp. 553–568, 2019.

[21] J. F. Fisac, M. Chen, C. J. Tomlin, and S. S. Sastry, Reach-avoid problems with time-varying dynamics, targets and constraints, in *HSCC 2015*, pp. 11–20, ACM, 2015.

[22] R. Yan, Z. Shi, and Y. Zhong, Reach-avoid games with two defenders and one attacker: An analytical approach, *IEEE Transactions on Cybernetics*, vol. 49, no. 3, pp. 1035–1046, 2018.

[23] J. Selvakumar and E. Bakolas, Evasion from a group of pursuers with a prescribed target set for the evader, in *ACC 2016*, pp. 155–160, July 2016.

[24] J. Selvakumar and E. Bakolas, Evasion with terminal constraints from a group of pursuers using a matrix game formulation, in *ACC 2017*, pp. 1604–1609, 2017.

[25] E. Bakolas and P. Tsiotras, Optimal synthesis of the Zermelo-Markov-Dubins problem in a constant drift field, *Journal of Optimization Theory and Applications*, vol. 156, no. 2, pp. 469–492, 2013.

[26] E. Bakolas, On the finite-time capture of a fast moving target, *Optimal Control Applications and Methods*, vol. 38, no. 5, pp. 778–794, 2017