Journal of Number Theory 210 (2020) 1–131

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

General Section

The local zeta function in enumerating quartic fields

Robert D. Hough [1]

Department of Mathematics, Stony Brook University, 100 Nicolls Road, Stony Brook, NY 11794, United States of America

A R T I C L E I N F O

Article history:
Received 26 July 2018
Received in revised form 20 November 2019
Accepted 30 December 2019
Available online 21 January 2020
Communicated by D. Goldfeld

Keywords:
Quartic number fields
Prehomogeneous vector space
Group action
Orbit decomposition
Exponential sum

A B S T R A C T

An exact formula is obtained for the Fourier transform of the local condition of maximality modulo primes $p > 3$ in the prehomogeneous vector space $2 \otimes \mathrm{Sym}^2(\mathbb{Z}_p^3)$ parametrizing quartic fields, thus solving the local 'quartic case' in enumerating quartic fields.

1. Introduction

In [9] Taniguchi and Thorne estimated the number of cubic fields with discriminant of size at most X, obtaining an asymptotic with a secondary main term and the current best known power saving error term. The method used the theory of Shintani zeta functions [7] enumerating the class numbers of binary cubic forms, together with a sieve sifting forms which are non-maximal at finitely many primes. An important ingredient in the sieve was an evaluation of the Fourier transform of the local indicator function of cubic rings

E-mail address: robert.hough@stonybrook.edu.

[1] Robert Hough is supported by NSF Grants DMS-1712682, "Probabilistic methods in discrete structures and applications," and DMS-1802336, "Analysis of Discrete Structures and Applications."

https://doi.org/10.1016/j.jnt.2019.12.017

over $\mathbb{Z}_p$ which are non-maximal [8], which showed that this signal is strongly localized. The purpose of this article is to obtain a corresponding exact formula for the Fourier transform of the indicator function of non-maximal quartic rings over $\mathbb{Z}_p$, thus solving the 'local quartic case' in enumerating number fields by discriminant. This formula may be used to obtain refined estimates for the equidistribution of the shapes of quartic fields paired with their cubic resolvent rings [5].

One reason that a refined error term is known for the count of cubic fields is that cubic fields may be identified with maximal cubic rings via the ring of integers, and cubic rings are parametrized by the prehomogeneous vector space of binary cubic forms, which has a rich algebraic structure. In the quartic case, there is a parallel structure in which pairs (Q, R) where Q is a quartic ring and R is its cubic resolvent ring are parametrized by the prehomogeneous vector space of pairs of ternary quadratic forms [2]. In [6] Sato and Shintani developed the theory of zeta functions enumerating class numbers of integral orbits in general prehomogeneous vector spaces. In [12] Wright and Yukie explain how zeta functions of prehomogeneous vector spaces may be used to enumerate étale quadratic, cubic, quartic and quintic number fields, and in [13] Yukie developed many of the analytic properties in the quartic case, identifying the order and location of the poles of the quartic zeta function. In [5] the author introduced an analogue of the zeta function in the quartic case, twisted by a cusp form evaluated at the lattice shape of each ring, and proved that the resulting object is entire. This avoided some technical difficulties in treating the quartic zeta function, where the residues are still unknown, and made it possible to obtain equidistribution results for the shape of quartic fields via the zeta function method. The current work permits obtaining refined estimates in the cuspidal spectrum of the shape of quartic fields, since an exact formula for the coefficients may be used in the dual zeta function which appears there.

In [1], [2], and [3] Bhargava obtained parameterizations of integral cubic, quartic and quintic rings, in particular giving local conditions modulo p^2 describing the maximality of such rings. Here the exact formula for the Fourier transform of the set of maximal quartic rings over $\mathbb{Z}_p$ is obtained by describing the relevant orbit structure of $\mathrm{GL}_3 \times \mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$ acting on pairs of ternary quadratic forms $2 \otimes \mathrm{Sym}^2((\mathbb{Z}/p^2\mathbb{Z})^3)$ and evaluating the orbital exponential sums. This method of evaluating the Fourier transform has the advantage that it extends to treat other local conditions besides the condition of maximality.

1.1. The space $2 \otimes \mathrm{Sym}^2(R^3)$

Let R be a commutative ring with unit. A quartic ring over R is a free rank 4 R-module with a ring structure. Elements of the space of ternary quadratic forms over R, $\mathrm{Sym}^2(R^3)$ are written

$$A = \begin{pmatrix} a & \frac{b}{2} & \frac{c}{2} \\ \frac{b}{2} & d & \frac{e}{2} \\ \frac{c}{2} & \frac{e}{2} & f \end{pmatrix}, \tag{1.1}$$

corresponding to the form

$$(u,v,w)A\begin{pmatrix}u\\v\\w\end{pmatrix} = au^2+buv+cuw+dv^2+evw+fw^2. \tag{1.2}$$

Forms are typically indicated by showing only those elements on or above the diagonal. There is a natural inner product on $\mathrm{Sym}^2(R^3)$,

$$[A,A'] = \mathrm{tr}\, A^tA' = aa' + \frac{bb'}{2} + \frac{cc'}{2} + dd' + \frac{ee'}{2} + ff'. \tag{1.3}$$

An element $g \in \mathrm{GL}_3(R)$ acts on $\mathrm{Sym}^2(R^3)$ by $g\cdot(A) = gAg^t$. Note that

$$[g\cdot A, (g^{-1})^t\cdot A'] = \mathrm{tr}\, gA^tg^t(g^{-1})^tA'g^{-1} = \mathrm{tr}\, gA^tA'g^{-1} = \mathrm{tr}\, A^tA' = [A,A']. \tag{1.4}$$

The space $V(R) = 2\otimes \mathrm{Sym}^2(R^3)$ consists of pairs of ternary quadratic forms (A,B). The group $G(R) = \mathrm{GL}_3(R)\times \mathrm{GL}_2(R)$ acts on $V(R)$ with the GL_3 factor acting on each ternary quadratic form separately, and the $\mathrm{GL}_2(R)$ factor forming linear combinations of the two forms, that is $(g_3, g_2 = \begin{pmatrix}a & b\\ c & d\end{pmatrix}) \in G(R)$ act by

$$(g_3,g_2)\cdot(A,B) = (ag_3Ag_3^t + bg_3Bg_3^t, cg_3Ag_3^t + dg_3Bg_3^t). \tag{1.5}$$

The inner product on $2\otimes\mathrm{Sym}^2(R^3)$ is

$$[(A,B),(A',B')] = \mathrm{tr}\begin{pmatrix}[A,A'] & [A,B']\\ [B,A'] & [B,B']\end{pmatrix} = [A,A'] + [B,B']. \tag{1.6}$$

Thus, with $(g^{-1})^t = ((g_3^{-1})^t, (g_2^{-1})^t)$,

$$[g\cdot(A,B), (g^{-1})^t\cdot(A',B')] = [(A,B),(A',B')]. \tag{1.7}$$

With this action, $2\otimes\mathrm{Sym}^2(R^3)$ is a prehomogeneous vector space, see [6].

In [2] the following theorem is proved parameterizing quartic rings.

Theorem *([2], Theorem 1). There is a canonical bijection between the set of* $\mathrm{GL}_3(\mathbb{Z})\times \mathrm{GL}_2(\mathbb{Z})$*-orbits on the space* $(2\otimes\mathrm{Sym}^2(\mathbb{Z}^3))^*$ *of pairs of integral ternary quadratic forms and the set of isomorphism classes of pairs* (Q,S)*, where* Q *is a quartic ring over* $\mathbb{Z}$ *and* S *is a cubic resolvent ring of* Q*.*

It is also verified that the condition that a quartic ring is maximal may be checked locally by tensoring with $\mathbb{Z}_p$ for each prime p, and that the condition that a quartic ring over $\mathbb{Z}_p$ is maximal can be checked modulo p^2. Call a pair of ternary quadratic forms over $\mathbb{Z}_p$ maximal if it is associated to quartic ring over $\mathbb{Z}_p$ which is maximal.

The basic object of study in this article are the *orbital exponential sums*, for $x, \xi \in V(\mathbb{Z}/p^2\mathbb{Z})$,[2]

$$S_{p^2}(x,\xi) = \sum_{g \in G(\mathbb{Z}/p^2\mathbb{Z})} e\left(\frac{1}{p^2}[g \cdot x, \xi]\right); \tag{1.8}$$

the Fourier transform of the indicator functions of maximal and non-maximal forms is a linear combination of such sums. The size and distribution of these sums for varying ξ describe the distribution of the orbit containing the point x within the space $V(\mathbb{Z}/p^2\mathbb{Z})$. In [10] the orbits of the space $V(\mathbb{Z}/p\mathbb{Z})$ are enumerated and the orbital exponential sums, for $x, \xi \in V(\mathbb{Z}/p\mathbb{Z})$,

$$S_p(x,\xi) = \sum_{g \in G(\mathbb{Z}/p\mathbb{Z})} e\left(\frac{1}{p}[g \cdot x, \xi]\right) \tag{1.9}$$

are calculated. The orbits are listed in the table below. Throughout ℓ denotes a non-square modulo p and

$$s(a,b,c,d) = (p-1)^a p^b (p+1)^c (p^2+p+1)^{\frac{d}{2}}. \tag{1.10}$$

The orbits labeled $\mathcal{O}_{1^4} - \mathcal{O}_4$ indicate the intersection type of the pair of ternary quadratic forms, while the remaining orbits follow the naming convention of [10].

Orbit	Representative	Orbit size	Stabilizer size
$\mathcal{O}_0$	$(0,0)$	1	$s(5,4,2,2)$
$\mathcal{O}_{D1^2}$	$(0,w^2)$	$s(1,0,1,2)$	$s(4,4,1,0)$
$\mathcal{O}_{D11}$	$(0,vw)$	$s(1,1,2,2)/2$	$2s(4,3,0,0)$
$\mathcal{O}_{D2}$	$(0,v^2-\ell w^2)$	$s(2,1,1,2)/2$	$2s(3,3,1,0)$
$\mathcal{O}_{Dns}$	$(0,u^2-vw)$	$s(2,2,1,2)$	$s(3,2,1,0)$
$\mathcal{O}_{Cs}$	(w^2,vw)	$s(2,1,2,2)$	$s(3,3,0,0)$
$\mathcal{O}_{Cns}$	(vw,uw)	$s(2,3,1,2)$	$s(3,1,1,0)$
$\mathcal{O}_{B11}$	(w^2,v^2)	$s(2,2,2,2)/2$	$2s(3,2,0,0)$
$\mathcal{O}_{B2}$	$(vw,v^2+\ell w^2)$	$s(3,2,1,2)/2$	$2s(2,2,1,0)$
$\mathcal{O}_{1^4}$	$(w^2,uw+v^2)$	$s(3,2,2,2)$	$s(2,2,0,0)$
$\mathcal{O}_{1^31}$	$(vw,uw+v^2)$	$s(3,3,2,2)$	$s(2,1,0,0)$
$\mathcal{O}_{1^21^2}$	(w^2,uv)	$s(2,4,2,2)/2$	$2s(3,0,0,0)$
$\mathcal{O}_{2^2}$	$(w^2,u^2-\ell v^2)$	$s(3,4,1,2)/2$	$2s(2,0,1,0)$
$\mathcal{O}_{1^211}$	(v^2-w^2,uw)	$s(3,4,2,2)/2$	$2s(2,0,0,0)$
$\mathcal{O}_{1^22}$	$(v^2-\ell w^2,uw)$	$s(3,4,2,2)/2$	$2s(2,0,0,0)$
$\mathcal{O}_{1111}$	$(uw-vw,uv-vw)$	$s(4,4,2,2)/24$	$24s(1,0,0,0)$
$\mathcal{O}_{112}$	$(vw,u^2-v^2-\ell w^2)$	$s(4,4,2,2)/4$	$4s(1,0,0,0)$
$\mathcal{O}_{22}$	$(vw,u^2-\ell v^2-\ell w^2)$	$s(4,4,2,2)/8$	$8s(1,0,0,0)$
$\mathcal{O}_{13}$	$(uw-v^2,B_3)$	$s(4,4,2,2)/3$	$3s(1,0,0,0)$
$\mathcal{O}_4$	$(uw-v^2,B_4)$	$s(4,4,2,2)/4$	$4s(1,0,0,0)$

(1.11)

[2] The notation $e(\cdot)$ indicates the additive character on $\mathbb{R}$, $e(x) = e^{2\pi i x}$. The notation, for positive integer m, $e_m(x) = e\left(\frac{x}{m}\right)$ is also used.

The items B_3 and B_4 indicate $B_3 = uv + a_3v^2 + b_3vw + c_3w^2$ and $B_4 = u^2 + a_4uv + b_4v^2 + c_4vw + d_4w^2$ where $X^3 + a_3X^2 + b_3X + c_3$ and $X^4 + a_4X^3 + b_4X^2 + c_4X + d_4$ are irreducible over $\mathbb{Z}/p\mathbb{Z}$.

In this article the orbits modulo p^2 relevant to the condition of maximality are identified, along with the relevant exponential sums. An illustrative theorem from the calculation is as follows. Let $\mathbf{1}_{\text{non-max}}$ be the indicator function of forms in $V(\mathbb{Z}/p^2\mathbb{Z})$ which correspond to quartic rings non-maximal at p.

