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Robust Time-Optimal Guidance in a Partially Uncertain Time-Varying Flow-Field

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Abstract In this paper, we address the problem of guiding an aerial or aquatic vehicle to a fixed target point in a partially uncertain flow-field. We assume that the motion of the vehicle is described by a point-mass linear kinematic model. In addition, the velocity of the flow-field, which is taken to be time varying, can be decomposed into two components: one which is known a priori, and another one which is uncertain and only a bound on its magnitude is known. We show that the guidance problem can be reformulated as an equivalent pursuit evasion game with time-varying affine dynamics. To solve the latter game, we propose an extension of a specialized solution approach which transforms the pursuit-evasion game (whose terminal time is free) via a special state transformation into a family of games with fixed terminal time. In addition, we provide a simple method to visualize the level sets of the value function of the game, along with the corresponding reachable sets. Furthermore, we compare our conservative game-theoretic solution with a pure optimal control solution for the special case in which the flow-field is perfectly known a priori.

Keywords Pursuit-evasion game · time-varying winds · minimum-time guidance · input constraints

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1 Introduction

More now than ever, mobile autonomous agents are desired for assisting or replacing human activity in potentially hazardous or complex situations. The ability to navigate successfully in uncertain environments is of significant interest for autonomous and semi-autonomous vehicles. In this paper, we focus on the problem of guiding an autonomous aquatic or aerial vehicle to a fixed known location in the presence of a partially uncertain flow-field induced by, for instance, oceanic currents and winds respectively. In particular, we assume that the velocity of the flow-field can be decomposed into two components, namely, one that is known a priori, and another one which is uncertain and only a bound on its magnitude is known a priori. Our objective is to characterize a *robust time-optimal* guidance law for the previous problem.

Accurate probabilistic models for the environmental disturbances that act upon an aquatic or an aerial vehicle are not always available. In the absence of such models, the problem of guiding a vehicle to a goal state cannot be addressed via optimal control techniques, which cannot account for the effect of the uncertain flow-field. By contrast, a game-theoretic formulation is well-suited to handle this type of uncertainty. Specifically, by formulating the guidance problem as a game, one can design robust guidance laws. In this formulation, the unknown component of the flow-field is assumed to be determined by a decision maker that is antagonistic to the vehicle. In particular, this adversary makes decisions that aim at delaying the arrival of the vehicle to its final destination as much as possible, if not preventing the latter from reaching its destination at all.

Given this central idea, we formulate the vehicle guidance problem as an affine zero-sum pursuit-evasion game between two agents, where the vehicle takes the role of the pursuer whereas the evader corresponds to a second fictitious player whose velocity is equal to the velocity of the flow-field. Because it is assumed that one component of the velocity of the flow-field is prescribed and known to the vehicle / pursuer, the evader will be confined to controlling only the part of its velocity that corresponds to the uncertain component of the velocity of the flow-field. In this set-up, the event of capture of the evader by the pursuer is equivalent to the vehicle arriving at the goal destination within a tolerance specified by the user.

Literature review: Two-player zero-sum pursuit evasion games (PEGs) have been well studied in literature [1–3]. Rufus Isaacs pioneered a great deal of work on differential games with an emphasis on PEGs. Isaacs' solution framework is centered around a particular partial differential equation, known as the Hamilton-Jacobi-Isaacs equation, the solution of which yields the optimal feedback strategies for the players [1]. An alternative approach to differential games including PEGs, which is built upon variational techniques, was proposed by Pontryagin [4]. Another solution concept is the so-called *stroboscopic strategy* introduced by Hàjek [5], in which a player has information about the other player's decision before making his own. An illustration of stroboscopic strategies in guidance problems can be found in [6].

In general, the time of termination or terminal time of the game may be fixed or free. A popular class of differential games with fixed terminal time is that of linear quadratic differential games, which seek the control inputs of two antagonistic players that will minimize and maximize a quadratic performance index, respectively, over a known and fixed time interval. Bernhard [7] provides necessary and sufficient conditions for the existence of a saddle point for linear-quadratic games in general, and computes the optimal feedback strategy by solving a certain Riccati equation. Two-player linear quadratic differential games with an uncertain dynamic model are addressed in [8]. The authors provide sufficient conditions for existence of closed loop strategies based on solutions to certain Riccati equations. Furthermore, Bernhard deals with the Isotropic Rocket problem in [9], where he investigates the optimality of Isaacs' solution and presents geometric results on the game of kind. Perelman et al. [10] formulate the problem of aircraft defense against a homing missile as a linear quadratic differential game, assuming perfect knowledge of the system, and they provide optimal cooperative strategies for all the players.

A specialized solution technique for a class of PEGs with free terminal time is presented by Gutman et al. [11]. The proposed technique associates a particular class of PEGs with free terminal time to a one-parameter family of differential games whose performance index consists of a single terminal cost term and whose final time is fixed (the final time is the parameter of this family of differential games). The open-loop solution to the latter family of problems is obtained by applying the variational approach for zero-sum differential games [3]. The authors also analyze the capture and avoidance zones of the original pursuit-evasion game. Game-theoretic approaches are used to address control problems in adversarial or uncertain environments in [10], [12], [13], [14], and [15]. In particular, Trottemant et al. [14] address the control design subject to model uncertainty for a missile guidance problem by utilizing a two-person game-theoretic approach and convex optimization techniques. Patsko et al. [15] have proposed numerical methods for the solution of an aircraft landing problem under severe wind disturbances formulated as a differential game. Further, Breitner and Pesch [12] solve the problem of shuttle re-entry under atmospheric uncertainty as a two-player differential game.

Zhao and Bryson [16] have addressed the aircraft control under uncertainty during takeoff in a downburst by formulating it as a two-objective (min and max) optimization problem. In addition, Bulirsch et al. [17] have addressed the aircraft landing problem in windshear based on a minimax optimal control problem formulation. Further, the minimum-time guidance problem in a deterministic flow-field for simpler kinematic models has received some attention in the literature. Combined analytical and computational techniques for the solution of the time-optimal guidance problem for a Dubins vehicle in constant wind are presented in [18–20]. A numerical solution approach to the problem of guiding a Dubins vehicle to a target set in a stochastically varying wind in minimum expected time is presented in [21]. In addition, the minimum-time guidance of a Newtonian particle (or point mass) in the presence of a known time-varying flow-field and fixed terminal position was addressed by Bakolas [22]. The results of [22] were extended by Bakolas and Marchidan in [23], who developed an algorithm for time-optimal guidance of a particle in a spatio-temporal field. Sun et al. [24] use level sets and reachability analysis to solve a multi-agent pursuit-evasion game in a dynamic flow-field under the assumption that all players follow simple motion dynamics (that is, the players can change the direction of their velocities instantaneously). The robust time-optimal guidance of a Newtonian particle in a partially uncertain but constant flow-field was solved by means of Isaacs' solution approach in our previous work [25].

Contributions: In this paper, we solve the problem of robust time-optimal guidance of a vehicle in a partially known time-varying flow-field. It should be highlighted at this point that assuming that the flow-field be completely adversarial will lead to an overly conservative solution. Because the flow-field is time-varying, the standard approaches presented in [1] and [11] cannot be directly applied to our problem. For this reason, we propose an extension of the solution approach proposed in [11], which is based on a special state transformation. This transformation not only allows us to calculate the robust time-optimal control input for the vehicle in a straightforward way but also simplifies the characterization of the level sets of the optimal value function of the robust time-optimal guidance problem. We illustrate our approach using numerical simulations for the specific example of guiding a Newtonian particle in a flow-field. The latter problem can be viewed as a variation of the classic Isotropic Rocket PEG of Isaacs [1]. At this point, we note that if the vehicle uses the game-theoretic control input at all times, that is, the vehicle acts as a *rational agent*, the time it will need to reach its goal position in the presence of an uncertain flow-field will be upper-bounded by the time of capture of the equivalent pursuit-evasion game. This is because in the game-theoretic formulation of the guidance problem, we assume that the uncertain component of the flow-field will have, at all times, the worst possible effect on the vehicle in terms of preventing the latter from reaching its goal destination as fast as possible. In practice, it is expected that the vehicle will arrive faster to its destination than what the game-theoretic solution predicts given that, for instance, the unknown component of the flow-field may turn out to be aiding the vehicle to reach its destination for certain time subintervals.

Organization: The contents of the paper are organized as follows. In Section 2, we formulate the robust time-optimal guidance problem which we associate with an equivalent two-player zero-sum differential game. In Section 3, we present in detail the solution approach to the two-player game. In Section 4, we apply our solution to a variation of the Isotropic Rocket PEG of Isaacs. The corresponding numerical simulations and observations are presented in Section 5. Finally, Section 6 concludes the paper with a summary of remarks.

2 Formulation of the Robust Time-Optimal Guidance Problem and its Equivalent Pursuit-Evasion Game

2.1 The Robust Time-Optimal Guidance Problem

Consider a vehicle V, whose equations of motion in the presence of a flow-field are given by the following state space model:

$$\dot{\boldsymbol{x}}_V(t) = \boldsymbol{A}_V(t)\boldsymbol{x}_V(t) + \boldsymbol{B}_V(t)\boldsymbol{u}_V(t) + \boldsymbol{E}_V(t)\boldsymbol{p}(t), \quad \boldsymbol{x}_V(t_0) = \boldsymbol{x}_V^0, \tag{1}$$

for all $t \in [t_0, T]$ with $t_0 \leq T < \infty$, where $\boldsymbol{x}_V(t) \in \mathbb{R}^{2n}$ and $\boldsymbol{x}_V^0 \in \mathbb{R}^{2n}$ denote, respectively, the state at time $t \in [t_0, T]$ and time $t = t_0$. In addition, $\boldsymbol{u}_V(t)$ denotes the control input at time t. It is assumed that $\boldsymbol{u}_V(\cdot) : [t_0, T] \to \mathcal{U} \subset \mathbb{R}^n$ is a piece-wise continuous function that attains values in the set $\mathcal{U} := \{\boldsymbol{u} \in \mathbb{R}^n : \ \|\boldsymbol{u}\| \leq 1\}$, for all $t \in [t_0, T]$ ($\|\cdot\|$ denotes the vector 2-norm in $\mathbb{R}^n$). It is assumed that the matrix-valued functions $\boldsymbol{E}_V(\cdot) : [t_0, T] \to \mathbb{R}^{2n\times n}$, $\boldsymbol{A}_V(\cdot) : [t_0, T] \to \mathbb{R}^{2n\times 2n}$ and $\boldsymbol{B}_V(\cdot) : [t_0, T] \to \mathbb{R}^{2n\times n}$ are piece-wise continuous. Furthermore, $\boldsymbol{p}(t) \in \mathbb{R}^n$ denotes the velocity of the partially known flow-field at time $t \in [t_0, T]$. In this work, we assume that

$$\boldsymbol{p}(t) := w_p \boldsymbol{v}_p(t) + \boldsymbol{p}_0(t), \quad \text{for all} \ \ t \in [t_0, T], \tag{2}$$

where $\boldsymbol{p}_0(t) \in \mathbb{R}^n$ denotes the known component of the velocity of the flow-field at time t. The term $w_p \boldsymbol{v}_p(t)$ corresponds to the unknown component of $\boldsymbol{p}(t)$. In particular, w_p is a known positive constant and $\boldsymbol{v}_p(\cdot) : [t_0, T] \to \mathcal{V} \subset \mathbb{R}^n$ is an unknown piece-wise continuous function that takes values in $\mathcal{V} := \{\boldsymbol{v} \in \mathbb{R}^n : \ \|\boldsymbol{v}\| \leq 1\}$. It follows that the magnitude of the uncertain component is upper bounded by w_p, that is, $\|w_p \boldsymbol{v}_p(t)\| \leq w_p$ for all $t \in [t_0, T]$. A schematic depiction of the vehicle guidance problem in a flow-field is shown in Fig. **??**.

