Distributed Partitioning Algorithms for Locational Optimization of Multi-Agent Networks in SE(2)

Efstathios Bakolas

Abstract—**This work is concerned with the development of distributed spatial partitioning algorithms for locational optimization problems involving networks of agents with planar rigid body dynamics subject to communication constraints. The domain of the problems we consider is a three-dimensional, non-flat manifold embedded in the state space of the agents, which we refer to as the terminal manifold. The approach we propose allows us to associate the partition of the three-dimensional terminal manifold, which is induced by a non-quadratic proximity metric and comprised of non-convex cells, with a one-parameter family of partitions of two-dimensional, flat manifolds, which are induced by (parametric) quadratic proximity metrics and comprised of convex polygonal cells. By exploiting the special structure of the parametric partitions, we develop distributed partitioning algorithms that converge in a finite number of steps. Subsequently, we utilize the solutions to the latter problems to solve a class of locational optimization problems over the terminal manifold. Numerical simulations that illustrate the capabilities of the proposed algorithms are also presented.**

I. Introduction

In this paper, we propose distributed algorithms for Voronoi-like partitioning and locational optimization problems involving networks of planar rigid bodies with limited communication capabilities. On one hand, the partitioning algorithms are intended to allow the agents of the network to compute their regions of influence over a three-dimensional non-flat manifold embedded in their six-dimensional state space in the presence of communication constraints. This three-dimensional manifold, which we refer to as the *terminal manifold*, consists of all the states that can be reached by the agents of the network with zero linear and angular velocities. The proposed partitioning algorithms yield a Voronoi-like partition of the terminal manifold, which is a subdivision of the latter into a finite collection of non-overlapping, but not necessarily convex, cells that are in one-to-one correspondence with the agents of the network (generators of this Voronoi-like partition). On the other hand, the locational optimization problem seeks for the optimal configurations of the agents of the network on the terminal manifold with respect to a relevant performance index.

At this point, we wish to emphasize that our goal is the development of partitioning and locational optimization algorithms that can be implemented in a *distributed* way. In particular, in the proposed framework, every agent will be able to perform the necessary computations for the characterization of its own cell independently from the other agents of the same network (for instance, no global grid of the terminal manifold will be employed). In addition, for these computations, the agents will rely on (local) information which can exchange with their neighbors in the topology induced by the Voronoi-like partition (two agents are neighbors, if the boundaries of their cells have a non-trivial intersection).

E. Bakolas is an Assistant Professor in the Department of Aerospace Engineering and Engineering Mechanics, The University of Texas at Austin, Austin, Texas 78712-1221, USA, Email: bakolas@austin.utexas.edu. This work was supported in part by the NSF under Grant 1562339.

One of the most well-studied locational optimization problems for multi-agent networks is the so-called coverage problem in the Euclidean plane, whose performance index is the expected value of the square of the Euclidean distance metric for a given density function. It turns out that the minimizers of this problem are the centroids of the cells of the standard Voronoi partition generated by the multi-agent network [1]–[5]. Despite the existence of a significant body of work on consensus-type problems for multi-agent networks evolving in $\mathbb{SE}(2)$ or $\mathbb{SE}(3)$ [6]–[9], as well as more abstract non-Euclidean spaces including connected, compact, and homogeneous manifolds [10]–[13], no significant efforts have been reported for addressing locational optimization problems in similar settings.

We wish to emphasize at this point that with the exception of standard Voronoi partitions of convex and compact subsets of a Euclidean space, whose proximity metric is the Euclidean distance, the development of distributed algorithms for Voronoi-like partitioning problems with non-Euclidean (generalized) proximity metrics and non-flat domains can be a complex task. This increased complexity can be mainly attributed to the fact that the latter partitions may be comprised of cells that are non-convex in general. In addition, the ability of a partitioning algorithm to be implemented in a distributed fashion hinges upon the ability of each agent to discover its neighbors in the topology of this partition (in this context, two agents are neighbors if the boundaries of their cells have a non-trivial intersection [14]) without having global knowledge of the partition a priori. In the case of standard Voronoi partitions, it is well known that each agent can discover its neighbors (in this special case, two agents are neighbors if their cells share a common edge, in two dimensions, or a common face, in higher dimensions) by means of simple distributed algorithms that exploit basic geometric properties of the standard Voronoi diagram and its dual, the Delaunay triangulation or graph [1], [15]–[17]. However, it is not obvious how the heading angles or the inertial properties of the agents, when the latter are modeled as rigid bodies, will affect both the structure of the cells that comprise the partition (these cells may not even be convex, as we have already mentioned) and their neighboring relations with the other agents from the same network.

Literature review: Voronoi partitions are useful tools for the development of distributed algorithms for control and optimization problems involving multi-agent networks and sensor networks [1], [2], [18]–[25]. Voronoi-like partitions whose proximity metrics do not solely stem from geometric considerations, such as the Euclidean distance, but encode instead information about the dynamics of the agents, which we will collectively refer to as *state-dependent* proximity metrics, have been studied extensively in our previous work (see, for instance, [14], [22], [26]–[29]). In particular, Refs. [27]–[29], which were originally motivated by [30], present partitioning algorithms that allow the agents of a network to compute their own cells from partitions that are induced by state-dependent proximity metrics independently from each other. In these references, however, communication constraints are not accounted. In addition, the applicability of the algorithms presented in these references is limited to partitioning problems over compact subsets of *flat* spaces (that is, linear or affine subspaces) with quadratic proximity metrics. The solutions of such problems turn out to be affine partitions comprised of convex polygonal cells whose combinatorial complexity is similar to that of standard Voronoi partitions.

Statement of contributions: The first objective of this work is the development of distributed algorithms for the computation of a Voronoi-like partition of the three-dimensional *terminal manifold*, whose proximity metric is taken to be the cost that will be incurred by an agent of the network for its transition to an arbitrary state in this manifold. This transition cost will be measured in terms of the decrease of a relevant (generalized) energy metric that occurs during the corresponding state transition. This proximity metric is non-quadratic and state-dependent. Consequently, the cells that comprise this Voronoi-like partition may be non-convex, in general, and they cannot be computed by directly applying any of the available techniques in the relevant literature [31], [32]. To address this difficult, at a first glance, partitioning problem, we propose an approach that is based on a special *embedding* technique. With this technique, the original partitioning problem is associated with a one-parameter family of partitioning problems whose domains are two-dimensional flat sub-manifolds of the terminal manifold and their proximity metrics are (parametric) quadratic functions. It turns out that the solution to each of these parametric partitioning problems corresponds to an affine partition comprised of convex polygonal cells with a modest combinatorial complexity. In this work, we exploit the special structure of these affine partitions in order to develop a novel distributed partitioning algorithm for their computation. The proposed algorithm, which leverages a certain optimization-based interpretation of the partitioning problem, finds exactly a representative sample of boundary points for any of the cells of the affine partition in a finite number of steps. This is in sharp contradistinction with the partitioning algorithms presented in our previous work [27]–[29], which can find boundary points of the cells that comprise an affine partition only asymptotically (practically, the algorithms proposed in these references can achieve accuracy that is comparable with that achieved by the algorithm proposed herein only after a significantly large number of iterations). After the solution to each parametric two-dimensional partitioning problem has been characterized, one can immediately obtain the solution to the original three-dimensional partitioning problem by stacking appropriately the former solutions next to each other and along the parameter axis. In this way, we characterize the three-dimensional and non-convex cells that comprise the partition of the (non-flat) terminal manifold, which are hard to compute directly, by repeatedly applying efficient algorithms for the computation of the convex polygonal cells that comprise each parametric affine partition. We wish to emphasize at this point that some of the key ideas and techniques of the proposed *finite steps* and *distributed* partitioning algorithms for affine partitions and in particular, their optimization-based philosophy, constitute fundamental contributions to partitioning problems for spatially distributed multi-agent networks, and their applicability could potentially be extended to more general classes of problems.

It is important to highlight that the partitioning algorithms proposed herein allow the agents of the network to compute their own cells independently from their teammates based on local information only (distributed partitioning algorithms). To achieve this, we present an iterative scheme that seeks to find a communication range for each agent of the network that is sufficiently large to allow the latter to communicate directly with its neighbors in the topology of the Voronoi-like partition, which is not known a priori. The main challenge here comes from the fact that the proximity metric that determines the topology of the Voronoi-like partition is different from the Euclidean distance that in turn determines whether two agents are close enough to communicate with each other or not.

The second objective of this work is to address a certain class of coverage-type locational optimization problems over the terminal manifold in a distributed way. To this aim, we use the proposed Voronoi-like partitioning algorithms in order to allow each agent of the network to find its minimizing state based on information that is encoded in its own cell. It turns out that the minimizing position of each agent is an appropriately weighted average of the optimal positions of a family of locational optimization problems whose domains correspond to two-dimensional, flat sub-manifolds of the (three-dimensional) terminal manifold. On the other hand, we show that the minimizing heading angle of each agent for the original locational optimization problem can be computed directly by solving a trigonometric algebraic equation over a compact interval, which admits a solution always.

Structure of the paper: The rest of the paper is organized as follows. Section II presents the formulation of the partitioning problem subject to communication constraints. In Section III, we embed the original partitioning problem into a one-parameter family of partitioning problems that can be addressed by means of distributed algorithms. Distributed solutions to coverage-type locational optimization problems are presented in Section IV. Section V presents numerical simulations, and finally, Section VI concludes the paper with a summary of remarks together with directions for future work.

II. Problem Formulation

A. Notation

We denote by $\mathbb{R}^n$ the set of n-dimensional real vectors and by $\mathbb{R}_{\geq 0}$ the set of non-negative real numbers. The set of integers and the set of non-negative integers are denoted by $\mathbb{Z}$ and $\mathbb{Z}_{\geq 0}$, respectively. We write $|\boldsymbol{\alpha}|$ to denote the 2-norm of a vector $\boldsymbol{\alpha} \in \mathbb{R}^n$. The unit circle in $\mathbb{R}^2$ will be denoted by $\mathbb{S}^1$, that is, $\mathbb{S}^1 := \{\boldsymbol{x} \in \mathbb{R}^2 : \ |\boldsymbol{x}| = 1\}$. Given a unit vector $\boldsymbol{e}$ in $\mathbb{R}^2$, we will write $\boldsymbol{e} \in \mathbb{S}^1$ (note the vector $\boldsymbol{e}$ is written in bold font). If ϑ is the angular parameter that corresponds to a unit vector $\boldsymbol{e} = \boldsymbol{e}(\vartheta)$ in $\mathbb{S}^1$ (for instance, $\boldsymbol{e}(\vartheta) = [\cos\vartheta, \ \sin\vartheta]^{\mathrm{T}}$), we will write $\vartheta \in \mathbb{S}^1$ instead of $\vartheta \in [2k\pi, (2k+1)\pi[$, for some $k \in \mathbb{Z}$, with a slight abuse of notation (note that the angular parameter ϑ is written in normal font). In addition, we write $\overline{\boldsymbol{B}}(\boldsymbol{x}; \eta)$ to denote the closed ball of radius $\eta > 0$ around $\boldsymbol{x} \in \mathbb{R}^n$, that is, $\overline{\boldsymbol{B}}(\boldsymbol{x}; \eta) := \{\boldsymbol{z} \in \mathbb{R}^n : \ |\boldsymbol{z} - \boldsymbol{x}| \leq \eta\}$. Given two (column) vectors $\boldsymbol{\alpha} \in \mathbb{R}^{n_1}$, $\boldsymbol{\beta} \in \mathbb{R}^{n_2}$, we denote by $\mathrm{col}(\boldsymbol{\alpha},\boldsymbol{\beta})$ the (n_1+n_2)-dimensional real (column) vector that corresponds to their concatenation. The notation can be extended in the natural way for the concatenation of three or more vectors. In addition, given a vector $\boldsymbol{c}\in\boldsymbol{\mathcal{Q}}\subseteq\mathbb{R}^{n_1+n_2}$, where $\boldsymbol{c}=\mathrm{col}(\boldsymbol{\alpha},\boldsymbol{\beta})$, $\boldsymbol{\alpha}\in\mathcal{A}\subseteq\mathbb{R}^{n_1}$ and $\boldsymbol{\beta}\in\mathcal{B}\subseteq\mathbb{R}^{n_2}$, we write $\boldsymbol{\alpha}=\pi_{\mathcal{A}}(\boldsymbol{c})$ and $\boldsymbol{\beta}=\pi_{\mathcal{B}}(\boldsymbol{c})$ (note that $\pi_{\mathcal{A}}(\cdot)$ and $\pi_{\mathcal{B}}(\cdot)$ are projection operators). Furthermore, $\mathrm{bd}(\mathcal{A})$ and $\mathrm{int}(\mathcal{A})$ denote, respectively, the boundary and the interior of a set $\mathcal{A}$. The relative boundary and the relative interior of a set $\mathcal{A}$ will be denoted by $\mathrm{rbd}(\mathcal{A})$ and $\mathrm{rint}(\mathcal{A})$, respectively. Given two points $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{R}^n$, we denote by $[\boldsymbol{\alpha},\boldsymbol{\beta}]$ the line segment connecting them (including the two endpoints), that is, $[\boldsymbol{\alpha},\boldsymbol{\beta}]:=\{\boldsymbol{x}\in\mathbb{R}^n:\ \boldsymbol{x}=t\boldsymbol{\alpha}+(1-t)\boldsymbol{\beta},\ 0\le t\le 1\}$. In addition, we denote by $]\boldsymbol{\alpha},\boldsymbol{\beta}]$ and $[\boldsymbol{\alpha},\boldsymbol{\beta}[$ the set $[\boldsymbol{\alpha},\boldsymbol{\beta}]\backslash\{\boldsymbol{\alpha}\}$ and the set $[\boldsymbol{\alpha},\boldsymbol{\beta}]\backslash\{\boldsymbol{\beta}\}$, respectively. Furthermore, given a symmetric matrix $\mathbf{P}=\mathbf{P}^{\mathrm{T}}\in\mathbb{R}^{n\times n}$, we denote by $\lambda_{\min}(\mathbf{P})$ and $\lambda_{\max}(\mathbf{P})$ its minimum and maximum (real) eigenvalues, respectively. Similarly, the minimum and the maximum singular values of a matrix $\mathbf{A}\in\mathbb{R}^{m\times n}$ are denoted by $\sigma_{\min}(\mathbf{A})$ and $\sigma_{\max}(\mathbf{A})$, respectively. Given two matrices $\mathbf{A}$ and $\mathbf{B}$, we denote by $\mathrm{bdiag}(\mathbf{A},\mathbf{B})$ their corresponding block diagonal matrix. Moreover, we write $\mathbb{P}_n$ and $\mathbb{K}_n$ to denote the convex (open) cone of $n\times n$ positive definite (symmetric) matrices and the set of $n\times n$ skew symmetric matrices, respectively. Finally, we will denote by $\mathbb{SO}(3)$ the rotation group for three-dimensional spaces.

B. Equations of Motion

We consider a team of n agents distributed in the configuration space $\boldsymbol{\mathcal{Q}}:=\boldsymbol{\mathcal{X}}\times\mathbb{S}^1$, where $\boldsymbol{\mathcal{X}}\subseteq\mathbb{R}^2$; we write $\mathbf{q}=\mathrm{col}(\mathbf{x},\vartheta)$ to denote a configuration vector in $\boldsymbol{\mathcal{Q}}$, where $\mathbf{x}=\pi_{\boldsymbol{\mathcal{X}}}(\mathbf{q})\in\boldsymbol{\mathcal{X}}$ and $\vartheta\in\mathbb{S}^1$. We assume that the i-th agent from the team, where $i\in\mathcal{I}_n:=\{1,\dots,n\}$, is initially located at $\boldsymbol{q}_i^0:=\mathrm{col}(\boldsymbol{x}_i^0,\theta_i^0)$, where $\boldsymbol{x}_i^0\in\boldsymbol{\mathcal{X}}$ and $\theta_i^0\in\mathbb{S}^1$ denote, respectively, its position and heading angle at time $t=0$ measured with respect to an inertial reference frame. It is assumed that $\boldsymbol{x}_i^0\neq\boldsymbol{x}_j^0$, for all $i,j\in\mathcal{I}_n$ with $i\neq j$. The joint vector of the initial positions of the n agents is denoted by $\boldsymbol{X}^0$, that is, $\boldsymbol{X}^0:=\mathrm{col}(\boldsymbol{x}_1^0,\dots,\boldsymbol{x}_n^0)$. The set comprised of the initial positions of the n agents will be denoted by $\{\boldsymbol{X}^0\}$, that is, $\{\boldsymbol{X}^0\}:=\{\boldsymbol{x}_i^0\in\boldsymbol{\mathcal{X}},\ i\in\mathcal{I}_n\}$. Note that $\boldsymbol{X}^0\in\boldsymbol{\mathcal{X}}^n$ whereas $\{\boldsymbol{X}^0\}\subsetneq\boldsymbol{\mathcal{X}}$. The joint vectors of the initial configurations and initial heading angles are denoted by $\boldsymbol{Q}^0$, where $\boldsymbol{Q}^0:=\mathrm{col}(\boldsymbol{q}_1^0,\dots,\boldsymbol{q}_n^0)$, and $\boldsymbol{\Theta}^0$, where $\boldsymbol{\Theta}^0:=\mathrm{col}(\theta_1^0,\dots,\theta_n^0)$, respectively. The set of initial configurations will be denoted by $\{\boldsymbol{Q}^0\}$, where $\{\boldsymbol{Q}^0\}:=\{\boldsymbol{q}_i^0\in\boldsymbol{\mathcal{Q}},\ i\in\mathcal{I}_n\}$. Note again that $\boldsymbol{Q}^0\in\boldsymbol{\mathcal{Q}}^n$ whereas $\{\boldsymbol{Q}^0\}\subsetneq\boldsymbol{\mathcal{Q}}$. It is also assumed that each agent has a prescribed initial velocity, which is denoted by $\boldsymbol{v}_i^0\in\mathbb{R}^3$. In particular, $\boldsymbol{v}_i^0:=\mathrm{col}(\boldsymbol{\nu}_i^0,w_i^0)\in\mathbb{R}^3$, where $\boldsymbol{\nu}_i^0\in\mathbb{R}^2$ and $w_i^0\in\mathbb{R}$ correspond, respectively, to the (initial) linear velocity and angular velocity of the i-th agent expressed in a body-fixed frame. In addition, we denote the initial state vector of the i-th agent by $\boldsymbol{z}_i^0$, where $\boldsymbol{z}_i^0:=\mathrm{col}(\boldsymbol{q}_i^0,\boldsymbol{v}_i^0)$. The state space of the i-th agent will be denoted by $\boldsymbol{\mathcal{Z}}$, where $\boldsymbol{\mathcal{Z}}:=\boldsymbol{\mathcal{X}}\times\mathbb{R}^3$. Again, the joint vector of initial states and the corresponding set are denoted by $\boldsymbol{Z}^0$, where $\boldsymbol{Z}^0\in\boldsymbol{\mathcal{Z}}^n$, and $\{\boldsymbol{Z}^0\}$, where $\{\boldsymbol{Z}^0\}\subsetneq\boldsymbol{\mathcal{Z}}$, respectively.

The kinematics of the i-th agent are described in an inertial frame by the following vector equation:

$$\dot{\boldsymbol{q}}_i=\mathbf{T}(\theta_i)\boldsymbol{v}_i,\qquad\qquad \boldsymbol{q}_i(0)=\boldsymbol{q}_i^0,\tag{1}$$

where $\boldsymbol{q}_i:=\mathrm{col}(\boldsymbol{x}_i,\theta_i)\in\boldsymbol{\mathcal{Q}}$ and $\boldsymbol{v}_i:=\mathrm{col}(\boldsymbol{\nu}_i,w_i)\in\mathbb{R}^3$ denote, respectively, its configuration (whose components are measured with respect to an inertial reference frame) and its velocity vector (whose components are measured with respect to a body-fixed reference frame) at time t. Note that for a given $\vartheta\in\mathbb{S}^1$, the rotation matrix $\mathbf{T}(\vartheta)\in\mathbb{SO}(3)$, where

$$\mathbf{T}(\vartheta):=\mathrm{bdiag}(\mathbf{T}_1(\vartheta),1),\ \mathbf{T}_1(\vartheta):=\begin{bmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{bmatrix},\tag{2}$$

acts upon $\boldsymbol{v}_i$, which is the velocity of the i-th agent in the body-fixed frame, to generate $\dot{\boldsymbol{q}}_i$, which is the i-th agent's velocity in the inertial frame. Note that in the kinematic model described by (1), the heading angle θ_i of the i-th agent does not necessarily match with the direction of its linear velocity.

Next, we express the dynamics of the i-th agent in the body-fixed frame as follows:

$$\mathbf{M}\dot{\boldsymbol{v}}_i+\mathbf{C}(\boldsymbol{v}_i)\boldsymbol{v}_i+\boldsymbol{g}(\boldsymbol{q}_i)=\boldsymbol{\tau}_i,\qquad \boldsymbol{v}_i(0)=\boldsymbol{v}_i^0,\tag{3}$$

with $\mathbf{M}:=\mathrm{diag}(m,m,J)\in\mathbb{P}_3$, where $m>0$ is the mass of each agent and $J>0$ is its moment of inertia, and $\mathbf{C}(\boldsymbol{v}_i)\in\mathbb{K}_3$, for all $\boldsymbol{v}_i\in\mathbb{R}^3$. In addition, the terms $\mathbf{C}(\boldsymbol{v}_i)\boldsymbol{v}_i$ and $\boldsymbol{g}(\boldsymbol{q}_i)$ correspond, respectively, to the resultant centripetal-Coriolis and gravitational forces / moments applied to the i-th agent [33], [34]. Furthermore, $\boldsymbol{\tau}_i$ denotes the control input of the i-th agent. Finally, we will denote the joint state vector of the i-th agent at time t by $\boldsymbol{z}_i$, where $\boldsymbol{z}_i:=\mathrm{col}(\boldsymbol{q}_i,\boldsymbol{v}_i)\in\boldsymbol{\mathcal{Z}}$.

C. Communication Among the Agents

It is assumed that the i-th agent can only communicate with the agents from the same network that lie within its "communication range," which is denoted by η_i. Given $\eta_i>0$, we denote by $\mathcal{N}_c(i,\eta_i)$ the index-set of all the agents of the network that lie within the communication range of the i-th agent, that is, $\mathcal{N}_c(i,\eta_i):=\{\ell\in\mathcal{I}_n\backslash\{i\}:\boldsymbol{x}_\ell^0\in\overline{\boldsymbol{B}}(\boldsymbol{x}_i^0;\eta_i)\}$. In particular, we assume that the i-th agent sends a message omni-directionally, which can reach any point within its communication range (*broadcast* communication), requesting any agent that received this message to send back a confirmation message. From that point onwards, the i-th agent can establish direct communication channels with any agents lying in $\overline{\boldsymbol{B}}(\boldsymbol{x}_i^0;\eta_i)$ so that it can directly exchange information with them (*point-to-point* communication). We will assume that each agent can determine the relative configurations of the agents lying in its communication range with respect to itself via exchange of relevant information. In addition, we will assume that this exchange of information can take place infinitely fast. In other words, we won't explicitly account for any communication delays although, we will briefly present ways that would allow us to account for such delays in practice.