Theorem 1. *For $p > 3$ the Fourier transform*

$$\widehat{\mathbf{1}_{\text{non-max}}}(\xi) = \sum_{x \in V(\mathbb{Z}/p^2\mathbb{Z})} \mathbf{1}_{\text{non-max}}(x) e_{p^2}([x, \xi])$$

is supported on the $\bmod p$ *orbits $\mathscr{O}_0$, $\mathscr{O}_{D1^2}$, $\mathscr{O}_{D11}$ and $\mathscr{O}_{D^2}$. It satisfies*

$$\begin{aligned}\left\|\widehat{\mathbf{1}_{\text{non-max}}}\right\|_1 &= 2p^{29} + 2p^{28} + 4p^{27} - 8p^{26} - 19p^{25} - 2p^{24} + 20p^{23} + 24p^{22} - 5p^{21} \\ &\quad - 17p^{20} - 5p^{19} + 3p^{18} + 2p^{17} - 2p^{16} + p^{15} + p^{14}, \\ \left\|\widehat{\mathbf{1}_{\text{non-max}}}\right\|_2^2 &= p^{46} + 2p^{45} + 2p^{44} - 3p^{43} - 4p^{42} - p^{41} + 3p^{40} + 3p^{39} - p^{38} - p^{37}, \\ \left|\operatorname{supp} \widehat{\mathbf{1}_{\text{non-max}}}\right| &= 2p^{15} + p^{14} - 2p^{13} - p^{12} + 2p^{10} - p^8.\end{aligned}$$

An exact formula for the Fourier transform is given following the calculation of the relevant orbital exponential sums in Section 6. For applications, an important feature of this calculation is that the Fourier transform has small support, and that the L^1 norm saves a power of $p^{1.5}$ compared to a naive application of Cauchy-Schwarz using the already strong bound for the support. In practice, the exact evaluation of the Fourier transform can quantify the intuition that the condition of maximality is captured significantly modulo p, with a small refinement modulo p^2. One potential application of the exact formula is to the enumeration of quartic fields. In this connection, a sieve is introduced to sift for maximal quartic rings. Dualizing the resulting sums introduces the Fourier transform of the condition of maximality, where an exact formula for the coefficients may be inserted, see for instance the author's preprint [5] treating the shape of quartic fields. In this article, the exact formula for the Fourier transform at 2 and 3 has been omitted since the classification of the orbits is a little different in these cases. The orbit classification may be performed using a computer algebra system in these cases.

One conceptual reason for the small size of the support is that if $\xi \not\equiv 0 \bmod p$, the orbital exponential sum $S_{p^2}(x, \xi)$ can be shown to vanish unless the $\bmod p$ orbit of x contains elements orthogonal to the p-adic tangent space of the orbit $\mathscr{O}_\xi$ at $\xi \bmod p$. A second key observation is that the $\bmod p^2$ orbits above a $\bmod p$ orbit $\mathscr{O}_x$ are in bijection with the orbits of the action of the $\bmod p$ stabilizer G_x acting on a quotient by the tangent space. These key observations are explained in Sections 3 and 4, prior to the determination of the relevant orbits and evaluation of the corresponding exponential sums.

This paper contains a significant number of explicit matrix calculations, and, while self-contained, is intended to be read along with supporting Mathematica files currently available from the author's homepage https://www.math.stonybrook.edu/~rdhough/. The corresponding Mathematica notebook [11] is indicated in **bold**. Small cases of the orbit decompositions were obtained in Magma [4].

1.2. Organization of the paper

The paper is organized as follows.

- Section 2 collects together the statements describing the maximality condition at p and the local densities calculated by Bhargava in [2].
- Section 3 introduces the concept of a p-adic tangent space to a pair of forms, and describes the theory regarding how this is used to calculate the orbital exponential sums. In particular, many of the sums are shown to vanish in this Section.
- Section 4 proves a bijection between the mod p^2 orbits above a given mod p orbit, and the orbits of the mod p stabilizer acting on a quotient space. The mod p^2 orbits are then classified using this relationship.
- Section 5 performs the orbital exponential sums in coordinates, using the classification of the orbits.
- Section 6 collects together the exponential sums to calculate the Fourier transform of the condition of maximality and proves Theorem 1.

1.3. Background and notation

Throughout, p denotes a fixed odd prime, $p > 3$ and ℓ denotes a fixed quadratic non-residue mod p. Given a commutative ring R with unit, $\mathrm{Sym}^2(R^n)$ denotes n-ary quadratic forms with coefficients in R, identified with $n \times n$ symmetric matrices with R coefficients. In this work, R will always be one of the rings $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Z}/p\mathbb{Z} = \mathbf{F}_p$ or $\mathbb{Z}/p^2\mathbb{Z}$. $M_n(R)$ denotes $n \times n$ matrices over R. $V(R)$ denotes the space $2 \otimes \mathrm{Sym}^2(R^3)$ of pairs (A, B) of ternary quadratic forms over R. The algebraic group $G(R) = \mathrm{GL}_3(R) \times \mathrm{GL}_2(R)$ acts on $V(R)$ with $(g_3, g_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in G(R)$ acting by

$$(g_3, g_2) \cdot (A, B) = (a g_3 A g_3^t + b g_3 B g_3^t, c g_3 A g_3^t + d g_3 B g_3^t). \tag{1.12}$$

Given a form x, write $\|x\|_\infty$ for the maximum absolute value of the coefficients. In the case that x is defined over $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}/p^2\mathbb{Z}$, choose representatives in $\mathbb{Z}$ for the coefficients which have minimal absolute value.

The following lemma recalls the classical orbit description of $\mathrm{GL}_2(\mathbf{F}_p) \times \mathrm{GL}_1(\mathbf{F}_p)$ acting on $\mathrm{Sym}^2(\mathbf{F}_p^2)$, in which the GL_2 factor acts by change of variable, and the GL_1 factor acts by multiplication by a scalar.

Lemma 1. *The action of* $\mathrm{GL}_2(\mathbf{F}_p) \times \mathrm{GL}_1(\mathbf{F}_p)$ *on the space* $\mathrm{Sym}^2(\mathbf{F}_p^2)$ *of binary quadratic forms in variables* u, v *has four orbits with representatives* $0, u^2, uv$, *and* $u^2 - \ell v^2$.

Proof. If the form is non-zero and reducible, map one linear factor to u. The second linear factor is either the same, which obtains u^2, or different, in which case it may be mapped to v. If the form is non-zero and irreducible, then the coefficient on u^2 is non-zero, and thus, by completing the square, the uv coefficient may be set to 0. After scaling the variables and multiplying by a scalar, the form $u^2 - \ell v^2$ is obtained. $\square$

A related smaller group action is also used.

Lemma 2. *When the subgroup* $\begin{pmatrix} r & \\ a & s \end{pmatrix} \times \mathrm{GL}_1(\mathbf{F}_p)$ *of* $\mathrm{GL}_2(\mathbf{F}_p) \times \mathrm{GL}_1(\mathbf{F}_p)$ *acts on the space* $\mathrm{Sym}^2(\mathbf{F}_p^2)$ *of binary quadratic forms in variables* u, v *there are six orbits with representatives* $0, u^2, v^2, u(u+v), uv, u^2 - \ell v^2$.

Proof. Given a non-zero quadratic form $x_1u^2 + x_2uv + x_3v^2$, if the x_1 coordinate is non-zero then any equivalent form will have non-vanishing u^2 coordinate since $r \neq 0$ and v may only be scaled by the action. In this case, make a change of variables in u to eliminate the x_2 coordinate. After scaling and dilating u and v, the form is equivalent to one of $u^2, u^2 - v^2$ or $u^2 - \ell v^2$, each of which is inequivalent since the first is a double line, the second is a quadratic reducible with distinct linear factors, and the third is irreducible. Under a change of coordinates $u^2 - v^2$ is equivalent to $u(u+v)$.

Those non-zero forms with $x_1 = 0$ are inequivalent to those above and contain v as a factor. After possibly making a change of variable in u and after dilating and scaling the form is equivalent to either uv or v^2, which are inequivalent. $\square$

Given a group G acting on a set $\mathscr{X}$, $G_x = \mathrm{Stab}_G(x)$ denotes the stabilizer in G of the point $x \in \mathscr{X}$. $\mathscr{O}_x = \{g \cdot x : g \in G\}$ denotes the orbit containing x. Given a pair of ternary quadratic forms x, $\mathscr{O}_x \bmod p$ indicates the orbit of $x \in V(\mathbf{F}_p)$ under the action of $G(\mathbf{F}_p)$, while $\mathscr{O}_x \bmod p^2$ indicates the orbit of x in $V(\mathbb{Z}/p^2\mathbb{Z})$ under the action of $G(\mathbb{Z}/p^2\mathbb{Z})$.

Given a positive integer m, the notations $e(x) = e^{2\pi i x}$ and $e_m(x) = e^{\frac{2\pi i x}{m}}$ are used for complex exponentials. The Legendre symbol is indicated $\left(\frac{\cdot}{p}\right)$, which satisfies

$$\sum_{a \bmod p} \left(\frac{a}{p}\right) = 0. \tag{1.13}$$

The quadratic Gauss sum modulo p is indicated

$$\tau = \sum_{n \in \mathbf{F}_p} e_p(n^2). \tag{1.14}$$

This satisfies, for $a \not\equiv 0 \bmod p$,

$$\tau \left(\frac{a}{p}\right) = \sum_{n \in \mathbf{F}_p} e_p(an^2). \tag{1.15}$$

Also, $\tau^2 = \left(\frac{-1}{p}\right) p$.

2. Conditions of maximality

A quartic ring over $\mathbb{Z}$ is maximal if it is not a proper subring of another quartic ring. Maximality is a condition which may be checked locally. In [2] it is shown that the condition of maximality may be checked modulo p^2, and congruences identifying maximal quartic rings are given. The conclusions of [2], pp. 1355-1360 are summarized in the following lemma.

Lemma 3. *A quartic ring maximal at p comes from one of the following orbits modulo p*

$$\mathscr{O}_{1111}, \mathscr{O}_{112}, \mathscr{O}_{13}, \mathscr{O}_{22}, \mathscr{O}_{4}, \mathscr{O}_{1^211}, \mathscr{O}_{1^22}, \mathscr{O}_{1^21^2}, \mathscr{O}_{2^2}, \mathscr{O}_{1^31}, \mathscr{O}_{1^4}. \tag{2.1}$$

Of the orbits listed, all elements of the $\bmod p$ *orbits*

$$\mathscr{O}_{1111}, \mathscr{O}_{112}, \mathscr{O}_{13}, \mathscr{O}_{22}, \mathscr{O}_{4} \tag{2.2}$$

are maximal. An element of $\mathscr{O}_{1^211}, \mathscr{O}_{1^22}, \mathscr{O}_{1^21^2}, \mathscr{O}_{1^31}, \mathscr{O}_{1^4}$ is non-maximal if and only if it is $G(\mathbb{Z}/p^2\mathbb{Z})$ equivalent to a form

$$\begin{pmatrix} 0 \bmod p^2 & 0 \bmod p & 0 \bmod p \\ & * & * \\ & & * \end{pmatrix} \begin{pmatrix} 0 \bmod p & * & * \\ & * & * \\ & & * \end{pmatrix}. \tag{2.3}$$

A form in $\mathscr{O}_{2^2}$ is non-maximal if and only if (A, B) is equivalent to a form

$$\begin{pmatrix} 0 \bmod p^2 & 0 \bmod p^2 & 0 \bmod p \\ & 0 \bmod p^2 & 0 \bmod p \\ & & * \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}. \tag{2.4}$$

Note that the forms A, B have been exchanged compared to [2].

Proof. This is stated in geometric terms, rather than as orbits under the group action in [2]. The two notions coincide except in the case of a single point of intersection of multiplicity 4. Note that the elements satisfying the geometric condition $T_p^{(1)}$ of [2] correspond to the orbit $\mathscr{O}_{1^4}$, and those satisfying the condition $T_p^{(2)}$ are the union of the two orbits $\mathscr{O}_{B11} \sqcup \mathscr{O}_{B2}$ and are non-maximal. Indeed, the representative $(w^2, uw + v^2)$ of the orbit $\mathscr{O}_{1^4}$ has $w^2 = 0$, $uw + v^2 = 0$ intersecting at a single point $(1, 0, 0)$ of multiplicity 4, and has $uw + v^2$ irreducible, thus satisfying the geometric condition of $T_p^{(1)}$, while the representatives (w^2, v^2) and $(vw, v^2 + \ell w^2)$ of $\mathscr{O}_{B11}$ and $\mathscr{O}_{B2}$ again have a single point $(1, 0, 0)$ of intersection of multiplicity 4, but are equivalent to forms with both forms reducible, meeting the geometric condition of $T_p^{(2)}$, see [2] p. 1353. Since $|\mathscr{O}_{1^4}| = (p-1)^3p^2(p+1)^2(p^2+p+1)$ and $|\mathscr{O}_{B11}|+|\mathscr{O}_{B2}| = (p-1)^2p^3(p+1)(p^2+p+1)$, these orbits make up the full count of elements in $T_p^{(1)}(1^4)$ and $T_p^{(2)}(1^4)$, see [2], Lemma 21. □

The density of maximal elements within the mod p orbits is also determined by [2] and is summarized in the following lemma.

Lemma 4. *The following table lists those* mod p *orbits which contain maximal forms, and the density of maximal forms within each such orbit.*

Orbit	$\mathscr{O}_{1111}$	$\mathscr{O}_{112}$	$\mathscr{O}_{13}$	$\mathscr{O}_{22}$	$\mathscr{O}_{4}$	$\mathscr{O}_{1^211}$	$\mathscr{O}_{1^22}$	$\mathscr{O}_{1^21^2}$	$\mathscr{O}_{2^2}$	$\mathscr{O}_{1^31}$	$\mathscr{O}_{1^4}$
Density	1	1	1	1	1	$\frac{p-1}{p}$	$\frac{p-1}{p}$	$\left(\frac{p-1}{p}\right)^2$	$\frac{p^2-1}{p^2}$	$\frac{p-1}{p}$	$\frac{p-1}{p}$

(2.5)

3. The orbital exponential sums

Let $\mathbf{1}_{\max}$ denote the indicator function on $V(\mathbb{Z}/p^2\mathbb{Z})$ of those forms corresponding to rings maximal at p, $\mathbf{1}_{\text{non-max}} = 1 - \mathbf{1}_{\max}$. For $\xi \in V(\mathbb{Z}/p^2\mathbb{Z})$, the Fourier transforms are

$$\widehat{\mathbf{1}_{\max}}(\xi) = \sum_{x \in V(\mathbb{Z}/p^2\mathbb{Z})} \mathbf{1}_{\max}(x) e_{p^2}([x, \xi]), \qquad \widehat{\mathbf{1}_{\text{non-max}}} = p^{24}\delta_0 - \widehat{\mathbf{1}_{\max}}. \tag{3.1}$$

Since the condition of maximality is invariant under the action of $G(\mathbb{Z}/p^2\mathbb{Z})$, the Fourier transform is the linear combination of orbital exponential sums.