In the state space model given by (1), the state vector at time t, which is denoted as $\boldsymbol{x}_V(t)$, is the concatenation of the position and velocity vectors of the vehicle V at time t, which are denoted by $\boldsymbol{r}_V(t)$ and $\boldsymbol{v}_V(t)$, respectively. Therefore, the state space will be either four-dimensional ($n = 2$), in the planar 2-D case, or six-dimensional ($n = 3$), in the 3-D case. In order to streamline the presentation, we will henceforth focus on the planar case ($n = 2$). Therefore, the position vector $\boldsymbol{r}_V(t)$ can be written as follows: $\boldsymbol{r}_V(t) = \boldsymbol{D}\boldsymbol{x}_V(t)$ where $\boldsymbol{D} = \begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{0} \end{bmatrix} \in \mathbb{R}^{2\times 4}$. Note that the 3-D case can be treated in a similar way after the necessary modifications have been carried out.

Now, let $\boldsymbol{r}_V^{\mathrm{T}} \in \mathbb{R}^2$ be a given target position and $\ell > 0$ a given constant, and let $\mathcal{T}_V(\boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot); t_0, \boldsymbol{x}_V^0, \mathcal{S}_\ell^{\mathrm{T}})$ denote the first time at which the system described by (1), which emanates from $\boldsymbol{x}_V^0$ at time t_0 and is driven by the control input $\boldsymbol{u}(\cdot)$, for a given piecewise continuous function $\boldsymbol{v}(\cdot)$ (particular realization of the uncertain component of the flow-field), reaches the terminal set $\mathcal{S}_\ell^{\mathrm{T}} := \{\boldsymbol{x}_V \in \mathbb{R}^4 : \ \|\boldsymbol{D}\boldsymbol{x}_V - \boldsymbol{r}_V^{\mathrm{T}}\| \leq \ell\}$. For the given initial condition $\boldsymbol{x}_V^0$, the min-max time of arrival is denoted by $T_V^* = T_V^*(t_0, \boldsymbol{x}_V^0; \mathcal{S}_\ell^{\mathrm{T}})$ with

$$T_V^*(t_0, \boldsymbol{x}_V^0; \mathcal{S}_\ell^{\mathrm{T}}) := \min_{\boldsymbol{u}(\cdot)} \max_{\boldsymbol{v}(\cdot)} \mathcal{T}_V(\boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot); t_0, \boldsymbol{x}_V^0, \mathcal{S}_\ell^{\mathrm{T}}), \tag{3}$$

where the operators min and max operate on all piecewise continuous inputs $\boldsymbol{u}(\cdot)$ and $\boldsymbol{v}(\cdot)$ taking values in $\mathcal{U}$ and $\mathcal{V}$, respectively, with the application of which the vehicle's trajectory $\boldsymbol{x}_V(\cdot)$ reaches the terminal set $\mathcal{S}_\ell^{\mathrm{T}}$ in finite time. Next we formulate the guidance problem in an uncertain (or more precisely, partially uncertain) flow-field as a robust time-optimal problem.

Problem 2.1 *Given $\ell > 0$, an initial state $\boldsymbol{x}_V^0 \in \mathbb{R}^4$ and a target terminal position $\boldsymbol{r}_V^{\mathrm{T}} \in \mathbb{R}^2$, find a piecewise continuous control input $\boldsymbol{u}_V^*(\cdot) : [t_0, T_V^*] \to \mathcal{U}$ that will steer the vehicle V, whose motion is described by* (1) *and which emanates from the state $\boldsymbol{x}_V^0$ at time t_0, to the terminal set $\mathcal{S}_\ell^{\mathrm{T}}$ at time $t = T_V^*$, where $T_V^* = T_V^*(t_0, \boldsymbol{x}_V^0; \mathcal{S}_\ell^{\mathrm{T}})$ is the min-ma x time of arrival that is defined in* (3).

2.2 Formulation of the Robust Time-Optimal Guidance Problem in an Uncertain Flow-Field as a Pursuit-Evasion Game in a Known Flow-Field

Next, we reformulate Problem 2.1 as a two-player zero-sum differential game. The motivation for this approach comes from the fact that one can think of the vehicle 'playing' against an adversary that controls the uncertain component of the flow-field so that the latter will have the worst possible effect on the former player (vehicle) in terms of preventing it from reaching its target location as fast as possible. In the inertial frame, and in the presence of a flow-field, the target is static. However, the target actually moves according to the flow-field in the reference frame attached to the vehicle itself. This moving target is referred to as the evader, while the vehicle takes on the role of the pursuer. Let P denote the pursuer and E the evader. The equation of motion for the pursuer in the inertial frame of reference is as follows:

$$\dot{\boldsymbol{x}}_P(t) = \boldsymbol{A}_P(t)\boldsymbol{x}_P + \boldsymbol{B}_P(t)\boldsymbol{u}_P(t), \quad \boldsymbol{x}_P(t_0) = \boldsymbol{x}_P^0, \tag{4}$$

where the state $\boldsymbol{x}_P(t)$ is equal to $\boldsymbol{x}_V(t)$ as in (1), and $\boldsymbol{A}_P(t)$, $\boldsymbol{B}_P(t)$ and $\boldsymbol{u}_P(t)$ are all equal to $\boldsymbol{A}_V(t)$, $\boldsymbol{B}_V(t)$, and $\boldsymbol{u}_V(t)$, respectively, for all $t \in [t_0, T]$. The evader's velocity is determined by the velocity of the flow-field as in (2). The equation of motion for the evader in the inertial frame is given by:

$$\dot{\boldsymbol{r}}_E(t) = -\boldsymbol{p}(t), \quad \boldsymbol{r}_E(t_0) = \boldsymbol{r}_E^0, \tag{5}$$

where $\boldsymbol{r}_E(t) \in \mathbb{R}^2$ is the position of the evader at time t. Note that in this set-up, the evader can directly control its velocity. Given the equations of motion for the two agents, it is sufficient to look at the relative position of one agent with respect to the other. This is because the termination of the game depends only on the relative position of the two players, as we will see later.

We consider a new state space model for the game whose state vector consists of the relative position of E with respect to P and the inertial velocity of P. In particular, at time t, the new state $\boldsymbol{x}(t) \in \mathbb{R}^4$ is defined as follows: $\boldsymbol{x}(t) := \left[(\boldsymbol{r}_E(t) - \boldsymbol{r}_P(t))' \quad \tilde{\boldsymbol{v}}_P'(t)\right]'$. The situation is illustrated in Fig. **??**. We will say that the state $\boldsymbol{x}(t)$ belongs to the *reduced* state space of the PEG. The equations of motion for the PEG in the reduced state space can be written as follows:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{B}(t)\boldsymbol{u}(t) + \boldsymbol{C}\boldsymbol{v}(t) + \boldsymbol{E}\boldsymbol{p}_0(t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}^0, \tag{6}$$

where, at each time t, $\boldsymbol{A}(t) \in \mathbb{R}^{4\times 4}$ denotes the new state matrix and $\boldsymbol{B}(t)$, $\boldsymbol{C} \in \mathbb{R}^{4\times 2}$ denote the control input matrices for the pursuer and the evader, respectively. The matrices can be constructed from (1)-(5). In particular, $\boldsymbol{A}(t) = \mathrm{bdiag}(-\boldsymbol{I}_2, \boldsymbol{I}_2)\boldsymbol{A}_P(t)$, where $\mathrm{bdiag}(\boldsymbol{M}_1, \ldots, \boldsymbol{M}_k)$ denotes the block diagonal matrices formed by matrices $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_k$ of appropriate and compatible dimensions, $\boldsymbol{B}(t) = \boldsymbol{B}_P(t)$, for all $t \in [t_0, T]$, $\boldsymbol{C} = -w_p\boldsymbol{D}'$ and $\boldsymbol{E} = -\boldsymbol{D}'$ (note that the matrices $\boldsymbol{C}$ and $\boldsymbol{E}$ are time-invariant). In addition, $\boldsymbol{u}(t)$ denotes the control input of the pursuer at time t, which satisfies $\boldsymbol{u}(t) = \boldsymbol{u}_V(t)$ for all $t \in [t_0, T]$. Finally, $\boldsymbol{v}(t)$ denotes the control input of the evader at time t and is equal to the unit vector that determines the direction of the unknown component of the flow-field $\boldsymbol{p}(t)$. Note that the functions $\boldsymbol{u}(\cdot)$ and $\boldsymbol{v}(\cdot)$, which are both defined over $[t_0, T]$, are piecewise continuous and attain values in, respectively, the compact and convex sets $\mathcal{U}$ and $\mathcal{V}$, which have already been defined.