It is interesting to note that if $\boldsymbol{\mathcal{X}}$ is a compact subset of $\mathbb{R}^2$, then there exists a closed ball $\overline{\boldsymbol{B}}(\mathbf{x}^\circ;\mu)$, centered at some point $\mathbf{x}^\circ\in\boldsymbol{\mathcal{X}}$ with radius $\mu>0$, such that $\boldsymbol{\mathcal{X}}\subseteq\overline{\boldsymbol{B}}(\mathbf{x}^\circ;\mu)$, from which it is easy to show that $\boldsymbol{\mathcal{X}}\subseteq\overline{\boldsymbol{B}}(\boldsymbol{x}_i^0;2\mu)$, for all $i\in\mathcal{I}_n$. The situation is illustrated in Fig. 1. Therefore, by taking $\eta_i=2\mu$, it is guaranteed that $\mathcal{N}_c(i,\eta_i)=\mathcal{I}_n\backslash\{i\}$. As we will see later on, requiring that $\eta_i\geq 2\mu$ for all $i\in\mathcal{I}_n$ may be an unnecessarily strong assumption. This is because in order for an agent to be able to compute its own cell from a Voronoi-like partition, it may not be necessary to communicate with every other agent from the same network. It is important to note that by maintaining a large communication range that can cover the whole position space, $\boldsymbol{\mathcal{X}}$, at each time, an agent may

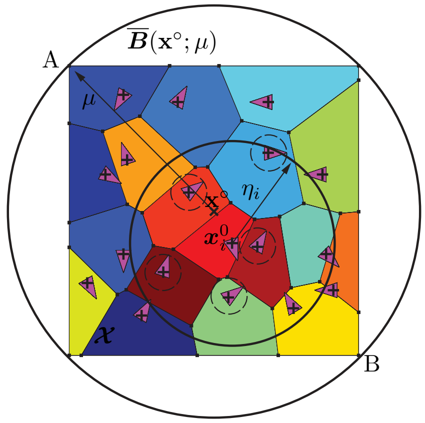


Fig. 1. Consider a square position space $\boldsymbol{\mathcal{X}}$ that is contained in a closed ball $\overline{\boldsymbol{B}}(\mathbf{x}^\circ;\mu)$. We observe that all the agents of the network can communicate with each other by maintaining a communication range that is greater than or equal to 2μ (note that this lower bound is attained when, for instance, two different agents are located, respectively, at the vertices A and B, where AB is a diagonal of the square domain). In most cases, however, the required communication range can be significantly smaller than 2μ.

be incurring an unnecessarily high communication cost (e.g., battery usage). This is clearly illustrated in Fig. 1 in which the communication range η_i of the i-th agent is sufficiently large to contain all of its neighbors in the topology of the Voronoi-like partition illustrated in the same figure (in this topology, two agents are neighbors if their polygonal cells share a common edge) and at the same time it is significantly smaller than 2μ. In this work, we will assume that all the agents can adjust their communication ranges so that the latter are sufficiently, but not unnecessarily, large to allow them to collect all the (local) information required for the computation of their own cells from the Voronoi-like partition. This perspective is in agreement with the paradigm for distributed computation of standard Voronoi partitions that was proposed in [2], [15]. The approach that we will employ in this work is, however, very different from the one adopted in these references.

D. The Partitioning Problem over the Terminal Manifold $\boldsymbol{\mathcal{T}}$

In this section, we will provide the exact formulation of the partitioning problem over the terminal manifold $\boldsymbol{\mathcal{T}}$. In simple words, our objective is to subdivide the terminal manifold $\boldsymbol{\mathcal{T}}$ into n non-overlapping regions, which we will refer to as Voronoi cells or simply cells. In addition, each cell will be uniquely associated with an agent of the network and in particular, it will exclusively consist of points in $\boldsymbol{\mathcal{T}}$ that are "closer" to its corresponding agent than to any other agent of the network. Here, the closeness between the agents and an arbitrary point in $\boldsymbol{\mathcal{T}}$ will be measured in terms of an appropriate *proximity* metric. In particular, for a given $\boldsymbol{z}_i^0 \in \{\boldsymbol{Z}^0\}$, we take the proximity (generalized) metric to be the function $\delta(\cdot;\boldsymbol{z}_i^0):\boldsymbol{\mathcal{T}}\to\mathbb{R}_{\geq 0}$ with

$$\delta(\mathbf{q};\boldsymbol{z}_i^0) := \boldsymbol{s}(\mathbf{q};\boldsymbol{z}_i^0)^{\mathrm{T}}\mathbf{M}\boldsymbol{s}(\mathbf{q};\boldsymbol{z}_i^0), \tag{4}$$

where $\mathbf{q} := \mathrm{col}(\mathbf{x},\vartheta)$ and

$$\boldsymbol{s}(\mathbf{q};\boldsymbol{z}_i^0) := \mathrm{col}(\boldsymbol{\Lambda}(\boldsymbol{x}_i^0-\mathbf{x}),\varepsilon(1-\cos(\theta_i^0-\vartheta))) + \mathbf{T}(\theta_i^0)\boldsymbol{v}_i^0. \tag{5}$$

In the last equation, $\boldsymbol{\Lambda}$ is a known diagonal matrix in $\mathbb{P}_2$ and ε is a known positive constant. The choice of the proximity metric is motivated by the fact that in Lyapunov-based analysis for steering problems in $\mathbb{SE}(2)$ or $\mathbb{SE}(3)$ [33]–[35], functions similar to δ are used as generalized energy metrics or Lyapunov candidate functions. Specifically, the quantity $\delta(\mathbf{q};\boldsymbol{z}_i^0)$ can be interpreted as the decrease of a relevant generalized energy that the i-th agent will incur for the transition from its initial state $\boldsymbol{z}_i^0$ to the state $\boldsymbol{z}_{\boldsymbol{\mathcal{T}}}(\mathbf{q})\in\boldsymbol{\mathcal{T}}$, for a given $\mathbf{q}\in\boldsymbol{\mathcal{Q}}$. By plugging (5) in (4), it follows that

$$\begin{aligned}\delta(\mathbf{q};\boldsymbol{z}_i^0) = {} & m|\boldsymbol{\Lambda}(\boldsymbol{x}_i^0-\mathbf{x})|^2 + \left|\mathbf{M}^{\frac{1}{2}}\boldsymbol{v}_i^0\right|^2 \\ & + \varepsilon^2 J(1-\cos(\theta_i^0-\vartheta))^2 \\ & + 2m(\boldsymbol{x}_i^0-\mathbf{x})^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{T}_1(\theta_i^0)\boldsymbol{\nu}_i^0 \\ & + 2\varepsilon J w_i^0(1-\cos(\theta_i^0-\vartheta)).\end{aligned} \tag{6}$$

It is important to note that the proximity metric δ is a non-quadratic function of $\mathbf{q}$. The exact formulation of the partitioning problem over $\boldsymbol{\mathcal{T}}$ is given next.

Problem 1: Partitioning Problem over $\boldsymbol{\mathcal{T}}$ Subject to Communication Constraints: Suppose that $\boldsymbol{\mathcal{X}}$ is a compact and convex polygonal subset of $\mathbb{R}^2$ which is contained in the closed ball $\overline{\boldsymbol{B}}(\mathbf{x}^\circ;\mu)$ of radius $\mu>0$ that is centered at some point $\mathbf{x}^\circ\in\boldsymbol{\mathcal{X}}$, and let $\boldsymbol{Z}^0 := \mathrm{col}(\boldsymbol{z}_1^0,\ldots,\boldsymbol{z}_n^0)$, where $\boldsymbol{z}_i^0 = \mathrm{col}(\boldsymbol{q}_i^0,\boldsymbol{v}_i^0)$, $i\in\mathcal{I}_n$, be the initial joint state of the multi-agent network in $\boldsymbol{\mathcal{Z}}^n$. In addition, let $\eta_i>0$ be the communication range of the i-th agent, which is an adjustable quantity for all $i\in\mathcal{I}_n$, and let $\boldsymbol{H} := \mathrm{col}(\eta_1,\ldots,\eta_n)$. Then, determine a collection of sets $\boldsymbol{\mathfrak{V}}(\boldsymbol{Z}^0;\boldsymbol{H}) := \{\boldsymbol{\mathfrak{V}}^i(\boldsymbol{z}_i^0;\eta_i),\ i\in\mathcal{I}_n\}$ of the terminal manifold $\boldsymbol{\mathcal{T}}$ such that:

(i) $\boldsymbol{\mathcal{T}} = \bigcup_{i\in\mathcal{I}_n}\boldsymbol{\mathfrak{V}}^i(\boldsymbol{z}_i^0;\eta_i)$,
(ii) $\mathrm{rint}(\boldsymbol{\mathfrak{V}}^i(\boldsymbol{z}_i^0;\eta_i))\cap\mathrm{rint}(\boldsymbol{\mathfrak{V}}^j(\boldsymbol{z}_j^0;\eta_j))=\varnothing$, for all $i,j\in\mathcal{I}_n$, $i\neq j$,
(iii) A state $\boldsymbol{z}_{\boldsymbol{\mathcal{T}}}(\mathbf{q})\in\boldsymbol{\mathcal{T}}$, where $\boldsymbol{z}_{\boldsymbol{\mathcal{T}}}(\mathbf{q}) := \mathrm{col}(\mathbf{q},\mathbf{0})$ and $\mathbf{q}\in\boldsymbol{\mathcal{Q}}$, belongs to $\boldsymbol{\mathfrak{V}}^i(\boldsymbol{z}_i^0;\eta_i)$ for some $i\in\mathcal{I}_n$, if, and only if, $\delta(\mathbf{q};\boldsymbol{z}_i^0)\leq\delta(\mathbf{q};\boldsymbol{z}_j^0)$, for all $j\in\mathcal{N}_c(i,\eta_i)$, where $\delta(\mathbf{q};\boldsymbol{z}_i^0)$ is defined in Eq. (6).

Remark 1 Note that the purpose of conditions (i) and (ii) is to ensure that the collection of sets $\boldsymbol{\mathfrak{V}}(\boldsymbol{Z}^0;\boldsymbol{H}) := \{\boldsymbol{\mathfrak{V}}^i(\boldsymbol{z}_i^0;\eta_i),\ i\in\mathcal{I}_n\}$ forms a partition of $\boldsymbol{\mathcal{T}}$ in the strict mathematical sense; in particular, condition (i) ensures that $\boldsymbol{\mathfrak{V}}(\boldsymbol{Z}^0;\boldsymbol{H})$ achieves complete covering of $\boldsymbol{\mathcal{T}}$ whereas condition (ii) guarantees that the cells comprising $\boldsymbol{\mathfrak{V}}(\boldsymbol{Z}^0;\boldsymbol{H})$ will not overlap with each other. It should also be noted that the presence of the communication constraints described in Section II-B are reflected in condition (iii) of Problem 1. By virtue of this condition, the i-th agent is confined to compare its proximity to a state $\boldsymbol{z}_{\boldsymbol{\mathcal{T}}}(\mathbf{q})\in\boldsymbol{\mathcal{T}}$ with the agents that lie within its communication range only.

Remark 2 Besides the existence of communication constraints, the facts that the proximity metric in Problem 1 is non-quadratic and its domain is non-flat make this problem challenging (for instance, the cells that comprise $\boldsymbol{\mathfrak{V}}(\boldsymbol{Z}^0;\boldsymbol{H})$ may be non-convex, in general). In particular, Problem 1 cannot be directly associated with any well-studied family of partitioning problems and it is not clear how it can be solved in a distributed way.

III. The One-Parameter Family of Partitioning Problems

In this section, we will address the original three-dimensional partitioning problem over the terminal manifold $\boldsymbol{\mathcal{T}}$ (Problem 1) by embedding it into a one-parameter family of two-dimensional partitioning problems. Specifically, the domain of each parametric problem is a two-dimensional sub-manifold $\boldsymbol{\mathcal{T}}_\vartheta$ of $\boldsymbol{\mathcal{T}}$, which consists of all states $\boldsymbol{z} \in \boldsymbol{\mathcal{T}}$ whose heading angle components are equal to a given $\vartheta \in \mathbb{S}^1$, that is, $\boldsymbol{\mathcal{T}}_\vartheta := \{\boldsymbol{z} \in \boldsymbol{\mathcal{T}} \subsetneq \boldsymbol{\mathcal{Z}} : \boldsymbol{z} = \boldsymbol{z}_\vartheta(\mathbf{x}),\ \mathbf{x} \in \boldsymbol{\mathcal{X}}\}$, where $\boldsymbol{z}_\vartheta(\mathbf{x}) := \mathrm{col}(\mathbf{x}, \vartheta, \mathbf{0})$. Note that for a given $\vartheta \in \mathbb{S}^1$, the two-dimensional manifold $\boldsymbol{\mathcal{T}}_\vartheta$ is homeomorphic, in the topological sense, to the manifold $\boldsymbol{\mathcal{Q}}_\vartheta := \{\mathbf{q} = \mathrm{col}(\mathbf{x}, \vartheta) \in \boldsymbol{\mathcal{Q}} :\ \mathbf{x} \in \boldsymbol{\mathcal{X}}\}$, which is in turn homeomorphic to $\boldsymbol{\mathcal{X}}$; we write $\boldsymbol{\mathcal{T}}_\vartheta \sim \boldsymbol{\mathcal{Q}}_\vartheta$ and $\boldsymbol{\mathcal{Q}}_\vartheta \sim \boldsymbol{\mathcal{X}}$. To address the parametric partitioning problem in $\boldsymbol{\mathcal{T}}_\vartheta$, for a given $\vartheta \in \mathbb{S}^1$, we will need a (generalized) proximity metric that, in contrast with δ, reflects the fact that the heading angle component of an arbitrary terminal state in $\boldsymbol{\mathcal{T}}_\vartheta$ is constant. In particular, given $\vartheta \in \mathbb{S}^1$ and $i \in \mathcal{I}_n$, we define the generalized distance of the i-th agent, which is emanating from the state $\boldsymbol{z}_i^0 \in \{\boldsymbol{Z}^0\}$ to a state $\boldsymbol{z}_\vartheta(\mathbf{x}) \in \boldsymbol{\mathcal{T}}_\vartheta$, to be the function $\delta_\vartheta(\cdot; \boldsymbol{z}_i^0) : \boldsymbol{\mathcal{X}} \to \mathbb{R}_{\geq 0}$ where

$$\delta_\vartheta(\mathbf{x}; \boldsymbol{z}_i^0) := \delta(\mathrm{col}(\mathbf{x}, \vartheta); \boldsymbol{z}_i^0). \tag{7}$$

In light of (6) and (7), we can write $\delta_\vartheta(\mathbf{x}; \boldsymbol{z}_i^0)$ more compactly as follows:

$$\delta_\vartheta(\mathbf{x}; \boldsymbol{z}_i^0) = |\boldsymbol{\Pi}^{\frac{1}{2}}(\boldsymbol{x}_i^0 - \mathbf{x})|^2 + 2(\boldsymbol{x}_i^0 - \mathbf{x})^{\mathrm{T}} \boldsymbol{r}_\vartheta^i + \sigma_\vartheta^i, \tag{8}$$

where $\boldsymbol{\Pi} := m\boldsymbol{\Lambda}^2$ and

$$\boldsymbol{r}_\vartheta^i := m\boldsymbol{\Lambda}\mathbf{T}_1(\theta_i^0)\boldsymbol{\nu}_i^0, \tag{9a}$$

$$\begin{aligned}\sigma_\vartheta^i &:= \varepsilon^2 J(1 - \cos(\theta_i^0 - \vartheta))^2 + |\mathbf{M}^{\frac{1}{2}} \boldsymbol{v}_i^0|^2 \\ &\quad + 2\varepsilon J w_i^0 (1 - \cos(\theta_i^0 - \vartheta)).\end{aligned} \tag{9b}$$

By completing the square in (8), we take

$$\begin{aligned}\delta_\vartheta(\mathbf{x}; \boldsymbol{z}_i^0) &= |\boldsymbol{\Pi}^{\frac{1}{2}}(\mathbf{x} - \boldsymbol{x}_i^0)|^2 - 2(\mathbf{x} - \boldsymbol{x}_i^0)^{\mathrm{T}} \boldsymbol{\Pi}^{\frac{1}{2}} \boldsymbol{\Pi}^{-\frac{1}{2}} \boldsymbol{r}_\vartheta^i + \sigma_\vartheta^i \\ &= |\boldsymbol{\Pi}^{\frac{1}{2}}(\mathbf{x} - \boldsymbol{x}_i^0 - \boldsymbol{\Pi}^{-1}\boldsymbol{r}_\vartheta^i)|^2 - |\boldsymbol{\Pi}^{-\frac{1}{2}} \boldsymbol{r}_\vartheta^i|^2 + \sigma_\vartheta^i.\end{aligned}$$

We immediately conclude that

$$\delta_\vartheta(\mathbf{x}; \boldsymbol{z}_i^0) = |\boldsymbol{\Pi}^{\frac{1}{2}}(\mathbf{x} - \boldsymbol{\xi}_\vartheta^i)|^2 + \mu_\vartheta^i, \tag{10}$$

$$\boldsymbol{\xi}_\vartheta^i := \boldsymbol{x}_i^0 + \boldsymbol{\Pi}^{-1}\boldsymbol{r}_\vartheta^i, \quad \mu_\vartheta^i := -|\boldsymbol{\Pi}^{-\frac{1}{2}} \boldsymbol{r}_\vartheta^i|^2 + \sigma_\vartheta^i. \tag{11}$$

Note that, because $\delta_\vartheta(\mathbf{x}; \boldsymbol{z}_i^0) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^2$, it is necessarily true that $\mu_\vartheta^i \in \mathbb{R}_{\geq 0}$ for all $i \in \mathcal{I}_n$, given that $\mu_\vartheta^i = \delta_\vartheta(\boldsymbol{\xi}_\vartheta^i; \boldsymbol{z}_i^0)$. In view of (10), we conclude that the proximity metric, δ_ϑ, for the parametric partitioning problem over the (two-dimensional and flat) sub-manifold $\boldsymbol{\mathcal{T}}_\vartheta$ of the (three-dimensional and non-flat) terminal manifold $\boldsymbol{\mathcal{T}}$ is a (parametric) quadratic function of $\mathbf{x}$, for any given $\vartheta \in \mathbb{S}^1$. Based on the previous observation and the fact that $\boldsymbol{\mathcal{T}}_\vartheta \sim \boldsymbol{\mathcal{Q}}_\vartheta \sim \boldsymbol{\mathcal{X}}$, it is expected (and will be proven later on) that the solution to the resulting partitioning problem will be a so-called *affine* partition [36]. Affine partitions of convex polygonal domains are comprised of convex polygonal cells and their combinatorial complexity is similar to that of standard Voronoi partitions [36].

Next, we formulate precisely the parametric partitioning problem over a sub-manifold $\boldsymbol{\mathcal{T}}_\vartheta$ of the terminal manifold $\boldsymbol{\mathcal{T}}$, for a given $\vartheta \in \mathbb{S}^1$.

Problem 2: Parametric Partitioning Problem over $\boldsymbol{\mathcal{T}}_\vartheta$ Subject to Communication Constraints: Suppose that $\boldsymbol{\mathcal{X}}$ is a convex and compact polygonal subset of $\mathbb{R}^2$ which is contained in the closed ball $\overline{\boldsymbol{B}}(\mathbf{x}^\circ; \mu)$ of radius $\mu > 0$ that is centered at some point $\mathbf{x}^\circ \in \boldsymbol{\mathcal{X}}$, and let $\vartheta \in \mathbb{S}^1$ be given. In addition, let $\boldsymbol{Z}^0 := \mathrm{col}(\boldsymbol{z}_1^0, \dots, \boldsymbol{z}_n^0)$, where $\boldsymbol{z}_i^0 = \mathrm{col}(\boldsymbol{q}_i^0,\ \boldsymbol{v}_i^0)$, $i \in \mathcal{I}_n$, be the initial joint state of the multi-agent network in $\boldsymbol{\mathcal{Z}}^n$. Furthermore, let $\eta_i > 0$ be the communication range of the i-th agent, which is an adjustable quantity for all $i \in \mathcal{I}_n$, and let also $\boldsymbol{H} := \mathrm{col}(\eta_1, \dots, \eta_n)$. Then, determine a collection of sets $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0; \boldsymbol{H}) := \{\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0; \eta_i),\ i \in \mathcal{I}_n\}$ of the sub-manifold $\boldsymbol{\mathcal{T}}_\vartheta$ of $\boldsymbol{\mathcal{T}}$ such that:

(i) $\boldsymbol{\mathcal{T}}_\vartheta = \bigcup_{i \in \mathcal{I}_n} \mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0; \eta_i)$,
(ii) $\mathrm{rint}(\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0; \eta_i)) \cap \mathrm{rint}(\mathfrak{V}_\vartheta^j(\boldsymbol{z}_j^0; \eta_j)) = \varnothing$, for all $i, j \in \mathcal{I}_n$, $i \neq j$,
(iii) A state $\boldsymbol{z}_\vartheta(\mathbf{x}) \in \boldsymbol{\mathcal{T}}_\vartheta$, where $\boldsymbol{z}_\vartheta(\mathbf{x}) := \mathrm{col}(\mathbf{x}, \vartheta, \mathbf{0})$ and $\mathbf{x} \in \boldsymbol{\mathcal{X}}$, belongs to the cell $\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0; \eta_i)$ for some $i \in \mathcal{I}_n$ if, and only if, $\delta_\vartheta(\mathbf{x}; \boldsymbol{z}_i^0) \leq \delta_\vartheta(\mathbf{x}; \boldsymbol{z}_j^0)$ for all $j \in \mathcal{N}_c(i, \eta_i)$, where $\delta_\vartheta(\mathbf{x}; \boldsymbol{z}_i^0)$ is defined in Eq. (10).