Definition 1. Given $x, \xi \in V(\mathbb{Z}/p^2\mathbb{Z})$, define their *orbital exponential sum*

$$S_{p^2}(x, \xi) = \sum_{g \in G(\mathbb{Z}/p^2\mathbb{Z})} e_{p^2}([g \cdot x, \xi]). \tag{3.2}$$

Define, also, the *unweighted orbital exponential sum*

$$\Sigma_{p^2}(x, \xi) = \sum_{x' \in \mathscr{O}_x \bmod p^2} e_{p^2}([x', \xi]) = \frac{S_{p^2}(x, \xi)}{|\operatorname{Stab}_{G(\mathbb{Z}/p^2\mathbb{Z})}(x)|}. \tag{3.3}$$

When $\xi = p\xi_1$ is an element of $pV(\mathbb{Z})$, the exponential sum reduces to the mod p exponential sum

$$S_{p^2}(x, \xi) = p^{13} S_p(x, \xi_1) = p^{13} \sum_{g \in G(\mathbb{Z}/p\mathbb{Z})} e_p([g \cdot x, \xi_1]). \tag{3.4}$$

These are obtained in [10].

As an intermediate step between the orbital exponential sums and the full Fourier transform of $\mathbf{1}_{\text{max}}$ and $\mathbf{1}_{\text{non-max}}$, exponential sums taken over those maximal elements above a mod p orbit are calculated.

Definition 2. Given $x \in V(\mathbb{Z}/p\mathbb{Z})$ and $\xi \in V(\mathbb{Z}/p^2\mathbb{Z})$, define the *maximal exponential sum*

$$\mathscr{M}(x,\xi) = \sum_{\substack{x' \in V(\mathbb{Z}/p^2\mathbb{Z}) \\ x' \in \mathscr{O}_x \bmod p \\ \text{maximal}}} e_{p^2}([x',\xi]). \tag{3.5}$$

Since the condition of maximality is $G(\mathbb{Z}/p^2\mathbb{Z})$-invariant, the maximal exponential sums may be expressed as a sum over unweighted orbital exponential sums,

$$\mathscr{M}(x,\xi) = \sum_{\substack{\mathscr{O}_{x'} \bmod p^2 : \mathscr{O}_{x'} \subset \mathscr{O}_x \bmod p \\ \text{maximal}}} \Sigma_{p^2}(x',\xi). \tag{3.6}$$

The following lemma reduces the number of orbits $\mathscr{O}_x \bmod p$ which need to be considered in determining the Fourier transforms of $\mathbf{1}_{\text{max}}$ and $\mathbf{1}_{\text{non-max}}$.

Lemma 5. *If $\xi \not\equiv 0 \bmod p$, then for all x,*

$$\sum_{\substack{x' \in V(\mathbb{Z}/p^2\mathbb{Z}) \\ x' \in \mathscr{O}_x \bmod p}} e_{p^2}\left([x',\xi]\right) = 0. \tag{3.7}$$

Similarly, if $x \not\equiv 0 \bmod p$, then for all ξ,

$$\sum_{\substack{\xi' \in V(\mathbb{Z}/p^2\mathbb{Z}) \\ \xi' \in \mathscr{O}_\xi \bmod p}} e_{p^2}\left([x,\xi']\right) = 0. \tag{3.8}$$

Thus

$$\mathscr{M}(x,\xi) = -\sum_{\substack{x' \in V(\mathbb{Z}/p^2\mathbb{Z}) \\ x' \in \mathscr{O}_x \bmod p \\ \textit{non-maximal}}} e_{p^2}([x',\xi]). \tag{3.9}$$

In particular, if $\mathscr{O}_x \bmod p$ contains only maximal, or only non-maximal elements, then for all $\xi \not\equiv 0 \bmod p$,

$$\mathscr{M}(x,\xi) = 0. \tag{3.10}$$

Proof. For the first claim, write

$$\sum_{\substack{x'\in V(\mathbb{Z}/p^2\mathbb{Z})\\ x'\in\mathscr{O}_x \bmod p}} e_{p^2}\left([x',\xi]\right) = \frac{1}{p^{12}} \sum_{x''\in V(\mathbb{Z}/p\mathbb{Z})} \sum_{\substack{x'\in V(\mathbb{Z}/p^2\mathbb{Z})\\ x'\in\mathscr{O}_x \bmod p}} e_{p^2}\left([x'+px'',\xi]\right) \tag{3.11}$$

$$= \sum_{\substack{x'\in V(\mathbb{Z}/p^2\mathbb{Z})\\ x'\in\mathscr{O}_x \bmod p}} e_{p^2}\left([x',\xi]\right) \frac{1}{p^{12}} \sum_{x''\in V(\mathbb{Z}/p\mathbb{Z})} e_p\left([x'',\xi]\right) = 0.$$

The claim with the roles of x and ξ reversed follows by symmetry.

The third statement is now obvious.

If $\mathscr{O}_x$ mod p contains only non-maximal elements then the sum defining $\mathscr{M}(x,\xi)$ is empty, hence 0, which combined with the second claim, proves the third claim. □

As a consequence of Lemmas 4 and 5, when $\xi \not\equiv 0 \bmod p$,

$$\widehat{\mathbf{1}_{\max}}(\xi) = \sum_x \mathscr{M}(x,\xi) \tag{3.12}$$

with x ranging over representatives for $\mathscr{O}_{1^211}, \mathscr{O}_{1^22}, \mathscr{O}_{1^21^2}, \mathscr{O}_{2^2}, \mathscr{O}_{1^31}, \mathscr{O}_{1^4}$ mod p.

3.1. Annihilator spaces

An important observation in passing from mod p orbits and orbital exponential sums to mod p^2 orbits and orbital exponential sums is that the group action may be linearized p-adically.

Definition 3. Given a form $x \in V(\mathbb{Z}/p^2\mathbb{Z})$, define the *annihilator subspace*, or *p-adic tangent space* V_x associated to x to be a subspace of $V(\mathbb{Z}/p^2\mathbb{Z})$ defined by

$$(I + pM_3(\mathbb{Z}/p\mathbb{Z}), I + pM_2(\mathbb{Z}/p\mathbb{Z})) \cdot x = x + pV_x. \tag{3.13}$$

Identify V_x naturally as a subspace of $V(\mathbb{Z}/p\mathbb{Z})$.

The notion of p-adic tangent space depends only on x mod p and not on the residue modulo p^2.

The orbits $\mathscr{O}_{1111}, \mathscr{O}_{112}, \mathscr{O}_{13}, \mathscr{O}_{22}, \mathscr{O}_4$, which have full dimension and contain only maximal elements, have annihilator subspaces equal to $V(\mathbb{Z}/p\mathbb{Z})$. This fact is not needed to calculate the Fourier transform and is not explicitly proved here. However, a verification is provided in **annihilator_spaces.nb**. The annihilator subspaces of orbit representatives of the remaining non-zero orbits are listed below, with computation postponed to Appendix A.

Orbit	Representative	Annihilator subspace
$\mathscr{O}_{D1^2}$	$(0, w^2)$	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & * \end{pmatrix} \begin{pmatrix} 0 & 0 & * \\ & 0 & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{D11}$	$(0, vw)$	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & * \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & * & * \\ & * & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{D2}$	$(0, v^2 - \ell w^2)$	$\begin{pmatrix} 0 & 0 & 0 \\ & z & 0 \\ & & -z\ell \end{pmatrix} \begin{pmatrix} 0 & * & * \\ & * & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{Dns}$	$(0, u^2 - vw)$	$\begin{pmatrix} z & 0 & 0 \\ & 0 & -\frac{z}{2} \\ & & 0 \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{Cs}$	(w^2, vw)	$\begin{pmatrix} 0 & 0 & z \\ & 0 & * \\ & & * \end{pmatrix} \begin{pmatrix} 0 & \frac{z}{2} & * \\ & * & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{Cns}$	(vw, uw)	$\begin{pmatrix} 0 & \frac{z}{2} & * \\ & y & * \\ & & * \end{pmatrix} \begin{pmatrix} z & \frac{y}{2} & * \\ & 0 & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{B11}$	(w^2, v^2)	$\begin{pmatrix} 0 & * & 0 \\ & * & * \\ & & * \end{pmatrix} \begin{pmatrix} 0 & 0 & * \\ & * & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{B2}$	$(vw, v^2 + \ell w^2)$	$\begin{pmatrix} 0 & \frac{z}{2} & \frac{y}{2} \\ & * & * \\ & & * \end{pmatrix} \begin{pmatrix} 0 & y & z\ell \\ & * & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{1^4}$	$(w^2, uw + v^2)$	$\begin{pmatrix} 0 & 0 & y + \frac{z}{2} \\ & z & * \\ & & * \end{pmatrix} \begin{pmatrix} y & * & * \\ & * & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{1^3 1}$	$(vw, uw + v^2)$	$\begin{pmatrix} 0 & \frac{z}{2} & * \\ & * & * \\ & & * \end{pmatrix} \begin{pmatrix} z & * & * \\ & * & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{1^2 1^2}$	(w^2, uv)	$\begin{pmatrix} 0 & * & * \\ & 0 & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{2^2}$	$(w^2, u^2 - \ell v^2)$	$\begin{pmatrix} z & 0 & * \\ & -z\ell & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{1^2 11}$	$(v^2 - w^2, uw)$	$\begin{pmatrix} 0 & * & * \\ & * & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{1^2 2}$	$(v^2 - \ell w^2, uw)$	$\begin{pmatrix} 0 & * & * \\ & * & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$

(3.14)

One important way in which the annihilator subspaces are used is to show that many of the orbital exponential sums vanish.

Lemma 6. *For $x, \xi \in V(\mathbb{Z}/p^2\mathbb{Z})$ the orbital exponential sum may be expressed*

$$S_{p^2}(x, \xi) = \sum_{g \in G(\mathbb{Z}/p^2\mathbb{Z})} e_{p^2}\left([g \cdot x, \xi]\right) \mathbf{1}(g \cdot x \in V_\xi^\perp \bmod p). \tag{3.15}$$

Proof. Since $(I + pM_3(\mathbb{Z}/p\mathbb{Z}), I + pM_2(\mathbb{Z}/p\mathbb{Z}))$ is a subgroup of $G(\mathbb{Z}/p^2\mathbb{Z})$, the orbital exponential sums may be written

$$\begin{aligned} S_{p^2}(x,\xi) &= \sum_{g \in G(\mathbb{Z}/p^2\mathbb{Z})} e_{p^2}\left([g\cdot x, \xi]\right) \\ &= \frac{1}{p^{13}} \sum_{g_1 \in M_3(\mathbb{Z}/p\mathbb{Z})\times M_2(\mathbb{Z}/p\mathbb{Z})} \sum_{g \in G(\mathbb{Z}/p^2\mathbb{Z})} e_{p^2}\left([(I+pg_1)g\cdot x, \xi]\right) \\ &= \sum_{g \in G(\mathbb{Z}/p^2\mathbb{Z})} \frac{1}{p^{13}} \sum_{g_1 \in M_3(\mathbb{Z}/p\mathbb{Z})\times M_2(\mathbb{Z}/p\mathbb{Z})} e_{p^2}\left([g\cdot x, (I+pg_1)^t\cdot \xi]\right) \end{aligned} \tag{3.16}$$

Since $(I + pg_1)^t \cdot \xi$ is uniform on $\xi + pV_\xi$ as g_1 runs over $M_3(\mathbb{Z}/p\mathbb{Z}) \times M_2(\mathbb{Z}/p\mathbb{Z})$, and since V_ξ is a vector space, the inner sum vanishes unless $g \cdot x \in V_\xi^\perp \bmod p$. Hence,

$$S_{p^2}(x,\xi) = \sum_{g \in G(\mathbb{Z}/p^2\mathbb{Z})} e_{p^2}\left([g\cdot x, \xi]\right) \mathbf{1}(g\cdot x \in V_\xi^\perp \bmod p). \quad \square \tag{3.17}$$

By symmetry, the condition $g^t \cdot \xi \in V_x^\perp \bmod p$ may also be imposed. In fact the conditions $g \cdot x \in V_\xi^\perp \bmod p$ and $g^t \cdot \xi \in V_x^\perp \bmod p$ are equivalent.

Lemma 7. *For any $x, \xi \in V(\mathbb{Z}/p^2\mathbb{Z})$ and $g \in G(\mathbb{Z}/p^2\mathbb{Z})$, $g \cdot x \in V_\xi^\perp$ if and only if $g^t \cdot \xi \in V_x^\perp$.*

Proof. One has

$$\begin{aligned} g\cdot x \in V_\xi^\perp &\Leftrightarrow \forall g_1 \in M_3 \times M_2,\ [g\cdot x, (I+pg_1)^t\cdot\xi] \equiv [g\cdot x, \xi] \bmod p^2 \\ &\Leftrightarrow \forall g_1 \in M_3 \times M_2,\ [(I+pg_1)g\cdot x, \xi] \equiv [x, g^t\cdot\xi] \bmod p^2 \\ &\Leftrightarrow \forall g_1 \in M_3 \times M_2,\ [g(I+pg^{-1}g_1g)\cdot x, \xi] \equiv [x, g^t\cdot\xi] \bmod p^2 \\ &\Leftrightarrow \forall g_1 \in M_3 \times M_2,\ [(I+pg^{-1}g_1g)\cdot x, g^t\cdot\xi] \equiv [x, g^t\cdot\xi] \bmod p^2 \\ &\Leftrightarrow g^t\cdot\xi \in V_x^\perp. \quad \square \end{aligned} \tag{3.18}$$

In addition to the notion of an annihilator subspace, a second useful notion in evaluating the exponential sums is that of an action set acting on a pair of forms.