The robust time-optimal guidance problem of the vehicle V in a partially known flow-field (Problem 2.1) can now be associated with an equivalent pursuit evasion game (PEG) between P and E in a known flow-field. It should be highlighted at this point that in the latter problem, there are two players, namely, the evader which tries to maximize the time of capture in the game, if not avoid capture completely, and the pursuer which tries to minimize the time of capture. For a given $\ell > 0$ (capture radius), the capture time, which is denoted as $\mathcal{T}(\boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot); t_0, \boldsymbol{x}^0, \Sigma_\ell)$, is defined as the first time at which the system described by (6), which emanates from $\boldsymbol{x}^0$ at time t_0 and is driven by the control inputs $\boldsymbol{u}(\cdot)$ and $\boldsymbol{v}(\cdot)$, reaches the terminal set $\Sigma_\ell := \{\boldsymbol{x} \in \mathbb{R}^4 : \|\boldsymbol{D}\boldsymbol{x}\| \leq \ell\}$. For the given initial condition $\boldsymbol{x}^0$, the optimal time of capture of the PEG is denoted by $T^* = T^*(t_0, \boldsymbol{x}^0; \Sigma_\ell)$, where

$$T^*(t_0, \boldsymbol{x}^0; \Sigma_\ell) := \min_{\boldsymbol{u}(\cdot)} \max_{\boldsymbol{v}(\cdot)} \mathcal{T}(\boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot); t_0, \boldsymbol{x}^0, \Sigma_\ell) = \mathcal{T}(\boldsymbol{u}^*(\cdot), \boldsymbol{v}^*(\cdot); t_0, \boldsymbol{x}^0, \Sigma_\ell)$$

and the following saddle point condition is satisfied:

$$\mathcal{T}(\boldsymbol{u}^*(\cdot), \boldsymbol{v}(\cdot); t_0, \boldsymbol{x}^0, \Sigma_\ell) \leq T^*(t_0, \boldsymbol{x}^0; \Sigma_\ell) \leq \mathcal{T}(\boldsymbol{u}(\cdot), \boldsymbol{v}^*(\cdot); t_0, \boldsymbol{x}^0, \Sigma_\ell),$$

for all piecewise continuous functions $\boldsymbol{u}(\cdot)$ and $\boldsymbol{v}(\cdot)$ taking values in $\mathcal{U}$ and $\mathcal{V}$, respectively, with the application of which the corresponding trajectory $\boldsymbol{x}(\cdot)$ of the system in (6) reaches the terminal set Σ_ℓ in finite time. In addition, it turns out that

$$\begin{aligned} T^*(t_0, \boldsymbol{x}^0; \Sigma_\ell) &= \min_{\boldsymbol{u}(\cdot)} \max_{\boldsymbol{v}(\cdot)} \mathcal{T}(\boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot); t_0, \boldsymbol{x}^0, \Sigma_\ell) \\ &= \max_{\boldsymbol{v}(\cdot)} \min_{\boldsymbol{u}(\cdot)} \mathcal{T}(\boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot); t_0, \boldsymbol{x}^0, \Sigma_\ell). \end{aligned} \tag{7}$$

The interchangeability of min and max operators follows from the fact that the terms in the expressions of the performance index and the dynamics of the game that contain $\boldsymbol{u}(\cdot)$ and $\boldsymbol{v}(\cdot)$ are independent of each other [1]. Next, we provide the precise formulation of the pursuit-evasion game.

Problem 2.2 *Given $\ell > 0$, find the optimal pair of piecewise continuous control inputs $(\boldsymbol{u}^*(\cdot), \boldsymbol{v}^*(\cdot))$ such that the resulting optimal trajectory $\boldsymbol{x}^*(\cdot) = \boldsymbol{x}^*(\cdot; t_0, \boldsymbol{x}^0)$ of the system described by* (6) *and emanating from $\boldsymbol{x}^0$ at time t_0 satisfies the capture condition at time $t = T^*$, that is, $\boldsymbol{x}^*(T^*) \in \Sigma_\ell$, where $T^* = T^*(t_0, \boldsymbol{x}^0; \Sigma_\ell)$ is the optimal time of capture that is defined in* (7).

Remark 2.1 In the formulation of Problem 2.2, we have taken the admissible set of control inputs to consist of open-loop controls (piecewise continuous functions of time). However, as we will see later on, it turns out that the open-loop control input that will solve the min-max problem will also admit a feedback representation a posteriori.

The optimal play of the evader corresponds to the situation in which the unknown component of the flow-field has the worst possible effect on the vehicle V, that is, it tries to delay the arrival of the latter to its destination as much as possible if not prevent it completely. Note that the capture of the evader by the pursuer in the game is equivalent to the vehicle V reaching the desired terminal set. Moreover, the optimal control inputs and the corresponding optimal time of capture obtained as the solution to Problem 2.2 yield the vehicle's control input as well as the optimal time of arrival which correspond to the solution to Problem 2.1. It should be emphasized again that the optimal time of arrival of Problem 2.1 corresponds to an upper bound of the actual time of arrival of the vehicle to its destination given that in practice, the flow-field may not have an adversarial effect on the vehicle at all times (it is likely, for instance, that during certain intervals of time the flow-field may even aid the vehicle to reach its destination faster).

3 Solution Method: Reformulation of the Zero-Sum Differential Game as a Differential Game with a Terminal Cost

The pursuit-evasion game with free terminal time formulated in Problem 2.2 is re-defined as a game whose performance index consists of only a terminal cost based on the approach proposed in [11]. For a game that starts at state $\boldsymbol{x}^0 \in \mathbb{R}^4$ at time t_0, with the control inputs $\boldsymbol{u}(\cdot)$ and $\boldsymbol{v}(\cdot)$ applied from time $t = t_0$ to time $t = T$, let the state trajectory be denoted by $\boldsymbol{x}(\cdot)$ (or $\boldsymbol{x}(\cdot; t_0, \boldsymbol{x}^0)$, when we want to highlight the initial time and state). Now, we define the performance index/terminal cost function

$$\mathcal{J}_x(\boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot); t_0, T, \boldsymbol{x}^0) := \|\boldsymbol{D}\boldsymbol{x}(T)\| = \|\boldsymbol{r}(T)\|, \tag{8}$$

where $T > 0$ is the time of termination of the game, which is assumed to be fixed. Let us recall that the two-player pursuit-evasion game terminates in capture when the relative distance between the two players becomes equal to the radius of capture for the first time (this follows from the formulation in Problem 2.2), and this time is unknown. Therefore, a pair $(\boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot))$ will make the game formulated in Problem 2.2 terminate in capture, if there exists a terminal time $T > 0$ such that $\mathcal{J}_x(\boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot); t_0, T, \boldsymbol{x}^0) = \ell$. Our conjecture, whose correctness will be proven later in this section, is that the smallest possible terminal time T^* such that

$$\mathcal{J}_x(\boldsymbol{u}^*(\cdot), \boldsymbol{v}^*(\cdot); t_0, T^*, \boldsymbol{x}^0) = \ell$$

corresponds to the optimal time of capture of the game formulated in Problem 2.2 and the pair $\boldsymbol{u}^*(\cdot), \boldsymbol{v}^*(\cdot))$ corresponds to the optimal (saddle-point) pair of the game.

Now, in order to obtain the solution to Problem 2.2, we will associate it with a family of games whose performance index is the terminal cost $\mathcal{J}_x$ defined in (8) and which are parameterized by the terminal time T (this terminal time is assumed to be fixed for each problem in the family). To set up this association, let us note that the state transition matrix of the system in (6) is given by $\boldsymbol{\Phi}_A(t, t_0)$, and that

$$\dot{\boldsymbol{\Phi}}_A(T, t) = -\boldsymbol{\Phi}_A(T, t)\boldsymbol{A}(t), \qquad \boldsymbol{\Phi}_A(T, T) = \boldsymbol{I}_4,$$

for all $t \in [t_0, T]$. Now, let us define the following state transformation $\boldsymbol{y}(t) := \boldsymbol{D}\boldsymbol{\Phi}_A(T, t)\boldsymbol{x}(t)$ for all $t \in [t_0, T]$. Then,

$$\dot{\boldsymbol{y}}(t) = -\boldsymbol{D}\boldsymbol{\Phi}_A(T, t)\boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{\Phi}_A(T, t)\dot{\boldsymbol{x}}(t), \qquad \boldsymbol{y}(t_0) = \boldsymbol{y}^0, \tag{9}$$

where $\boldsymbol{y}^0 = \boldsymbol{D}\boldsymbol{\Phi}_A(T, t_0)\boldsymbol{x}^0 \in \mathbb{R}^2$. In view of (6), (9) becomes

$$\dot{\boldsymbol{y}}(t) = \boldsymbol{\mathcal{B}}(t; T)\boldsymbol{u}(t) + \boldsymbol{\mathcal{C}}(t; T)\boldsymbol{v}(t) + \boldsymbol{e}(t; T), \qquad \boldsymbol{y}(t_0) = \boldsymbol{y}^0, \tag{10}$$

where

$$\boldsymbol{\mathcal{B}}(t; T) := \boldsymbol{D}\boldsymbol{\Phi}_A(T, t)\boldsymbol{B}(t), \tag{11a}$$
$$\boldsymbol{\mathcal{C}}(t; T) := \boldsymbol{D}\boldsymbol{\Phi}_A(T, t)\boldsymbol{C}, \tag{11b}$$
$$\boldsymbol{e}(t; T) := \boldsymbol{D}\boldsymbol{\Phi}_A(T, t)\boldsymbol{E}\boldsymbol{p}_0(t), \tag{11c}$$

for all $t \in [t_0, T]$.

From the definition of $\boldsymbol{y}(t)$, it follows readily that $\|\boldsymbol{y}(T)\| = \|\boldsymbol{D}\boldsymbol{x}(T)\|$. The expression for the terminal cost in terms of the transformed state is given by

$$\mathcal{J}_y(\boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot); t_0, T, \boldsymbol{y}^0) = \|\boldsymbol{y}(T)\|. \tag{12}$$

Note that, if the parameter/terminal time T that appears in the performance index $\mathcal{J}_y$ is such that $\|\boldsymbol{y}(T)\| = \ell$, then $\|\boldsymbol{D}\boldsymbol{x}(T)\| = \|\boldsymbol{y}(T)\| = \ell$, which means that the capture condition of the game (Problem 2.2) is satisfied at $t = T$.