A. Analysis of the Partitioning Problem over $\boldsymbol{\mathcal{T}}_\vartheta$ in the Absence of Communication Constraints

Next, we propose an algorithmic solution technique for Problem 2 in the absence of communication constraints, that is, for the special case when $\eta_i \geq 2\mu$, for all $i \in \mathcal{I}_n$, where μ is defined as in the formulation of Problem 2. For this special case, we will simplify the notation used in Problem 2 by denoting the collection of sets as $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0) = \{\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0),\ i \in \mathcal{I}_n\}$. In the absence of communication constraints, condition (iii) of Problem 2 will have to change accordingly. In particular, for a given $\vartheta \in \mathbb{S}^1$, a state $\boldsymbol{z} = \boldsymbol{z}_\vartheta(\mathbf{x}) \in \boldsymbol{\mathcal{T}}_\vartheta$, where $\boldsymbol{z}_\vartheta(\mathbf{x}) := \mathrm{col}(\mathbf{x}, \vartheta, \mathbf{0})$ and $\mathbf{x} \in \boldsymbol{\mathcal{X}}$, will belong to $\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0)$ if, and only if, $\delta_\vartheta(\mathbf{x}; \boldsymbol{z}_i^0) \leq \delta_\vartheta(\mathbf{x}; \boldsymbol{z}_j^0)$, for all $j \in \mathcal{I}_n \backslash \{i\}$. This means that, in the absence of communication constraints, the i-th agent will have to compare its transition cost to a state $\boldsymbol{z}_\vartheta(\mathbf{x}) \in \boldsymbol{\mathcal{T}}_\vartheta$ with all the agents of the network always (compare with the formulation of Problem 2). We will say that, in the presence of communication constraints, any agent has a smaller set of competitors than it has in the absence of communication constraints.

Lemma 1: Let $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0) = \{\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0),\ i \in \mathcal{I}_n\}$ denote the solution to Problem 2, when $\eta_i \geq 2\mu$ for all $i \in \mathcal{I}_n$. Then, $\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0) \subseteq \mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0; \eta_i)$, for all $\eta_i \in [0, 2\mu[$ and $\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0) = \mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0; \eta_i)$ for all $\eta_i \geq 2\mu$.

It is interesting to note that Lemma 1 brings to light an important issue regarding the well-posedness of Problem 2. Consider, for instance, the scenario in which no other agent lies within the communication range of the i-th agent besides itself. In this case, the i-th agent has no competitors and will conclude incorrectly that its own cell, $\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0; \eta_i)$, coincides with the whole sub-manifold $\boldsymbol{\mathcal{T}}_\vartheta$. Along the same lines, the fact that $\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0; \eta_i) \supseteq \mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0)$ for all $\eta_i \in [0, 2\mu[$, which in turn implies that $\cup_{i \in \mathcal{I}_n} \mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0; \eta_i) \supseteq \cup_{i \in \mathcal{I}_n} \mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0)$, for all $\eta_i \in [0, 2\mu[$, suggests the existence of two possibilities:
Case 1: Some of the cells of $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0; \boldsymbol{H})$ have overlapping (relative) interiors, which is a violation of condition (ii) of Problem 2.
Case 2: The collection of sets $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0)$ has "coverage holes" in the sense that $\cup_{i \in \mathcal{I}_n} \mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0) \subsetneq \boldsymbol{\mathcal{T}}_\vartheta$, which is a violation of condition (i) of Problem 2.

However, as we will see later in this section, $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0)$ is an affine partition, where the word partition should be understood in the strict mathematical sense, and as such it does not have any coverage holes. In other words, in the absence of communication constraints, that is, when $\eta_i \geq 2\mu$ for all $i \in \mathcal{I}_n$, Problem 2 will always be well-posed. Therefore, Case 2 will never occur. By contrast, Case 1 is likely to occur, that is, the cells that comprise $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0; \boldsymbol{H})$ may overlap with each other (in which case, the collection of sets $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0; \boldsymbol{H})$ does not form a partition of $\boldsymbol{\mathcal{T}}_\vartheta$ in the strict mathematical sense).

On the other hand, as we have mentioned before, requiring that $\eta_i \geq 2\mu$ for all $i \in \mathcal{I}_n$ can be a very conservative condition in many cases. This is because in order for the i-th agent to be able to compute its own cell in $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0; \boldsymbol{H})$, which is the solution of Problem 2 in the general case, when communication constraints come into play (that is, the condition $\eta_i \geq 2\mu$ does not necessarily hold true for all $i \in \mathcal{I}_n$), it suffices to have a communication range that will cover all of its neighbors "in the topology" of $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0)$, which is the solution of Problem 2 in the absence of communication constraints (that is, when $\eta_i \geq 2\mu$ for all $i \in \mathcal{I}_n$). Recall that the i-th and the j-th agents of the network are neighbors in the topology of $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0)$, if, and only if, the boundaries of their corresponding cells have a non-trivial intersection. Finding a lower bound on the communication range of each agent so that Problem 2 is well-posed without requiring $\eta_i \geq 2\mu$ for all $i \in \mathcal{I}_n$, will be the topic of Section III-C. In this section, we will focus on the computation of the solution to Problem 2 with no communication constraints; this problem, as we have already mentioned, is always well-posed.

Before we proceed to the description of the proposed algorithm, we will present and examine the key features of the solution to Problem 2 in the absence of communication constraints. To this aim, let us consider a pair of generators $(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0) \in \{\boldsymbol{Z}^0\} \times \{\boldsymbol{Z}^0\}$, $i \neq j$. Their corresponding bisector with respect to the generalized metric δ_ϑ, which is denoted as $\mathcal{B}(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0; \delta_\vartheta)$, will consist of all states in $\boldsymbol{\mathcal{T}}_\vartheta$ that are equidistant from $\boldsymbol{z}_i^0$ and $\boldsymbol{z}_j^0$ with respect to δ_ϑ, that is, the states $\boldsymbol{z}_\vartheta(\mathrm{x}) \in \boldsymbol{\mathcal{T}}_\vartheta$, where $\mathrm{x} \in \boldsymbol{\mathcal{X}}$ satisfies the following equation:

$$\delta_\vartheta(\mathrm{x}; \boldsymbol{z}_i^0) = \delta_\vartheta(\mathrm{x}; \boldsymbol{z}_j^0). \tag{12}$$

It turns out that $\mathcal{B}(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0; \delta_\vartheta)$ is a line segment that lies in $\boldsymbol{\mathcal{T}}_\vartheta$.

Proposition 1: For a given $\vartheta \in \mathbb{S}^1$, the bisector $\mathcal{B}(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0; \delta_\vartheta)$ corresponding to the pair of generators $(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0) \in \{\boldsymbol{Z}^0\} \times \{\boldsymbol{Z}^0\}$, $i \neq j$, with respect to the generalized metric δ_ϑ is the loci of all states $\boldsymbol{z}_\vartheta(\mathrm{x}) \in \boldsymbol{\mathcal{T}}_\vartheta$, where $\mathrm{x} \in \boldsymbol{\mathcal{X}}$ satisfies the following equation:

$$\mathrm{x}^{\mathrm{T}} \boldsymbol{\gamma}_\vartheta(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0) = \zeta_\vartheta(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0), \tag{13}$$

with

$$\boldsymbol{\gamma}_\vartheta(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0) := 2\boldsymbol{\Pi}(\boldsymbol{\xi}_\vartheta^i - \boldsymbol{\xi}_\vartheta^j), \tag{14a}$$

$$\zeta_\vartheta(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0) := |\boldsymbol{\Pi}^{\frac{1}{2}} \boldsymbol{\xi}_\vartheta^i|^2 - |\boldsymbol{\Pi}^{\frac{1}{2}} \boldsymbol{\xi}_\vartheta^j|^2 + \mu_\vartheta^i - \mu_\vartheta^j, \tag{14b}$$

where $\boldsymbol{\xi}_\vartheta^i$, $\boldsymbol{\xi}_\vartheta^j$, μ_ϑ^i and μ_ϑ^j satisfy (11).

The derivation of (13) follows readily after substitution of (10) in (12). Note that the left hand side of Eq. (13) defines a linear functional over $\boldsymbol{\mathcal{X}}$. Therefore, the same equation describes a particular level set of a linear functional, which implies [37] that the collection of all points x that satisfy this equation corresponds to a straight line in $\boldsymbol{\mathcal{X}}$. This straight line is orthogonal to the vector $\boldsymbol{\gamma}_\vartheta(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0)$ and its distance from the origin $\mathrm{x} = \mathbf{0}$ (point O in Fig. 2) is equal to $|\zeta_\vartheta(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0)|$. Alternatively, the projection of $\mathcal{B}(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0; \delta_\vartheta)$, for a given $\vartheta \in \mathbb{S}^1$, into $\boldsymbol{\mathcal{X}}$ corresponds to the collection of all points that belong to the intersection of the c-level sets of $\delta_\vartheta(\cdot; \boldsymbol{z}_i^0)$ and $\delta_\vartheta(\cdot; \boldsymbol{z}_j^0)$, that is, the sets $\{\mathrm{x} \in \boldsymbol{\mathcal{X}} : \ \delta_\vartheta(\mathrm{x}; \boldsymbol{z}_i^0) = c\}$ and $\{\mathrm{x} \in \boldsymbol{\mathcal{X}} : \ \delta_\vartheta(\mathrm{x}; \boldsymbol{z}_j^0) = c\}$, respectively, for all $c > 0$. Note that these level sets are ellipsoids centered at $\boldsymbol{\xi}_\vartheta^i$ and $\boldsymbol{\xi}_\vartheta^j$, respectively. It is interesting to note that the projection of $\mathcal{B}(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0; \delta_\vartheta)$ into $\boldsymbol{\mathcal{X}}$ divides the latter into two compact sets (assuming that $\boldsymbol{\mathcal{X}}$ is also compact), namely $\boldsymbol{\mathcal{C}}_{ij}$ and $\boldsymbol{\mathcal{C}}_{ji}$, which have non-overlapping interiors. In particular, the set $\boldsymbol{\mathcal{C}}_{ij}$ consists of all points $\mathrm{x} \in \boldsymbol{\mathcal{X}}$ for which $\delta_\vartheta(\mathrm{x}; \boldsymbol{z}_i^0) \leq \delta_\vartheta(\mathrm{x}; \boldsymbol{z}_j^0)$ whereas $\boldsymbol{\mathcal{C}}_{ji}$ consists of all points $\mathrm{x} \in \boldsymbol{\mathcal{X}}$ for which $\delta_\vartheta(\mathrm{x}; \boldsymbol{z}_i^0) \geq \delta_\vartheta(\mathrm{x}; \boldsymbol{z}_j^0)$. Fig. 2 illustrates the key points of the previous discussion.

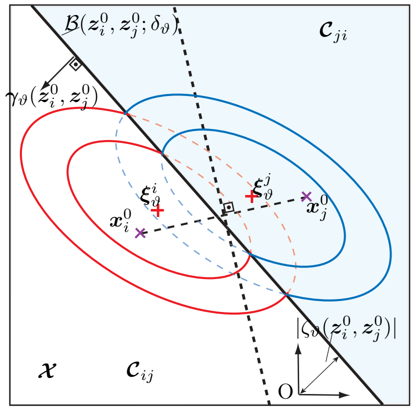


Fig. 2. The projection of the bisector $\mathcal{B}(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0; \delta_\vartheta)$ into $\boldsymbol{\mathcal{X}}$ is a line segment, which is neither (by default) perpendicular to the line segment $[\boldsymbol{x}_i^0, \boldsymbol{x}_j^0]$ (note that the points $\boldsymbol{x}_i^0$ and $\boldsymbol{x}_j^0$ are the projections of $\boldsymbol{z}_i^0$ and $\boldsymbol{z}_j^0$ into $\boldsymbol{\mathcal{X}}$, respectively) nor it passes through the midpoint of the same segment, as does the bisector of $\boldsymbol{x}_i^0$ and $\boldsymbol{x}_j^0$ in a standard Voronoi partition, which is shown in dashed line. It also corresponds to the collection of all points in $\boldsymbol{\mathcal{X}}$ that belong to the intersections of the c-level sets of $\delta_\vartheta(\cdot; \boldsymbol{z}_i^0)$ and $\delta_\vartheta(\cdot; \boldsymbol{z}_j^0)$, for all $c \geq 0$.

In light of the previous discussion, we will now associate the solution to the parametric partitioning problem (Problem 2) with an *affine* partition of $\boldsymbol{\mathcal{X}} \subsetneq \mathbb{R}^2$.

Proposition 2: Let $\vartheta \in \mathbb{S}^1$ be given, and let $\boldsymbol{Z}^0 := \mathrm{col}(\boldsymbol{z}_1^0, \ldots, \boldsymbol{z}_n^0)$, where $\boldsymbol{z}_i^0 = \mathrm{col}(\boldsymbol{q}_i^0, \boldsymbol{v}_i^0)$ and $\boldsymbol{q}_i^0 = \mathrm{col}(\boldsymbol{x}_i^0, \theta_i^0)$. In addition, let $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0) := \{\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0), \ i \in \mathcal{I}_n\}$ denote the solution to Problem 2, when $\eta_i \geq 2\mu$ for all $i \in \mathcal{I}_n$. Then, $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0)$ is an affine partition with combinatorial complexity $\Theta(n)$[1].

Proof: In light of Eq. (13), the bisector $\mathcal{B}(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0; \delta_\vartheta)$, which is the image of the set $\{\mathrm{x} \in \boldsymbol{\mathcal{X}} : \ \mathrm{x}^{\mathrm{T}} \boldsymbol{\gamma}_\vartheta(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0) = \zeta_\vartheta(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0)\}$ under the function $\boldsymbol{z}_\vartheta(\cdot)$, is a straight line that lies in $\boldsymbol{\mathcal{T}}_\vartheta$, for all the pairs of distinct generators $(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0) \in \{\boldsymbol{Z}^0\} \times \{\boldsymbol{Z}^0\}$, $i \neq j$. In addition, for any state $\boldsymbol{z}_\vartheta(\mathrm{x}) \in \boldsymbol{\mathcal{T}}_\vartheta$ that does not belong to the bisector of any pair of distinct generators, there is a unique index $\ell \in \mathcal{I}_n$ such that

[1]We denote by $\Theta(f(n))$ the set of functions $F : \mathbb{Z}_{>0} \to [0, \infty)$ for which there exist c_1, $c_2 > 0$ and $n_0 \in \mathbb{Z}_{>0}$ such that $c_1 f(n) \leq F(n) \leq c_2 f(n)$, for all $n \geq n_0$.

$\delta_\vartheta(\mathrm{x}; \boldsymbol{z}_\ell^0) < \delta_\vartheta(\mathrm{x}; \boldsymbol{z}_k^0)$, for all $k \in \mathcal{I}_n \backslash \{\ell\}$. We conclude that $\mathfrak{V}(\boldsymbol{Z}^0)$ is an affine partition in $\boldsymbol{\mathcal{T}}_\vartheta$. The result on the combinatorial complexity of $\mathfrak{V}(\boldsymbol{Z}^0)$ follows immediately from Theorem 18.2.3 in [36, p. 439]. ■

Note that the partition $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0) := \{\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0),\ i \in \mathcal{I}_n\}$ of $\boldsymbol{\mathcal{T}}_\vartheta$ is homeomorphic, in the topological sense, to a partition $\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0) = \{\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0),\ i \in \mathcal{I}_n\}$ of $\boldsymbol{\mathcal{X}}$ (recall that $\boldsymbol{\mathcal{X}} \sim \boldsymbol{\mathcal{T}}_\vartheta$), where $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0) := \pi_{\boldsymbol{\mathcal{X}}}(\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0))$. Practically, this means that instead of computing $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0)$, we should compute $\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0)$, which is the affine partition generated by the point-set $\{\boldsymbol{\Xi}_\vartheta\} := \{\boldsymbol{\xi}_\vartheta^i \in \mathbb{R}^2,\ i \in \mathcal{I}_n\}$ with respect to the (generalized) proximity metric δ_ϑ. Despite the fact that the set $\{\boldsymbol{\Xi}_\vartheta\}$ is the new set of generators, we will continue writing $\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0)$ instead of $\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{\Xi}_\vartheta)$ in order to emphasize the correspondence between the cells of $\boldsymbol{\mathcal{X}}_\vartheta$ and the agents of the network, which are originally located at the point-set $\{\boldsymbol{Z}^0\} \in \boldsymbol{\mathcal{Z}}$. In our previous work [28], [29], we have proposed partitioning algorithms that allow the agents of a network to compute approximations of the boundaries of their own cells independently from their teammates. The approach proposed in these references suggests that the characterization of the boundary $\mathrm{bd}(\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0))$ of the cell $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0)$ of the i-th agent can be achieved with the application of a bisection search algorithm over a family of rays $\{\Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e}),\ \boldsymbol{e} \in \mathbb{S}^1\}$ that emanate from point $\boldsymbol{\xi}_\vartheta^i$ and cover $\boldsymbol{\mathcal{X}}$. In particular, the goal of the line search algorithm is to find the intersection of the ray $\Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e})$ and the (unknown) boundary $\mathrm{bd}(\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0))$ of $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0)$. In this way, one obtains a convenient parametrization of $\mathrm{bd}(\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0))$ in terms of the unit vector $\boldsymbol{e} \in \mathbb{S}^1$.

Next, we will present the key ideas of the approach presented in [28], [29] by adopting, however, an optimization point of view in lieu of the geometric perspective utilized therein. This more abstract perspective will help us set the scene for the development of a novel *exact* partitioning algorithm, which will be presented in Section III-B. First, we will make the mild assumption[2] that $\{\boldsymbol{\Xi}_\vartheta\} \subsetneq \mathrm{int}(\boldsymbol{\mathcal{X}})$. Next, we consider the following two (mutually exclusive) cases:

Case 1: Point $\boldsymbol{\xi}_\vartheta^i$ does not belong to the interior of $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0)$. In this case, it is not guaranteed that the intersection of the ray $\Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e})$ and the boundary $\mathrm{bd}(\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0))$ will be non-empty for all $\boldsymbol{e} \in \mathbb{S}^1$. If, for some $\boldsymbol{e} \in \mathbb{S}^1$, this intersection is non-empty, then it will either be comprised of two (unknown) points, which are denoted by $\boldsymbol{x}_\vartheta^\triangledown(i;\boldsymbol{e})$ and $\boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e})$, or will correspond to a whole edge of the convex polygon $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0)$, denoted as $[\boldsymbol{x}_\vartheta^\triangledown(i;\boldsymbol{e}), \boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e})]$ (a singular case occurs when $\boldsymbol{x}_\vartheta^\triangledown(i;\boldsymbol{e}) \equiv \boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e})$, which implies that the intersection $\mathrm{bd}(\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0)) \cap \Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e})$ is a singleton and in particular, a vertex of the convex polygon $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0)$). In both of these two subcases, we have that

$$\boldsymbol{x}_\vartheta^\triangledown(i;\boldsymbol{e}) := \boldsymbol{\xi}_\vartheta^i + \underline{\varrho}_\vartheta(i;\boldsymbol{e},\mathcal{I}_n)\boldsymbol{e}, \tag{15a}$$

$$\boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e}) := \boldsymbol{\xi}_\vartheta^i + \overline{\varrho}_\vartheta(i;\boldsymbol{e},\mathcal{I}_n)\boldsymbol{e}, \tag{15b}$$

where

$$\underline{\varrho}_\vartheta(i;\boldsymbol{e},\mathcal{I}_n) := \inf \mathfrak{R}_\vartheta(i;\boldsymbol{e},\mathcal{I}_n), \tag{16a}$$

$$\overline{\varrho}_\vartheta(i;\boldsymbol{e},\mathcal{I}_n) := \sup \mathfrak{R}_\vartheta(i;\boldsymbol{e},\mathcal{I}_n), \tag{16b}$$

$$\mathfrak{R}_\vartheta(i;\boldsymbol{e},\mathcal{I}_n) := \{\varrho \geq 0 :\ \boldsymbol{\xi}_\vartheta^i + \varrho\boldsymbol{e} \in \boldsymbol{\mathcal{X}} \text{ and } \delta_\vartheta(\boldsymbol{\xi}_\vartheta^i + \varrho\boldsymbol{e}; \boldsymbol{z}_i^0) \leq \min_{j\in\mathcal{I}_n} \delta_\vartheta(\boldsymbol{\xi}_\vartheta^i + \varrho\boldsymbol{e}; \boldsymbol{z}_j^0)\}. \tag{16c}$$

In view of (15a)-(16c), it follows that $\boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e})$ (resp., $\boldsymbol{x}_\vartheta^\triangledown(i;\boldsymbol{e})$) enjoys the following two properties: 1) it is the point in the segment of $\Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e})$ contained in $\boldsymbol{\mathcal{X}}$ that is the furthest (resp., nearest) to $\boldsymbol{\xi}_\vartheta^i$, in terms of the Euclidean distance, and 2) it is closer to the i-th agent, in terms of the proximity metric δ_ϑ, than to any other agent from the same network. The characterization of $\boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e})$ and $\boldsymbol{x}_\vartheta^\triangledown(i;\boldsymbol{e})$ is based on the observation that at these two points, the *continuous* function $\mathcal{D}_\vartheta^i(\cdot;\mathcal{I}_n) : \boldsymbol{\mathcal{X}} \to \mathbb{R}$, where

$$\mathcal{D}_\vartheta^i(\mathrm{x};\mathcal{I}_n) := \delta_\vartheta(\mathrm{x};\boldsymbol{z}_i^0) - \min_{j\in\mathcal{I}_n\backslash\{i\}} \delta_\vartheta(\mathrm{x};\boldsymbol{z}_j^0), \tag{17}$$

should change sign as one traverses the ray $\Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e})$ without exiting $\boldsymbol{\mathcal{X}}$ (except from some special cases). In particular, we have that $\mathcal{D}_\vartheta^i(\mathrm{x};\mathcal{I}_n) > 0$ for all $\mathrm{x} \in [\boldsymbol{\xi}_\vartheta^i, \boldsymbol{x}_\vartheta^\triangledown(i;\boldsymbol{e})[$, and $\mathcal{D}_\vartheta^i(\mathrm{x};\mathcal{I}_n) < 0$ for all $\mathrm{x} \in]\boldsymbol{x}_\vartheta^\triangledown(i;\boldsymbol{e}), \boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e})[$. The sign of $\mathcal{D}_\vartheta^i$ will change one more time, if $\boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e}) \neq \boldsymbol{x}'_\vartheta(i;\boldsymbol{e},\boldsymbol{\mathcal{X}})$, where $\boldsymbol{x}'_\vartheta(i;\boldsymbol{e},\boldsymbol{\mathcal{X}})$ is the point of intersection of $\Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e})$ with $\mathrm{bd}(\boldsymbol{\mathcal{X}})$, that is, $\{\boldsymbol{x}'_\vartheta(i;\boldsymbol{e},\boldsymbol{\mathcal{X}})\} = \Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e}) \cap \mathrm{bd}(\boldsymbol{\mathcal{X}})$; in this last case, we have that $\mathcal{D}_\vartheta^i(\mathrm{x};\mathcal{I}_n) > 0$ for all $\mathrm{x} \in]\boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e}), \boldsymbol{x}'_\vartheta(i;\boldsymbol{e},\boldsymbol{\mathcal{X}})]$.