Definition 4. Given forms $x, \xi \in V(\mathbb{Z}/p\mathbb{Z})$, define the *action set*

$$G_{x,\xi} = \{g \in G(\mathbb{Z}/p\mathbb{Z}) : g^t \cdot \xi \in V_x^\perp \bmod p\}. \tag{3.19}$$

Note that, by Lemma 6, the condition $G_{x,\xi} \neq \emptyset$ is a necessary condition for $S_{p^2}(x,\xi) \neq 0$, a fact which restricts consideration to a small number of $\bmod p$ orbit pairings.

Lemma 8. *The following table lists all pairs $(\mathscr{O}_x, \mathscr{O}_\xi)$ of non-zero orbits modulo p such that $\mathscr{O}_x$ contains both maximal and non-maximal elements, and such that $x \in \mathscr{O}_x$, $\xi \in \mathscr{O}_\xi$ have $G_{x,\xi} \neq \emptyset$.*

$\mathscr{O}_x$	$\mathscr{O}_\xi$
$\mathscr{O}_{1^4}$	$\mathscr{O}_{D1^2}, \mathscr{O}_{D11}, \mathscr{O}_{1^4}$
$\mathscr{O}_{1^31}$	$\mathscr{O}_{D1^2}, \mathscr{O}_{Cs}$
$\mathscr{O}_{1^21^2}$	$\mathscr{O}_{D1^2}, \mathscr{O}_{D11}, \mathscr{O}_{D2}$
$\mathscr{O}_{2^2}$	$\mathscr{O}_{D11}, \mathscr{O}_{D2}$
$\mathscr{O}_{1^211}$	$\mathscr{O}_{D1^2}$
$\mathscr{O}_{1^22}$	$\mathscr{O}_{D1^2}$

(3.20)

Proof. The annihilator subspaces $V_{1^22}, V_{1^211}, V_{2^2}$ and $V_{1^21^2}$ associated to the standard representatives in the corresponding orbits annihilate the second quadratic form, see the last four rows of (3.14). Thus these orbits may be paired only with frequencies ξ from orbits of type D, which have representatives for which the second quadratic form vanishes. In the first quadratic form, each of these spaces annihilate all coefficients associated to the w variable, and hence $G_{x,\xi} = \emptyset$ unless ξ is equivalent to a D orbit with a representative that depends only on u and v. This eliminates $\mathscr{O}_{Dns}$. The spaces $V_{1^22}^\perp, V_{1^211}^\perp$ are one dimensional and contain only a double line, so $\mathscr{O}_{1^22}$ and $\mathscr{O}_{1^211}$ may be paired only with $\mathscr{O}_{D1^2}$. The orbit $\mathscr{O}_{D1^2}$ may not be paired with $\mathscr{O}_{2^2}$ since a double line has coefficients on u^2, v^2 which are related by a square. Since a quadratic form $\ell u^2 + auv + v^2$ may be made reducible or irreducible by adjusting a, both $\mathscr{O}_{D11}$ and $\mathscr{O}_{D2}$ may be paired with $\mathscr{O}_{2^2}$. The pairs $(u^2, 0)$, $(u^2 - v^2, 0)$ and $(u^2 - \ell v^2, 0)$ show that all D orbits besides $\mathscr{O}_{Dns}$ may be paired with $\mathscr{O}_{1^21^2}$.

When x is the standard representative for $\mathscr{O}_{1^31}$, $G_{x,\xi} \neq \emptyset$ implies that $\mathscr{O}_\xi$ has a representative of the form $(c_1u^2 + 2c_2uv, -c_2u^2)$, with c_1, c_2 scalars. Using the G action, one may have $c_1 \neq 0$ ($\mathscr{O}_{D1^2}$) or $c_2 \neq 0$ ($\mathscr{O}_{Cs}$) but not both, since if $c_2 \neq 0$ then a multiple of the second form may be added to the first to force $c_1 = 0$. Thus these are all of the pairings with $G_{x,\xi} \neq \emptyset$.

When x is the standard representative for $\mathscr{O}_{1^4}$, $G_{x,\xi} \neq \emptyset$ implies that $\mathscr{O}_\xi$ has a representative of the form

$$(c_1u^2 + c_2uv + c_3(v^2 - 2uw), 2c_3u^2) \tag{3.21}$$

with c_1, c_2, c_3 scalars. When $c_3 = 0$ one may obtain $\mathscr{O}_{D1^2}$ and $\mathscr{O}_{D11}$ by taking either c_1 or c_2 non-zero. When $c_3 \neq 0$ one may impose $c_1 = 0$ by adding a multiple of the second form to the first, and $c_2 = 0$ by replacing w with $w + av$ for some scalar a. The $c_3 \neq 0$ case obtains $\xi \in \mathscr{O}_{1^4}$. □

The transpose of the action sets for each pair contained in Lemma 8 are listed in the following table, with the proofs given in Appendix B.

$(\mathscr{O}_x, \mathscr{O}_\xi)$	x_0	ξ_0	$G^t_{x,\xi}$	$\lvert G_{x,\xi}\rvert$
$(\mathscr{O}_{1^211}, \mathscr{O}_{D1^2})$	$\begin{pmatrix}0&0&0\\&1&0\\&&-1\end{pmatrix}\begin{pmatrix}0&0&\frac12\\&0&0\\&&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$\begin{pmatrix}*&*&*\\&*&*\\&*&*\end{pmatrix}\begin{pmatrix}*&*\\&*\end{pmatrix}$	$(p-1)^5p^4(p+1)$
$(\mathscr{O}_{1^22}, \mathscr{O}_{D1^2})$	$\begin{pmatrix}0&0&0\\&1&0\\&&-\ell\end{pmatrix}\begin{pmatrix}0&0&\frac12\\&0&0\\&&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$\begin{pmatrix}*&*&*\\&*&*\\&*&*\end{pmatrix}\begin{pmatrix}*&*\\&*\end{pmatrix}$	$(p-1)^5p^4(p+1)$
$(\mathscr{O}_{2^2}, \mathscr{O}_{D11})$	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}1&0&0\\&-\ell&0\\&&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$\begin{pmatrix}a&c\lambda\ell&*\\c&a\lambda&*\\&&*\end{pmatrix}\begin{pmatrix}*&*\\&*\end{pmatrix}$, $\lambda\in\mathbf{F}_p^\times$, $(a,c)\in\mathbf{F}_p^2\setminus\{(0,0)\}$	$(p-1)^5p^3(p+1)$
$(\mathscr{O}_{2^2}, \mathscr{O}_{D2})$	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}1&0&0\\&-\ell&0\\&&0\end{pmatrix}$	$\begin{pmatrix}\ell&\beta&0\\&1&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$ $\ell u^2+2\beta uv+v^2$ irred.	$\begin{pmatrix}a&b&*\\c&d&*\\&&*\end{pmatrix}\begin{pmatrix}*&*\\&*\end{pmatrix}$, $\left[(\ell u^2+2\beta uv+v^2), \begin{pmatrix}a&c\\b&d\end{pmatrix}\cdot(u^2-\ell v^2)\right]=0$	$(p-1)^4p^3(p+1)^2$
$(\mathscr{O}_{1^21^2}, \mathscr{O}_{D1^2})$	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&\frac12&0\\&0&0\\&&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$\begin{pmatrix}*&*&*\\&*&*\\&*&*\end{pmatrix}\begin{pmatrix}*&*\\&*\end{pmatrix}$ $\sqcup\begin{pmatrix}&*&*\\ *&*&*\\&*&*\end{pmatrix}\begin{pmatrix}*&*\\&*\end{pmatrix}$	$2(p-1)^5p^4(p+1)$
$(\mathscr{O}_{1^21^2}, \mathscr{O}_{D11})$	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&\frac12&0\\&0&0\\&&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\&-1&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$\begin{pmatrix}*&*&*\\ *&*&*\\&&*\end{pmatrix}\begin{pmatrix}*&*\\&*\end{pmatrix}$ $u+v\mapsto\alpha u+\beta v$ $u-v\mapsto\lambda(\alpha u-\beta v)$	$(p-1)^6p^3$
$(\mathscr{O}_{1^21^2}, \mathscr{O}_{D2})$	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&\frac12&0\\&0&0\\&&0\end{pmatrix}$	$\begin{pmatrix}-\ell&0&0\\&1&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$\begin{pmatrix}a&c\ell&*\\c\lambda&a\lambda&*\\&&*\end{pmatrix}\begin{pmatrix}*&*\\&*\end{pmatrix}$, $\lambda\in\mathbf{F}_p^\times$, $(a,c)\in\mathbf{F}_p^2\setminus\{(0,0)\}$	$(p-1)^5p^3(p+1)$
$(\mathscr{O}_{1^31}, \mathscr{O}_{D1^2})$	$\begin{pmatrix}0&0&0\\&0&\frac12\\&&0\end{pmatrix}\begin{pmatrix}0&0&\frac12\\&1&0\\&&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$\begin{pmatrix}*&*&*\\&*&*\\&*&*\end{pmatrix}\begin{pmatrix}*&*\\&*\end{pmatrix}$	$(p-1)^5p^4(p+1)$
$(\mathscr{O}_{1^31}, \mathscr{O}_{Cs})$	$\begin{pmatrix}0&0&0\\&0&\frac12\\&&0\end{pmatrix}\begin{pmatrix}0&0&\frac12\\&1&0\\&&0\end{pmatrix}$	$\begin{pmatrix}0&-1&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}1&0&0\\&0&0\\&&0\end{pmatrix}$	$\begin{pmatrix}a&*&*\\&b&*\\&&*\end{pmatrix}\begin{pmatrix}c&*\\&\frac{bc}{a}\end{pmatrix}$	$(p-1)^4p^4$
$(\mathscr{O}_{1^4}, \mathscr{O}_{D1^2})$	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&\frac12\\&1&0\\&&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$\begin{pmatrix}*&*&*\\&*&*\\&*&*\end{pmatrix}\begin{pmatrix}*&*\\&*\end{pmatrix}$	$(p-1)^5p^4(p+1)$
$(\mathscr{O}_{1^4}, \mathscr{O}_{D11})$	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&\frac12\\&1&0\\&&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$\begin{pmatrix}*&*&*\\&*&*\\&&*\end{pmatrix}\begin{pmatrix}*&*\\&*\end{pmatrix}$ $\sqcup\begin{pmatrix}*&*&*\\ *&&*\\&&*\end{pmatrix}\begin{pmatrix}*&*\\&*\end{pmatrix}$	$2(p-1)^5p^4$
$(\mathscr{O}_{1^4}, \mathscr{O}_{1^4})$	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&\frac12\\&1&0\\&&0\end{pmatrix}$	$\begin{pmatrix}0&0&-1\\&1&0\\&&0\end{pmatrix}\begin{pmatrix}2&0&0\\&0&0\\&&0\end{pmatrix}$	$\begin{pmatrix}a&*&*\\&b&*\\&&c\end{pmatrix}\begin{pmatrix}d&*\\&e\end{pmatrix}$, $b^2=ac, b^2d=a^2e$	$(p-1)^3p^4$

(3.22)

The introduction of the action set permits a simplified representation of the orbital exponential sums.

Lemma 9. *Let $x = x_0 + px_1$, $\xi = \xi_0 + p\xi_1$ from the above table with $\|x_0\|_\infty, \|\xi_0\|_\infty \le \frac{p}{2}$. The orbital exponential sum $S_{p^2}(x,\xi)$ has evaluation $S_{p^2}(x,\xi) = p^{13}\mathscr{S}(x,\xi)$ with*

$$\mathscr{S}(x,\xi) = \sum_{g\in G^t_{x,\xi}} e_p\left([x_1, g\cdot\xi_0] + [x_0, g\cdot\xi_1]\right). \tag{3.23}$$

Proof. By examination of (3.22), for $g \in G^t_{x,\xi}$, $[x_0, g\cdot\xi_0] = 0$. Here we use that x_0 and ξ_0 are equal modulo p^2 to their representative given in (3.22), due to the constraint on the maximum of the coefficients. Hence

$$\begin{aligned} S_{p^2}(x,\xi) &= \sum_{g\in G(\mathbb{Z}/p^2\mathbb{Z}),\, g \bmod p\in G^t_{x,\xi}} e_{p^2}\left([x_0+px_1, g\cdot(\xi_0+p\xi_1)]\right) \qquad (3.24)\\ &= \sum_{g\in G(\mathbb{Z}/p^2\mathbb{Z}),\, g \bmod p\in G^t_{x,\xi}} e_p\left([x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0]\right) \end{aligned}$$

$$= p^{13} \sum_{g \in G^t_{x,\xi} \bmod p} e_p\left([x_0, g\cdot \xi_1] + [x_1, g \cdot \xi_0]\right)$$
$$= p^{13} \mathscr{S}(x,\xi),$$

which proves the claim. □

Lemmas 5, 8 and 9 demonstrate that in order to obtain $\widehat{\mathbf{1}_{\mathrm{max}}}$ and $\widehat{\mathbf{1}_{\text{non-max}}}$ at $\xi \not\equiv 0 \bmod p$ it suffices to consider orbital exponential sums $S(x,\xi)$ in which $\mathscr{O}_x \bmod p$ contains both maximal and non-maximal elements, and in which $\mathscr{O}_\xi$ appears paired with $\mathscr{O}_x$ in Lemma 8. In particular, the maximal or non-maximal $\bmod\, p^2$ orbits above $\mathscr{O}_{1^4}, \mathscr{O}_{1^31}, \mathscr{O}_{1^21^2}, \mathscr{O}_{2^2}, \mathscr{O}_{1^211}$ and $\mathscr{O}_{1^22} \bmod p$ need to be determined, and all $\bmod\, p^2$ orbits need to be determined above $\mathscr{O}_{D1^2}, \mathscr{O}_{D11}, \mathscr{O}_{D2}, \mathscr{O}_{Cs}$ and $\mathscr{O}_{1^4} \bmod p$. This classification is performed in the next Section.