Problem 3.1 *Given $\ell > 0$, find the smallest parameter $T^* \geq t_0$ (terminal time) and the corresponding optimal pair of piecewise continuous control inputs $(\bar{\boldsymbol{u}}(\cdot), \bar{\boldsymbol{v}}(\cdot))$ taking values in $\mathcal{U} \times \mathcal{V}$ such that* (i) *the performance index/terminal cost $\mathcal{J}_y$ evaluated along $(\bar{\boldsymbol{u}}(\cdot), \bar{\boldsymbol{v}}(\cdot))$ satisfies the following equation:*

$$\begin{aligned}\mathcal{J}_y(\bar{\boldsymbol{u}}(\cdot), \bar{\boldsymbol{v}}(\cdot); t_0, T^*, \boldsymbol{y}^0) &= \min_{\boldsymbol{u}(\cdot)} \max_{\boldsymbol{v}(\cdot)} \mathcal{J}_y(\boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot); t_0, T^*, \boldsymbol{y}^0) \\ &=: \bar{\mathcal{J}}_y(t_0, \boldsymbol{y}^0; T^*),\end{aligned} \tag{13}$$

and in addition, (ii) $\|\bar{\boldsymbol{y}}(T^*)\| = \ell$ *and* $\|\bar{\boldsymbol{y}}(t)\| > \ell$ *for all* $t \in [t_0, T^*)$, *where* $\bar{\boldsymbol{y}}(\cdot)$ *denotes the optimal trajectory of the system described by* (10) *that emanates from $\boldsymbol{y}^0$ at time t_0 and is generated with the application of the optimal pair of inputs $(\bar{\boldsymbol{u}}(\cdot), \bar{\boldsymbol{v}}(\cdot))$.*

According to the formulation of Problem 3.1, capture can be achieved no earlier than the time $t = T^*$ when both players apply their corresponding optimal inputs. Next, we establish the equivalence between the solution to Problem 3.1 and the solution to Problem 2.2 and subsequently characterize the optimal pair of inputs under the following well-known assumption due to Gutman [11]:

Assumption 1. *There exist piecewise continuous functions $b(\cdot; T) : [t_0, T] \to [0, \infty[$ and $c(\cdot; T) : [t_0, T] \to [0, \infty[$, where $b(t; T) = 0$ or $c(t; T) = 0$ only at a finite number of points in $[t_0, T]$ at most, such that the matrices $\boldsymbol{\mathcal{B}}(t; T)$ and $\boldsymbol{\mathcal{C}}(t; T)$ satisfy, respectively, the following equations:*

$$\boldsymbol{\mathcal{B}}(t; T)\boldsymbol{\mathcal{B}}'(t; T) = b^2(t; T)\boldsymbol{I}_2, \quad \boldsymbol{\mathcal{C}}(t; T)\boldsymbol{\mathcal{C}}'(t; T) = c^2(t; T)\boldsymbol{I}_2, \tag{14}$$

for all $t \in [t_0, T]$.

Remark 3.1 The previous assumption is intuitive in the sense that in many practical situations, the matrices $\boldsymbol{\mathcal{B}}(t; T)$ and $\boldsymbol{\mathcal{C}}(t; T)$ do not vanish except for, possibly, a finite number of instants of time. This is true in practical situations, characteristic examples of which are pursuit-evasion games involving vehicles with single or double-integrator type dynamics in the presence of flow-fields, or lack thereof, missile-target systems and spacecraft in orbit, to name but a few. It should be highlighted that Assumption 1 will significantly simplify the computation of the optimal pair of control inputs, as we will see later on.

Lemma 3.1 *For any $t \in [t_0, T]$, $\boldsymbol{\mathcal{B}}(t; T) = \boldsymbol{0}$ if and only if $b(t; T) = 0$. Similarly, $\boldsymbol{\mathcal{C}}(t; T) = \boldsymbol{0}$ if and only if $c(t; T) = 0$.*

Proof. If $b(t; T) = 0$, then in the light of (14), it follows that $\boldsymbol{\mathcal{B}}(t; T)\boldsymbol{\mathcal{B}}'(t; T) = \boldsymbol{0}$. Because $\mathrm{rank}(\boldsymbol{\mathcal{B}}(t; T)) = \mathrm{rank}(\boldsymbol{\mathcal{B}}(t; T)\boldsymbol{\mathcal{B}}'(t; T)) = \mathrm{rank}(\boldsymbol{0}) = 0$, we immediately conclude that $\boldsymbol{\mathcal{B}}(t; T) = \boldsymbol{0}$. Conversely, if $\boldsymbol{\mathcal{B}}(t; T) = \boldsymbol{0}$, then $\boldsymbol{\mathcal{B}}(t; T)\boldsymbol{\mathcal{B}}'(t; T) = \boldsymbol{0}$, which implies, in view of (14), that $b(t; T) = 0$. □

Theorem 3.1 *Problem 2.2 admits a solution if and only if Problem 3.1 is solvable. Furthermore, if Problem 2.2 admits a solution, then the optimal pair* *of min-max control inputs $(\boldsymbol{u}^*(\cdot), \boldsymbol{v}^*(\cdot))$ which constitutes the solution to Problem 2.2 is equal to the optimal pair of control inputs $(\bar{\boldsymbol{u}}(\cdot), \bar{\boldsymbol{v}}(\cdot))$ that solves Problem 3.1. In particular, provided that Assumption 1 holds true, the optimal pair of inputs are given by*

$$\boldsymbol{u}^*(t) = \bar{\boldsymbol{u}}(t) = -\frac{\boldsymbol{\mathcal{B}}'(t;T^*)\bar{\boldsymbol{z}}(t)}{\|\boldsymbol{\mathcal{B}}'(t;T^*)\bar{\boldsymbol{z}}(t)\|}, \tag{15a}$$

$$\boldsymbol{v}^*(t) = \bar{\boldsymbol{v}}(t) = \frac{\boldsymbol{\mathcal{C}}'(t;T^*)\bar{\boldsymbol{z}}(t)}{\|\boldsymbol{\mathcal{C}}'(t;T^*)\bar{\boldsymbol{z}}(t)\|}, \tag{15b}$$

for all $t \in [t_0, T^*]$, where $\bar{\boldsymbol{z}}(t) := \bar{\boldsymbol{y}}(t) + \int_t^{T^*} \boldsymbol{e}(\eta;T^*)\mathrm{d}\eta$, and T^* is the min-max time of capture, which is defined in (7), and is also equal to the optimal parameter of Problem 3.1. In particular, T^* corresponds to the smallest non-negative root of the following equation in T:

$$\|\boldsymbol{y}^0 + \int_{t_0}^{T} \boldsymbol{e}(\eta;T)\mathrm{d}\eta\| + \int_{t_0}^{T} \dot{\gamma}(\bar{\boldsymbol{z}}(\eta))\mathrm{d}\eta = \ell, \tag{16}$$

where $\dot{\gamma}(\bar{\boldsymbol{z}}(t)) = \frac{\bar{\boldsymbol{z}}'(t)}{\|\bar{\boldsymbol{z}}(t)\|}\big(\boldsymbol{\mathcal{B}}(t;T^*)\bar{\boldsymbol{u}}(t) + \boldsymbol{\mathcal{C}}(t;T^*)\bar{\boldsymbol{v}}(t)\big)$ for all $t \in [t_0, T^*]$.

Proof. : Let $\boldsymbol{y}(t) = \boldsymbol{y}_{\mathrm{sol}}(t; \boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot), t_0, \boldsymbol{y}^0)$, with $\boldsymbol{y}^0 = \boldsymbol{y}(t_0)$, denote the solution to (10) at time $t \in [t_0, T]$ for a given initial state $\boldsymbol{y}^0$ at time t_0 and for a given pair of control inputs $(\boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot))$ that are applied over the time interval $[t_0, t)$. Let us also consider the function $\omega(\cdot) : \mathbb{R}^2 \to [0, \infty[$ with $\omega(\boldsymbol{y}) := \|\boldsymbol{y}\|$, whose time derivative along the trajectory $\boldsymbol{y}(\cdot)$ is given, in light of (10), by

$$\begin{aligned}\dot{\omega}(\boldsymbol{y}(t)) &= \boldsymbol{\xi}'(t)\dot{\boldsymbol{y}}(t) \\ &= \boldsymbol{\xi}'(t)\left[\boldsymbol{\mathcal{B}}(t;T)\boldsymbol{u}(t) + \boldsymbol{\mathcal{C}}(t;T)\boldsymbol{v}(t) + \boldsymbol{e}(t;T)\right],\end{aligned} \tag{17}$$

for all $t \in [t_0, T]$, where $\boldsymbol{\xi}(t) := \boldsymbol{y}(t)/\|\boldsymbol{y}(t)\|$. Then, the terminal cost, starting from $\boldsymbol{y} = \boldsymbol{y}(t)$ at time $t \in [t_0, T]$, is given by

$$\begin{aligned}\mathcal{J}_y(\boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot); t, T, \boldsymbol{y}(t)) = \|\boldsymbol{y}(T)\| &= \omega(\boldsymbol{y}(t)) + \int_t^T \dot{\omega}(\boldsymbol{y}(\eta))\mathrm{d}\eta \\ &= \|\boldsymbol{y}(t)\| + \int_t^T \dot{\omega}(\boldsymbol{y}(\eta))\mathrm{d}\eta.\end{aligned} \tag{18}$$

Therefore, the optimal terminal cost from $\boldsymbol{y}(t)$ at time t, which is denoted as $\bar{\mathcal{J}}_y(t, \boldsymbol{y}(t); T)$, is defined as follows:

$$\begin{aligned}\bar{\mathcal{J}}_y(t, \boldsymbol{y}(t); T) &:= \min_{\boldsymbol{u}(\cdot)} \max_{\boldsymbol{v}(\cdot)} \mathcal{J}_y(\boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot); t, T, \boldsymbol{y}(t)) \\ &= \|\boldsymbol{y}(t)\| + \min_{\boldsymbol{u}(\cdot)} \max_{\boldsymbol{v}(\cdot)} \int_t^T \dot{\omega}(\boldsymbol{y}(\eta))\mathrm{d}\eta.\end{aligned}$$

Let $(\bar{\boldsymbol{u}}_T(\cdot), \bar{\boldsymbol{v}}_T(\cdot))$ denote the corresponding optimal (min-max) pair of inputs and $\bar{\boldsymbol{y}}_T(\cdot)$ the corresponding optimal trajectory, with

$$\bar{\boldsymbol{y}}_T(t_1) = \boldsymbol{y}_{\mathrm{sol}}(t_1; \bar{\boldsymbol{u}}_T(\cdot), \bar{\boldsymbol{v}}_T(\cdot), t, \boldsymbol{y}(t))$$

for $t_1 \in [t, T]$. By definition,

$$\bar{\mathcal{J}}_y(t, \boldsymbol{y}(t); T) = \|\bar{\boldsymbol{y}}_T(T)\| = \|\boldsymbol{y}(t)\| + \int_t^T \dot{\omega}(\bar{\boldsymbol{y}}_T(\eta))\mathrm{d}\eta. \tag{19}$$