If, on the other hand, the intersection of $\Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e})$ and $\mathrm{bd}(\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0))$ is empty, then $\mathfrak{R}_\vartheta(i;\boldsymbol{e},\mathcal{I}_n) = \varnothing$ and we set $\underline{\varrho}_\vartheta(i;\boldsymbol{e},\mathcal{I}_n) := +\infty$ and $\overline{\varrho}_\vartheta(i;\boldsymbol{e},\mathcal{I}_n) := -\infty$; in addition, both $\boldsymbol{x}_\vartheta^\triangledown(i;\boldsymbol{e})$ and $\boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e})$ will be assigned null values. Intuitively, when the intersection of $\Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e})$ and $\mathrm{bd}(\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0))$ is empty, we have that $\mathcal{D}_\vartheta^i(\mathrm{x};\mathcal{I}_n) > 0$ for all $\mathrm{x} \in \Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e}) \cap \boldsymbol{\mathcal{X}}$, that is, the cost that the i-th agent will incur to reach any state $\boldsymbol{z}_\vartheta(\mathrm{x}) \in \boldsymbol{\mathcal{T}}_\vartheta$ with $\mathrm{x} \in [\boldsymbol{\xi}_\vartheta^i, \boldsymbol{x}'_\vartheta(i;\boldsymbol{e},\boldsymbol{\mathcal{X}})]$ is strictly greater than the cost that will be incurred by at least one different agent from the same network to reach the same state.

Case 2: Point $\boldsymbol{\xi}_\vartheta^i$ is an interior point of $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0)$. In this case, the intersection of $\Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e})$ and $\mathrm{bd}(\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0))$ will be a singleton, namely $\{\boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e})\}$, for all $\boldsymbol{e} \in \mathbb{S}^1$; we also set $\boldsymbol{x}_\vartheta^\triangledown(i;\boldsymbol{e}) := \boldsymbol{\xi}_\vartheta^i$. In this case, $\mathcal{D}_\vartheta^i(\mathrm{x};\mathcal{I}_n) < 0$ for all $\mathrm{x} \in [\boldsymbol{\xi}_\vartheta^i, \boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e})[$ and, if in addition $\boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e}) \neq \boldsymbol{x}'_\vartheta(i;\boldsymbol{e},\boldsymbol{\mathcal{X}})$, then $\mathcal{D}_\vartheta^i(\mathrm{x};\mathcal{I}_n) > 0$ for all $\mathrm{x} \in]\boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e}), \boldsymbol{x}'_\vartheta(i;\boldsymbol{e},\boldsymbol{\mathcal{X}})]$.

The two cases that we previously described are illustrated in Fig. 3. We observe therein that $\boldsymbol{\xi}_\vartheta^i \notin \mathrm{int}(\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0))$, and consequently, for different $\boldsymbol{e} \in \mathbb{S}^1$, the intersection $\mathrm{bd}(\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0)) \cap \Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e})$ will either be empty or consist of two points (except from the singular cases in which the ray $\Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e})$ passes through vertex A or vertex B). By contrast, $\boldsymbol{\xi}_\vartheta^j \in \mathrm{int}(\boldsymbol{\mathcal{X}}_\vartheta^j(\boldsymbol{z}_j^0))$, which means that $\mathrm{bd}(\boldsymbol{\mathcal{X}}_\vartheta^j(\boldsymbol{z}_j^0)) \cap \Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e})$ will be a singleton for all $\boldsymbol{e} \in \mathbb{S}^1$.

As shown in [27]–[29], in which similar classes of partitioning problems were considered, one can utilize simple bisection search algorithms to characterize $\boldsymbol{x}_\vartheta^\triangledown(i;\boldsymbol{e})$ and $\boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e})$. Such algorithms generate sequences of "query" points that will eventually converge within the prescribed error tolerance to the desired, unknown points of interest. Because, as is stressed in [16], there are many practical problems in which a Voronoi-

[2]Note that by removing this rather mild assumption, our analysis will change only but slightly, yet the presentation will become more complex since we will have to discuss separately a list of singular cases that are of low interest in practice.

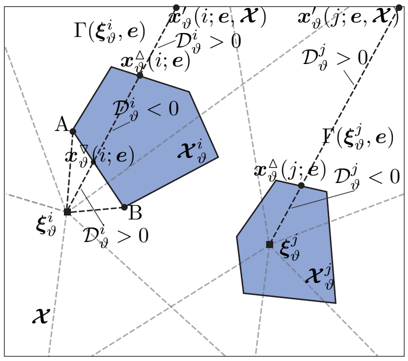


Fig. 3. Point $\boldsymbol{\xi}_\vartheta^j$ belongs to the interior of $\boldsymbol{\mathcal{X}}_\vartheta^j$, which implies that the intersection of the ray $\Gamma(\boldsymbol{\xi}_\vartheta^j, \boldsymbol{e})$ with the boundary $\mathrm{bd}(\boldsymbol{\mathcal{X}}_\vartheta^j)$ of the cell $\boldsymbol{\mathcal{X}}_\vartheta^j$ will be a singleton for all $\boldsymbol{e} \in \mathbb{S}^1$. On the other hand, point $\boldsymbol{\xi}_\vartheta^i$ does not belong to $\boldsymbol{\mathcal{X}}_\vartheta^i$, and in this case, the intersection of $\Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e})$ with $\mathrm{bd}(\boldsymbol{\mathcal{X}}_\vartheta^i)$ can 1) be empty, 2) consist of two points or 3) be a single point (vertex of a cell), namely point A or point B.

like partition of a given set have to be known with high accuracy, the tolerance error for the bisection search algorithms that seek for $\boldsymbol{x}_\vartheta^\triangledown(i; \boldsymbol{e})$ and $\boldsymbol{x}_\vartheta^\vartriangle(i; \boldsymbol{e})$ should be very small or, equivalently, a large number of "query" points have to be generated during the iterative process. It is also possible that the bisection search may end up performing vacuous searches along rays that do not intersect with the boundary of the cell of interest, in which cases $\boldsymbol{x}_\vartheta^\triangledown(i; \boldsymbol{e})$ and $\boldsymbol{x}_\vartheta^\vartriangle(i; \boldsymbol{e})$ should be assigned null values, as we have already explained. Note that during a vacuous search, the bisection search algorithm, which is an exhaustive and consistent algorithm, will generate the maximum number of query points (maximum number of steps) that the prescribed error tolerance dictates before it returns a null output. As we will see next, a careful analysis based on the interpretation of the problems of finding $\boldsymbol{x}_\vartheta^\triangledown(i; \boldsymbol{e})$ and $\boldsymbol{x}_\vartheta^\vartriangle(i; \boldsymbol{e})$ as optimization problems, which we discussed before, will reveal that it is possible to characterize these points exactly at a finite number of steps, which is in sharp contradistinction with the bisection-based algorithm, which can characterize the same points only asymptotically.

B. An Exact and Finite-Steps Algorithm for the Partitioning Problem over $\boldsymbol{\mathcal{T}}_\vartheta$ in the Absence of Communication Constraints

Motivated by the previous discussion, we will next propose a partitioning algorithm that characterizes exactly and in a finite number of steps the boundary points of the i-th cell of the Voronoi-like partition $\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0)$ of $\boldsymbol{\mathcal{X}}$, for any $i \in \mathcal{I}_n$. This algorithm aims at finding the boundary points of the cell $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0)$ by solving the two optimization problems that are described in terms of Eqs. (16a)-(16c). The main idea of the proposed approach lies in the fact that the solutions to the previous optimization problems correspond to two points, namely, $\boldsymbol{x}_\vartheta^\triangledown(i; \boldsymbol{e})$ and $\boldsymbol{x}_\vartheta^\vartriangle(i; \boldsymbol{e})$, at which, as we have mentioned, the sign of a certain continuous function changes as one traverses the ray $\Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e})$ without exiting $\boldsymbol{\mathcal{X}}$ (except from some special cases); in other words, the points $\boldsymbol{x}_\vartheta^\triangledown(i; \boldsymbol{e})$ and $\boldsymbol{x}_\vartheta^\vartriangle(i; \boldsymbol{e})$ should be roots of the latter function. Next we describe the process for finding these two points by leveraging the previous remarks.

First we observe that if, for a given $\boldsymbol{e} \in \mathbb{S}^1$ and $i \in \mathcal{I}_n$, the set $\mathfrak{R}_\vartheta(i; \boldsymbol{e}, \mathcal{I}_n)$, which is defined in Eq. (16c), is non-empty, then $\underline{\varrho}_\vartheta(i; \boldsymbol{e}, \mathcal{I}_n)$ and $\overline{\varrho}_\vartheta(i; \boldsymbol{e}, \mathcal{I}_n)$, which are defined in (16a) and (16b), respectively, will be (finite) non-negative numbers that belong necessarily to the set $\mathcal{R}_\vartheta(i; \boldsymbol{e}, \mathcal{I}_n) \cup \{|\boldsymbol{x}'_\vartheta(i; \boldsymbol{e}, \boldsymbol{\mathcal{X}}) - \boldsymbol{\xi}_\vartheta^i|\}$, where $|\boldsymbol{x}'_\vartheta(i; \boldsymbol{e}, \boldsymbol{\mathcal{X}}) - \boldsymbol{\xi}_\vartheta^i|$ is equal to the length of the line segment that corresponds to the restriction of $\Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e})$ in $\boldsymbol{\mathcal{X}}$ (as we have already mentioned, $\boldsymbol{x}'_\vartheta(i; \boldsymbol{e}, \boldsymbol{\mathcal{X}})$ denotes the point of intersection of $\Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e})$ and $\mathrm{bd}(\boldsymbol{\mathcal{X}})$) and $\mathcal{R}_\vartheta(i; \boldsymbol{e}, \mathcal{I}_n)$ denotes the set of non-negative numbers that satisfy the following equation (in ϱ):

$$\delta_\vartheta(\boldsymbol{\xi}_\vartheta^i + \varrho \boldsymbol{e}; \boldsymbol{z}_i^0) = \delta_\vartheta(\boldsymbol{\xi}_\vartheta^i + \varrho \boldsymbol{e}; \boldsymbol{z}_j^0), \tag{18}$$

for all $j \in \mathcal{I}_n \backslash \{i\}$; we denote each of these solutions by $\varrho_\vartheta(i; j, \boldsymbol{e})$ and thus, $\mathcal{R}_\vartheta(i; \boldsymbol{e}, \mathcal{I}_n) := \{\varrho_\vartheta(i; k, \boldsymbol{e}) \in \mathbb{R}_{\geq 0},\ k \in \mathcal{I}_n \backslash \{i\}\}$. Note that, in view of (18), all points $\boldsymbol{y} \in \boldsymbol{\mathcal{X}}$ with $\boldsymbol{y} = \boldsymbol{\xi}_\vartheta^i + \varrho \boldsymbol{e}$, where $\rho \in \mathcal{R}_\vartheta(i; \boldsymbol{e}, \mathcal{I}_n)$, will also belong to the bisector $\mathcal{B}(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0; \delta_\vartheta)$ that corresponds to the pair $(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0) \in \{\boldsymbol{Z}^0\} \times \{\boldsymbol{Z}^0\}$. Equation (18) can also be written as follows:

$$\begin{aligned} \varrho^2 |\boldsymbol{\Pi}^{\frac{1}{2}} \boldsymbol{e}|^2 + \mu_\vartheta^i = \varrho^2 |\boldsymbol{\Pi}^{\frac{1}{2}} \boldsymbol{e}|^2 + |\boldsymbol{\Pi}^{\frac{1}{2}} (\boldsymbol{\xi}_\vartheta^i - \boldsymbol{\xi}_\vartheta^j)|^2 \\ + 2\varrho (\boldsymbol{\xi}_\vartheta^i - \boldsymbol{\xi}_\vartheta^j)^{\mathrm{T}} \boldsymbol{\Pi} \boldsymbol{e} + \mu_\vartheta^j, \end{aligned} \tag{19}$$

from which it follows that $\varrho = \varrho_\vartheta(i; j, \boldsymbol{e})$ satisfies the following linear equation:

$$\alpha_\vartheta(i; j, \boldsymbol{e}) \varrho + \beta_\vartheta(i, j) = 0, \quad j \in \mathcal{I}_n \backslash \{i\}, \tag{20}$$

where $\alpha_\vartheta(i; j, \boldsymbol{e}) := 2(\boldsymbol{\xi}_\vartheta^i - \boldsymbol{\xi}_\vartheta^j)^{\mathrm{T}} \boldsymbol{\Pi} \boldsymbol{e}$ and $\beta_\vartheta(i, j) := \mu_\vartheta^j - \mu_\vartheta^i + |\boldsymbol{\Pi}^{\frac{1}{2}} (\boldsymbol{\xi}_\vartheta^i - \boldsymbol{\xi}_\vartheta^j)|^2$. Therefore,

$$\varrho_\vartheta(i; j, \boldsymbol{e}) = \frac{\mu_\vartheta^i - \mu_\vartheta^j - |\boldsymbol{\Pi}^{\frac{1}{2}} (\boldsymbol{\xi}_\vartheta^i - \boldsymbol{\xi}_\vartheta^j)|^2}{2(\boldsymbol{\xi}_\vartheta^i - \boldsymbol{\xi}_\vartheta^j)^{\mathrm{T}} \boldsymbol{\Pi} \boldsymbol{e}}, \tag{21}$$

provided that the vector $\boldsymbol{\Pi}(\boldsymbol{\xi}_\vartheta^i - \boldsymbol{\xi}_\vartheta^j)$ is not orthogonal to the vector $\boldsymbol{e}$; otherwise, in the definition of the set $\mathcal{R}_\vartheta(i; \boldsymbol{e}, \mathcal{I}_n)$ we should replace $\mathcal{I}_n \backslash \{i\}$ with $\mathcal{I}_n \backslash \{i, j\}$. Note that the vector $\boldsymbol{\Pi}(\boldsymbol{\xi}_\vartheta^i - \boldsymbol{\xi}_\vartheta^j)$ is orthogonal to $\boldsymbol{e}$ if, and only if, the vector $\boldsymbol{\gamma}_\vartheta(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0)$, which is defined in (14a), is orthogonal to $\boldsymbol{e}$. Since the vector $\boldsymbol{\gamma}_\vartheta(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0)$ is orthogonal to the bisector $\mathcal{B}(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0; \delta_\vartheta)$ (refer to the discussion following Proposition 1), we conclude (assuming $\boldsymbol{\xi}_\vartheta^i \neq \boldsymbol{\xi}_\vartheta^j$) that $\boldsymbol{\Pi}(\boldsymbol{\xi}_\vartheta^i - \boldsymbol{\xi}_\vartheta^j)$ is orthogonal to the vector $\boldsymbol{e}$ if, and only if, $\boldsymbol{e}$ is parallel to the bisector $\mathcal{B}(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0; \delta_\vartheta)$. Obviously, in the latter case the intersection of the ray $\Gamma(\boldsymbol{\xi}_\vartheta^i, \boldsymbol{e})$ and the bisector $\mathcal{B}(\boldsymbol{z}_i^0, \boldsymbol{z}_j^0; \delta_\vartheta)$ will be empty, and consequently, Eq. (18) will have no solution.

In addition, we should remove from $\mathcal{I}_n$ each index j for which either $\varrho_\vartheta(i; j, \boldsymbol{e}) \leq 0$ or the corresponding point $\boldsymbol{y} = \boldsymbol{\xi}_\vartheta^i + \varrho_\vartheta(i; j, \boldsymbol{e}) \boldsymbol{e}$ does not belong to the interior of $\boldsymbol{\mathcal{X}}$, that is, $\varrho_\vartheta(i; j, \boldsymbol{e}) \geq |\boldsymbol{x}'_\vartheta(i; \boldsymbol{e}, \boldsymbol{\mathcal{X}}) - \boldsymbol{\xi}_\vartheta^i|$. Finally, if there exists a non-empty set $\mathcal{J}$, where $\mathcal{J} \subsetneq \mathcal{I}_n \backslash \{i, j\}$, such that $\varrho_\vartheta(i; j, \mathcal{I}_n) = \varrho_\vartheta(i; \ell, \mathcal{I}_n)$ for all $\ell \in \mathcal{J}$, then all the indices in $\mathcal{J}$ should be removed from $\mathcal{I}_n$, to avoid any duplicates. After removing all these indices, which we refer to as *inadmissible*, we obtain an index set $\mathcal{I}'_n = \mathcal{I}'_n(i; \boldsymbol{e}, \vartheta)$ (henceforth, we will simply write $\mathcal{I}'_n$ to avoid the notational clutter), which is a subset of $\mathcal{I}_n \backslash \{i\}$. Let us assume that $\mathcal{I}'_n$ contains $N = N(i; \boldsymbol{e}, \vartheta)$ elements with $1 \leq N \leq n - 1$ (later on, we will separately discuss the special case in which $\mathcal{I}'_n = \varnothing$ or, equivalently, $N = 0$ when necessary). We subsequently associate the index set $\mathcal{I}_N = \mathcal{I}_N(i, \boldsymbol{e}, \vartheta) := \{1, \ldots, N\}$ to the index-set $\{j_1, \ldots, j_N\}$, which in turn corresponds to the permutation of $\mathcal{I}'_n$ with

$$0 < \varrho_\vartheta(i; j_1, \boldsymbol{e}) < \cdots < \varrho_\vartheta(i; j_N, \boldsymbol{e}) < |\boldsymbol{x}'_\vartheta(i; \boldsymbol{e}, \boldsymbol{\mathcal{X}}) - \boldsymbol{\xi}_\vartheta^i|,$$

where $\mathrm{index}(\cdot; i)$ is a bijective mapping from $\mathcal{I}_N$ to $\mathcal{I}'_n$ such that $j_k := \mathrm{index}(k; i)$, for $k \in \mathcal{I}_N$. Furthermore, to each $\varrho_\vartheta(i; j_k, \boldsymbol{e})$ with $k \in \mathcal{I}_N$, we associate the following point:

$$\boldsymbol{x}_\vartheta(i; j_k, \boldsymbol{e}) := \boldsymbol{\xi}^i_\vartheta + \varrho_\vartheta(i; j_k, \boldsymbol{e})\boldsymbol{e}, \quad k \in \mathcal{I}_N. \tag{22}$$

In addition, for a given $\boldsymbol{e} \in \mathbb{S}^1$, we define the function $\Delta^i_\vartheta(\cdot; \boldsymbol{e}) : \Gamma(\boldsymbol{\xi}^i_\vartheta, \boldsymbol{e}) \cap \boldsymbol{\mathcal{X}} \to \mathbb{R}$, where

$$\Delta^i_\vartheta(\mathbf{x}; \boldsymbol{e}) := \delta_\vartheta(\mathbf{x}; \boldsymbol{z}^0_i) - \min_{k \in \mathcal{I}'_n} \delta_\vartheta(\mathbf{x}; \boldsymbol{z}^0_k). \tag{23}$$

A key observation that our algorithm will exploit is that $\Delta^i_\vartheta(\cdot; \boldsymbol{e})$ is a continuous function whose sign is preserved over each of the following line segments: $]\boldsymbol{\xi}^i_\vartheta, \boldsymbol{x}_\vartheta(i; j_1, \boldsymbol{e})[$ and $]\boldsymbol{x}_\vartheta(i; j_k, \boldsymbol{e}), \boldsymbol{x}_\vartheta(i; j_{k+1}, \boldsymbol{e})[$, for all $k \in \mathcal{I}_N \cap [1, N-1]$, and $]\boldsymbol{x}_\vartheta(i; j_N, \boldsymbol{e}), \boldsymbol{x}'_\vartheta(i; \boldsymbol{e}, \boldsymbol{\mathcal{X}})[$. Note that the endpoints of the previous intervals correspond to candidate roots of the equation $\Delta^i_\vartheta = 0$. For this reason together with the continuity of Δ^i_ϑ, it follows that the sign of Δ^i_ϑ does not change over each of these intervals (sign changes can occur only at the endpoints of the previous intervals that are roots of the equation $\Delta^i_\vartheta = 0$). Let us now consider the following two cases[3]:

Case 1: If $\Delta^i_\vartheta(\mathbf{x}; \boldsymbol{e}) \geq 0$ for any $\mathbf{x} \in]\boldsymbol{\xi}^i_\vartheta, \boldsymbol{x}_\vartheta(i; j_1, \boldsymbol{e})[$ (and thus, as we have already explained, for all $\boldsymbol{x} \in]\boldsymbol{\xi}^i_\vartheta, \boldsymbol{x}_\vartheta(i; j_1, \boldsymbol{e})[$, in view of the continuity of Δ^i_ϑ), then $\boldsymbol{\xi}^i_\vartheta$ will not belong to the interior of $\boldsymbol{\mathcal{X}}^i_\vartheta$. Thus, it is possible that the intersection of Γ and $\mathrm{bd}(\boldsymbol{\mathcal{X}}^i_\vartheta)$ will be empty in which case both $\boldsymbol{x}^\triangledown_\vartheta$ and $\boldsymbol{x}^\vartriangle_\vartheta$ will be assigned null values. Note that in this last case, $\Delta^i_\vartheta(\mathbf{x}; \boldsymbol{e}) > 0$ for all $\mathbf{x} \in]\boldsymbol{\xi}^i_\vartheta, \boldsymbol{x}'_\vartheta[$.

If the intersection of Γ and $\mathrm{bd}(\boldsymbol{\mathcal{X}}^i_\vartheta)$ is non-empty, or equivalently, it is not true that $\Delta^i_\vartheta(\mathbf{x}; \boldsymbol{e}) > 0$ for all $\mathbf{x} \in]\boldsymbol{\xi}^i_\vartheta, \boldsymbol{x}'_\vartheta[$, then this intersection will either consist of two points, namely $\boldsymbol{x}^\triangledown_\vartheta$ and $\boldsymbol{x}^\vartriangle_\vartheta$, or it will correspond to a whole edge of $\boldsymbol{\mathcal{X}}^i_\vartheta$ that is denoted by $[\boldsymbol{x}^\triangledown_\vartheta, \boldsymbol{x}^\vartriangle_\vartheta]$. (When $\boldsymbol{x}^\triangledown_\vartheta \equiv \boldsymbol{x}^\vartriangle_\vartheta$, we have a singular case in which the previous edge is condensed to a single vertex of $\boldsymbol{\mathcal{X}}^i_\vartheta$). Next, we describe how to characterize $\boldsymbol{x}^\triangledown_\vartheta$ and $\boldsymbol{x}^\vartriangle_\vartheta$. In particular, if $k \in \mathcal{I}_N \cap [1, N-1]$, is the smallest index for which $\Delta^i_\vartheta(\mathbf{x}; \boldsymbol{e}) < 0$ for any $\mathbf{x} \in]\boldsymbol{x}_\vartheta(i; j_k, \boldsymbol{e}), \boldsymbol{x}_\vartheta(i; j_{k+1}, \boldsymbol{e})[$, then we set $j^\triangledown := j_k$. If there is no such index k, the only possibility is that $\Delta^i_\vartheta(\mathbf{x}; \boldsymbol{e}) < 0$ for any $\mathbf{x} \in]\boldsymbol{x}_\vartheta(i; j_N, \boldsymbol{e}), \boldsymbol{x}'_\vartheta[$, in which case we set $j^\triangledown(i; \boldsymbol{e}) := j_N$ and $\boldsymbol{x}^\triangledown_\vartheta := \boldsymbol{x}_\vartheta(i; j^\triangledown(i; \boldsymbol{e}), \boldsymbol{e})$.