4. The quotient group action and mod p^2 orbits

The following Section reduces the determination of the $\bmod\, p^2$ orbits above a $\bmod\, p$ orbit with representative x to the determination of the orbits of a quotient space under the $\bmod\, p$ stabilizer of x. Then the necessary $\bmod\, p^2$ orbits are classified. The stabilizer groups of the following orbit representatives are needed below.

Orbit	Repr.	Stabilizer	Stab. size
$\mathscr{O}_{D1^2}$	$(0, w^2)$	$\begin{pmatrix} a & b & \\ c & d & \\ e & f & t \end{pmatrix} \begin{pmatrix} x & \\ y & \frac{1}{t^2} \end{pmatrix}$, $x, t \in (\mathbb{Z}/p\mathbb{Z})^\times, e, f, y \in \mathbb{Z}/p\mathbb{Z}, ad - bc \neq 0$	$s(4,4,1,0)$
$\mathscr{O}_{D11}$	$(0, vw)$	$\begin{pmatrix} r & & \\ a & s & \\ b & & t \end{pmatrix} \begin{pmatrix} x & \\ y & \frac{1}{st} \end{pmatrix} \sqcup \begin{pmatrix} r & & \\ a & & s \\ b & t & \end{pmatrix} \begin{pmatrix} x & \\ y & \frac{1}{st} \end{pmatrix}$, $x, r, s, t \in (\mathbb{Z}/p\mathbb{Z})^\times, a, b, y \in \mathbb{Z}/p\mathbb{Z}$	$2s(4,3,0,0)$
$\mathscr{O}_{D2}$	$(0, v^2 - \ell w^2)$	$\begin{pmatrix} r & & \\ a & c & e \\ b & \pm e\ell & \pm c \end{pmatrix} \begin{pmatrix} x & \\ y & \frac{1}{c^2 - e^2\ell} \end{pmatrix}$, $r, x \in (\mathbb{Z}/p\mathbb{Z})^\times, a, b, y \in \mathbb{Z}/p\mathbb{Z}$, $(c, e) \neq (0,0) \in (\mathbb{Z}/p\mathbb{Z})^2$	$2s(3,3,1,0)$
$\mathscr{O}_{Cs}$	(w^2, vw)	$\begin{pmatrix} r & & \\ a & s & \\ b & c & t \end{pmatrix} \begin{pmatrix} \frac{1}{t^2} & \\ \frac{-c}{st^2} & \frac{1}{st} \end{pmatrix}$, $r, s, t \in (\mathbb{Z}/p\mathbb{Z})^\times, a, b, c \in \mathbb{Z}/p\mathbb{Z}$	$s(3,3,0,0)$
$\mathscr{O}_{1^4}$	$(w^2, uw + v^2)$	$\begin{pmatrix} s & & \\ -\frac{2as}{t} & t & \\ b & a & \frac{t^2}{s} \end{pmatrix} \begin{pmatrix} \frac{s^2}{t^4} & \\ -\left(a^2 + \frac{bt^2}{s}\right)\frac{s^2}{t^6} & \frac{1}{t^2} \end{pmatrix}$, $s, t \in (\mathbb{Z}/p\mathbb{Z})^\times, a, b \in \mathbb{Z}/p\mathbb{Z}$	$s(2,2,0,0)$

Orbit	Repr.	Stabilizer	Stab. size
$\mathscr{O}_{1^31}$	$(vw, uw + v^2)$	$\begin{pmatrix} s & & \\ a & t & \\ & & \frac{t^2}{s} \end{pmatrix} \begin{pmatrix} \frac{s}{t^3} & \\ \frac{-a}{t^3} & \frac{1}{t^2} \end{pmatrix}$, $s, t \in (\mathbb{Z}/p\mathbb{Z})^\times, a \in \mathbb{Z}/p\mathbb{Z}$	$s(2,1,0,0)$
$\mathscr{O}_{1^21^2}$	(w^2, uv)	$\begin{pmatrix} r & & \\ & s & \\ & & t \end{pmatrix} \begin{pmatrix} \frac{1}{t^2} & \\ & \frac{1}{rs} \end{pmatrix} \sqcup \begin{pmatrix} & r & \\ s & & \\ & & t \end{pmatrix} \begin{pmatrix} \frac{1}{t^2} & \\ & \frac{1}{rs} \end{pmatrix}$, $r, s, t \in (\mathbb{Z}/p\mathbb{Z})^\times$	$2s(3,0,0,0)$
$\mathscr{O}_{2^2}$	$(w^2, u^2 - \ell v^2)$	$\begin{pmatrix} c & e & \\ \pm e\ell & \pm c & \\ & & s \end{pmatrix} \begin{pmatrix} \frac{1}{s^2} & \\ & \frac{1}{c^2 - e^2\ell} \end{pmatrix}$, $(c, e) \neq (0,0) \in (\mathbb{Z}/p\mathbb{Z})^2, s \in (\mathbb{Z}/p\mathbb{Z})^\times$	$2s(2,0,1,0)$
$\mathscr{O}_{1^211}$	$(v^2 - w^2, uw)$	$\begin{pmatrix} s & & \\ & \pm t & \\ & & t \end{pmatrix} \begin{pmatrix} \frac{1}{t^2} & \\ & \frac{1}{st} \end{pmatrix}$, $s, t \in (\mathbb{Z}/p\mathbb{Z})^\times$	$2s(2,0,0,0)$
$\mathscr{O}_{1^22}$	$(v^2 - \ell w^2, uw)$	$\begin{pmatrix} s & & \\ & \pm t & \\ & & t \end{pmatrix} \begin{pmatrix} \frac{1}{t^2} & \\ & \frac{1}{st} \end{pmatrix}$, $s, t \in (\mathbb{Z}/p\mathbb{Z})^\times$	$2s(2,0,0,0)$

(4.1)

The stabilizers may be checked by verifying that the claimed stabilizer does fix the form and comparing the size with the orbit sizes in (1.11), see **mod_p_orbit_stabilizer.nb** for the verification.

Lemma 10. *For each standard orbit representative x of an orbit*

$$\mathscr{O}_{D1^2}, \mathscr{O}_{D11}, \mathscr{O}_{D2}, \mathscr{O}_{Cs}, \mathscr{O}_{1^4}, \mathscr{O}_{1^31}, \mathscr{O}_{1^21^2}, \mathscr{O}_{2^2}, \mathscr{O}_{1^211}, \mathscr{O}_{1^22} \tag{4.2}$$

of $V(\mathbb{Z}/p\mathbb{Z})$ listed in (4.3), the stabilizer subgroup G_x acts on $V(\mathbb{Z}/p\mathbb{Z})/V_x$. The orbits of $V(\mathbb{Z}/p^2\mathbb{Z})$ under $G(\mathbb{Z}/p^2\mathbb{Z})$ above $\mathscr{O}_x \bmod p$ are in bijection with the orbits of $V(\mathbb{Z}/p\mathbb{Z})/V_x$ under G_x.

Proof. For each orbit representative, let P_x be the algebraic group listed in the table below. Note that $P_x(\mathbb{Z}/p\mathbb{Z}) \supset G_x$.

Orbit	Repr.	P_x	V_x
$\mathscr{O}_{D1^2}$	$(0, w^2)$	$\begin{pmatrix} * & * & \\ * & * & \\ * & * & * \end{pmatrix} \begin{pmatrix} * & \\ * & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & * \end{pmatrix} \begin{pmatrix} 0 & 0 & * \\ & 0 & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{D11}$	$(0, vw)$	$\begin{pmatrix} * & & \\ * & * & * \\ * & * & * \end{pmatrix} \begin{pmatrix} * & \\ * & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & * \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & * & * \\ & * & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{D2}$	$(0, v^2 - \ell w^2)$	$\begin{pmatrix} * & & \\ * & * & * \\ * & * & * \end{pmatrix} \begin{pmatrix} * & \\ * & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ & z & 0 \\ & & -z\ell \end{pmatrix} \begin{pmatrix} 0 & * & * \\ & * & * \\ & & * \end{pmatrix}$

Orbit	Repr.	P_x	V_x
$\mathscr{O}_{Cs}$	(w^2, vw)	$\begin{pmatrix} * & & \\ * & * & \\ * & * & * \end{pmatrix} \begin{pmatrix} * & \\ * & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & z \\ & 0 & * \\ & & * \end{pmatrix} \begin{pmatrix} 0 & \frac{z}{2} & * \\ & * & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{1^4}$	$(w^2, uw + v^2)$	$\begin{pmatrix} * & & \\ * & * & \\ * & * & * \end{pmatrix} \begin{pmatrix} * & \\ * & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & y + \frac{z}{2} \\ & z & * \\ & & * \end{pmatrix} \begin{pmatrix} y & * & * \\ & * & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{1^3 1}$	$(vw, uw + v^2)$	$\begin{pmatrix} * & & \\ * & * & \\ & & * \end{pmatrix} \begin{pmatrix} * & \\ * & * \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{z}{2} & * \\ & * & * \\ & & * \end{pmatrix} \begin{pmatrix} z & * & * \\ & * & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{1^2 1^2}$	(w^2, uv)	$\begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix} \begin{pmatrix} * & \\ & * \end{pmatrix}$	$\begin{pmatrix} 0 & * & * \\ & 0 & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{2^2}$	$(w^2, u^2 - \ell v^2)$	$\begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix} \begin{pmatrix} * & \\ & * \end{pmatrix}$	$\begin{pmatrix} z & 0 & * \\ & -z\ell & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{1^2 11}$	$(v^2 - w^2, uw)$	$\begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \begin{pmatrix} * & \\ & * \end{pmatrix}$	$\begin{pmatrix} 0 & * & * \\ & * & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$
$\mathscr{O}_{1^2 2}$	$(v^2 - \ell w^2, uw)$	$\begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \begin{pmatrix} * & \\ & * \end{pmatrix}$	$\begin{pmatrix} 0 & * & * \\ & * & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$

(4.3)

By examination, over $\mathbb{Q}$, $P_x(\mathbb{Q}) \cdot x \subset V_x(\mathbb{Q})$, see **action_contained_tangent_space.nb**.

Over $\mathbb{Z}/p\mathbb{Z}$, G_x preserves V_x since if $g \in G_x$ and $y \in V_x$ then for some $m \in M_3(\mathbb{Z}/p\mathbb{Z}) \times M_2(\mathbb{Z}/p\mathbb{Z})$,

$$py = (I + pm) \cdot x - x \tag{4.4}$$

and thus

$$pg \cdot y = g \cdot (I + pm) \cdot x - g \cdot x = (I + pgmg^{-1})(g \cdot x) - (g \cdot x) \in pV_{g \cdot x} = pV_x. \tag{4.5}$$

Thus G_x acts on V/V_x.

To obtain the orbits over $\mathbb{Z}/p^2\mathbb{Z}$, given a form x_0 act by G to bring it to a form $x + px'$ where x is one of the standard representative forms listed in the table, considered to be a form with integer coefficients. Next, by acting by $(I + pM_3(\mathbb{Z}/p\mathbb{Z}), I + pM_2(\mathbb{Z}/p\mathbb{Z}))$, consider this to be an element in $x + p \cdot (V(\mathbb{Z}/p\mathbb{Z})/V_x)$ where V_x is the annihilator subspace of x. Given $g \in G(\mathbb{Z}/p^2\mathbb{Z})$ with $g \cdot x \equiv x \bmod p$, then $g \in G_x < P_x \bmod p$, so write $g = g_0 + pg_1$ with $g_0 \in P_x(\mathbb{Z})$. Let $m \in M_3(\mathbb{Z}/p\mathbb{Z}) \times M_2(\mathbb{Z}/p\mathbb{Z})$ be such that $g = g_0(I + pm)$. Calculate

$$\begin{aligned} g \cdot (x + px') &\equiv g_0 \cdot ((I + pm) \cdot x + px') \bmod pV_x \\ &\equiv g_0 \cdot (x + px' + pV_x) \bmod pV_x \\ &\equiv g_0 \cdot x + pg_0 \cdot x' \bmod pV_x. \end{aligned} \tag{4.6}$$

Notice that $g_0 \cdot x \in V_x(\mathbb{Z})$ and hence $g_0 \cdot x - x \in pV_x(\mathbb{Z})$. Hence

$$\begin{aligned} g \cdot (x + px') &\equiv x + (g_0 \cdot x - x) + pg_0 \cdot x' \bmod pV_x \\ &\equiv x + pg_0 \cdot x' \bmod pV_x. \end{aligned} \tag{4.7}$$

It follows that $x + px_1 \equiv x + px_2$ under $G(\mathbb{Z}/p^2\mathbb{Z})$ if and only if $x_1 \equiv x_2 \bmod V_x$ under G_x. $\square$

The following lemma determines the size of the $\bmod p^2$ stabilizers in terms of the size of the stabilizer of the G_x action on V/V_x.