The optimal parameter $T = T^*$ (optimal terminal time) for Problem 3.1 is the smallest non-negative root of the following equation:

$$\ell = \|\boldsymbol{y}(t)\| + \int_t^T \dot{\omega}(\bar{\boldsymbol{y}}_T(\eta))\mathrm{d}\eta. \tag{20}$$

Then the optimal pair of inputs that solves Problem 3.1, which is denoted by $(\bar{\boldsymbol{u}}(\cdot), \bar{\boldsymbol{v}}(\cdot))$, is defined as follows:

$$(\bar{\boldsymbol{u}}(t), \bar{\boldsymbol{v}}(t)) = (\bar{\boldsymbol{u}}_{T^*}(t), \bar{\boldsymbol{v}}_{T^*}(t)), \qquad \text{for all} \;\; t \in [t_0, T^*].$$

Next, we proceed with the characterization of the optimal pair of inputs. To this aim, we introduce a new variable $\boldsymbol{z}(t)$, where

$$\boldsymbol{z}(t) := \boldsymbol{y}(t) + \int_t^{T^*} \boldsymbol{e}(\eta; T^*)\mathrm{d}\eta, \tag{21}$$

which implies that $\dot{\boldsymbol{z}}(t) = \dot{\boldsymbol{y}}(t) - \boldsymbol{e}(t; T^*)$, from which it follows that

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{\mathcal{B}}(t; T^*)\boldsymbol{u}(t) + \boldsymbol{\mathcal{C}}(t; T^*)\boldsymbol{v}(t), \qquad \boldsymbol{z}(t_0) = \boldsymbol{z}^0, \tag{22}$$

where $\boldsymbol{z}^0 := \boldsymbol{y}^0 + \int_{t_0}^{T^*} \boldsymbol{e}(\eta; T^*)\mathrm{d}\eta$. Now we define a function $\gamma(\cdot) : \mathbb{R}^2 \to [0, \infty[$, where $\gamma(\boldsymbol{z}) := \|\boldsymbol{z}\|$, whose time derivative along the trajectory $\boldsymbol{z}(\cdot)$ is given by

$$\dot{\gamma}(\boldsymbol{z}(t)) = \frac{\boldsymbol{z}'(t)}{\|\boldsymbol{z}(t)\|}\big(\boldsymbol{\mathcal{B}}(t; T^*)\boldsymbol{u}(t) + \boldsymbol{\mathcal{C}}(t; T^*)\boldsymbol{v}(t)\big). \tag{23}$$

Let us also define the cost-to-go function from $\boldsymbol{z}(t)$ at time t as follows:

$$\mathcal{J}_z(\boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot); t, T^*, \boldsymbol{z}(t)) = \|\boldsymbol{z}(T^*)\| = \|\boldsymbol{z}(t)\| + \int_t^{T^*} \dot{\gamma}(\boldsymbol{z}(\eta))\mathrm{d}\eta. \tag{24}$$

Then, the optimal terminal cost is given by:

$$\begin{aligned} \bar{\mathcal{J}}_z(t, \boldsymbol{z}(t); T^*) &:= \min_{\boldsymbol{u}(\cdot)} \max_{\boldsymbol{v}(\cdot)} \mathcal{J}_z(\boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot); t, T^*, \boldsymbol{z}(t)) \\ &= \|\boldsymbol{z}(t)\| + \min_{\boldsymbol{u}(\cdot)} \max_{\boldsymbol{v}(\cdot)} \int_t^{T^*} \dot{\gamma}(\boldsymbol{z}(\eta))\mathrm{d}\eta. \end{aligned}$$

Let $(\bar{\boldsymbol{u}}_z(\cdot), \bar{\boldsymbol{v}}_z(\cdot))$ denote the corresponding optimal min-max pair of inputs and $\bar{\boldsymbol{z}}(\cdot)$ the corresponding optimal trajectory of the system described by (22) that emanates from state $\boldsymbol{z}(t)$. By its definition, the optimal cost-to-go can also be written as follows:

$$\bar{\mathcal{J}}_z(t,\boldsymbol{y}(t);T^*) = \|\bar{\boldsymbol{z}}(T^*)\| = \|\boldsymbol{z}(t)\| + \int_t^{T^*} \dot{\gamma}(\bar{\boldsymbol{z}}(\eta))\mathrm{d}\eta, \tag{25}$$

where $\bar{\boldsymbol{z}}(T^*)$ is the terminal state of the optimal trajectory $\bar{\boldsymbol{z}}(\cdot)$. We claim that

$$(\bar{\boldsymbol{u}}(t),\bar{\boldsymbol{v}}(t)) = (\bar{\boldsymbol{u}}_z(t),\bar{\boldsymbol{v}}_z(t)), \quad \text{for all} \quad t\in[t_0,T^*].$$

Let us now solve the min-max optimization of the terminal cost $\|\boldsymbol{z}(T^*)\|$, subject to the dynamics in (22) and the input constraints $\boldsymbol{u}(t)\in\mathcal{U}$ and $\boldsymbol{v}(t)\in\mathcal{V}$ for all $t\in[t_0,T^*]$. To this aim, we consider the Hamiltonian $\mathcal{H}(\cdot):[t_0,T^*]\times\mathbb{R}^2\times\mathcal{U}\times\mathcal{V}\times\mathbb{R}^2$ with

$$\mathcal{H}(t,\boldsymbol{z},\boldsymbol{u},\boldsymbol{v},\boldsymbol{\lambda}) = \boldsymbol{\lambda}'[\boldsymbol{\mathcal{B}}(t;T^*)\boldsymbol{u}+\boldsymbol{\mathcal{C}}(t;T^*)\boldsymbol{v}], \tag{26}$$

where $\boldsymbol{\lambda}$ is the co-state. Then, in view of the two player extension of the maximum principle (refer to Theorem 2, pg. 349 in [3]) we have that

$$\mathcal{H}(t,\bar{\boldsymbol{z}}(t),\bar{\boldsymbol{u}}_z(t),\bar{\boldsymbol{v}}_z(t),\bar{\boldsymbol{\lambda}}(t)) = \min_{\boldsymbol{u}\in\mathcal{U}}\max_{\boldsymbol{v}\in\mathcal{V}}\mathcal{H}(t,\bar{\boldsymbol{z}}(t),\boldsymbol{u},\boldsymbol{v},\bar{\boldsymbol{\lambda}}(t)), \tag{27}$$

where $\bar{\boldsymbol{\lambda}}(\cdot)$ satisfies the following equation:

$$\dot{\bar{\boldsymbol{\lambda}}}(t) = -\mathcal{H}'_{\boldsymbol{z}}(t,\bar{\boldsymbol{z}}(t),\bar{\boldsymbol{u}}_z(t),\bar{\boldsymbol{v}}_z(t),\bar{\boldsymbol{\lambda}}(t)) = \boldsymbol{0}, \quad \text{for all} \quad t\in[t_0,T^*], \tag{28a}$$

$$\bar{\boldsymbol{\lambda}}(T^*) = \frac{\mathrm{d}\|\boldsymbol{z}\|}{\mathrm{d}\boldsymbol{z}}\bigg|_{\boldsymbol{z}=\bar{\boldsymbol{z}}(T^*)} = \frac{\bar{\boldsymbol{z}}'(T^*)}{\|\bar{\boldsymbol{z}}(T^*)\|}. \tag{28b}$$

Equations (28a)-(28b) imply that the co-state $\bar{\boldsymbol{\lambda}}(\cdot)$ is a constant unit vector. In particular,

$$\bar{\boldsymbol{\lambda}}(t) = \bar{\boldsymbol{\lambda}}(T^*) = \frac{\bar{\boldsymbol{z}}(T^*)}{\|\bar{\boldsymbol{z}}(T^*)\|}, \quad \text{for all} \quad t\in[t_0,T^*]. \tag{29}$$

In addition, the optimal pair of control inputs that solve the corresponding min-max optimization problem can be independently chosen as follows:

$$\bar{\boldsymbol{u}}_z(t) = -\frac{\boldsymbol{\mathcal{B}}'(t;T^*)\bar{\boldsymbol{\lambda}}(t)}{\|\boldsymbol{\mathcal{B}}'(t;T^*)\bar{\boldsymbol{\lambda}}(t)\|}, \tag{30a}$$

$$\bar{\boldsymbol{v}}_z(t) = \frac{\boldsymbol{\mathcal{C}}'(t;T^*)\bar{\boldsymbol{\lambda}}(t)}{\|\boldsymbol{\mathcal{C}}'(t;T^*)\bar{\boldsymbol{\lambda}}(t)\|}, \tag{30b}$$

for all $t\in[t_0,T^*]$ for which $\boldsymbol{\mathcal{B}}(t;T^*)\neq\boldsymbol{0}$ and $\boldsymbol{\mathcal{C}}(t;T^*)\neq\boldsymbol{0}$. When this condition is satisfied, the denominators of the expressions of $\bar{\boldsymbol{u}}_z(t)$ and $\bar{\boldsymbol{v}}_z(t)$ that appear in (30a) and (30b), respectively, are non-zero. Indeed, if $\boldsymbol{\mathcal{B}}(t;T^*)\neq\boldsymbol{0}$, then $\boldsymbol{\mathcal{B}}(t;T^*)\boldsymbol{\mathcal{B}}'(t;T^*) = b^2(t;T^*)\boldsymbol{I}_2 > \boldsymbol{0}$ by virtue of Assumption 1. Therefore, the null space of $\boldsymbol{\mathcal{B}}'(t;T^*)$, which is denoted by $\mathcal{N}(\boldsymbol{\mathcal{B}}'(t;T^*))$, is necessarily trivial, that is, $\mathcal{N}(\boldsymbol{\mathcal{B}}'(t;T^*)) = \{\boldsymbol{0}\}$; and since the co-state vector $\bar{\boldsymbol{\lambda}}(t)$ has unit magnitude according to (29), $\bar{\boldsymbol{\lambda}}(t)\notin\mathcal{N}(\boldsymbol{\mathcal{B}}'(t;T^*))$ for all $t\in[t_0,T^*]$

for which $\boldsymbol{\mathcal{B}}(t;T^*) \neq \mathbf{0}$. Similarly, we can prove that $\bar{\boldsymbol{\lambda}}(t) \notin \mathcal{N}(\boldsymbol{\mathcal{C}}'(t;T^*))$ for all $t \in [t_0, T^*]$ with $\boldsymbol{\mathcal{C}}(t;T^*) \neq \mathbf{0}$.
Note that the exact values of the optimal inputs $\bar{\boldsymbol{u}}_z(t)$ and $\bar{\boldsymbol{v}}_z(t)$ at any (isolated) points of $t \in [t_0, T^*]$ that either $\boldsymbol{\mathcal{B}}(t;T^*)$ and $\boldsymbol{\mathcal{C}}(t;T^*) = \mathbf{0}$ (the number of such points is finite, at most) are irrelevant to the solution of Problem 3.1 and can be chosen to be any vectors in $\mathcal{U}$ and $\mathcal{V}$, respectively.