We continue with the characterization of $\boldsymbol{x}^\vartriangle_\vartheta$. To this aim, we seek for the smallest index $\ell \in \mathcal{I}_N \cap [k, N-1]$ such that $\Delta^i_\vartheta(\mathbf{x}; \boldsymbol{e}) > 0$ for any (and thus for all) $\mathbf{x} \in]\boldsymbol{x}_\vartheta(i; j_\ell, \boldsymbol{e}), \boldsymbol{x}_\vartheta(i; j_{\ell+1}, \boldsymbol{e})[$. If such index ℓ exists, we set $j^\vartriangle(i; \boldsymbol{e}) := j_\ell$ and $\boldsymbol{x}^\vartriangle_\vartheta := \boldsymbol{x}_\vartheta(i; j^\vartriangle(i; \boldsymbol{e}), \boldsymbol{e})$. Otherwise, there are two possibilities. The first one is that $\Delta^i_\vartheta(\mathbf{x}; \boldsymbol{e}) > 0$ for any $\mathbf{x} \in]\boldsymbol{x}_\vartheta(i; j_N, \boldsymbol{e}), \boldsymbol{x}'_\vartheta[$, in which case we set $j^\vartriangle(i; \boldsymbol{e}) := j_N$ and $\boldsymbol{x}^\vartriangle_\vartheta := \boldsymbol{x}_\vartheta(i; j^\vartriangle(i; \boldsymbol{e}), \boldsymbol{e})$. The second one is that $\Delta^i_\vartheta(\mathbf{x}; \boldsymbol{e}) < 0$ for any $\mathbf{x} \in]\boldsymbol{x}^\triangledown_\vartheta, \boldsymbol{x}'_\vartheta[$, in which case we set $\boldsymbol{x}^\vartriangle_\vartheta := \boldsymbol{x}'_\vartheta$.

Case 2: If, on the other hand, $\Delta^i_\vartheta(\mathbf{x}; \boldsymbol{e}) < 0$ for any (and thus for all) $\mathbf{x} \in [\boldsymbol{\xi}^i_\vartheta, \boldsymbol{x}_\vartheta(i; j_1, \boldsymbol{e})[$, then $\boldsymbol{\xi}^i_\vartheta$ will belong to the interior of $\boldsymbol{\mathcal{X}}^i_\vartheta$. In this case, we set $\boldsymbol{x}^\triangledown_\vartheta := \boldsymbol{\xi}^i_\vartheta$ and in order to find $\boldsymbol{x}^\vartriangle_\vartheta$, we need to find first the smallest index $k \in \mathcal{I}_N \cap [1, N-1]$ such that $\Delta^i_\vartheta(\mathbf{x}; \boldsymbol{e}) > 0$ for any $\mathbf{x} \in]\boldsymbol{x}_\vartheta(i; j_k, \boldsymbol{e}), \boldsymbol{x}_\vartheta(i; j_{k+1}, \boldsymbol{e})[$. If such k exists, we set $j^\vartriangle(i; \boldsymbol{e}) := j_k$ and $\boldsymbol{x}^\vartriangle_\vartheta := \boldsymbol{x}_\vartheta(i; j^\vartriangle(i; \boldsymbol{e}), \boldsymbol{e})$. Otherwise, we check if $\Delta^i_\vartheta(\mathbf{x}; \boldsymbol{e}) > 0$ for any $\mathbf{x} \in]\boldsymbol{x}_\vartheta(i; j_N, \boldsymbol{e}), \boldsymbol{x}'_\vartheta[$, in which case we set $j^\vartriangle(i; \boldsymbol{e}) := j_N$ and $\boldsymbol{x}^\vartriangle_\vartheta := \boldsymbol{x}_\vartheta(i; j^\vartriangle(i; \boldsymbol{e}), \boldsymbol{e})$. Finally, if $\Delta^i_\vartheta(\mathbf{x}; \boldsymbol{e}) < 0$ for any $\mathbf{x} \in]\boldsymbol{\xi}^i_\vartheta, \boldsymbol{x}'_\vartheta[$, we set $\boldsymbol{x}^\vartriangle_\vartheta := \boldsymbol{x}'_\vartheta$.

In the special case when $\mathcal{I}'_n = \varnothing$, we consider again two cases. Specifically, if it holds true that

$$\mu^i_\vartheta = \delta(\boldsymbol{\xi}^i_\vartheta; \boldsymbol{z}^0_i) < \min_{\ell \in \mathcal{I}_n \setminus \{i\}} \delta(\boldsymbol{\xi}^i_\vartheta; \boldsymbol{z}^0_\ell),$$

then $\boldsymbol{x}^\triangledown_\vartheta := \boldsymbol{\xi}^i_\vartheta$ and $\boldsymbol{x}^\vartriangle_\vartheta = \boldsymbol{x}'_\vartheta$. Otherwise, both $\boldsymbol{x}^\vartriangle_\vartheta$ and $\boldsymbol{x}^\triangledown_\vartheta$ will be assigned null values.

The pseudo-code of the algorithm that we just described is given in Algorithm 1. The outputs of Algorithm 1 are the two points, $\boldsymbol{x}^\triangledown_\vartheta(i; \boldsymbol{e})$ and $\boldsymbol{x}^\vartriangle_\vartheta(i; \boldsymbol{e})$, for a given $i \in \mathcal{I}_n$ and a given unit vector $\boldsymbol{e}$. To obtain a polygonal approximation of the boundary $\mathrm{bd}(\boldsymbol{\mathcal{X}}^i_\vartheta(\boldsymbol{z}^0_i))$ of the cell $\boldsymbol{\mathcal{X}}^i_\vartheta(\boldsymbol{z}^0_i)$, we will utilize a finite grid $\mathbb{E}$ over $\mathbb{S}^1$, where $\mathbb{E} := \{\boldsymbol{e}_k,\ k \in \mathbb{Z}_{\geq 0} \cap [1, \overline{k}]\}$

Algorithm 1 Exact Line Search Algorithm

procedure EXLSEARCH
Input data: $\boldsymbol{Z}^0$, ϑ
Input variables: i, $\boldsymbol{e}$, $\mathcal{I}_n$
Output variables: $\boldsymbol{x}^\triangledown_\vartheta(i; \boldsymbol{e})$, $\boldsymbol{x}^\vartriangle_\vartheta(i; \boldsymbol{e})$
$\mathcal{I}'_n \leftarrow \mathcal{I}_n \setminus \{i\}$
  for $j \in \mathcal{I}'_n$ **do**
    if j is inadmissible **then**
      $\mathcal{I}'_n \leftarrow \mathcal{I}'_n \setminus \{j\}$
  $\mathcal{I}_N := \{1, \ldots, \mathrm{length}(\mathcal{I}'_n)\}$
  if $\mathcal{I}_N = \varnothing$ **then**
    if $\mu^i_\vartheta < \min_{\ell \in \mathcal{I}_n \setminus \{i\}} \delta_\vartheta(\boldsymbol{\xi}^i_\vartheta; \boldsymbol{z}^0_\ell)$ **then**
      $\boldsymbol{x}^\triangledown_\vartheta(i; \boldsymbol{e}) \leftarrow \boldsymbol{\xi}^i_\vartheta$; $\boldsymbol{x}^\vartriangle_\vartheta(i; \boldsymbol{e}) \leftarrow \boldsymbol{x}'_\vartheta(i; \boldsymbol{e}, \boldsymbol{\mathcal{X}})$
    else
      $\boldsymbol{x}^\triangledown_\vartheta(i; \boldsymbol{e}) \leftarrow$ null; $\boldsymbol{x}^\vartriangle_\vartheta(i; \boldsymbol{e}) \leftarrow$ null
    return
  for $\ell \in \mathcal{I}_N$ **do**
    $j_\ell \leftarrow \mathrm{index}(\ell; i)$
    compute $\boldsymbol{x}_\vartheta(i; j_\ell, \boldsymbol{e})$ via (22)
    if $\ell = 1$ **then**
      $\mathbf{x} \leftarrow (\boldsymbol{x}_\vartheta(i; j_1, \boldsymbol{e}) + \boldsymbol{\xi}^i_\vartheta)/2$
    else
      $\mathbf{x} \leftarrow (\boldsymbol{x}_\vartheta(i; j_\ell, \boldsymbol{e}) + \boldsymbol{x}_\vartheta(i; j_{\ell-1}, \boldsymbol{e}))/2$
    $\mathfrak{D}^i_\ell \leftarrow \Delta^i_\vartheta(\mathbf{x}; \boldsymbol{e})$
  $\mathfrak{D}^i_{N+1} \leftarrow \Delta^i_\vartheta((\boldsymbol{x}_\vartheta(i; j_N, \boldsymbol{e}) + \boldsymbol{x}'_\vartheta(i; \boldsymbol{e}, \boldsymbol{\mathcal{X}}))/2; \boldsymbol{e})$
  if $\mathfrak{D}^i_1 < 0$ **then**
    $\boldsymbol{x}^\triangledown_\vartheta(i; \boldsymbol{e}) \leftarrow \boldsymbol{\xi}^i_\vartheta$
    $j^\vartriangle \leftarrow \min\{\ell \in \mathcal{I}_N \cup \{N+1\} :\ \mathfrak{D}^i_\ell > 0\} - 1$
    if $j^\vartriangle \neq$ null **then**
      $\boldsymbol{x}^\vartriangle_\vartheta(i; \boldsymbol{e}) \leftarrow \boldsymbol{x}_\vartheta(i; \mathrm{index}(j^\vartriangle; i), \boldsymbol{e})$
    else
      $\boldsymbol{x}^\vartriangle_\vartheta(i; \boldsymbol{e}) \leftarrow \boldsymbol{x}'_\vartheta(i; \boldsymbol{e}, \boldsymbol{\mathcal{X}})$
  else
    $j^\triangledown \leftarrow \min\{\ell \in \cup\{N+1\} :\ \mathfrak{D}^i_\ell < 0\} - 1$
    if $j^\triangledown \neq$ null **then**
      $\boldsymbol{x}^\triangledown_\vartheta(i; \boldsymbol{e}) \leftarrow \boldsymbol{x}_\vartheta(i; \mathrm{index}(j^\triangledown; i), \boldsymbol{e})$
      $j^\vartriangle \leftarrow \min\{\ell \in \mathcal{I}_N \cap [j^\triangledown, N] :\ \mathfrak{D}^i_{\ell+1} > 0\}$
      if $j^\vartriangle \neq$ null **then**
        $\boldsymbol{x}^\vartriangle_\vartheta(i; \boldsymbol{e}) \leftarrow \boldsymbol{x}_\vartheta(i; \mathrm{index}(j^\vartriangle; i), \boldsymbol{e})$
      else
        $\boldsymbol{x}^\vartriangle_\vartheta(i; \boldsymbol{e}) \leftarrow \boldsymbol{x}'_\vartheta(i; \boldsymbol{e}, \boldsymbol{\mathcal{X}})$
    else
      $\boldsymbol{x}^\triangledown_\vartheta(i; \boldsymbol{e}) \leftarrow$ null; $\boldsymbol{x}^\vartriangle_\vartheta(i; \boldsymbol{e}) \leftarrow$ null

[3]In the discussion of these two cases and in order to avoid the notational clutter, we will remove the arguments from the following variables: $\boldsymbol{x}^\triangledown_\vartheta$, $\boldsymbol{x}^\vartriangle_\vartheta$, $\boldsymbol{x}'_\vartheta$, $\boldsymbol{\mathcal{X}}^i_\vartheta$ and Γ, given that both i, $\boldsymbol{\xi}^i_\vartheta$ and $\boldsymbol{e}$ will be fixed.

and $\overline{k}$ is a positive integer (design parameter). The idea is to characterize $\boldsymbol{x}_\vartheta^\triangledown(i;\boldsymbol{e})$ and $\boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e})$ for each $\boldsymbol{e} \in \mathbb{E}$ and remove all points $\boldsymbol{x}_\vartheta^\triangledown(i;\boldsymbol{e})$ that coincide with $\boldsymbol{\xi}_\vartheta^i$. The implementation details of this idea are given in Algorithm 2. Note that the polygonal approximation of the cell $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0)$ obtained in this way is exact at $\boldsymbol{x}_\vartheta^\triangledown(i;\boldsymbol{e})$ and $\boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e})$ for each $\boldsymbol{e} \in \mathbb{E}$ and is computed in a finite number of steps, in contrast with our previous work [27]–[29], in which these boundary points were only characterized asymptotically via a bisection search algorithm.

Algorithm 2 Independent Computation of a Cell by its Associated Agent

procedure CELLCOMP
Input data: $\boldsymbol{Z}^0$, ϑ
Input variables: i, $\mathbb{E}$, $\mathcal{I}_n$
Output variables: $\mathrm{bd}(\boldsymbol{\mathcal{X}}_\vartheta^i)$
  $\boldsymbol{S}_0 \leftarrow \varnothing$
  for each $k \in \{1, \ldots, \overline{k}\}$ **do**
    $\boldsymbol{e} \leftarrow \boldsymbol{e}_k$
    $\{\boldsymbol{x}_\vartheta^\triangledown(i;\boldsymbol{e}), \boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e})\} =$ EXLSEARCH$(i, \boldsymbol{e}_k, \mathcal{I}_n; \boldsymbol{Z}^0, \vartheta)$
    $\boldsymbol{S}_k \leftarrow \boldsymbol{S}_{k-1}$
    if $\boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e}) \neq \mathrm{null}$ **then**
      $\boldsymbol{S}_k \leftarrow \boldsymbol{S}_k \cup \{\boldsymbol{x}_\vartheta^\vartriangle(i;\boldsymbol{e})\}$
      if $\boldsymbol{x}_\vartheta^\triangledown(i;\boldsymbol{e}) \neq \boldsymbol{\xi}_\vartheta^i$ **then**
        $\boldsymbol{S}_k \leftarrow \boldsymbol{S}_k \cup \{\boldsymbol{x}_\vartheta^\triangledown(i;\boldsymbol{e})\}$
  $\mathrm{bd}(\boldsymbol{\mathcal{X}}_\vartheta^i) \leftarrow \boldsymbol{S}_{\overline{k}}$

C. Analysis of the Partitioning Problem in the Presence of Communication Constraints

Next, we present a distributed algorithm that solves Problem 2 in the presence of communication constraints, that is, when it is not necessarily true that $\eta_i \geq 2\mu$, for all $i \in \mathcal{I}_n$[4]. In this case, the i-th agent of the network may not be able to correctly characterize the minimum or the maximum of $\mathfrak{R}_\vartheta(i;\boldsymbol{e},\mathcal{I}_n)$ given that it may not be in position to exchange information with all the other agents. This in turn would imply that the i-th agent may not be able to correctly compute the boundary points of its own cell. We will henceforth denote by $\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0;\boldsymbol{H})$, where $\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0;\boldsymbol{H}) := \{\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i), i \in \mathcal{I}_n\}$ and $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i) := \pi_{\boldsymbol{\mathcal{X}}}(\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i))$ for $i \in \mathcal{I}_n$, the partition of $\boldsymbol{\mathcal{X}}$ for a given $\vartheta \in \mathbb{S}^1$ in the presence of communication constraints. As before, the cells of $\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0;\boldsymbol{H})$ are homeomorphic, in the topological sense, to the cells of $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0;\boldsymbol{H})$. It should be noted that the i-th agent is now confined to search for the minimum and the maximum of the following set:

$$\mathfrak{R}_\vartheta(i;\boldsymbol{e},\mathcal{N}_c(i,\eta_i)) := \{\varrho \in \mathbb{R}_{\geq 0} : \ \boldsymbol{\xi}_\vartheta^i + \varrho\boldsymbol{e} \in \boldsymbol{\mathcal{X}} \quad \text{and} \quad \delta_\vartheta(\boldsymbol{\xi}_\vartheta^i + \varrho\boldsymbol{e};\boldsymbol{z}_i^0) \leq \min_{j \in \mathcal{N}_c(i,\eta_i)} \delta_\vartheta(\boldsymbol{\xi}_\vartheta^j + \varrho\boldsymbol{e};\boldsymbol{z}_j^0)\},$$

in lieu of $\mathfrak{R}_\vartheta(i;\boldsymbol{e},\mathcal{I}_n)$, which is defined in (16c). Now, let us henceforth denote by $\mathcal{N}(i;\mathfrak{V}_\vartheta(\boldsymbol{Z}^0))$ the index set of the neighbors of the i-th agent in the topology of the partition $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0)$ when $\eta_i \geq 2\mu$ (no communication constraints). Note that the neighbors of the i-th agent in the topology of $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0)$ are the agents whose cells share a common edge with $\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0)$, which is a convex polygonal cell.

Before we proceed any further, it is important to stress that the outcome of Algorithm 2 would remain the same, even if we have replaced $\boldsymbol{Z}^0$ (input data) with the joint state vector that corresponds to the concatenation of the initial states of the i-th agent and its neighbors in the topology of the partition $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0)$. As we have previously explained, the neighbors of the i-th agent are the only agents involved in the necessary computations for the characterization of its cell, $\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0)$. Because of this fact, one can claim that Algorithm 2 can be easily implemented in a distributed way. The important nuance here is that the i-th agent is in no position to know its neighbors without having computed its own cell first; however, an agent cannot compute its own cell in a distributed way without knowing its neighbors, which leads us to a cyclic argument. Therefore, in the presence of communication constraints (in which case it is not necessarily true that $\eta_i \geq 2\mu$), the i-th agent will have to adjust its communication range, η_i, so that it includes at least the agents that correspond to its neighbors in the topology of $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0)$ (which is the partition that solves Problem 2, when $\eta_i \geq 2\mu$ for all $i \in \mathcal{I}_n$). The objective of the i-th agent is to use a communication range $\eta_i \in [0, 2\mu]$ that strikes a balance between being sufficiently large to allow it to communicate with its neighbors and being as small as possible in order to keep the incurred communication cost low. In other words, the i-th agent is seeking for a communication range $\eta_i \in [0, 2\mu]$ such that $\mathcal{N}_c(i,\eta_i) \supseteq \mathcal{N}(i;\mathfrak{V}_\vartheta(\boldsymbol{Z}^0))$, which in turn implies that $\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i) = \mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0)$, as we show next.

Proposition 3: Let $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0) := \{\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0), \ i \in \mathcal{I}_n\}$ denote the Voronoi-like partition that solves Problem 2, when $\eta_i \geq 2\mu$ for all $i \in \mathcal{I}_n$ (absence of communication constraints). If, for a given $i \in \mathcal{I}_n$, there exists $\eta_i^\star \in]0, 2\mu]$ such that $\mathcal{N}_c(i,\eta_i^\star) \supseteq \mathcal{N}(i;\mathfrak{V}_\vartheta(\boldsymbol{Z}^0))$, then $\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i) = \mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0)$, for all $\eta_i \geq \eta_i^\star$.

Proof: It suffices to show that

$$\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i) := \pi_{\boldsymbol{\mathcal{X}}}(\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i)) = \pi_{\boldsymbol{\mathcal{X}}}(\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0)) =: \boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0),$$

for any $\eta_i \geq \eta_i^\star$. By its definition, every cell of the affine partition $\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0)$ can be written as the intersection of $n-1$ closed half-spaces confined in $\boldsymbol{\mathcal{X}}$, which is by hypothesis a compact and convex polygonal set. In particular, the cell $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0)$, $i \in \mathcal{I}_n$, can be written as follows [36]: $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0) = \cap_{j \in \mathcal{I}_n \backslash \{i\}} \boldsymbol{\mathcal{C}}_{ij}$, where $\boldsymbol{\mathcal{C}}_{ij} := \{\mathrm{x} \in \boldsymbol{\mathcal{X}} : \delta_\vartheta(\mathrm{x};\boldsymbol{z}_i^0) \leq \delta_\vartheta(\mathrm{x};\boldsymbol{z}_j^0)\}$ for $j \in \mathcal{I}_n \backslash \{i\}$. We conclude that $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0)$ is a convex polygon which may only share its edges with its neighboring cells, which implies that $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0) = \cap_{j \in \mathcal{N}(i;\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0))} \boldsymbol{\mathcal{C}}_{ij}$, where $\mathcal{N}(i;\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0)) = \mathcal{N}(i;\mathfrak{V}_\vartheta(\boldsymbol{Z}^0))$. Therefore,

$$\cap_{j \in \mathcal{I}_n \backslash \{i\}} \boldsymbol{\mathcal{C}}_{ij} = \cap_{j \in \mathcal{J}} \boldsymbol{\mathcal{C}}_{ij} = \cap_{j \in \mathcal{N}(i;\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0))} \boldsymbol{\mathcal{C}}_{ij},$$

for any index-set $\mathcal{J}$ with $\mathcal{N}(i;\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0)) \subseteq \mathcal{J} \subseteq \mathcal{I}_n \backslash \{i\}$. The result follows readily by taking $\mathcal{J} := \mathcal{N}_c(i,\eta_i^\star)$. ■

In view of Proposition 3, the inclusion $\mathcal{N}_c(i,\eta_i) \supseteq \mathcal{N}(i;\mathfrak{V}_\vartheta(\boldsymbol{Z}^0))$ implies that

$$\overline{\varrho}_\vartheta(i;\boldsymbol{e},\mathcal{N}_c(i,\eta_i)) = \overline{\varrho}_\vartheta(i;\boldsymbol{e},\mathcal{I}_n), \tag{24a}$$

$$\underline{\varrho}_\vartheta(i;\boldsymbol{e},\mathcal{N}_c(i,\eta_i)) = \underline{\varrho}_\vartheta(i;\boldsymbol{e},\mathcal{I}_n). \tag{24b}$$

It should be stressed here that the i-th agent needs to be in position to determine whether the inclusion $\mathcal{N}_c(i,\eta_i) \supseteq \mathcal{N}(i;\mathfrak{V}_\vartheta(\boldsymbol{Z}^0))$ holds true or not, via a relevant stopping criterion whose verification is solely relied on (local) information obtained from agents lying within its communication range.

[4]The reader interested in applications in which the condition $\eta_i \geq 2\mu$ holds true for all $i \in \mathcal{I}_n$ (in such applications, the communication constraints will not play a significant role in the partitioning problem) may skip this section and go directly to Section III-D.

The following proposition stresses an important *monotonicity* property enjoyed by Voronoi-like partitions, which will prove useful in the subsequent analysis.