Lemma 11. *Let x be a standard representative for one of the orbits*

$$\mathscr{O}_{D1^2}, \mathscr{O}_{D11}, \mathscr{O}_{D2}, \mathscr{O}_{Cs}, \mathscr{O}_{1^4}, \mathscr{O}_{1^31}, \mathscr{O}_{1^21^2}, \mathscr{O}_{2^2}, \mathscr{O}_{1^211}, \mathscr{O}_{1^22} \tag{4.8}$$

taken over $\mathbb{Z}$ and let $x' = x + px_1$. Then the following relationship between stabilizers holds

$$\left|\mathrm{Stab}_{G(\mathbb{Z}/p^2\mathbb{Z})}(x')\right| = p^{13-\dim V_x} \left|\mathrm{Stab}_{G_x}(x_1)\right|. \tag{4.9}$$

In particular, the orbit sizes are related by

$$\left|G(\mathbb{Z}/p^2\mathbb{Z}) \cdot x'\right| = p^{\dim V_x} \cdot \left|G(\mathbb{Z}/p\mathbb{Z}) \cdot x \bmod p\right| \cdot \left|G_x \cdot x_1 \bmod V_x\right|, \tag{4.10}$$

and the density of the orbit $G(\mathbb{Z}/p^2\mathbb{Z}) \cdot x'$ within $\mathscr{O}_x \bmod p$ is equal to the density of $G_x \cdot x_1 \bmod V_x$ in V/V_x,

$$\frac{\left|G(\mathbb{Z}/p^2\mathbb{Z}) \cdot x'\right|}{p^{12}\left|G(\mathbb{Z}/p\mathbb{Z}) \cdot x\right|} = \frac{\left|G_x \cdot x_1 \bmod V_x\right|}{\left|V/V_x\right|}. \tag{4.11}$$

Proof. Let $g \in G(\mathbb{Z}/p^2\mathbb{Z})$ stabilize x', so $g \cdot x \equiv x \bmod p$ and write $g = g_0 + pg_1$ with $g_0 \in P_x(\mathbb{Z})$. Factor, on the left this time, $g = (I + pm) \cdot g_0$, and write

$$\begin{aligned} (I + pm)g_0 \cdot (x + px_1) &= (I + pm)(g_0 \cdot x + pg_0 \cdot x_1) \\ &\equiv (I + pm) \cdot (x + (g_0 \cdot x - x) + pg_0 \cdot x_1) \bmod p^2 \\ &\equiv ((I + pm) \cdot x + (g_0 \cdot x - x) + pg_0 \cdot x_1) \bmod p^2. \end{aligned} \tag{4.12}$$

Since $(I + pm) \cdot x - x \in pV_x$ and $g_0 \cdot x - x \in pV_x$, it follows that $g_0 \cdot x_1 \equiv x_1 \bmod V_x$. As m varies in the 13 dimensional vector space $M_3(\mathbb{Z}/p\mathbb{Z}) \times M_2(\mathbb{Z}/p\mathbb{Z})$, the map $(I + pm) \cdot x - x \in pV_x$ is a linear map onto, with kernel of dimension $13 - \dim V_x$. Hence after choosing g_0 in $\left|\mathrm{Stab}_{G_x}(x_1)\right|$ ways, there are $p^{13-\dim V_x}$ ways to choose m.

To obtain the relations among the orbit sizes, calculate, using the Orbit-Stabilizer Theorem

$$\begin{aligned}\left|G(\mathbb{Z}/p^2\mathbb{Z})\cdot x'\right| &= \frac{|G(\mathbb{Z}/p^2\mathbb{Z})|}{\left|\mathrm{Stab}_{G(\mathbb{Z}/p^2\mathbb{Z})}(x')\right|} \\ &= \frac{|G(\mathbb{Z}/p^2\mathbb{Z})|p^{\dim V_x - 13}}{|\mathrm{Stab}_{G_x}(x_1)|} \\ &= p^{\dim V_x}\frac{|G(\mathbb{Z}/p\mathbb{Z})|}{\left|\mathrm{Stab}_{G(\mathbb{Z}/p\mathbb{Z})}(x)\right|}\frac{|G_x|}{|\mathrm{Stab}_{G_x}(x_1)|} \\ &= p^{\dim V_x}\cdot|G(\mathbb{Z}/p\mathbb{Z})\cdot x \bmod p|\cdot|G_x\cdot x_1 \bmod V_x|\,.\end{aligned} \tag{4.13}$$

The relation among densities is obtained by rearranging the relation among orbit sizes. □

4.1. The $\bmod p^2$ orbits relevant to maximality

First the $\bmod p^2$ orbits are considered above $\bmod p$ orbits containing both maximal and non-maximal orbits. Recall the density of maximal elements in orbits of this type,

Orbit	Density	$\dim(V/V_x)$
$\mathscr{O}_{1^4}$	$\frac{p-1}{p}$	3
$\mathscr{O}_{1^31}$	$\frac{p-1}{p}$	2
$\mathscr{O}_{1^21^2}$	$\left(\frac{p-1}{p}\right)^2$	2
$\mathscr{O}_{2^2}$	$\frac{p^2-1}{p^2}$	2
$\mathscr{O}_{1^211}$	$\frac{p-1}{p}$	1
$\mathscr{O}_{1^22}$	$\frac{p-1}{p}$	1

(4.14)

In the case of $\mathscr{O}_{2^2}, \mathscr{O}_{1^211}$ and $\mathscr{O}_{1^22}$ the non-maximal orbits are most easily classified.

Lemma 12. *The non-maximal elements above the orbits $\mathscr{O}_{2^2}, \mathscr{O}_{1^211}$ and $\mathscr{O}_{1^22}$ consist in a single $G(\mathbb{Z}/p^2\mathbb{Z})$ orbit corresponding to the singleton 0 orbit in V/V_x.*

Proof. In each case, the zero orbit has an element which satisfies the congruence conditions of Lemma 3, so these orbits all consist of non-maximal elements. By Lemma 11, these orbits already make up the full density of non-maximal elements (see the table above). □

4.1.1. Case of $\mathscr{O}_{1^4}$

The orthogonal complement to $V_x = \begin{pmatrix} 0 & 0 & y+\frac{z}{2} \\ & z & * \\ & & * \end{pmatrix}\begin{pmatrix} y & * & * \\ & * & * \\ & & * \end{pmatrix}$ is spanned by $\{v_1, v_2, v_3\}$,

$$\begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ & 1 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}. \tag{4.15}$$

The stabilizer G_x is given by

$$G_x = \begin{pmatrix} s & & \\ \frac{-2as}{t} & t & \\ b & a & \frac{t^2}{s} \end{pmatrix} \begin{pmatrix} \frac{s^2}{t^4} & \\ -\left(a^2 + \frac{bt^2}{s}\right)\frac{s^2}{t^6} & \frac{1}{t^2} \end{pmatrix}. \tag{4.16}$$

Lemma 13. *The orbits and stabilizers of V/V_x under the G_x action are summarized below.*

Orbit representative	*Orbit size*	*Stabilizer*
$\begin{pmatrix} \epsilon & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$, $\epsilon \in \mathbf{F}_p^\times/\{x^4 : x \in \mathbf{F}_p^\times\}$	$\frac{(p-1)p^2}{\#\{\epsilon^4=1\}}$	$\begin{pmatrix} s & 0 & 0 \\ 0 & s\varepsilon & 0 \\ 0 & 0 & s\varepsilon^2 \end{pmatrix} \begin{pmatrix} \frac{1}{s^2} & 0 \\ 0 & \frac{1}{s^2\varepsilon^2} \end{pmatrix}$, $\varepsilon \in \mathbf{F}_p^\times/\{x^4 : x \in \mathbf{F}_p^\times\}$
$\begin{pmatrix} 0 & \epsilon & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$, $\epsilon \in \mathbf{F}_p^\times/\{x^3 : x \in \mathbf{F}_p^\times\}$	$\frac{(p-1)p}{\#\{\epsilon^3=1\}}$	$\begin{pmatrix} s & 0 & 0 \\ 0 & s\varepsilon & 0 \\ b & 0 & s\varepsilon^2 \end{pmatrix} \begin{pmatrix} \frac{1}{s^2\varepsilon} & 0 \\ -\frac{b}{s^3\varepsilon} & \frac{1}{s^2\varepsilon^2} \end{pmatrix}$, $\varepsilon \in \mathbf{F}_p^\times/\{x^3 : x \in \mathbf{F}_p^\times\}$
$\begin{pmatrix} 0 & 0 & -1 \\ & 1 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$\frac{p-1}{2}$	$\begin{pmatrix} s & 0 & 0 \\ -2a\varepsilon & s\varepsilon & 0 \\ b & a & s \end{pmatrix} \begin{pmatrix} \frac{1}{s^2} & 0 \\ -\frac{a^2}{s^4} - \frac{b}{s^3} & \frac{1}{s^2} \end{pmatrix}$, $\varepsilon = \pm 1$
$\begin{pmatrix} 0 & 0 & -\ell \\ & \ell & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 2\ell & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$\frac{p-1}{2}$	$\begin{pmatrix} s & 0 & 0 \\ -2a\varepsilon & s\varepsilon & 0 \\ b & a & s \end{pmatrix} \begin{pmatrix} \frac{1}{s^2} & 0 \\ -\frac{a^2}{s^4} - \frac{b}{s^3} & \frac{1}{s^2} \end{pmatrix}$, $\varepsilon = \pm 1$
$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	1	G_x

(4.17)

The orbits of the first type correspond to the maximal orbits under the $G(\mathbb{Z}/p^2\mathbb{Z})$ action.

Proof. See **O14_stabilizers.nb**, where the calculation in this lemma is performed in Mathematica.

To check the stabilizer of the orbits of the first type, note that a is forced to be 0 by considering the uv coefficient, which then forces b to be 0 by considering the uw coefficient. The u^2 coefficient becomes $\frac{s^4}{t^4}\epsilon$, which obtains the relation $\frac{s^4}{t^4} = 1$. The inequivalence of orbits having differing ϵ is checked similarly.

To calculate the stabilizer of the second orbit, note that a is forced to be 0, since it is not possible to produce a u^2 term which would be required to avoid a v_3 component. Evidently b is arbitrary, and the relation $\frac{s^3}{t^3} = 1$ is obtained by considering the uv coefficient. This also proves the inequivalence of orbits having differing ϵ. For the final two orbits, note that the span of v_3 is preserved by the action. The square and non-square leading coefficients are inequivalent. Since the remaining orbits all have differing sizes, each of the orbits listed is inequivalent from the others.

Those orbits not of the first type make up a $\frac{1}{p}$ fraction of all of the elements of V/V_x. Thus at least one, and thus all orbits of the first type are maximal, and these exhaust the maximal orbits by density. □

4.1.2. Case of $\mathscr{O}_{1^31}$

The orthogonal complement to $V_x = \begin{pmatrix} 0 & \frac{z}{2} & * \\ & * & * \\ & & * \end{pmatrix} \begin{pmatrix} z & * & * \\ & * & * \\ & & * \end{pmatrix}$ is spanned by

$$v_1 = \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}. \tag{4.18}$$

The acting group is

$$G_x = \begin{pmatrix} s & & \\ a & t & \\ & & \frac{t^2}{s} \end{pmatrix} \begin{pmatrix} \frac{s}{t^3} & \\ -\frac{a}{t^3} & \frac{1}{t^2} \end{pmatrix}. \tag{4.19}$$

Lemma 14. *The maximal orbits above $\mathscr{O}_{1^31}$ in the $G(\mathbb{Z}/p^2\mathbb{Z})$ correspond to the orbits of*

$$\begin{pmatrix} \epsilon & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \qquad \epsilon \in \mathbf{F}_p^\times/\{x^3 : x \in \mathbf{F}_p^\times\} \tag{4.20}$$

in the action on V/V_x. In the action on V/V_x, each of these orbits has size $\frac{(p-1)p}{\#\{\epsilon^3=1\}}$.

Proof. When G_x acts it maps v_1 to

$$\begin{pmatrix} \frac{s^3}{t^3} & \frac{as^2}{t^3} & * \\ & * & * \\ & & * \end{pmatrix} \begin{pmatrix} -\frac{as^2}{t^3} & * & * \\ & * & * \\ & & * \end{pmatrix} \tag{4.21}$$

and thus the stabilizer of λv_1 satisfies $s^3 = t^3$ and $a = 0$, see **O131_stabilizers.nb**. Hence there are $\#\{\epsilon^3 = 1\}$ orbits ϵv_1 with $\epsilon \in \mathbf{F}_p^\times/\{x^3 : x \in \mathbf{F}_p^\times\}$, each of size $\frac{(p-1)p}{\#\{\epsilon^3=1\}}$. These orbits together are maximal or non-maximal, and hence by density they are maximal and exhaust the maximal orbits. □

4.1.3. Case of $\mathscr{O}_{1^21^2}$

The annihilated subspace is $V_x = \begin{pmatrix} 0 & * & * \\ & 0 & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$. The acting group is

$$G_x = \begin{pmatrix} r & & \\ & s & \\ & & t \end{pmatrix} \begin{pmatrix} \frac{1}{t^2} & \\ & \frac{1}{rs} \end{pmatrix} \sqcup \begin{pmatrix} & r & \\ s & & \\ & & t \end{pmatrix} \begin{pmatrix} \frac{1}{t^2} & \\ & \frac{1}{rs} \end{pmatrix}. \tag{4.22}$$

Lemma 15. *The orbits of the G_x action on V/V_x in the case $x \in \mathscr{O}_{1^21^2}$ are given in the following table*

$$\begin{array}{|l|l|l|l|} \hline \textit{Orbit representative} & \textit{Orbit size} & \textit{Stabilizer size} & \textit{Type} \\ \hline \begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} & p-1 & 2(p-1)^2 & \textit{Non-maximal} \\ \hline \begin{pmatrix} 0 & 0 & 0 \\ & \ell & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} & p-1 & 2(p-1)^2 & \textit{Non-maximal} \\ \hline \begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} & \frac{(p-1)^2}{4} & 8(p-1) & \textit{Maximal} \\ \hline \begin{pmatrix} \ell & 0 & 0 \\ & \ell & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} & \frac{(p-1)^2}{4} & 8(p-1) & \textit{Maximal} \\ \hline \begin{pmatrix} 1 & 0 & 0 \\ & \ell & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} & \frac{(p-1)^2}{2} & 4(p-1) & \textit{Maximal} \\ \hline \end{array} \tag{4.23}$$

Proof. According to the congruence condition, the orbits with representatives

$$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ & \ell & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \tag{4.24}$$

are both non-maximal. These orbits are inequivalent, since the non-zero elements scale by a square under the action. Each has stabilizer

$$\begin{pmatrix} s & & \\ & t & \\ & & \pm t \end{pmatrix} \begin{pmatrix} \frac{1}{t^2} & \\ & \frac{1}{st} \end{pmatrix}. \tag{4.25}$$

Hence both orbits have size $p-1$. Together with the zero orbit, these exhaust the non-maximal orbits, by density.