Next, motivated by the fact that $\bar{\boldsymbol{\lambda}}(T^*) = \bar{\boldsymbol{z}}(T^*)/\|\bar{\boldsymbol{z}}(T^*)\|$, we argue that $\bar{\boldsymbol{\lambda}}(t) = \bar{\boldsymbol{z}}(t)/\|\bar{\boldsymbol{z}}(t)\|$ for all $t \in [t_0, T^*]$, in which case, we have that

$$\bar{\boldsymbol{\lambda}}(t) = \bar{\boldsymbol{\lambda}}(T^*) = \frac{\bar{\boldsymbol{z}}(T^*)}{\|\bar{\boldsymbol{z}}(T^*)\|} = \frac{\bar{\boldsymbol{z}}(t)}{\|\bar{\boldsymbol{z}}(t)\|}, \quad \text{for all} \quad t \in [t_0, T^*]. \tag{31}$$

To prove that (31) holds true, we first observe that (28a), which is in general a linear ordinary differential equation, with the boundary condition given by (28b), admits a unique solution. Therefore, it suffices to show that $\bar{\boldsymbol{\lambda}}(t) = \bar{\boldsymbol{z}}(t)/\|\bar{\boldsymbol{z}}(t)\|$ is the solution to (28a)-(28b), for all $t \in [t_0, T^*]$. To this aim, we first compute the (total) time derivative of $\boldsymbol{z}/\|\boldsymbol{z}\|$ along the optimal trajectory $\bar{\boldsymbol{z}}(\cdot)$, which is given by:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\bar{\boldsymbol{z}}(t)}{\|\bar{\boldsymbol{z}}(t)\|}\right) &= \frac{1}{\|\bar{\boldsymbol{z}}(t)\|^2}\left(\|\bar{\boldsymbol{z}}(t)\|\dot{\bar{\boldsymbol{z}}}(t) - \frac{1}{\|\bar{\boldsymbol{z}}(t)\|}\big(\bar{\boldsymbol{z}}(t)\bar{\boldsymbol{z}}'(t)\big)\dot{\bar{\boldsymbol{z}}}(t)\right)\\ &= \frac{1}{\|\bar{\boldsymbol{z}}(t)\|}\left(\boldsymbol{I}_2 - \frac{1}{\|\bar{\boldsymbol{z}}(t)\|^2}\big(\bar{\boldsymbol{z}}(t)\bar{\boldsymbol{z}}'(t)\big)\right)\dot{\bar{\boldsymbol{z}}}(t),\end{aligned} \tag{32}$$

for all $t \in [t_0, T^*]$. In view of (14) and (30a)-(30b), along the optimal trajectory, (22) becomes

$$\begin{aligned}\dot{\bar{\boldsymbol{z}}}(t) &= \boldsymbol{\mathcal{B}}(t;T^*)\bar{\boldsymbol{u}}_z(t) + \boldsymbol{\mathcal{C}}(t;T^*)\bar{\boldsymbol{v}}_z(t)\\ &= -\frac{\boldsymbol{\mathcal{B}}(t;T^*)\boldsymbol{\mathcal{B}}'(t;T^*)\bar{\boldsymbol{\lambda}}(t)}{\|\boldsymbol{\mathcal{B}}'(t;T^*)\bar{\boldsymbol{\lambda}}(t)\|} + \frac{\boldsymbol{\mathcal{C}}(t;T^*)\boldsymbol{\mathcal{C}}'(t;T^*)\bar{\boldsymbol{\lambda}}(t)}{\|\boldsymbol{\mathcal{C}}'(t;T^*)\bar{\boldsymbol{\lambda}}(t)\|}\\ &= -\frac{b^2(t,T^*)\bar{\boldsymbol{\lambda}}(t)}{|b(t,T^*)|\|\bar{\boldsymbol{\lambda}}(t)\|} + \frac{c^2(t;T^*)\bar{\boldsymbol{\lambda}}(t)}{|c(t;T^*)|\|\bar{\boldsymbol{\lambda}}(t)\|}\\ &= -|b(t,T^*)|\frac{\bar{\boldsymbol{\lambda}}(t)}{\|\bar{\boldsymbol{\lambda}}(t)\|} + |c(t;T^*)|\frac{\bar{\boldsymbol{\lambda}}(t)}{\|\bar{\boldsymbol{\lambda}}(t)\|}\\ &= -\Big(b(t,T^*) - c(t;T^*)\Big)\bar{\boldsymbol{\lambda}}(t),\end{aligned}$$

since $\|\bar{\boldsymbol{\lambda}}(t)\| = 1$ for all $t \in [t_0, T^*]$, and $|b(t,T^*)| = b(t,T^*)$ and $|c(t,T^*)| = c(t,T^*)$, in view of Assumption 1. After substituting the last expression of $\dot{\bar{\boldsymbol{z}}}(t)$ in (32), we obtain

$$\begin{aligned}\dot{\bar{\boldsymbol{\lambda}}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\bar{\boldsymbol{z}}(t)}{\|\bar{\boldsymbol{z}}(t)\|}\right) &= -\frac{b(t,T^*) - c(t;T^*)}{\|\bar{\boldsymbol{z}}(t)\|}\left(\boldsymbol{I}_2 - \frac{1}{\|\bar{\boldsymbol{z}}(t)\|^2}\big(\bar{\boldsymbol{z}}(t)\bar{\boldsymbol{z}}'(t)\big)\right)\frac{\bar{\boldsymbol{z}}(t)}{\|\bar{\boldsymbol{z}}(t)\|}\\ &= -\frac{b(t,T^*) - c(t;T^*)}{\|\bar{\boldsymbol{z}}(t)\|^2}\left(\bar{\boldsymbol{z}}(t) - \bar{\boldsymbol{z}}(t)\frac{\bar{\boldsymbol{z}}'(t)\bar{\boldsymbol{z}}(t)}{\|\bar{\boldsymbol{z}}(t)\|^2}\right)\\ &= \mathbf{0},\end{aligned}$$

and at time $t = T^*$, $\bar{\boldsymbol{\lambda}}(T^*) = \bar{z}(T^*)/\|\bar{z}(T^*)\|$. Hence, the solution to (28a)-(28b) is given by $\bar{\boldsymbol{\lambda}}(t) = \bar{z}(t)/\|\bar{z}(t)\|$ for all $t \in [t_0, T^*]$, and (31) holds true as we have claimed. Now, in light of (31), the optimal control pair in (30a)-(30b) can be written as follows:

$$\begin{aligned}\bar{\boldsymbol{u}}_z(t) &= -\frac{\boldsymbol{\mathcal{B}}'(t;T^*)(\bar{z}(t)/\|\bar{z}(t)\|)}{\|\boldsymbol{\mathcal{B}}'(t;T^*)(\bar{z}(t)/\|\bar{z}(t)\|)\|} \\ &= -\frac{\boldsymbol{\mathcal{B}}'(t;T^*)\bar{z}(t)}{\|\boldsymbol{\mathcal{B}}'(t;T^*)\bar{z}(t)\|},\end{aligned} \tag{33a}$$

and similarly,

$$\bar{\boldsymbol{v}}_z(t) = \frac{\boldsymbol{\mathcal{C}}'(t;T^*)\bar{z}(t)}{\|\boldsymbol{\mathcal{C}}'(t;T^*)\bar{z}(t)\|}, \tag{33b}$$

for all $t \in [t_0, T^*]$, for which $\boldsymbol{\mathcal{B}}(t;T^*) \neq \mathbf{0}$ and $\boldsymbol{\mathcal{C}}(t;T^*) \neq \mathbf{0}$. Note that when $\boldsymbol{\mathcal{B}}(t;T^*) \neq \mathbf{0}$ and $\boldsymbol{\mathcal{C}}(t;T^*) \neq \mathbf{0}$, the expressions of $\bar{\boldsymbol{u}}_z(t)$ and $\bar{\boldsymbol{v}}_z(t)$ given in (33a) and (33b), respectively, are well defined. This is because when $\boldsymbol{\mathcal{B}}(t;T^*) \neq \mathbf{0}$, then $\mathcal{N}(\boldsymbol{\mathcal{B}}'(t;T^*)) = \{\mathbf{0}\}$, as we have shown in the discussion following (30a)-(30b). In addition, because $\|\bar{z}(t)\| \geq \ell > 0$ (and thus $\bar{z}(t) \neq \mathbf{0}$) for all $t \in [t_0, T^*]$, we conclude that $\|\boldsymbol{\mathcal{B}}'(t;T^*)\bar{z}(t)\| > 0$ for all $t \in [t_0, T^*]$. Similarly, we can prove that $\|\boldsymbol{\mathcal{C}}'(t;T^*)\bar{z}(t)\| > 0$ for all $t \in [t_0, T^*]$.