Proposition 4: Let $\mathcal{N}(i;\mathfrak{V}_{\vartheta}(\boldsymbol{Z}^0))$ denote the index-set of the neighbors of the i-th agent in the topology of $\mathfrak{V}_{\vartheta}(\boldsymbol{Z}^0)$, for a given $\vartheta \in \mathbb{S}^1$, and let $\mathcal{N}' \subseteq \mathcal{N}'' \subseteq \mathcal{N}(i;\mathfrak{V}_{\vartheta}(\boldsymbol{Z}^0))$. In addition, let $\{\boldsymbol{Z}'\} := \{\boldsymbol{z}_j^0 \in \{\boldsymbol{Z}^0\},\ j \in \mathcal{N}'\} \cup \{\boldsymbol{z}_i^0\}$ and $\{\boldsymbol{Z}''\} := \{\boldsymbol{z}_j^0 \in \{\boldsymbol{Z}^0\},\ j \in \mathcal{N}''\} \cup \{\boldsymbol{z}_i^0\}$, and let $\boldsymbol{Z}'$ and $\boldsymbol{Z}''$ denote the corresponding joint vectors in $\boldsymbol{\mathcal{Z}}^{n_1}$ and $\boldsymbol{\mathcal{Z}}^{n_2}$, respectively, where $n_1 := \mathrm{card}(\mathcal{N}') + 1$ and $n_2 := \mathrm{card}(\mathcal{N}'') + 1$. Moreover, let $\mathfrak{V}_{\vartheta}(\boldsymbol{Z}')$ and $\mathfrak{V}_{\vartheta}(\boldsymbol{Z}'')$ denote the partitions that solve Problem 2 when $\eta_i \geq 2\mu$ for all $i \in \mathcal{I}_n$, with $\boldsymbol{Z}'$ and $\boldsymbol{Z}''$ in lieu of $\boldsymbol{Z}^0$, respectively. In addition, let $\mathfrak{V}_{\vartheta}(\boldsymbol{z}_i^0|\boldsymbol{Z}')$ and $\mathfrak{V}_{\vartheta}(\boldsymbol{z}_i^0|\boldsymbol{Z}'')$ denote, respectively, the cells from the partition $\mathfrak{V}_{\vartheta}(\boldsymbol{Z}')$ and $\mathfrak{V}_{\vartheta}(\boldsymbol{Z}'')$ that are associated with the agent emanating from the state $\boldsymbol{z}_i^0$. Then,

$$\mathfrak{V}_{\vartheta}(\boldsymbol{z}_i^0|\boldsymbol{Z}') \supseteq \mathfrak{V}_{\vartheta}(\boldsymbol{z}_i^0|\boldsymbol{Z}'') \supseteq \mathfrak{V}_{\vartheta}^i(\boldsymbol{z}_i^0). \tag{25}$$

Remark 3 Proposition 4 implies that the cell associated with a particular agent will either remain the same or expand, if any agents are removed from its network. This result is quite intuitive given that the more agents the network has, the harder would be for any of its agents to “claim” that a particular state in $\boldsymbol{\mathcal{T}}_{\vartheta}$ is closer to them than to any of their teammates (since there are more “competitors”). It is interesting to note that Lemma 1 is a direct consequence of Proposition 4.

Next, we propose an algorithm that will allow the i-th agent to discover all of its neighboring agents in the topology of $\mathfrak{V}_{\vartheta}(\boldsymbol{Z}^0)$ by adjusting appropriately its communication range. To this aim, let $\eta_i = \eta_i^0 > 0$ be the initial communication range of the i-th agent and let $\mathfrak{V}_{\vartheta}^i(\boldsymbol{z}_i^0;\eta_i^0)$ be its corresponding cell. Now let $\eta_i^1 > \eta_i^0$ be its communication range at stage $k = 1$ which is such that $\mathcal{N}_c(i,\eta_i^1) \supsetneq \mathcal{N}_c(i,\eta_i^0)$. Let $\ell \in \mathcal{N}_c(i,\eta_i^1)\backslash\mathcal{N}_c(i,\eta_i^0)$. Then the ℓ-th agent, which was not within the communication range of the i-th agent at stage $k = 0$, can directly exchange information with the latter via a communication channel established at stage $k = 1$. In addition, let us assume that the ℓ-th agent is closer, in terms of the metric δ_{ϑ}, to at least one of the points in the relative boundary of $\mathfrak{V}_{\vartheta}^i(\boldsymbol{z}_i^0;\eta_i^0)$. Then, we claim that the communication range η_i^0 is not sufficiently large to allow the i-th agent to exchange information with all of its neighbors in the topology of $\mathfrak{V}_{\vartheta}(\boldsymbol{Z}^0)$, that is, $\mathcal{N}_c(i,\eta_i^0) \subsetneq \mathcal{N}(i;\mathfrak{V}_{\vartheta}(\boldsymbol{Z}^0))$. Next, we prove this claim.

Proposition 5: Let $\eta_i^{k+1} > \eta_i^k > 0$ and suppose that $\mathcal{N}_c(i,\eta_i^{k+1}) \supsetneq \mathcal{N}_c(i,\eta_i^k)$ for some $k \in \mathbb{Z}_{\geq 0}$. If there is $\ell \in \mathcal{N}_c(i,\eta_i^{k+1})\backslash\mathcal{N}_c(i,\eta_i^k)$ such that $\delta_{\vartheta}(\mathrm{x};\boldsymbol{z}_\ell^0) < \delta_{\vartheta}(\mathrm{x};\boldsymbol{z}_i^0)$ for some $\mathrm{x} \in \mathrm{bd}(\boldsymbol{\mathcal{X}}_{\vartheta}^i(\boldsymbol{z}_i^0;\eta_i^k))$, then $\mathcal{N}_c(i,\eta_i^k) \subsetneq \mathcal{N}(i;\mathfrak{V}_{\vartheta}(\boldsymbol{Z}^0))$.

Proof: By hypothesis, there is a state $\boldsymbol{z}_{\vartheta}(\mathrm{x}) \in \mathrm{rbd}(\mathfrak{V}_{\vartheta}^i(\boldsymbol{z}_i^0;\eta_i^k)) \subseteq \mathfrak{V}_{\vartheta}^i(\boldsymbol{z}_i^0;\eta_i^k)$ such that $\delta_{\vartheta}(\mathrm{x};\boldsymbol{z}_\ell^0) < \delta_{\vartheta}(\mathrm{x};\boldsymbol{z}_i^0)$. Therefore, $\boldsymbol{z}_{\vartheta}(\mathrm{x}) \notin \mathfrak{V}_{\vartheta}^i(\boldsymbol{z}_i^0)$, and we conclude immediately that $\mathfrak{V}_{\vartheta}^i(\boldsymbol{z}_i^0;\eta_i^k) \supsetneq \mathfrak{V}_{\vartheta}^i(\boldsymbol{z}_i^0)$. This in turn implies that $\mathcal{N}_c(i,\eta_i^k) \subseteq \mathcal{N}(i;\mathfrak{V}_{\vartheta}(\boldsymbol{Z}^0))$ in light of Proposition 4. We claim that actually $\mathcal{N}_c(i,\eta_i^k) \subsetneq \mathcal{N}(i;\mathfrak{V}_{\vartheta}(\boldsymbol{Z}^0))$ for if $\mathcal{N}_c(i,\eta_i^k) = \mathcal{N}(i;\mathfrak{V}_{\vartheta}(\boldsymbol{Z}^0))$, then we would have $\mathfrak{V}_{\vartheta}^i(\boldsymbol{z}_i^0;\eta_i^k) = \mathfrak{V}_{\vartheta}^i(\boldsymbol{z}_i^0)$, which contradicts the fact that $\mathfrak{V}_{\vartheta}^i(\boldsymbol{z}_i^0;\eta_i^k) \supsetneq \mathfrak{V}_{\vartheta}^i(\boldsymbol{z}_i^0)$, which we have already proved. We conclude that $\mathcal{N}_c(i,\eta_i^k) \subsetneq \mathcal{N}(i;\mathfrak{V}_{\vartheta}(\boldsymbol{Z}^0))$. ■

The upshot of Proposition 5 is that, at stage k, the i-th agent should increase its communication range to $\eta_i^{k+1} > \eta_i^k$, if it is not true that all the points in the relative boundary of its own cell, which was computed at stage k, are closer to it than to any other agent that would lie within the closed ball $\overline{\boldsymbol{B}}(\boldsymbol{x}_i^0;\eta_i^{k+1})$. The previous observation leads naturally to the following update law for the i-th agent's communication range:

$$\eta_i^{k+1} = \min\{\gamma\eta_i^k, 2\mu\}, \qquad k \in \mathbb{Z}_{\geq 0}, \tag{26}$$

where $\gamma > 1$ (typically, $\gamma = 2$). Note that otherwise, that is, if $\delta_{\vartheta}(\mathrm{x};\boldsymbol{z}_\ell^0) \geq \delta_{\vartheta}(\mathrm{x};\boldsymbol{z}_i^0)$ for every $\mathrm{x} \in \mathrm{bd}(\boldsymbol{\mathcal{X}}_{\vartheta}^i(\boldsymbol{z}_i^0;\eta_i^k))$ and for any $\ell \in \mathcal{N}(i,\eta_i^{k+1})\backslash\mathcal{N}_c(i,\eta_i^k)$, we will not be able to conclude with certainty that $\mathcal{N}_c(i,\eta_i^k) \supseteq \mathcal{N}(i;\mathfrak{V}_{\vartheta}(\boldsymbol{Z}^0))$. The i-th agent will have to keep increasing its communication range until it successfully discovers all of its neighbors in the topology of $\mathfrak{V}_{\vartheta}(\boldsymbol{Z}^0)$. The occurrence of this event should be checked at each step via a relevant stopping criterion.

Unfortunately, the techniques and the stopping criteria used in [1], [15] for the discovery of the neighbors of an agent in the topology of the standard Voronoi partition cannot be used in our case. This is because the index-sets $\mathcal{N}_c(i,\eta_i)$ and $\mathcal{N}(i;\mathfrak{V}_{\vartheta}(\boldsymbol{Z}^0))$ are not induced by the same metric; in particular, the first one is induced by the Euclidean distance whereas the second one by the (generalized) proximity metric δ_{ϑ}. Next, we show how one can account for this “metric mismatch.” To this aim, let $\bar{\delta}_{\vartheta}^i(\eta_i)$ denote the maximum value of $\delta_{\vartheta}(\cdot;\boldsymbol{z}_i^0)$ over $\boldsymbol{\mathcal{X}}_{\vartheta}^i(\boldsymbol{z}_i^0;\eta_i)$, that is,

$$\bar{\delta}_{\vartheta}^i(\eta_i) := \max\{\delta_{\vartheta}(\mathrm{x};\boldsymbol{z}_i^0) :\ \mathrm{x} \in \boldsymbol{\mathcal{X}}_{\vartheta}^i(\boldsymbol{z}_i^0;\eta_i)\}. \tag{27}$$

Because in the formulation of the partitioning problem over $\boldsymbol{\mathcal{T}}_{\vartheta}$ the position space $\boldsymbol{\mathcal{X}}$ is assumed to be a convex and compact polygonal set, all the cells of $\boldsymbol{\mathcal{X}}_{\vartheta}(\boldsymbol{Z}^0;\boldsymbol{H})$ will also be compact and convex polygons. Therefore, the restriction of the convex quadratic function δ_{ϑ} in the cell $\boldsymbol{\mathcal{X}}_{\vartheta}^i(\boldsymbol{z}_i^0;\eta_i)$ will always attain its maximum value in $\mathrm{bd}(\boldsymbol{\mathcal{X}}_{\vartheta}^i(\boldsymbol{z}_i^0;\eta_i))$, and specifically, at one (or more) of its vertices [38], for all $i \in \mathcal{I}_n$. Now let $\bar{\mathrm{x}}_{\vartheta}(\eta_i)$ be the corresponding maximizer, which is not necessarily unique. Unless $\bar{\mathrm{x}}_{\vartheta}(\eta_i)$ belongs to the boundary of $\boldsymbol{\mathcal{X}}$, there exists at least one $j \in \mathcal{N}_c(i,\eta_i)$ such that the latter point is equidistant from the i-th and the j-th agents in terms of δ_{ϑ}, that is, $\delta_{\vartheta}(\bar{\mathrm{x}}_{\vartheta}(\eta_i);\boldsymbol{z}_j^0) = \delta_{\vartheta}(\bar{\mathrm{x}}_{\vartheta}(\eta_i);\boldsymbol{z}_i^0) = \bar{\delta}_{\vartheta}^i(\eta_i)$. In this case, it is also true that $\bar{\mathrm{x}}_{\vartheta}(\eta_i) \in \mathrm{bd}(\overline{\boldsymbol{\mathcal{E}}}_{\vartheta}^i(\eta_i)) \cap \mathrm{bd}(\overline{\boldsymbol{\mathcal{E}}}_{\vartheta}^j(\eta_i))$, where $\overline{\boldsymbol{\mathcal{E}}}_{\vartheta}^{\ell}(\eta_i) := \{\mathrm{x} \in \boldsymbol{\mathcal{X}} : \delta_{\vartheta}(\mathrm{x};\boldsymbol{z}_\ell^0) \leq \bar{\delta}_{\vartheta}^i(\eta_i)\}$, for $\ell = i, j$, are the $\bar{\delta}_{\vartheta}^i(\eta_i)$-sub-level sets of $\delta_{\vartheta}(\cdot;\boldsymbol{z}_i^0)$ and $\delta_{\vartheta}(\cdot;\boldsymbol{z}_j^0)$, respectively, which are (closed) ellipsoids in $\boldsymbol{\mathcal{X}}$ centered at $\boldsymbol{\xi}_{\vartheta}^i$ and $\boldsymbol{\xi}_{\vartheta}^j$. Note that $\overline{\boldsymbol{\mathcal{E}}}_{\vartheta}^j(\eta_i) = \{\mathrm{x} \in \boldsymbol{\mathcal{X}} :\ |\boldsymbol{\Pi}^{\frac{1}{2}}(\boldsymbol{\xi}_{\vartheta}^j - \mathrm{x})|^2 \leq \bar{\delta}_{\vartheta}^i(\eta_i) - \mu_{\vartheta}^j\}$, which implies that $\overline{\boldsymbol{\mathcal{E}}}_{\vartheta}^j(\eta_i) \subseteq \{\mathrm{x} \in \boldsymbol{\mathcal{X}} :\ |\boldsymbol{\Pi}^{\frac{1}{2}}(\boldsymbol{\xi}_{\vartheta}^j - \mathrm{x})|^2 \leq \bar{\delta}_{\vartheta}^i(\eta_i)\} =: \overline{\boldsymbol{E}}_{\vartheta}^j(\eta_i)$ (the previous inclusion follows readily in view of $\mu_{\vartheta}^j \geq 0$). In the light of the Rayleigh quotient inequality together with the definition of $\overline{\boldsymbol{E}}_{\vartheta}^j(\eta_i)$, we have that

$$\bar{\delta}_{\vartheta}^i(\eta_i) \geq |\boldsymbol{\Pi}^{\frac{1}{2}}(\boldsymbol{\xi}_{\vartheta}^j - \mathrm{x})|^2 \geq \lambda_{\min}(\boldsymbol{\Pi})|\boldsymbol{\xi}_{\vartheta}^j - \mathrm{x}|^2,$$

for all $\mathrm{x} \in \overline{\boldsymbol{E}}_{\vartheta}^j(\eta_i)$, which in turn implies that

$$|\boldsymbol{\xi}_{\vartheta}^j - \mathrm{x}| \leq \sqrt{\bar{\delta}_{\vartheta}^i(\eta_i)/\lambda_{\min}(\boldsymbol{\Pi})} =: \psi_{\vartheta}^i(\eta_i), \tag{28}$$

for all $\mathrm{x} \in \overline{\boldsymbol{E}}_{\vartheta}^j(\eta_i)$. It follows immediately that $\overline{\boldsymbol{B}}(\boldsymbol{\xi}_{\vartheta}^i;\psi_{\vartheta}^i(\eta_i)) \supseteq \overline{\boldsymbol{E}}_{\vartheta}^j(\eta_i)$. Now, let us consider the stripe $\boldsymbol{\mathcal{S}}$ that is formed by the collection of all balls of radius $\psi_{\vartheta}^i(\eta_i)$ which are translations of $\overline{\boldsymbol{B}}(\boldsymbol{\xi}_{\vartheta}^i;\psi_{\vartheta}^i(\eta_i))$ and are tangent to the boundary of $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i)$, as is illustrated in Fig. 4 (in this figure, the balls of the stripe $\boldsymbol{\mathcal{S}}$ are depicted with dashed lines). Note that any generator $\boldsymbol{\xi}_\vartheta^\ell \in \{\boldsymbol{\Xi}_\vartheta\}$ that does not belong to $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i)$ and whose distance from $\mathrm{bd}(\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i))$, in terms of $\delta_\vartheta(\cdot;\boldsymbol{z}_\ell^0)$, is less than or equal to $\bar{\delta}_\vartheta^i(\eta_i)$ will belong to $\boldsymbol{\mathcal{S}}$. In addition, the stripe $\boldsymbol{\mathcal{S}}$ will be contained itself in the closed ball $\overline{\boldsymbol{B}}(\boldsymbol{x}_i^0;\bar{\psi}_\vartheta^i(\eta_i))$, which is centered at the initial location of the i-th agent and has radius

$$\bar{\psi}_\vartheta^i(\eta_i) := \psi_\vartheta^i(\eta_i) + \bar{d}_\vartheta^i(\eta_i), \tag{29}$$

where $\bar{d}_\vartheta^i(\eta_i)$ denotes the maximum (Euclidean) distance between $\boldsymbol{x}_i^0$ and the boundary of $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i)$, that is,

$$\bar{d}_\vartheta^i(\eta_i) := \max\{|\boldsymbol{x}_i^0 - \mathbf{x}|,\ \mathbf{x} \in \mathrm{bd}(\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i))\}.$$

Again, the maximum in the previous expression will be attained at one (or more) of the vertices of the convex and compact polygon $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i)$ [38].

Proposition 6: The closed ball $\overline{\boldsymbol{B}}(\boldsymbol{x}_i^0;\eta_i)$ will contain the point-set $\{\boldsymbol{\Xi}_\vartheta^i\} := \{\boldsymbol{\xi}_\vartheta^\ell,\ \ell \in \mathcal{N}(i;\mathfrak{V}_\vartheta(\boldsymbol{Z}^0))\}$, provided that $\eta_i \geq \min\{\bar{\psi}_\vartheta^i(\eta_i), 2\mu\}$, where $\bar{\psi}_\vartheta^i(\eta_i)$ is defined in (29).

Proof: Let $\boldsymbol{\xi}_\vartheta^\ell \in \{\boldsymbol{\Xi}_\vartheta^i\}$. By definition, the cell $\boldsymbol{\mathcal{X}}_\vartheta^\ell(\boldsymbol{z}_\ell^0) \in \boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0)$ that is associated with $\boldsymbol{\xi}_\vartheta^\ell$ will share a common edge with the cell $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0) \in \boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0)$. Consequently, there exists a point $\mathbf{x} \in \mathrm{bd}(\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0))$ that is "equidistant" with respect to δ_ϑ from the i-th and the ℓ-th agents, that is, $\delta_\vartheta(\mathbf{x};\boldsymbol{z}_\ell^0) = \delta_\vartheta(\mathbf{x};\boldsymbol{z}_i^0)$. Now let $\bar{\bar{\delta}}_\vartheta^i := \max_{\boldsymbol{y}\in\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0)} \delta_\vartheta(\boldsymbol{y};\boldsymbol{z}_i^0)$. We claim that $\bar{\bar{\delta}}_\vartheta^i = \bar{\delta}_\vartheta^i(2\mu)$, where $\bar{\delta}_\vartheta^i(\cdot)$ is defined in (27). To prove this, we first show that the non-negative (and thus lower bounded) function $\bar{\delta}_\vartheta^i(\cdot)$ is a non-increasing function of η_i over $[0,2\mu]$. To this aim, it suffices to note that given two communication ranges, namely η_i' and η_i, with $\eta_i' \geq \eta_i$, we have that $\mathcal{N}_c(i,\eta_i) \subseteq \mathcal{N}_c(i,\eta_i')$, which in turn implies that $\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i') \subseteq \mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i)$, in view of Proposition 4. We also know that $\mathcal{N}_c(i,2\mu) \supseteq \mathcal{N}(i;\mathfrak{V}_\vartheta(\boldsymbol{Z}^0)) = \mathcal{N}(i;\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0))$, which implies that $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0;2\mu) = \boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0)$. Then, in view of the monotone convergence theorem from real analysis, it follows readily that $\max_{\boldsymbol{y}\in\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0)} \delta_\vartheta(\boldsymbol{y};\boldsymbol{z}_i^0) = \bar{\bar{\delta}}_\vartheta^i = \bar{\delta}_\vartheta^i(2\mu) = \lim_{k\to\infty} \bar{\delta}_\vartheta^i(\eta_i^k)$, where $\{\eta_i^k\}_{k\in\mathbb{Z}_{\geq 0}}$ is a non-increasing sequence of positive numbers such that $\lim_{k\to\infty}\eta_i^k = 2\mu$. The fact that $\bar{\delta}_\vartheta^i(\eta_i)$ is a non-increasing function of η_i also implies that for any $\eta_i \in [0,2\mu]$ and for the same point $\mathbf{x} \in \mathrm{bd}(\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0)) \subsetneq \boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i)$, which is not necessarily a boundary point of $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i)$, it holds that

$$\delta_\vartheta(\mathbf{x};\boldsymbol{z}_\ell^0) \leq \bar{\bar{\delta}}_\vartheta^i = \bar{\delta}_\vartheta^i(2\mu) \leq \bar{\delta}_\vartheta^i(\eta_i), \quad \text{for all } \eta_i \in [0,2\mu].$$

However, if $\delta_\vartheta(\mathbf{x};\boldsymbol{z}_\ell^0) \leq \bar{\delta}_\vartheta^i(\eta_i)$, then $\mathbf{x}$ belongs to the ellipsoid $\overline{\boldsymbol{E}}_\vartheta^\ell(\eta_i)) := \{\mathbf{x} \in \boldsymbol{\mathcal{X}} :\ |\boldsymbol{\Pi}^{\frac{1}{2}}(\boldsymbol{\xi}_\vartheta^\ell - \mathbf{x})|^2 \leq \bar{\delta}_\vartheta^i(\eta_i)\}$, which is contained necessarily in $\boldsymbol{\mathcal{S}} \cup \boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i)$. Because, $\overline{\boldsymbol{E}}_\vartheta^\ell(\eta_i) \subseteq (\boldsymbol{\mathcal{S}} \cup \boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i)) \subseteq \overline{\boldsymbol{B}}(\boldsymbol{x}_i^0;\bar{\psi}_\vartheta^i(\eta_i))$, we conclude that $\boldsymbol{\xi}_\vartheta^\ell \in \overline{\boldsymbol{B}}(\boldsymbol{x}_i^0;\bar{\psi}_\vartheta^i(\eta_i))$. ■

Before we proceed any further, we will need the following lemma.

Lemma 2: Let $\overline{\nu} > 0$ and $\overline{w} > 0$ be such that $|\boldsymbol{\nu}_i^0| \leq \overline{\nu}$ and $|w_i^0| \leq \overline{w}$, for all $i \in \mathcal{I}_n$, respectively. Then, $|\boldsymbol{r}_\vartheta^i| \leq \overline{r}$, where $\overline{r} := \lambda_{\max}(\boldsymbol{\Lambda})m\overline{\nu}$, for all $i \in \mathcal{I}_n$ and all $\vartheta \in \mathbb{S}^1$.