The stabilizer of the orbits with representatives

$$\begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \qquad \begin{pmatrix} \ell & 0 & 0 \\ & \ell & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \tag{4.26}$$

is

$$\begin{pmatrix} t & & \\ & t\epsilon_1 & \\ & & t\epsilon_2 \end{pmatrix} \begin{pmatrix} \frac{1}{t^2} & \\ & \frac{1}{t^2\epsilon_1} \end{pmatrix} \sqcup \begin{pmatrix} & t & \\ t\epsilon_1 & & \\ & & t\epsilon_2 \end{pmatrix} \begin{pmatrix} \frac{1}{t^2} & \\ & \frac{1}{t^2\epsilon_1} \end{pmatrix}, \qquad \epsilon_1, \epsilon_2 \in \{\pm 1\}. \tag{4.27}$$

In the case of $\begin{pmatrix} 1 & 0 & 0 \\ & \ell & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$ the stabilizer is

$$\begin{pmatrix} t & & \\ & t\epsilon_1 & \\ & & t\epsilon_2 \end{pmatrix} \begin{pmatrix} \frac{1}{t^2} & \\ & \frac{1}{t^2\epsilon_1} \end{pmatrix}, \qquad \epsilon_1, \epsilon_2 \in \{\pm 1\} \tag{4.28}$$

since the square and non-square coordinate cannot be exchanged. These exhaust the maximal orbits, by density. □

4.2. The $\bmod p^2$ orbits which appear as frequencies

The remaining orbits of V/V_x are classified projectively, which suffices in calculating the Fourier transform, see Lemma 21 below. Equivalently, the acting group is taken to be $G_x \times \mathrm{GL}_1$, in which GL_1 acts by multiplication by a scalar.

4.2.1. Case of $\mathscr{O}_{Cs}$

The acting group is

$$G_x \times \mathrm{GL}_1 = \begin{pmatrix} r & & \\ a & s & \\ b & c & t \end{pmatrix} \begin{pmatrix} \frac{\lambda}{t^2} & \\ \frac{-c\lambda}{st^2} & \frac{\lambda}{st} \end{pmatrix}. \tag{4.29}$$

The action is on V/V_x,

$$V_x = \begin{pmatrix} 0 & 0 & z \\ & 0 & * \\ & & * \end{pmatrix} \begin{pmatrix} 0 & \frac{z}{2} & * \\ & * & * \\ & & * \end{pmatrix}. \tag{4.30}$$

Lemma 16. *When x is the standard representative of $\mathscr{O}_{Cs} \bmod p$, there are 11 orbits of $G_x \times \mathrm{GL}_1$ acting on V/V_x, listed in the following table, together with the orbit size, and the relations on $G_x \times \mathrm{GL}_1$ describing each orbit stabilizer.*

Orbit representative	*Orbit size*	*Stabilizer relations*
$\begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)p^3$	$a = b = c = 0, r^2\lambda = t^2$
$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$p-1$	$s^2\lambda = t^2$
$\begin{pmatrix} 0 & 0 & 1 \\ & 1 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)^2$	$s^2\lambda = t^2, r\lambda = t$
$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)^2p$	$s^2\lambda = t^2, r^2\lambda = st, a = 0$
$\begin{pmatrix} 1 & \frac{1}{2} & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$\frac{(p-1)^2p^3}{2}$	$b = c = 0, a(a+s) = 0$ *if* $a = 0$: $s = r, r^2\lambda = t^2$ *if* $a + s = 0$: $a = r, r^2\lambda = t^2$

Orbit representative	*Orbit size*	*Stabilizer relations*
$\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)p^2$	$a=0, c=0, rs\lambda = t^2$
$\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)^2p^2$	$a=0, c=0, rs\lambda = t^2, r^2\lambda = st$
$\begin{pmatrix} 1 & 0 & 0 \\ & -\ell & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$\frac{(p-1)^2p^3}{2}$	$a=b=c=0, s=\pm r, r^2\lambda = t^2$
$\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$p-1$	$r\lambda = t$
$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)p$	$a=0, r^2\lambda = st$
$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	1	–

(4.31)

Proof. See **OCs_stabilizers.nb**, where the calculations in this lemma are performed in Mathematica.

Identify the upper left two by two elements of the first quadratic form with $\mathrm{Sym}^2(\mathbf{F}_p^2)$. The group $\begin{pmatrix} r & \\ a & s \end{pmatrix} \times \mathrm{GL}_1$ acts on $\mathrm{Sym}^2(\mathbf{F}_p^2)$ with six orbits, having representatives 0, u^2, v^2, $u(u+v)$, uv, $u^2 - \ell v^2$, see Lemma 2. In addition to one of these in the first quadratic form, there can be included either a uw term in the first quadratic form, or a u^2 term in the second.

The 0 form can include either uw in the first form, or u^2 in the second, but not both, since if the u^2 coefficient is non-zero in the second quadratic form, making the transformation $u \mapsto u + av$ can reduce the pair of forms to one equivalent to $(0, u^2)$ mod V_x.

The u^2 orbit cannot include either term, since subtracting a multiple of the first form from the second eliminates a u^2 term in the second quadratic form, and replacing $u \mapsto u + bw$ can eliminate the uw term in the first.

The v^2 orbit can include either term, but not both by the same argument as in the case of 0.

$u(u+v)$ cannot include either term for the same reason that u^2 cannot.

uv can include the u^2 term in the second form, but not a uw term, which could be eliminated by replacing $v \mapsto v + cw$.

$u^2 - \ell v^2$ cannot include either term by the same argument as in the u^2 case.

The above give the 11 orbits listed. The stabilizer conditions have been verified to guarantee stabilization of the form. If any of the stabilizer groups were larger than the one listed, or if any of the 11 orbits listed were equivalent, then there would not be sufficiently many points to cover the space, so that it follows that the 11 orbits are inequivalent and the stabilizers are correct. □

4.2.2. Case of $\mathscr{O}_{D2}$

The acting group is

$$G_x \times \mathrm{GL}_1 = \begin{pmatrix} x & & \\ a & c & e \\ b & \pm e\ell & \pm c \end{pmatrix} \begin{pmatrix} r & \\ s & t \end{pmatrix}. \tag{4.32}$$

The action is on the space V/V_x with

$$V_x = \begin{pmatrix} 0 & 0 & 0 \\ & z & 0 \\ & & -z\ell \end{pmatrix} \begin{pmatrix} 0 & * & * \\ & * & * \\ & & * \end{pmatrix}. \tag{4.33}$$

Lemma 17. *In the case of $\mathscr{O}_{D2}$, the action of $G_x \times \mathrm{GL}_1$ on V/V_x has eleven orbits, listed with a representative, size and stabilizer in the following table.*

	Orbit representative	*Orbit size*	*Stabilizer*
1	$\begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)p^3$	$\begin{pmatrix} x & 0 & 0 \\ 0 & c & e \\ 0 & \pm e\ell & \pm c \end{pmatrix} \begin{pmatrix} \frac{1}{x^2} & \\ & t \end{pmatrix}$
2	$\begin{pmatrix} 1 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$\frac{(p-1)^2 p^3 (p+1)}{2}$	$\begin{pmatrix} x & & \\ & x & \\ & & \pm x \end{pmatrix} \begin{pmatrix} \frac{1}{x^2} & \\ & t \end{pmatrix} \sqcup \begin{pmatrix} x & & \\ 2x & -x & \\ & & \pm x \end{pmatrix} \begin{pmatrix} \frac{1}{x^2} & \\ & t \end{pmatrix}$
$3a$	$\begin{pmatrix} 1 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$, $(-1 = \square)$	$\frac{(p-1)^2 p^3 (p+1)}{2}$	$\begin{pmatrix} x & & \\ & \pm x & \\ & & \pm x \end{pmatrix} \begin{pmatrix} \frac{1}{x^2} & \\ & t \end{pmatrix} \sqcup \begin{pmatrix} x & & \\ & \pm\sqrt{-1}x & \\ & & \mp\sqrt{-1}x \end{pmatrix} \begin{pmatrix} \frac{1}{x^2} & \\ & t \end{pmatrix}$
$3b$	$\begin{pmatrix} 1 & 0 & 0 \\ & 1 & k \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$, $\ell - 4k^2 = \square, (-1 \neq \square)$	$\frac{(p-1)^2 p^3 (p+1)}{2}$	$\star$
4	$\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)p^2(p+1)$	$\begin{pmatrix} x & & \\ & c & \\ & & \pm c \end{pmatrix} \begin{pmatrix} \frac{1}{xc} & \\ s & t \end{pmatrix}$
5	$\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)^2p^2(p+1)$	$\begin{pmatrix} x & & \\ & c & \\ & & \pm c \end{pmatrix} \begin{pmatrix} \frac{1}{xc} & \\ s & \frac{1}{x^2} \end{pmatrix}$
6	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$\frac{(p-1)(p+1)}{2}$	$\begin{pmatrix} x & & \\ a & c & \\ b & & \pm c \end{pmatrix} \begin{pmatrix} \frac{1}{c^2} & \\ s & t \end{pmatrix} \sqcup \begin{pmatrix} x & & \\ a & & e \\ b & \pm e\ell & \end{pmatrix} \begin{pmatrix} \frac{1}{e^2\ell} & \\ s & t \end{pmatrix}$

	Orbit representative	*Orbit size*	*Stabilizer*
7	$\begin{pmatrix}0&0&0\\&1&0\\&&0\end{pmatrix}\begin{pmatrix}1&0&0\\&0&0\\&&0\end{pmatrix}$	$\frac{(p-1)^2(p+1)}{2}$	$\begin{pmatrix}x&&\\a&c&\\b&&\pm c\end{pmatrix}\begin{pmatrix}\frac{1}{c^2}&\\s&\frac{1}{x^2}\end{pmatrix} \sqcup \begin{pmatrix}x&&\\a&&e\\b&\pm e\ell&\end{pmatrix}\begin{pmatrix}\frac{1}{e^2\ell}&\\s&\frac{1}{x^2}\end{pmatrix}$
$8a$	$\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$, $(-1=\square)$	$\frac{(p-1)(p+1)}{2}$	$\begin{pmatrix}x&&\\a&c&\\b&&\pm c\end{pmatrix}\begin{pmatrix}\frac{\pm 1}{c^2}&\\s&t\end{pmatrix} \sqcup \begin{pmatrix}x&&\\a&&e\\b&\pm e\ell&\end{pmatrix}\begin{pmatrix}\frac{\pm 1}{e^2\ell}&\\s&t\end{pmatrix}$
$8b$	$\begin{pmatrix}0&0&0\\&1&k\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$, $\ell-4k^2=\square, (-1\neq\square)$	$\frac{(p-1)(p+1)}{2}$	$\star$
$9a$	$\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}\begin{pmatrix}1&0&0\\&0&0\\&&0\end{pmatrix}$, $(-1=\square)$	$\frac{(p-1)^2(p+1)}{2}$	$\begin{pmatrix}x&&\\a&c&\\b&&\pm c\end{pmatrix}\begin{pmatrix}\frac{\pm 1}{c^2}&\\s&\frac{1}{x^2}\end{pmatrix} \sqcup \begin{pmatrix}x&&\\a&&e\\b&\pm e\ell&\end{pmatrix}\begin{pmatrix}\frac{\pm 1}{e^2\ell}&\\s&\frac{1}{x^2}\end{pmatrix}$
$9b$	$\begin{pmatrix}0&0&0\\&1&k\\&&0\end{pmatrix}\begin{pmatrix}1&0&0\\&0&0\\&&0\end{pmatrix}$, $\ell-4k^2=\square, (-1\neq\square)$	$\frac{(p-1)^2(p+1)}{2}$	$\star$
10	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}1&0&0\\&0&0\\&&0\end{pmatrix}$	$p-1$	$\begin{pmatrix}x&&\\a&c&e\\b&\pm e\ell&\pm c\end{pmatrix}\begin{pmatrix}r&\\s&\frac{1}{x^2}\end{pmatrix}$
11	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	1	$G_x \times \mathrm{GL}_1$

(4.34)

Descriptions of the $\star$'d cases are given below.

Proof. To prove correctness, it suffices to verify the stabilizers and prove that the orbits of the same size are inequivalent, since then all points are accounted for.

Verification that equal sized orbits are inequivalent To prove that orbits 2 and 3 are inequivalent, consider the action

$$\begin{pmatrix}x&&\\a&c&e\\b&\pm e\ell&\pm c\end{pmatrix}\begin{pmatrix}r&\\s&t\end{pmatrix}\cdot(u^2+2uv,0) \bmod V_x. \tag{4.35}$$

The first quadratic form becomes

$$r(xu+av+bw)(xu+(a+2c)v+(b\pm 2e\ell)w). \tag{4.36}$$

To fix the u^2 coefficient, $r = \frac{1}{x^2}$. For the uv and uw terms to vanish, $(a + 2c) = -a$ and $(b \pm 2e\ell) = -b$. Hence the v^2 and w^2 terms become

$$\frac{1}{x^2}(-a^2v^2 - b^2w^2) \equiv -\frac{1}{x^2}(a^2 + b^2\ell^{-1})v^2 \bmod (v^2 - \ell w^2). \tag{4.37}$$

The vw coefficient is $\frac{-2ab}{x^2}$. When $-1 = \square$, $a^2 + b^2\ell^{-1} = 0$ if and only if $a = b = 0$, but this causes the acting matrix to be singular, a contradiction. When $-1 \neq \square$ substitute $a = \frac{-a}{x}$, $b = \frac{b}{x}$ so that the system to be solved is $ab = k$, $a^2+b^2\ell^{-1} = -1$ or $a^2+\frac{k^2}{a^2\ell} = -1$, or $a^4+a^2+\frac{k^2}{\ell} = 0$. This has no solution, since the discriminant $1-\frac{4k^2}{\ell} \neq \square \Leftrightarrow \ell-4k^2 = \square$.