At any time $t \in [t_0, T^*]$ that either $\boldsymbol{\mathcal{B}}(t;T^*) = \mathbf{0}$ or $\boldsymbol{\mathcal{C}}(t;T^*) = \mathbf{0}$, the corresponding optimal input can be chosen to be any vector in $\mathcal{U}$ or $\mathcal{V}$, respectively. Further, it turns out that the expressions of $\bar{\boldsymbol{u}}_z(t)$ and $\bar{\boldsymbol{v}}_z(t)$ given in (33a) and (33b), respectively, ensure that the closed-loop dynamics, which are described by the following equation:

$$\dot{\bar{z}}(t) = \frac{\boldsymbol{\mathcal{B}}(t;T^*)\boldsymbol{\mathcal{B}}'(t;T^*)\bar{z}(t)}{\|\boldsymbol{\mathcal{B}}'(t;T^*)\bar{z}(t)\|} + \frac{\boldsymbol{\mathcal{C}}(t;T^*)\boldsymbol{\mathcal{C}}'(t;T^*)\bar{z}(t)}{\|\boldsymbol{\mathcal{C}}'(t;T^*)\bar{z}(t)\|}, \tag{34}$$

are well-defined for all $t \in [t_0, T^*]$. This is because the two terms that appear in the right hand side of (34) are well-defined for all $t \in [t_0, T^*]$, even when $\boldsymbol{\mathcal{B}}(t;T^*) = \mathbf{0}$ or $\boldsymbol{\mathcal{C}}(t;T^*) = \mathbf{0}$. In particular, in view of Assumption 1, we have that

$$\frac{\boldsymbol{\mathcal{B}}(t;T^*)\boldsymbol{\mathcal{B}}'(t;T^*)\bar{z}(t)}{\|\boldsymbol{\mathcal{B}}'(t;T^*)\bar{z}(t)\|} = \frac{\boldsymbol{\mathcal{B}}(t;T^*)\boldsymbol{\mathcal{B}}'(t;T^*)\bar{z}(t)}{\sqrt{\bar{z}'(t)\boldsymbol{\mathcal{B}}(t;T^*)\boldsymbol{\mathcal{B}}'(t;T^*)\bar{z}(t)}} = \frac{|b(t;T^*)|\bar{z}(t)}{\|\bar{z}(t)\|},$$

which is well defined given that $\bar{z}(t) \geq \ell > 0$ for all $t \in [t_0, T^*]$. If $\boldsymbol{\mathcal{B}}(t;T^*) = \mathbf{0}$ for some $t \in [t_0, T]$, then in the light of Lemma 3.1 we have that $b(t;T^*) = 0$. Thus,

$$\frac{\boldsymbol{\mathcal{B}}(t;T^*)\boldsymbol{\mathcal{B}}'(t;T^*)\bar{z}(t)}{\|\boldsymbol{\mathcal{B}}'(t;T^*)\bar{z}(t)\|} = \mathbf{0},$$

for all $t \in [t_0, T^*]$ with $\boldsymbol{\mathcal{B}}(t;T^*) = \mathbf{0}$. The proof for the well-posedness of the second term that appear in the right hand side of (34) is similar, and we will omit it.

Furthermore, in view of (21), it follows that $\boldsymbol{z}(T^*) = \boldsymbol{y}(T^*)$, and so the game terminates in capture when $\|\boldsymbol{z}(T^*)\| = \|\boldsymbol{y}(T^*)\| = \ell$. The optimal terminal time T^* corresponds to the smallest real positive value that satisfies $\bar{\mathcal{J}}_z(t, T^*, \boldsymbol{z}(t)) = \ell$, where $\bar{\mathcal{J}}_z(t, T^*, \boldsymbol{z}(t))$ is defined in (25). Furthermore, the optimal pair of open-loop control inputs $(\bar{\boldsymbol{u}}_z(\cdot), \bar{\boldsymbol{v}}_z(\cdot))$ corresponding to T^* coincides with the optimal pair $(\bar{\boldsymbol{u}}(\cdot), \bar{\boldsymbol{v}}(\cdot))$ that solves Problem 3.1.

Now, we argue that the optimal solution to Problem 3.1 is also the optimal solution to Problem 2.2. This follows readily after observing that with the application of the pair of control inputs $(\bar{\boldsymbol{u}}_z(\cdot), \bar{\boldsymbol{v}}_z(\cdot))$ to both the systems given in (6) and (10) with initial conditions $\boldsymbol{x}_0$ and $\boldsymbol{y}_0 = \boldsymbol{D}\boldsymbol{\Phi}_A(T, t_0)\boldsymbol{x}_0$ respectively at time t_0, over the time interval $[t_0, T]$, the two systems will reach terminal states $\boldsymbol{x}^*(T)$ and $\bar{\boldsymbol{y}}(T)$, respectively, with $\bar{\boldsymbol{y}}(T) = \boldsymbol{D}\boldsymbol{x}^*(T)$, for all $T \geq t_0$. Consequently, $T = T^*$ is the first time for which the following equation

$$\|\bar{\boldsymbol{y}}(T^*)\| = \|\boldsymbol{D}\boldsymbol{x}^*(T^*)\| = \ell$$

holds true, which in turn implies that at time $t = T^*$, $\boldsymbol{x}^*(T^*) \in \Sigma_\ell$. Hence, the optimal pair of control inputs for Problem 2.2 is given by (15). This concludes the proof of Theorem 3.1. □

Remark 3.2 Because the optimal pair of control inputs at each time t depend on the transformed state $\boldsymbol{z}$, these control inputs can be implemented as realizations of the corresponding pair of feedback min-max control laws.

Remark 3.3 It is preferable to use (16) in order to find the optimal time of capture T^* instead of 20). This is because the integrand in the latter equation is, in view of (17), explicitly dependent on the trajectory $\bar{\boldsymbol{y}}(\cdot)$ because of the $\boldsymbol{e}(\cdot; T^*)$ term. By contrast, the integrand in (16) does not depend explicitly on $\bar{\boldsymbol{z}}(\cdot)$, and the terminal time T^* is easily obtained.

4 Example: Guidance of a Newtonian particle in a Partially Known Flow-Field

In this section, we apply the previous results to a specific example, which is the problem of guiding a vehicle in a partially known flow-field in the plane. The vehicle's motion is described in terms of a simple point-mass kinematic model (Newtonian particle).

The state space model for the vehicle in the flow-field is given by (1) where $n = 2$ with

$$\boldsymbol{A}_V = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_2 \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{B}_V = F \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_2 \end{bmatrix}, \quad \boldsymbol{E}_V = \boldsymbol{D}',$$

where $F > 0$ is a known constant. Note that all the matrices in the previous state space model are time-invariant.

Here, we take the known component $\boldsymbol{p}_0(t)$ of the velocity $\boldsymbol{p}(t)$ of the partially unknown time-varying flow-field, which is defined in (2), to be a bounded and periodic function of time. In particular, the known component of the flow-field $\boldsymbol{p}_0(t) = \sin(k\pi t)\boldsymbol{\mu}$, where $k > 0, \boldsymbol{\mu} \in \mathbb{R}^2$. As we have already explained, the guidance problem of the Newtonian particle in a partially known flow-field is equivalent to a two-player pursuit-evasion game in a known flow-field. The two-player game is a variation of Isaacs' Isotropic Rocket game [1]. The vehicle's motion is described in terms of a Newtonian point mass model, which Isaacs has referred to as the Isotropic Rocket, which is a term that we will also use henceforth. In the Isotropic Rocket PEG, the pursuer controls its acceleration vector whose magnitude is an a priori known positive constant whereas the evader controls directly the direction of its velocity vector, whose magnitude (i.e., its speed) is assumed to be constant.

At each time t, the state vector $\boldsymbol{x}(t)$, that belongs to the reduced state space of the Isotropic Rocket pursuit-evasion game, is defined as follows: $\boldsymbol{x}(t) := [\boldsymbol{r}'(t)\ \tilde{\boldsymbol{v}}_P'(t)]' \in \mathbb{R}^4$, where $\boldsymbol{r}(t) := \boldsymbol{r}_E(t) - \boldsymbol{r}_P(t)$ is the relative position of E with respect to P and $\tilde{\boldsymbol{v}}_P(t)$ is the velocity of P at time t. The time-invariant state-space model is given by the following matrices in (6):

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & -\boldsymbol{I}_2 \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{B} = F \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_2 \end{bmatrix}, \quad \boldsymbol{C} = -w_p \begin{bmatrix} \boldsymbol{I}_2 \\ \boldsymbol{0} \end{bmatrix} \quad \boldsymbol{E} = -\boldsymbol{D}'.$$

For any terminal time T, the state transition matrix corresponding to the state matrix $\boldsymbol{A}$ is given by:

$$\boldsymbol{\Phi}_A(T,t) = \begin{bmatrix} \boldsymbol{I}_2 & -(T-t)\boldsymbol{I}_2 \\ \boldsymbol{0} & \boldsymbol{I}_2 \end{bmatrix}.$$

Then by definition, $\boldsymbol{\mathcal{B}}(t;T) = \boldsymbol{D}\boldsymbol{\Phi}_A(T,t)\boldsymbol{B}(t) = -F(T-t)\boldsymbol{I}_2$. Similarly, $\boldsymbol{\mathcal{C}}(t;T) = -w_p\boldsymbol{I}_2$, and $\boldsymbol{e}(t;T) = -\boldsymbol{p}_0(t)$. The expression for the transformed variable $\boldsymbol{y}(t)$ is also obtained as follows:

$$\boldsymbol{y}(t) = \boldsymbol{r}(t) - (T-t)\tilde{\boldsymbol{v}}_P(t). \tag{35}$$

Note that for all $t \in [t_0, T^*]$, where T^* is the min-max time of capture, $\bar{\boldsymbol{z}}(t)$ depends on $\boldsymbol{p}_0(t)$ and the values of the min-max control inputs $\boldsymbol{u}^*(t)$ and $\boldsymbol{v}^*(t)$ can be computed using (33a) and (33b), respectively. Then, at time t, the min-max value for $\dot{\gamma}(\boldsymbol{z}(t))$, which is denoted by $\dot{\gamma}(\bar{\boldsymbol{z}}(t))$, is given by:

$$\dot{\gamma}(\bar{\boldsymbol{z}}(t)) = -F(T^*-t) + w_p,$$

and (16) becomes:

$$\ell = \|\boldsymbol{z}^0\| + \int_{t_0}^{T^*} (-F(T^*-\eta) + w_p)\mathrm{d}\eta. \tag{36}$$

It is notable at this point that whether the vehicle V can reach its target location or not depends on the initial conditions as well as the parameters of the problem. There is a particular region of the state space for which there exists a positive real value of T^* that satisfies (36). This region is called the *capturable region*. For the vehicle V, the capturable region represents the set of states that are reachable under the given flow-field conditions and control limits. If (36) does not admit such a solution, the evader has a possibility of escape, which means that there exists no admissible control that the pursuer can employ to guarantee capture. Note that in the latter case of no guaranteed capture, we cannot guarantee that we can guide V to the desired terminal set in the partially known flow-field. In particular, from (36), $w_p^2 < 2F\ell$ is a sufficient condition for the existence of an admissible control that will guide V to the desired terminal set in the partially known flow-field.