Proof: In view of Eq. (9a), we have that $|\boldsymbol{r}_\vartheta^i| \leq m|\boldsymbol{\Lambda}\mathbf{T}_1(\theta_i^0)\boldsymbol{\nu}_i^0| \leq m\sigma_{\max}(\boldsymbol{\Lambda})|\mathbf{T}_1(\theta_i^0)\boldsymbol{\nu}_i^0| = m\lambda_{\max}(\boldsymbol{\Lambda})|\boldsymbol{\nu}_i^0| \leq \lambda_{\max}(\boldsymbol{\Lambda})m\overline{\nu}$, for all $i \in \mathcal{I}_n$, where we have used the facts that $\sigma_{\max}(\boldsymbol{\Lambda}) = \sqrt{\lambda_{\max}(\boldsymbol{\Lambda}^2)} = \sqrt{\lambda_{\max}^2(\boldsymbol{\Lambda})} = \lambda_{\max}(\boldsymbol{\Lambda})$, given that $\boldsymbol{\Lambda}$ is a diagonal matrix in $\mathbb{P}_2$, and $|\mathbf{T}_1(\theta_i^0)\boldsymbol{\nu}_i^0| = |\boldsymbol{\nu}_i^0|$, given that $\mathbf{T}_1(\theta_i^0)$ is an orthogonal matrix. ■

Next, we present a condition that will serve as the *stopping criterion* of the iterative process for the discovery of the neighbors of the i-th agent.

Proposition 7: Let $\overline{\nu}$, $\overline{w}$, and $\overline{r}$ be positive constants defined as in Lemma 2 and let $\vartheta \in \mathbb{S}^1$ be given. In addition, suppose that for a given $\eta_i > 0$, we have that $\eta_i \geq \mathrm{crlb}_\vartheta^i(\eta_i)$ (crlb: communication range lower bound), where

$$\mathrm{crlb}_\vartheta^i(\eta_i) := \min\{\bar{\psi}_\vartheta^i(\eta_i) + \overline{r}/\lambda_{\min}(\boldsymbol{\Pi}), 2\mu\}, \tag{30}$$

where $\bar{\psi}_\vartheta^i(\eta_i)$ is defined as in (29). Then, the closed ball $\overline{\boldsymbol{B}}(\boldsymbol{x}_i^0;\eta_i)$ contains all the neighbors of the i-th agent in the topology of the partition $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0)$, which corresponds to the solution of Problem 2 when $\eta_i \geq 2\mu$, for all $i \in \mathcal{I}_n$, that is, $\mathcal{N}_c(i,\eta_i) \supseteq \mathcal{N}(i;\mathfrak{V}_\vartheta(\boldsymbol{Z}^0))$, which also implies that $\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0;\eta_i) = \mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0)$.

Proof: In view of Proposition 6 and its proof, it follows that $\boldsymbol{\xi}_\vartheta^j \in \overline{\boldsymbol{B}}(\boldsymbol{x}_i^0;\bar{\psi}_\vartheta^i(\eta_i))$ for each $j \in \mathcal{N}(i;\mathfrak{V}_\vartheta(\boldsymbol{Z}^0))$ given that $\eta_i \geq \mathrm{crlb}_\vartheta^i(\eta_i) \geq \min\{\bar{\psi}_\vartheta^i(\eta_i), 2\mu\}$, which in turn holds true in the light of (30). Consequently,

$$\begin{aligned}|\boldsymbol{x}_i^0 - \boldsymbol{x}_j^0| &= |\boldsymbol{x}_i^0 - \boldsymbol{\xi}_\vartheta^j + \boldsymbol{\xi}_\vartheta^j - \boldsymbol{x}_j^0| \leq |\boldsymbol{x}_i^0 - \boldsymbol{\xi}_\vartheta^j| + |\boldsymbol{\xi}_\vartheta^j - \boldsymbol{x}_j^0| \\ &\leq \bar{\psi}_\vartheta^i(\eta_i) + |\boldsymbol{\Pi}^{-1}\boldsymbol{r}_\vartheta^j| \\ &\leq \bar{\psi}_\vartheta^i(\eta_i) + \sigma_{\max}(\boldsymbol{\Pi}^{-1})|\boldsymbol{r}_\vartheta^j| \\ &\leq \bar{\psi}_\vartheta^i(\eta_i) + \overline{r}/\lambda_{\min}(\boldsymbol{\Pi}),\end{aligned}$$

where in the previous derivation we have also used the triangle inequality together with Lemma 2, and the fact that $\sigma_{\max}(\boldsymbol{\Pi}^{-1}) = 1/\sigma_{\min}(\boldsymbol{\Pi}) = 1/\lambda_{\min}(\boldsymbol{\Pi})$, which holds true given that $\boldsymbol{\Pi}$ is a diagonal matrix in $\mathbb{P}_2$ (note that in this case, $\sigma_{\min}(\boldsymbol{\Pi}) = \sqrt{\lambda_{\min}(\boldsymbol{\Pi}^2)} = \sqrt{\lambda_{\min}^2(\boldsymbol{\Pi})} = \lambda_{\min}(\boldsymbol{\Pi})$). We conclude that if $\eta_i \geq \mathrm{crlb}_\vartheta^i(\eta_i)$, where $\mathrm{crlb}_\vartheta^i(\eta_i)$ is defined as in (30), then the closed ball $\overline{\boldsymbol{B}}(\boldsymbol{x}_i^0;\eta_i)$ will contain the set $\{\boldsymbol{x}_\ell^0,\ \ell \in \mathcal{N}(i;\mathfrak{V}_\vartheta(\boldsymbol{Z}^0))\}$, from which we immediately conclude that $\mathcal{N}_c(i,\eta_i) \supseteq \mathcal{N}(i;\mathfrak{V}_\vartheta(\boldsymbol{Z}^0))$. ■

Remark 4 It is important to note that (30) is an implicit inequality, given that η_i appears at both of its sides.

The step-by-step description of the distributed algorithm for the computation of the (boundary of) cell $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0)$ is given in Algorithm 3. This algorithm also provides a value $\overline{\eta}_\vartheta^i > 0$ such that the inequality $\eta_i \geq \mathrm{crlb}_\vartheta^i(\eta_i)$ holds true for the communication range $\eta_i = \overline{\eta}_\vartheta^i$.

Remark 5 In the proposed approach, the i-th agent computes its own cell independently from the other agents of the same network while discovering *in parallel* its neighbors in the topology of the Voronoi-like partition. In addition, we have made the assumption that the i-th agent can essentially execute Algorithm 3 "instantaneously." It is actually not difficult to explicitly account for the effect of the time period between the execution of two consecutive while loops of Algorithm 3 on the required communication range of the i-th agent. To this aim, let us denote by $\bar{\tau}$ an upper bound on this time period. Then, the update law for the communication range of the i-th agent at stage $k+1$ should be given by

$$\eta_i^{k+1} = \min\{\gamma\eta_i^k + \overline{\delta\eta}, 2\mu\}, \tag{31}$$

Algorithm 3 Distributed Algorithm for the Independent Computation of the i-th Cell of $\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{\mathcal{Z}}^0; \boldsymbol{H})$ by the i-th Agent

procedure DISTRCELLCOMP
Input data: $\boldsymbol{Z}^0$, ϑ
Input variables: i, $\mathbb{E}$, η_i, γ
Output variables: $\mathrm{bd}(\boldsymbol{\mathcal{X}}^i_\vartheta)$, $\bar{\eta}^i_\vartheta$
    $k \leftarrow 0$
    $\eta^k_i \leftarrow \eta_i$
    $\{\boldsymbol{Z}^0_k\} \leftarrow \{\boldsymbol{z}^0_\ell \in \{\boldsymbol{Z}^0\} : \ \ell \in \mathcal{N}_c(i, \eta^k_i)\} \cup \{\boldsymbol{z}^0_i\}$
    $\mathrm{bd}(\boldsymbol{\mathcal{X}}^i_\vartheta(\boldsymbol{Z}^0_k)) = \text{CELLCOMP}(i, \mathbb{E}, \mathcal{N}_c(i, \eta^k_i); \boldsymbol{Z}^0_k, \vartheta)$
    compute $\mathrm{crlb}^i_\vartheta(\eta^k_i)$ via (30)
    while $\eta^k_i < \mathrm{crlb}^i_\vartheta(\eta^k_i)$ **do**
        $\eta^{k+1}_i \leftarrow \gamma \eta^k_i$
        $\{\boldsymbol{Z}^0_{k+1}\} \leftarrow \{\boldsymbol{z}^0_\ell \in \{\boldsymbol{Z}^0\} : \ \ell \in \mathcal{N}_c(i, \eta^{k+1}_i)\} \cup \{\boldsymbol{z}^0_i\}$
        $\mathrm{bd}(\boldsymbol{\mathcal{X}}^i_\vartheta(\boldsymbol{Z}^0_{k+1})) = \text{CELLCOMP}(i, \mathbb{E}, \mathcal{N}_c(i, \eta^{k+1}_i); \boldsymbol{Z}^0_{k+1}, \vartheta)$
        compute $\mathrm{crlb}^i_\vartheta(\eta^{k+1}_i)$ via (30)
        $k \leftarrow k + 1$
    $\mathrm{bd}(\boldsymbol{\mathcal{X}}^i_\vartheta) \leftarrow \mathrm{bd}(\boldsymbol{\mathcal{X}}^i_\vartheta(\boldsymbol{Z}^0_k))$
    $\bar{\eta}^i_\vartheta \leftarrow \eta^k_i$

where $\overline{\delta\eta} := 2\bar{\nu}\bar{\tau}$ with $\bar{\nu}$ be defined as in Proposition 7. Note that the correction term, $\overline{\delta\eta}$, corresponds to the maximum increase on the relative distance between the i-th agent and any other agent from the same network that can take place within $\bar{\tau}$ units of time.

D. The Partitioning Problem over the Terminal Manifold $\boldsymbol{\mathcal{T}}$

After having addressed the partitioning problems over the terminal manifold $\boldsymbol{\mathcal{T}}_\vartheta$ for each $\vartheta \in \mathbb{S}^1$ (Problem 2), the solution to the partitioning problem over $\boldsymbol{\mathcal{T}}$ (Problem 1) will follow readily by stacking the solutions to the parametric problems next to each other as the parameter ϑ runs through $\mathbb{S}^1$. In particular, in the special case in which no communication constraints are enforced, that is, $\eta_i \geq 2\mu$ for all $i \in \mathcal{I}_n$, we have that the solution to Problem 2 is simply the partition $\mathfrak{V}(\boldsymbol{Z}^0) := \{\mathfrak{V}^i(\boldsymbol{z}^0_i), \ i \in \mathcal{I}_n\}$, where $\mathfrak{V}^i(\boldsymbol{z}^0_i) := \cup_{\vartheta \in \mathbb{S}^1}[\mathfrak{V}^i_\vartheta(\boldsymbol{z}^0_i) \times \{\vartheta\}]$, $i \in \mathcal{I}_n$. In the presence of communication constraints, there is one additional step that needs to be made, namely to find a uniform lower bound on η_i that is independent of ϑ. In particular, Proposition 7 together with Algorithm 3 allow us to characterize a positive number $\bar{\eta}^i_\vartheta$ such that $\mathfrak{V}^i_\vartheta(\boldsymbol{z}^0_i; \eta_i) = \mathfrak{V}^i_\vartheta(\boldsymbol{z}^0_i)$ for all $\eta_i \geq \bar{\eta}^i_\vartheta$ and for a given $\vartheta \in \mathbb{S}^1$. Then, the maximum of $\bar{\eta}^i_\vartheta$ over $\mathbb{S}^1$, which is always attained and is denoted by $\eta^\star_i$, will be such that $\mathfrak{V}^i(\boldsymbol{z}^0_i; \eta_i) = \mathfrak{V}^i(\boldsymbol{z}^0_i)$ for all $\eta_i \geq \eta^\star_i$.

Remark 6 In order to simplify the presentation, it will be henceforth assumed that the communication range of the i-th agent, η_i, satisfies the following inequality: $\eta_i \geq \eta^\star_i$, for all $i \in \mathcal{I}_n$, and consequently, there will be no need to distinguish between the partition $\mathfrak{V}(\boldsymbol{Z}^0)$ and the partition $\mathfrak{V}(\boldsymbol{Z}^0; \boldsymbol{H})$ and their cells.

IV. Coverage-type Locational Optimization in $\boldsymbol{\mathcal{T}}_\vartheta$ and $\boldsymbol{\mathcal{T}}$

In this section, we address a class of coverage-type locational optimization problems for multi-agent networks with planar rigid body dynamics based on a "divide and conquer" approach that leverages the proposed Voronoi-like partitions of $\boldsymbol{\mathcal{T}}_\vartheta$ and $\boldsymbol{\mathcal{T}}$.

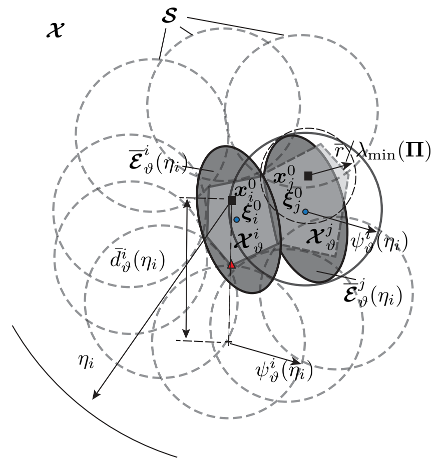


Fig. 4. The i-th agent will increase its communication range until it discovers all of its neighbors in the topology of $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0)$ (or, equivalently, in the topology of $\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0)$). To simplify this illustration, we have assumed that $\mu^i_\vartheta = \mu^j_\vartheta = 0$, which implies that the ellipsoids $\overline{\boldsymbol{\mathcal{E}}}^i_\vartheta(\eta_i)$, $\overline{\boldsymbol{\mathcal{E}}}^j_\vartheta(\eta_i)$ and $\overline{\boldsymbol{E}}^j_\vartheta(\eta_i)$ are all equal modulo a linear translation.

A. Locational Optimization over $\boldsymbol{\mathcal{T}}_\vartheta$

In a nutshell, the locational optimization problem over $\boldsymbol{\mathcal{T}}_\vartheta$, for a given $\vartheta \in \mathbb{S}^1$, seeks for the joint position vector of the network, $\boldsymbol{X}^\star := \mathrm{col}(\boldsymbol{x}^\star_1, \ldots, \boldsymbol{x}^\star_n)$, that minimizes the performance index $\mathcal{H}_\vartheta(\cdot) : \boldsymbol{\mathcal{X}}^n \to \mathbb{R}_{\geq 0}$ with

$$\mathcal{H}_\vartheta(\boldsymbol{X}^0) := \sum_{i \in \mathcal{I}_n} \int_{\boldsymbol{\mathcal{X}}^i_\vartheta} \delta_\vartheta(\mathrm{x}; \boldsymbol{z}_\vartheta(\boldsymbol{x}^0_i))\phi_1(\mathrm{x})\mathrm{d}\mathrm{x}, \tag{32}$$

where $\boldsymbol{\mathcal{X}}^i_\vartheta := \pi_{\boldsymbol{\mathcal{X}}}(\mathfrak{V}^i_\vartheta)$ and $\phi_1(\cdot) : \boldsymbol{\mathcal{X}} \to \mathbb{R}_{\geq 0}$ is a continuous and non-negative function (density function over $\boldsymbol{\mathcal{X}}$). Note that in the formulation of the locational optimization problem over $\boldsymbol{\mathcal{T}}_\vartheta$, the joint vector $\boldsymbol{X}^\star$ corresponds to the concatenation of the initial positions of the agents of the network that are optimal in the following sense: if the i-th agent is located at the configuration $\mathrm{col}(\boldsymbol{x}^\star_i, \vartheta)$ with zero angular and linear velocities, for each $i \in \mathcal{I}_n$, then $\mathcal{H}_\vartheta$ will attain its minimum value. Next, we characterize the minimizers of $\mathcal{H}_\vartheta$. In the light of (10), we have that

$$\begin{aligned}\mathcal{H}_\vartheta(\boldsymbol{X}^0) = &\sum_{i \in \mathcal{I}_n} \int_{\boldsymbol{\mathcal{X}}^i_\vartheta} |\boldsymbol{\Pi}^{\frac{1}{2}}(\mathrm{x} - \boldsymbol{\xi}^i_\vartheta)|^2 \phi_1(\mathrm{x})\mathrm{d}\mathrm{x} \\ &+ \sum_{i \in \mathcal{I}_n} \int_{\boldsymbol{\mathcal{X}}^i_\vartheta} \mu^i_\vartheta \phi_1(\mathrm{x})\mathrm{d}\mathrm{x},\end{aligned} \tag{33}$$

where $\boldsymbol{\xi}^i_\vartheta$ and μ^i_ϑ satisfy the respective equations in Eq. (11) after the following substitutions: $\theta^0_i = \vartheta$, $\boldsymbol{\nu}^0_i = \boldsymbol{0}$, and $w^0_i = 0$ (these substitutions are made in order to account for the fact that the states of all the agents of the network are confined to the two-dimensional sub-manifold $\boldsymbol{\mathcal{T}}_\vartheta$). It follows that $\boldsymbol{\xi}^i_\vartheta = \boldsymbol{x}^0_i$ and $\mu^i_\vartheta = 0$. Therefore,

$$\mathcal{H}_\vartheta(\boldsymbol{X}^0) = \sum_{i \in \mathcal{I}_n} \int_{\boldsymbol{\mathcal{X}}^i_\vartheta} \phi_1(\mathrm{x})|\boldsymbol{\Pi}^{\frac{1}{2}}(\mathrm{x} - \boldsymbol{x}^0_i)|^2\mathrm{d}\mathrm{x}. \tag{34}$$

The minimizer of $\mathcal{H}_\vartheta(\boldsymbol{X}^0)$, which is denoted as $\boldsymbol{X}^\star$ satisfies the first-order necessary condition for optimality [37], [39]:

$$(\partial\mathcal{H}_\vartheta(\boldsymbol{X}^0)/\partial\boldsymbol{x}_i^0)\boldsymbol{d}_i \geq 0, \quad i \in \mathcal{I}_n, \tag{35}$$

for any feasible direction $\boldsymbol{d}_i \in \mathbb{R}^2$ of $\boldsymbol{\mathcal{X}}$ at $\boldsymbol{x}_i^0$. Because the set $\boldsymbol{\mathcal{X}}$ is convex (and compact), according to the formulation of Problem 2, we can replace $\boldsymbol{d}_i$ in (35) with the vector $\mathbf{x} - \boldsymbol{x}_i^0$. In addition, if $\boldsymbol{x}_i^\star$ is an interior point of $\boldsymbol{\mathcal{X}}$, then (35) becomes

$$\partial\mathcal{H}_\vartheta(\boldsymbol{X}^0)/\partial\boldsymbol{x}_i^0 = \mathbf{0}, \quad i \in \mathcal{I}_n. \tag{36}$$

In the light of the discussion in [3, pp. 128-131], one can show that

$$\big(\partial\mathcal{H}_\vartheta(\boldsymbol{X}^0)/\partial\boldsymbol{x}_i^0\big)^{\mathrm{T}} = -\int_{\boldsymbol{\mathcal{X}}_\vartheta^i} 2\phi_1(\mathbf{x})\boldsymbol{\Pi}(\mathbf{x} - \boldsymbol{x}_i^0)\mathrm{d}\mathbf{x}. \tag{37}$$

Thus, the solution of (36) is given by

$$\boldsymbol{x}_i^\star = (\int_{\boldsymbol{\mathcal{X}}_\vartheta^i} \phi_1(\mathbf{x})\mathbf{x}\mathrm{d}\mathbf{x})/\Phi_1(\boldsymbol{\mathcal{X}}_\vartheta^i) =: \boldsymbol{x}_{\mathrm{cm}|\vartheta}^i, \tag{38}$$

for $i \in \mathcal{I}_n$, where $\Phi_1(\boldsymbol{\mathcal{X}}_\vartheta^i) := \int_{\boldsymbol{\mathcal{X}}_\vartheta^i} \phi_1(\mathbf{x})\mathrm{d}\mathbf{x}$ and $\boldsymbol{x}_{\mathrm{cm}|\vartheta}^i$ denotes the centroid of $\boldsymbol{\mathcal{X}}_\vartheta^i$ with respect to the density function $\phi_1(\cdot)$. Note that by its definition, which is given in (38), the centroid $\boldsymbol{x}_{\mathrm{cm}|\vartheta}^i$ lies in the interior of the convex cell $\boldsymbol{\mathcal{X}}_\vartheta^i$.

Proposition 8: The function $\mathcal{H}_\vartheta(\cdot) : \boldsymbol{\mathcal{X}}^n \to \mathbb{R}_{\geq 0}$, where $\mathcal{H}_\vartheta(\boldsymbol{X}^0)$ satisfies (32) for $\boldsymbol{X}^0 \in \boldsymbol{\mathcal{X}}^n$, attains its minimum value at the joint position vector of the network $\boldsymbol{X}^\star := \mathrm{col}(\boldsymbol{x}_{\mathrm{cm}|\vartheta}^1, \ldots, \boldsymbol{x}_{\mathrm{cm}|\vartheta}^n)$, where $\boldsymbol{x}_{\mathrm{cm}|\vartheta}^i$ is the centroid of $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0)$ with respect to $\phi_1(\cdot)$ that satisfies Eq. (38), for all $i \in \mathcal{I}_n$.

Proof: The (strict) convexity of $|\boldsymbol{\Pi}^{\frac{1}{2}}(\mathbf{x} - \boldsymbol{x}_i^0)|^2$ as a function of $\boldsymbol{x}_i^0$, implies that $\mathcal{H}_\vartheta(\cdot)$ is a convex function of $\boldsymbol{X}^0$ (see, for instance, [40, p. 79]). The fact that the centroid $\boldsymbol{x}_{\mathrm{cm}|\vartheta}^i$ of the convex cell $\boldsymbol{\mathcal{X}}_\vartheta^i(\boldsymbol{z}_i^0)$, which is by its definition an interior point of the latter cell and consequently of the domain $\boldsymbol{\mathcal{X}}$, is the unique solution to the equation $\partial\mathcal{H}_\vartheta(\boldsymbol{X}^0)/\partial\boldsymbol{x}_i^0 = \mathbf{0}$, for all $i \in \mathcal{I}_n$, implies that $\boldsymbol{X}^\star = \mathrm{col}(\boldsymbol{x}_{\mathrm{cm}|\vartheta}^1, \ldots, \boldsymbol{x}_{\mathrm{cm}|\vartheta}^n)$ is the unique global minimizer of the convex function $\mathcal{H}_\vartheta(\cdot)$ in $\boldsymbol{\mathcal{X}}^n$. Note that the fact that $\boldsymbol{x}_{\mathrm{cm}|\vartheta}^i$ is an isolated solution to (36) in the interior of $\boldsymbol{\mathcal{X}}$ precludes the existence of a boundary point of $\boldsymbol{\mathcal{X}}$ that satisfies (35) and is also a minimizer of $\mathcal{H}_\vartheta(\cdot)$ (the set of minimizers of $\mathcal{H}_\vartheta(\cdot)$ is necessarily a convex, and thus connected, set [39]). ■

Remark 7 The upshot of Proposition 8 is that the projection of the minimizing state of each agent into $\boldsymbol{\mathcal{X}}$ corresponds to the centroid of its associated cell in the affine partition $\boldsymbol{\mathcal{X}}_\vartheta$. This result mirrors the solution to popular coverage-type locational optimization problems addressed in the literature (see [3] and references therein). On the other hand, the solution to the locational optimization problem over $\boldsymbol{\mathcal{T}}$ is much more interesting, as we will see next.