Finally, for the special case in which the known component of the flow-field is constant, the optimal value function of the game and the trajectories subsequently obtained match the solution to the Isotropic Rocket problem using Isaacs' method of *retrogressive path equations* [1].

5 Numerical Simulations: Newtonian Particle in Partially Known Flow-Field

In this section, we illustrate our proposed solution approach for the example problem using numerical simulations. We choose the values $t_0 = 0, F = 1, w_p = 0.4$, and $\ell = 0.1$, $\boldsymbol{\mu} = [1 \;\; 1]'$ and $k = 0.5$.

First, let us look in detail into the solution of the two-player game. Since an analytical expression for the time of capture is available from (36), we can deduce that the isochrones for this problem are actually circles whose center and radius are functions of time. This enables us to geometrically construct the level sets of the time of capture in the pursuit evasion game. A sample capture trajectory is shown in Fig. **??** in the inertial frame and the corresponding level sets for the pursuer (vehicle) are shown in Fig. **??**, projected on the relative position space.

Fig. **??** illustrates the effect of varying the magnitude of the unknown component w_p on the level sets of the time of capture function, for a given initial state and periodic wind. It is clear that the reachable sets of the pursuer in the relative state space are different in each case, and in particular, as the magnitude of w_p increases, the reachable set of the pursuer shrinks accordingly. Similarly, the dependence of the reachable sets on the (constant) magnitude F of the vehicle's acceleration is illustrated in Fig. **??**. As expected, when F is increased, the reachable set of the pursuer in the relative state space will also increase. Also, the higher the value of F, the lesser the time that the pursuer will need to reach a particular location.

Remark 5.1 The underlying dependence of the game-theoretic solution on the parameters F and w_p can be used to calculate the minimum required F to maneuver the vehicle to a chosen target location in the presence of the flow-field, given a particular value of w_p. For instance, in the presence of the flow-field with $k = 0.5$, $w_p = 0.4$, $\ell = 0.1$, $\boldsymbol{\mu} = [1 \;\; 1]'$, and maximum allowed terminal time of $T^* = 1.5$, with the initial state of the pursuit-evasion game being $\boldsymbol{x} = [2.631 \;\; -0.778 \;\; 1 \;\; -1]'$, we require at least $F = 1.5$ for the capture of the evader, that is, for the vehicle to reach its desired target location in the presence of the antagonistic flow-field.

Now, we consider the solution to the problem of guiding the vehicle in a partially known flow-field. The game-theoretic estimate of the time taken by the vehicle V to reach the origin in a flow-field is an upper-bound on the actual minimum time required should the flow-field be known a priori or act favorable to V. This is because the game-theoretic framework always assumes the worst intention of the adversary. In our case, this assumption means a component of the flow-field is unknown but bounded, and actively tries to prevent, if possible, or delay the progress of the vehicle towards its target.

For instance, let us examine a case where the flow-field component that is unknown a priori is not completely adversarial, in particular, for all $t \in [0, T]$, let $\boldsymbol{p}(t) = \boldsymbol{p}_0(t) + 0.5 w_p (\cos t - \sin t)[1 \quad -1]'$. Fig. **??** shows the trajectory of the vehicle V to the target in this flow-field, which is computed by utilizing the game-theoretic approach, for a given initial condition. The direction of the flow-field component that is not completely known a priori is shown along the trajectory in blue. The trajectories of the players in the equivalent pursuit-evasion game assuming a completely adversarial flow-field component is shown in Fig. **??**.

At this point, we wish to highlight an observation about the game-theoretic time of capture. Let us assume that the vehicle (pursuer) always acts optimally based on the state information available to it. When the evader engages in optimal play, the min-max time of capture $T^*(t_0, \boldsymbol{x}^0; \Sigma_\ell)$ remains constant over the duration of the pursuit-evasion game, that is, $T^*(t, \boldsymbol{x}^*(t); \Sigma_\ell) = T^*(t_0, \boldsymbol{x}^0; \Sigma_\ell)$ and its value needs to be computed only once at time $t = t_0$. However, when the evader acts sub-optimally (as in the case of an unknown flow-field that is not completely adversarial), the game-theoretic time of capture changes instantaneously as the game is played. In this case, at each time instant t, we must re-compute the value of $T^*(t, \boldsymbol{x}(t); \Sigma_\ell)$ along the ensuing non-optimal trajectory $\boldsymbol{x}(\cdot)$ and then compute the optimal control input for the vehicle (pursuer) because, in general, $T^*(t, \boldsymbol{x}(t); \Sigma_\ell) \neq T^*(t_0, \boldsymbol{x}^0; \Sigma_\ell)$.

Further, at any given time $t \in [t_0, T^*]$, and for a given state $\boldsymbol{x}(t)$, it is important to distinguish between the min-max time of capture $T^*(t, \boldsymbol{x}(t); \Sigma_\ell)$, which is the first time instant at which capture occurs, and the time-to-go $T_{go}(t, \boldsymbol{x}(t); \Sigma_\ell)$, which is the time remaining before capture. The two are related; the time-to-go $T_{go}(t, \boldsymbol{x}(t); \Sigma_\ell) = T^*(t, \boldsymbol{x}(t); \Sigma_\ell) - t$. Note that if $t_0 = 0$, $T_{go}(t_0, \boldsymbol{x}^0; \Sigma_\ell) = T^*(t_0, \boldsymbol{x}^0; \Sigma_\ell)$.

The difference between the game-theoretic terminal time and the actual time taken by the vehicle to reach the target is illustrated in Fig. **??**. In Fig. **??**, the evolution of the time-to-go when the unknown component of the flow-field is fully adversarial is shown as a green line. It is clearly seen that the game-theoretic time-to-go is an upper bound for the optimal time-to-go when the unknown flow-field component is not completely adversarial (shown in deep blue). Another case where the flow-field is completely known a priori is shown in light blue in the same figure. The evolution of the distance between the pursuer and the evader (vehicle and the target) shows similar behavior as shown in Fig. **??**.

Further, the existence of a solution to the game-theoretic formulation of the guidance problem in a partially unknown flow-field is only a sufficient condition for the vehicle V to be able to navigate to the target in that flow-field. This condition can be very conservative in some cases. In reality, the flow-field is not expected to have an adversarial effect at all times on the vehicle V as is assumed in the game-theoretic framework (it is possible that at some time subintervals the flow-field may even be favorable to the vehicle V). Consequently, it is possible for V to reach the target in finite time even in cases where the game-theoretic formulation of the guidance problem does not have a solution. Fig. **??** illustrates the vehicle's trajectories emanating from two different initial states generated with the application of the minimax guidance law proposed in this paper and the nonlinear feedback guidance law proposed in [26]. The feedback guidance law proposed in [26] is derived based on backstepping nonlinear control design techniques without taking explicitly into account neither input constraints nor optimal performance specifications. For the purpose of this comparison, we have normalized the computed control input so that it satisfies the same norm bound as the minimax control proposed in this work. The target position of the vehicle in Fig. **??** is the origin and it assumed that the winds are periodic. After comparing the two guidance laws for approximately 10^2 random initial conditions, we found that for nearly 70% of the cases, the vehicle on average takes 57% lesser time to reach the origin when it uses the game-theoretic guidance law. We must state here that due to its conservative nature, the game-theoretic approach to the guidance problem may not admit a solution for some initial conditions, from which, however, the vehicle can successfully reach the origin by utilizing the backstepping feedback control law.

Returning to the solution of the two-player game, we note that at any time $t \in [t_0, T^*]$, the relative magnitude of the known component of the flow-field $\|\boldsymbol{p}_0(t)\|$ with respect to the magnitude bound w_p of the unknown component does not affect the capturability of the evader in the two-player game. However, it has an effect on the time of capture in the game. Let $\lambda \in [0, 1]$ denote the ratio of the magnitude bound of the known component of the flow-field to the bound on its total magnitude, that is, $\lambda = (\max_{t\in[t_0,T^*]} \|\boldsymbol{p}_0(t)\|)/(\max_{t\in[t_0,T^*]} \|\boldsymbol{p}(t)\|)$.

In Fig. **??**, for a given initial condition and for different values of λ, we compare the min-max time of capture (in the game theoretic formulation). Note that as the ratio λ decreases, the magnitude of the unknown component increases and so does the time estimate for the vehicle V to reach the target. In Fig. **??**, the values of parameters were chosen as $F = 3$ and $\ell = 0.4$ with the same values of $\boldsymbol{\mu}$ and k as before.

6 Conclusions

In this paper, we have used game-theoretic tools to address the problem of guiding a vehicle, such as an aquatic or aerial vehicle, in a partially known time-varying flow-field induced by currents or winds, respectively. Our approach involves the reformulation of the guidance problem to an equivalent two-player zero-sum game between the vehicle and a fictitious player that can control the velocity of the flow-field. After expressing the equations of motion in the so-called reduced state space, we have developed a semi-analytic solution to the guidance problem for the case in which the known component of the flow-field is time-varying. Our proposed solution method for the case of time-varying flow-fields is an advancement over the method in [11], while being considerably simpler than Isaacs' framework [1]. The applicability of our approach has been demonstrated on the problem of guiding a Newtonian particle in a flow-field with a known periodic component.

In addition, we have computed the worst-case reachable sets of the vehicle by exploiting the structure of the level sets of the value function of the game. This is simpler than constructing the reachable sets by solving the generalized Hamilton-Jacobi-Isaacs equation for the two-player game. We argue that the game-theoretic approach is ideal for solving guidance problems in the absence of probabilistic models of the uncertainty in the environment. In particular, the capture time based on the game-theoretic formulation is an upper bound on the actual time required by the vehicle to reach the target position. This is because when the flow-field is either completely known a priori or its unknown component is not completely adversarial, the vehicle requires less time to reach the target than determined in the adversarial game. Hence with knowledge of a bound on the uncertain component of the flow-field, the existence of a solution to the two-player game can be used as a certificate for the existence of a solution to the vehicle guidance problem.

The framework presented in this paper can be extended to address the guidance problem for a vehicle in a spatio-temporal flow-field that can be locally approximated by a time-varying affine flow-field. We also plan to extend the applicability of the proposed solution framework to robust time-optimal problems for systems that do not satisfy Assumption 1, such as problems in which it is desirable to drive both the position and the velocity of a vehicle to prescribed goal vectors.

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