B. Locational Optimization over the Terminal Manifold $\boldsymbol{\mathcal{T}}$

Next, we analyze and address the locational optimization problem over $\boldsymbol{\mathcal{T}}$. To this aim, we consider a continuous and non-negative function $\phi(\cdot) : \boldsymbol{\mathcal{Q}} \to \mathbb{R}_{\geq 0}$, which will play the role of the density function over $\boldsymbol{\mathcal{Q}}$. To facilitate the presentation, we will assume that for any $\mathbf{q} \in \boldsymbol{\mathcal{Q}}$, $\phi(\mathbf{q}) = \phi_1(\mathbf{x})\phi_2(\vartheta)$, where $\phi_1(\cdot) : \boldsymbol{\mathcal{X}} \to \mathbb{R}_{\geq 0}$ and $\phi_2(\cdot) : \mathbb{S}^1 \to \mathbb{R}_{\geq 0}$ are known continuous, non-negative functions. In addition, $\phi_2(\cdot)$ is a 2π-periodic function.

We will be seeking for the joint configuration of the network, $\boldsymbol{Q}^\star := \mathrm{col}(\boldsymbol{q}_1^\star, \ldots, \boldsymbol{q}_n^\star) \in \boldsymbol{\mathcal{Q}}^n$, that minimizes the performance index $\mathcal{H}(\cdot) : \boldsymbol{\mathcal{Q}}^n \to \mathbb{R}_{\geq 0}$, where

$$\mathcal{H}(\boldsymbol{Q}^0) := \sum_{i \in \mathcal{I}_n} \int_{\boldsymbol{\mathcal{Q}}^i} \delta(\mathbf{q}; \boldsymbol{z}_{\boldsymbol{\mathcal{T}}}(\boldsymbol{q}_i^0))\phi(\mathbf{q})\mathrm{d}\mathbf{q}, \tag{39}$$

where $\boldsymbol{\mathcal{Q}}^i := \pi_{\boldsymbol{\mathcal{Q}}}(\mathfrak{V}^i(\boldsymbol{z}_i^0))$ or, equivalently, $\boldsymbol{\mathcal{Q}}^i := \cup_{\vartheta \in \mathbb{S}^1}\big[\boldsymbol{\mathcal{X}}_\vartheta^i \times \{\vartheta\}\big]$, where $\boldsymbol{\mathcal{X}}_\vartheta^i := \pi_{\boldsymbol{\mathcal{X}}}(\mathfrak{V}_\vartheta^i(\boldsymbol{z}_i^0))$. It follows immediately that

$$\begin{aligned}\mathcal{H}(\boldsymbol{Q}^0) = &\sum_{i \in \mathcal{I}_n} \int_{\boldsymbol{\mathcal{Q}}^i} |\boldsymbol{\Pi}^{\frac{1}{2}}(\mathbf{x} - \boldsymbol{\xi}_\vartheta^i)|^2\phi(\mathbf{q})\mathrm{d}\mathbf{q} \\ &+ \sum_{i \in \mathcal{I}_n} \int_{\boldsymbol{\mathcal{Q}}^i} \mu_\vartheta^i\phi(\mathbf{q})\mathrm{d}\mathbf{q},\end{aligned} \tag{40}$$

where $\boldsymbol{\xi}_\vartheta^i$ and μ_ϑ^i are defined in the respective equations in (11), after substituting $\boldsymbol{\nu}_i^0 = 0$ and $w_i^0 = 0$ therein. It follows that $\boldsymbol{\xi}_\vartheta^i = \boldsymbol{x}_i^0$, and $\mu_\vartheta^i = \varepsilon^2 J(1 - \cos(\theta_i^0 - \vartheta))^2$. In addition, we have that $\mathcal{H}(\boldsymbol{Q}^0)$ can be written as follows: $\mathcal{H}(\boldsymbol{Q}^0) = \mathcal{H}_1(\boldsymbol{X}^0) + \mathcal{H}_2(\boldsymbol{\Theta}^0)$, where

$$\mathcal{H}_1(\boldsymbol{X}^0) := \sum_{i \in \mathcal{I}_n} \int_{\boldsymbol{\mathcal{Q}}^i} \phi(\mathbf{q})|\boldsymbol{\Pi}^{\frac{1}{2}}(\mathbf{x} - \boldsymbol{x}_i^0)|^2\mathrm{d}\mathbf{q}, \tag{41a}$$

$$\mathcal{H}_2(\boldsymbol{\Theta}^0) := \sum_{i \in \mathcal{I}_n} \int_{\boldsymbol{\mathcal{Q}}^i} \phi(\mathbf{q})\varepsilon^2 J(1 - \cos(\theta_i^0 - \vartheta))^2\mathrm{d}\mathbf{q}. \tag{41b}$$

In view of (41a) and the discussion in [3, pp. 128-131], it follows that

$$\begin{aligned}&\big(\partial\mathcal{H}(\boldsymbol{Q}^0)/\partial\boldsymbol{x}_i^0\big)^{\mathrm{T}} = \big(\partial\mathcal{H}_1(\boldsymbol{X}^0)/\partial\boldsymbol{x}_i^0\big)^{\mathrm{T}} \\ &= -\int_{\boldsymbol{\mathcal{Q}}^i} 2\phi(\mathbf{q})\boldsymbol{\Pi}(\mathbf{x} - \boldsymbol{x}_i^0)\mathrm{d}\mathbf{q} \\ &= -2\boldsymbol{\Pi}\int_{\mathbb{S}^1} \phi_2(\vartheta)\int_{\boldsymbol{\mathcal{X}}_\vartheta^i} \phi_1(\mathbf{x})\mathbf{x}\mathrm{d}\mathbf{x}\mathrm{d}\vartheta + 2\big(\int_{\boldsymbol{\mathcal{Q}}^i} \phi(\mathbf{q})\mathrm{d}\mathbf{q}\big)\,\boldsymbol{\Pi}\boldsymbol{x}_i^0 \\ &= -2\boldsymbol{\Pi}\int_{\mathbb{S}^1} \phi_2(\vartheta)\Phi_1(\boldsymbol{\mathcal{X}}_\vartheta^i)\boldsymbol{x}_{\mathrm{cm}|\vartheta}^i\mathrm{d}\vartheta + 2\Phi(\boldsymbol{\mathcal{Q}}^i)\boldsymbol{\Pi}\boldsymbol{x}_i^0,\end{aligned} \tag{42}$$

where $\Phi(\boldsymbol{\mathcal{Q}}^i) := \int_{\boldsymbol{\mathcal{Q}}^i} \phi(\mathbf{q})\mathrm{d}\mathbf{q} = \int_{\mathbb{S}^1} \Phi_1(\boldsymbol{\mathcal{X}}_\vartheta^i)\phi_2(\vartheta)\mathrm{d}\vartheta$ (total "mass" of $\boldsymbol{\mathcal{Q}}^i$). Given that $\Phi(\boldsymbol{\mathcal{Q}}^i) > 0$, when $\boldsymbol{\mathcal{Q}}^i \subsetneq \boldsymbol{\mathcal{Q}}$ has a non-empty interior, we immediately conclude that the solution to the equation $\partial\mathcal{H}(\boldsymbol{Q}^0)/\partial\boldsymbol{x}_i^0 = \mathbf{0}$ is given by

$$\boldsymbol{x}_i^\star = (\int_{\mathbb{S}^1} \Phi_1(\boldsymbol{\mathcal{X}}_\vartheta^i)\phi_2(\vartheta)\boldsymbol{x}_{\mathrm{cm}|\vartheta}^i\mathrm{d}\vartheta)/\Phi(\boldsymbol{\mathcal{Q}}^i). \tag{43}$$

Furthermore, in view of (41), we have that

$$\begin{aligned}\partial\mathcal{H}(\boldsymbol{Q}^0)/\partial\theta_i^0 &= \partial\mathcal{H}_2(\boldsymbol{\Theta}^0)/\partial\theta_i^0 \\ &= \int_{\mathbb{S}^1}\int_{\boldsymbol{\mathcal{X}}_\vartheta^i} \phi_1(\mathbf{x})\phi_2(\vartheta)(\partial\mu_\vartheta^i/\partial\theta_i^0)\mathrm{d}\mathbf{x}\mathrm{d}\vartheta \\ &= \int_{\mathbb{S}^1} \Phi_1(\boldsymbol{\mathcal{X}}_\vartheta^i)\phi_2(\vartheta)(\partial\mu_\vartheta^i/\partial\theta_i^0)\mathrm{d}\vartheta.\end{aligned} \tag{44}$$

Therefore, $\partial\mathcal{H}(\boldsymbol{Q}^0)/\partial\theta_i^0 = 0$ is equivalent to

$$\int_{\mathbb{S}^1} \Phi_1(\boldsymbol{\mathcal{X}}_\vartheta^i)\phi_2(\vartheta)(1 - \cos(\theta_i^0 - \vartheta))\sin(\theta_i^0 - \vartheta)\mathrm{d}\vartheta = 0. \tag{45}$$

By applying standard trigonometric identities, Eq. (45) can be written as follow:

$$\sin(\theta_i^0)\int_{\mathbb{S}^1}\Phi_1(\boldsymbol{\mathcal{X}}_\vartheta^i)\phi_2(\vartheta)\cos(\vartheta)\mathrm{d}\vartheta$$
$$-\cos(\theta_i^0)\int_{\mathbb{S}^1}\Phi_1(\boldsymbol{\mathcal{X}}_\vartheta^i)\phi_2(\vartheta)\sin(\vartheta)\mathrm{d}\vartheta$$
$$-{}^1\!/\!_2\sin(2\theta_i^0)\int_{\mathbb{S}^1}\Phi_1(\boldsymbol{\mathcal{X}}_\vartheta^i)\phi_2(\vartheta)\cos(2\vartheta)\mathrm{d}\vartheta$$
$$+{}^1\!/\!_2\cos(2\theta_i^0)\int_{\mathbb{S}^1}\Phi_1(\boldsymbol{\mathcal{X}}_\vartheta^i)\phi_2(\vartheta)\sin(2\vartheta)\mathrm{d}\vartheta=0. \quad (46)$$

By using the following identity: $A\cos\theta + B\sin\theta = C\cos(\theta - \chi)$, where $C := \sqrt{A^2+B^2}$ and $\tan\chi = B/A$, Eq. (46) can be written as follows:

$$C_1\cos(\theta_i^0-\chi_1) - C_2\cos(2\theta_i^0-\chi_2) = 0, \quad (47)$$

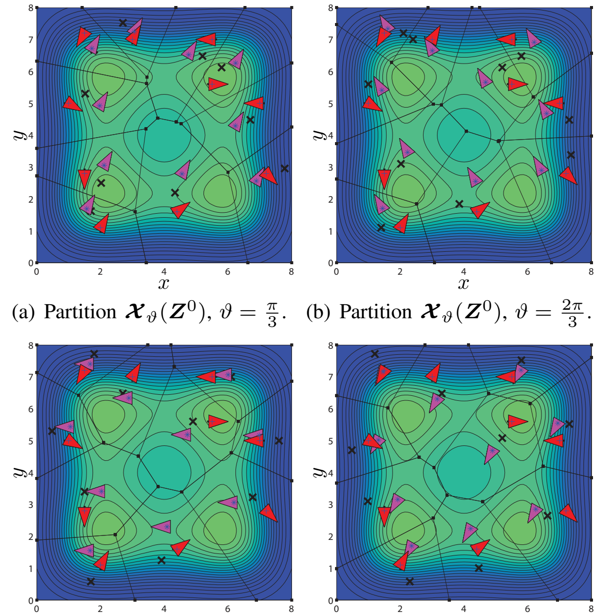
where the positive constants C_1 and C_2 and the angles χ_1 and χ_2 can be computed accordingly. It is easy to see that (47) will always have a solution in $[0, 2\pi]$ given that the graphs of the functions $f_i(\cdot) : [0, 2\pi] \to \mathbb{R}$, $i = 1, 2$, with values $f_1(\vartheta) := C_1\cos(\vartheta - \chi_1)$ and $f_2(\vartheta) := C_2\cos(2\vartheta - \chi_2)$ will always intersect. To see this, simply note that $f_1(\cdot)$ and $f_2(\cdot)$ are, respectively, 2π-periodic and π-periodic functions, from which it follows that both of them attain every single value in the intervals $[-C_1, C_1]$ and $[-C_2, C_2]$ as ϑ runs through $[0, 2\pi]$. If $C_1 > C_2$ (the case $C_1 < C_2$ can be treated similarly), then there will be $\vartheta_1, \vartheta_2 \in [0, 2\pi]$ with $\vartheta_1 \neq \vartheta_2$ such that $f_1(\vartheta_1) \in [-C_1, -C_2[$, which implies that $f_1(\vartheta_1) < f_2(\vartheta_1)$, and $f_1(\vartheta_2) \in]C_2, C_1]$, which implies that $f_1(\vartheta_2) > f_2(\vartheta_2)$. Consequently, the sign of the expression on the left hand side of Eq. (47) will necessarily change from negative at $\vartheta = \vartheta_1$ to positive at $\vartheta = \vartheta_2$ (or vice versa). This implies the existence of a root of Eq. (47) in $[0, 2\pi]$. Finally, if $C_1 = C_2$, then Eq. (47) becomes: $\theta_i^0 - \chi_1 = \pm(2\theta_i^0 - \chi_2 + 2k\pi)$, which always has a solution in $[0, 2\pi]$ for some $k \in \mathbb{Z}$.

Proposition 9: The function $\mathcal{H}(\cdot) : \boldsymbol{\mathcal{Q}}^n \to \mathbb{R}_{\geq 0}$, where $\mathcal{H}(\boldsymbol{Q}^0)$ is given in (39), attains its minimum value at the joint configuration $\boldsymbol{Q}^\star := \mathrm{col}(\boldsymbol{q}_1^\star, \dots, \boldsymbol{q}_n^\star)$ with $\boldsymbol{q}_i^\star = \mathrm{col}(\boldsymbol{x}_i^\star, \vartheta_i^\star)$, where $\boldsymbol{x}_i^\star$ is defined in Eq. (43) and $\vartheta_i^\star$ belongs necessarily to the non-empty subset of the compact interval $[0, 2\pi]$ that is comprised of the roots of Eq. (45), for all $i \in \mathcal{I}_n$.

Remark 8 Proposition 9 implies that the position component of the minimizer of $\mathcal{H}$ that is associated with the i-th agent corresponds to the weighted average of the centroids of the cells of this agent from the solution to each parametric partitioning problem in $\boldsymbol{\mathcal{T}}_\vartheta$, for $\vartheta \in \mathbb{S}^1$. This result is intuitive. On the other hand, one can find the optimal heading angle of the i-th agent by solving a single trigonometric equation, namely Eq. (45), in the compact interval $[0, 2\pi]$, (the latter equation always admits a solution, as we have already shown).

V. Numerical Simulations

For our simulations, we consider a network of ten agents whose initial positions, heading angles, and linear and angular velocities are chosen randomly. For these simulations, we have used the following data: $J = 0.1$, $m = 1$, $\varepsilon = 0.5$, and $\boldsymbol{\Lambda} := 0.5\mathbf{I}_2$. The density function $\phi_1(\cdot)$ in $\boldsymbol{\mathcal{X}}$ was taken to be $\phi_1(x, y) = \exp(0.1((x-4)^2 + (y-5)^2 - 0.15((x-4)^4 + (y-5)^4)))$, whereas $\phi_2(\vartheta) \equiv 1$ (no preference is attached to a particular terminal heading angle). Finally, the set $\boldsymbol{\mathcal{X}}$ is taken to be the square domain $[0, 8] \times [0, 8]$.



(a) Partition $\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0)$, $\vartheta = \frac{\pi}{3}$. (b) Partition $\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0)$, $\vartheta = \frac{2\pi}{3}$.

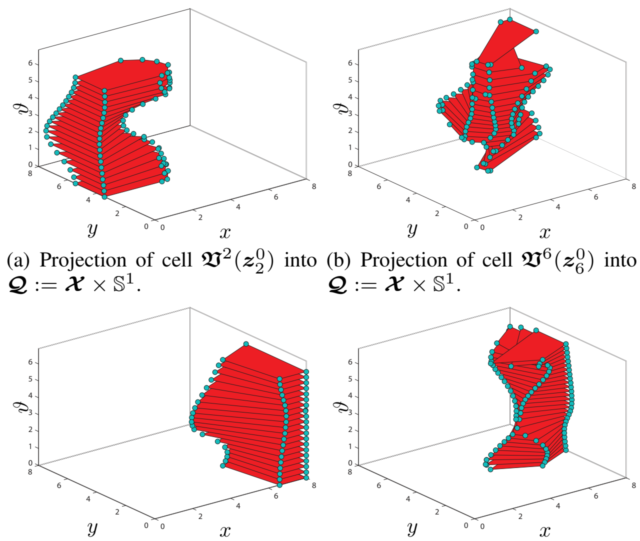
(c) Partition $\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0)$, $\vartheta = \pi$. (d) Partition $\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0)$, $\vartheta = \frac{4\pi}{3}$.

Fig. 5. The partition $\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0)$ of $\boldsymbol{\mathcal{X}}$, which is generated by a set of ten planar rigid bodies (their configurations in $\boldsymbol{\mathcal{Q}}$ correspond to the red triangles), may change significantly for different $\vartheta \in \mathbb{S}^1$. In these figures, the magenta triangles correspond to the minimizing configurations of $\mathcal{H}_\vartheta$ in $\boldsymbol{\mathcal{Q}}_\vartheta$, whereas the "$*$" markers correspond to the centroids of the cells of $\boldsymbol{\mathcal{X}}_\vartheta(\boldsymbol{Z}^0)$ with respect to the density function $\phi_1(\cdot)$, whose contours are also illustrated. Finally, the "$\times$" markers denote the points from the set $\boldsymbol{\Xi}_\vartheta := \{\boldsymbol{\xi}_\vartheta^i,\ i \in \mathcal{I}_{10}\}$.

Figure 5 illustrates the projection of the cells of the partition $\mathfrak{V}_\vartheta(\boldsymbol{Z}^0)$ of $\boldsymbol{\mathcal{T}}_\vartheta$ into $\boldsymbol{\mathcal{X}}$, when $\vartheta = \pi/3$ (Fig. 5(a)), $\vartheta = 2\pi/3$ (Fig. 5(b)), $\vartheta = \pi$ (Fig. 5(c)) and $\vartheta = 4\pi/3$ (Fig. 5(d)). Figure 6 shows the three-dimensional view of different cells of $\mathfrak{V}(\boldsymbol{Z}^0)$. As we can see in this figure, the three-dimensional (non-convex) cell of each agent corresponds to the outcome of stacking together the two-dimensional (convex) cells of the same agent from the solutions to the one-parameter family of two-dimensional partitioning problems over $\boldsymbol{\mathcal{T}}_\vartheta$, as the parameter ϑ runs through $\mathbb{S}^1$.

VI. Conclusion

In this paper, we have developed distributed algorithms for spatial partitioning and locational optimization problems for multi-agent networks in $SE(2)$. Two of the distinctive features of the problems considered herein is that 1) their domain is a non-flat manifold embedded in a higher-dimensional ambient space and 2) the proximity metric that measures the distance between an agent and a state in the latter manifold is a non-quadratic function. The key idea of our approach was to embed the original partitioning problem into a one-parameter family of problems whose domains have the required linear structure and their proximity metrics are parametric quadratic functions. In our future work, we plan to extend the ideas and techniques proposed in this work to partitioning and deployment problems involving heterogeneous multi-agent networks such as networks whose agents have different dynamics.



(a) Projection of cell $\mathfrak{V}^2(\boldsymbol{z}_2^0)$ into $\boldsymbol{\mathcal{Q}} := \boldsymbol{\mathcal{X}} \times \mathbb{S}^1$.
(b) Projection of cell $\mathfrak{V}^6(\boldsymbol{z}_6^0)$ into $\boldsymbol{\mathcal{Q}} := \boldsymbol{\mathcal{X}} \times \mathbb{S}^1$.
(c) Projection of cell $\mathfrak{V}^8(\boldsymbol{z}_8^0)$ into $\boldsymbol{\mathcal{Q}} := \boldsymbol{\mathcal{X}} \times \mathbb{S}^1$.
(d) Projection of cell $\mathfrak{V}^{10}(\boldsymbol{z}_{10}^0)$ into $\boldsymbol{\mathcal{Q}} := \boldsymbol{\mathcal{X}} \times \mathbb{S}^1$.

Fig. 6. Three-dimensional plots of the sets $\boldsymbol{\mathcal{Q}}^2(\boldsymbol{z}_2^0)$, $\boldsymbol{\mathcal{Q}}^6(\boldsymbol{z}_6^0)$, $\boldsymbol{\mathcal{Q}}^8(\boldsymbol{z}_8^0)$, and $\boldsymbol{\mathcal{Q}}^{10}(\boldsymbol{z}_{10}^0)$, which correspond to the projections of the cells $\mathfrak{V}^2(\boldsymbol{z}_2^0)$, $\mathfrak{V}^6(\boldsymbol{z}_6^0)$, $\mathfrak{V}^8(\boldsymbol{z}_8^0)$, and $\mathfrak{V}^{10}(\boldsymbol{z}_{10}^0)$, respectively, into the configuration space $\boldsymbol{\mathcal{Q}} := \boldsymbol{\mathcal{X}} \times \mathbb{S}^1$.

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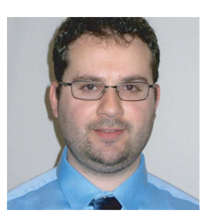

Efstathios Bakolas received his Diploma in Mechanical Engineering with highest honors from the National Technical University of Athens, Greece, in 2004 and his M.S. and Ph.D. degrees in Aerospace Engineering from the Georgia Institute of Technology, Atlanta, in 2007 and 2011, respectively. He is currently an assistant professor in the Department of Aerospace Engineering and Engineering Mechanics at The University of Texas at Austin. His research is mainly focused on optimal control theory (deterministic and stochastic) and the theory and applications of autonomous agents and multi-agent networks. Other interests include differential and dynamic games and optimization-based control.