To check that orbits 6 and 8 are inequivalent, acting on the representative in 6 maps the first quadratic form to

$$r \cdot (c^2v^2 \pm 2ce\ell vw + e^2\ell^2w^2) \equiv r \cdot ((c^2 + e^2\ell)v^2 \pm 2ce\ell vw) \bmod v^2 - \ell w^2. \tag{4.38}$$

When $-1 = \square$, $c^2 + e^2\ell = 0$ if and only if $c = e = 0$, which makes the acting matrix singular. When $-1 \neq \square$, there is no solution to

$$\frac{\pm ce\ell}{c^2 + e^2\ell} = k \Leftrightarrow c^2k \pm ce\ell + e^2\ell k = 0 \tag{4.39}$$

in $(c, e) \neq (0, 0)$ since $\ell - 4k^2 = \square$ implies that the discriminant $\ell^2 - 4\ell k^2 \neq \square$. The inequivalence of orbits 7 and 9 follows similarly.

Calculation of stabilizers The following calculations are performed in **OD2_stabilizers.nb**. Throughout the acting group is assumed given in coordinates as

$$\begin{pmatrix} x & & \\ a & c & e \\ b & \pm e\ell & \pm c \end{pmatrix} \begin{pmatrix} r & \\ s & t \end{pmatrix} \tag{4.40}$$

and recall $V_x = \begin{pmatrix} 0 & 0 & 0 \\ & z & 0 \\ & & -z\ell \end{pmatrix} \begin{pmatrix} 0 & * & * \\ & * & * \\ & & * \end{pmatrix}$.

(1.) The pair of forms are $(u^2, 0)$. The u^2 coefficients force $s = 0$, $r = \frac{1}{x^2}$. The uv and uw coefficients force $a = b = 0$.

(2.) The pair of forms are $(u^2 + 2uv, 0)$. The u^2 coefficients force $s = 0$, $r = \frac{1}{x^2}$. The GL_3 action carries

$$u(u + 2v) \mapsto (xu + av + bw)(xu + (a + 2c)v + (b + 2d)w), \qquad d = \pm 2e\ell \tag{4.41}$$

To make the uw term 0, $x(2b + 2d) = 0$ so $b = -d$. To make the vw term 0, $a \cdot (2b + 2d) + 2bc = 0$ so $b = 0$ or $c = 0$.

- (Case 1, $b = 0$): In this case $d = 0$ so $(xu + av)(xu + (a + 2c)v) = su(u + 2v)$, and $s = \frac{1}{x^2}$. If $a = 0$ then $c = x$ which gives the solutions

$$\begin{pmatrix} x & & \\ & x & \\ & & \pm x \end{pmatrix} \begin{pmatrix} \frac{1}{x^2} & \\ & t \end{pmatrix}. \tag{4.42}$$

If $a \neq 0$ then $a = 2x$ and $c = -x$ which obtains the solutions

$$\begin{pmatrix} x & & \\ 2x & -x & \\ & & \pm x \end{pmatrix} \begin{pmatrix} \frac{1}{x^2} & \\ & t \end{pmatrix}. \tag{4.43}$$

- (Case 2, $b \neq 0$, $b = -d$, $c = 0$): In this case

$$\begin{aligned} (xu + av + bw)(xu + av - bw) &= x^2u^2 + 2axuv + a^2v^2 - b^2w^2 \\ &\equiv x^2u^2 + 2axuv + (a^2\ell - b^2)w^2 \bmod v^2 - \ell w^2. \end{aligned} \tag{4.44}$$

Since $b \neq 0$, $a^2\ell - b^2 \neq 0$, so this does not produce solutions.

(3a.) The pair of forms are $(u^2 + 2vw, 0)$. The u^2 coefficients force $r = \frac{1}{x^2}$ and $s = 0$. The uv and uw coefficients force $a = b = 0$. The GL_3 action carries

$$\begin{aligned} 2vw \mapsto & 2(cv \pm e\ell w)(ev \pm cw) \\ &= 2cev^2 \pm 2(c^2 + e^2\ell)vw + 2ce\ell w^2 \\ &\equiv 4cev^2 \pm 2(c^2 + e^2\ell)vw \bmod v^2 - \ell w^2 \end{aligned} \tag{4.45}$$

Thus $ce = 0$. The case $c = 0$ may be ruled out, since the ratio between the u^2 and $2vw$ coefficient becomes a non-square in this case. The case $e = 0$, $c \neq 0$ obtains the stabilizer claimed.

(3b.) The pair of forms are $(u^2 + v^2 + 2kvw, 0)$. The u^2 coefficients force $r = \frac{1}{x^2}$ and $s = 0$. The uv and uw coefficients force $a = b = 0$. Under the GL_3 action,

$$\begin{aligned} v(v + 2kw) \mapsto & (cv \pm e\ell w)((cv \pm e\ell w) + 2k(ev \pm cw)) \\ &\equiv (c^2 + 4cek + e^2\ell)v^2 \pm (2c^2k + 2ce\ell + 2e^2\ell k)vw \bmod (v^2 - \ell w^2). \end{aligned} \tag{4.46}$$

To stabilize,

$$k(c^2 + 4cek + e^2\ell) = \pm(c^2k + ce\ell + e^2\ell k). \tag{4.47}$$

The plus case leads to $(4k^2 - \ell)ce = 0$, or $ce = 0$. Note that $c = 0$, $e \neq 0$ does not lead to a solution, since it is not possible that $e^2\ell = x^2$. This obtains two solutions, with $e = 0$ and $c = \pm x$.

Solving the quadratic equation, the minus case obtains

$$\frac{c}{e} = \frac{-(4k^2+\ell) \pm (4k^2-\ell)}{4k}, \quad \Leftrightarrow \quad \frac{c}{e} = -2k \text{ or } -\frac{\ell}{2k}. \tag{4.48}$$

Subject to the quadratic condition $2c^2k^2+(4k^2+\ell)ce+2e^2\ell k = 0$, the form becomes

$$-\frac{1}{2k}\left(\ell - 4k^2\right) cev^2 - (\ell - 4k^2)cevw. \tag{4.49}$$

Thus the condition to be a stabilizer becomes,

$$-\frac{1}{2k}\left(\ell - 4k^2\right) ce = x^2. \tag{4.50}$$

Since $\ell - 4k^2 = \square$, $\frac{c}{e} = -2k$ leads to a pair of solutions. Meanwhile $\frac{c}{e} = -\frac{\ell}{2k}$ leads to none, since its product with $-2k$ is not a square.

(4.) The pair of forms are $(2uv, 0)$. Under the GL_3 action,

$$2uv \mapsto 2(xu + av + bw)(cv \pm e\ell w). \tag{4.51}$$

Considering the uw coefficient forces $e = 0$. Considering the vw and v^2 coefficients forces $a = b = 0$. The uv coefficient gives $r = \frac{1}{xc}$.

(5.) The pair of forms are $(2uv, u^2)$. This is the same as (4.), except that now $t = \frac{1}{x^2}$.

(6.) The pair of forms are $(v^2, 0)$. The GL_3 actions carries $v^2 \mapsto (cv \pm e\ell w)^2$, so the vw coefficient implies $ce = 0$. This leads to the two cases given.

(7.) The pair of forms are (v^2, u^2). This is the same as (6.), with the additional requirement that now $t = \frac{1}{x^2}$.

(8a.) The pair of forms are $(2vw, 0)$. Under the GL_3 action,

$$2vw \mapsto 2(cv \pm e\ell w)(ev \pm cw) \equiv 4cev^2 \pm 2(c^2 + e^2\ell)vw \bmod v^2 - \ell w^2. \tag{4.52}$$

Hence $ce = 0$, which leads to the two solutions given.

(8b.) The pair of forms are $(v^2 + 2kvw, 0)$. Under the GL_3 action,

$$\begin{aligned} v(v+2kw) \mapsto& (cv \pm e\ell w)((cv \pm e\ell w) + 2k(ev \pm cw)) \\ &\equiv (c^2 + 4cek + e^2\ell)v^2 \pm (2c^2k + 2ce\ell + 2e^2\ell k)vw \bmod (v^2 - \ell w^2). \end{aligned} \tag{4.53}$$

To stabilize,

$$k(c^2 + 4cek + e^2\ell) = \pm(c^2k + ce\ell + e^2\ell k). \tag{4.54}$$

The plus case leads to $(4k^2 - \ell)ce = 0$, or $ce = 0$. Both $c \neq 0$ and $e \neq 0$ lead to solutions. Solving the quadratic equation, the minus case obtains

$$\frac{c}{e} = \frac{-(4k^2 + \ell) \pm (4k^2 - \ell)}{4k}. \tag{4.55}$$

This gives two further solutions.

(9a.) The pair of forms are $(2vw, u^2)$. This case is the same as (8a.) with the further restriction $t = \frac{1}{x^2}$.

(9b.) The pair of forms are $(v^2 + 2kvw, u^2)$. This case is the same as (8b.) with the further restriction $t = \frac{1}{x^2}$.

(10.) The pair of forms are $(0, u^2)$. The only restriction is $t = \frac{1}{x^2}$.

(11.) The pair of forms are $(0, 0)$. There is no constraint. $\square$

4.2.3. Case of $\mathscr{O}_{D11}$

The acting group is

$$G_x \times \mathrm{GL}_1 = \begin{pmatrix} * & & \\ * & * & \\ * & & * \end{pmatrix} \begin{pmatrix} * & \\ * & * \end{pmatrix} \sqcup \begin{pmatrix} * & & \\ * & & * \\ * & * & \end{pmatrix} \begin{pmatrix} * & \\ * & * \end{pmatrix} \tag{4.56}$$

of size $2(p-1)^5p^3$. This acts on V/V_x with

$$V_x = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & * \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & * & * \\ & * & * \\ & & * \end{pmatrix}. \tag{4.57}$$

Lemma 18. *In the case of $\mathscr{O}_{D11}$ the space V/V_x splits into 20 orbits with representatives, sizes and stabilizers given in the following table.*

	Orbit representative	*Orbit size*	*Stabilizer*
1	$\begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)p^3$	$\begin{pmatrix} r & & \\ & * & \\ & & * \end{pmatrix} \begin{pmatrix} \frac{1}{r^2} & \\ & * \end{pmatrix} \sqcup \begin{pmatrix} r & & \\ & & * \\ & * & \end{pmatrix} \begin{pmatrix} \frac{1}{r^2} & \\ & * \end{pmatrix}$
2	$\begin{pmatrix} 1 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)^2p^3$	$\begin{pmatrix} r & & \\ & r & \\ & & * \end{pmatrix} \begin{pmatrix} \frac{1}{r^2} & \\ & * \end{pmatrix} \sqcup \begin{pmatrix} r & & \\ 2r & -r & \\ & & * \end{pmatrix} \begin{pmatrix} \frac{1}{r^2} & \\ & * \end{pmatrix}$

	Orbit representative	*Orbit size*	*Stabilizer*
3	$\begin{pmatrix}1&1&1\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$\frac{(p-1)^3p^3}{4}$	$\begin{pmatrix}r&&\\&r&\\&&r\end{pmatrix}\begin{pmatrix}\frac{1}{r^2}&\\&*\end{pmatrix}$ $\sqcup\begin{pmatrix}r&&\\&&r\\&r&\end{pmatrix}\begin{pmatrix}\frac{1}{r^2}&\\&*\end{pmatrix}$ $\sqcup\begin{pmatrix}r&&\\2r&-r&\\2r&&-r\end{pmatrix}\begin{pmatrix}\frac{1}{r^2}&\\&*\end{pmatrix}$ $\sqcup\begin{pmatrix}r&&\\2r&&-r\\2r&-r&\end{pmatrix}\begin{pmatrix}\frac{1}{r^2}&\\&*\end{pmatrix}$ $\sqcup\begin{pmatrix}r&&\\2r&-r&\\&&r\end{pmatrix}\begin{pmatrix}\frac{1}{r^2}&\\&*\end{pmatrix}$ $\sqcup\begin{pmatrix}r&&\\2r&&-r\\&r&\end{pmatrix}\begin{pmatrix}\frac{1}{r^2}&\\&*\end{pmatrix}$ $\sqcup\begin{pmatrix}r&&\\&r&\\2r&&-r\end{pmatrix}\begin{pmatrix}\frac{1}{r^2}&\\&*\end{pmatrix}$ $\sqcup\begin{pmatrix}r&&\\&&r\\2r&-r&\end{pmatrix}\begin{pmatrix}\frac{1}{r^2}&\\&*\end{pmatrix}$
4	$\begin{pmatrix}1&0&0\\&-\ell&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$(p-1)^2p^3$	$\begin{pmatrix}r&&\\&\pm r&\\&&*\end{pmatrix}\begin{pmatrix}\frac{1}{r^2}&\\&*\end{pmatrix}$
5	$\begin{pmatrix}1&0&1\\&-\ell&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$\frac{(p-1)^3p^3}{2}$	$\begin{pmatrix}r&&\\&\pm r&\\&&r\end{pmatrix}\begin{pmatrix}\frac{1}{r^2}&\\&*\end{pmatrix}$ $\sqcup\begin{pmatrix}r&&\\&\pm r&\\2r&&-r\end{pmatrix}\begin{pmatrix}\frac{1}{r^2}&\\&*\end{pmatrix}$
6	$\begin{pmatrix}1&0&0\\&-\ell&0\\&&-\ell\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$\frac{(p-1)^3p^3}{4}$	$\begin{pmatrix}r&&\\&r\epsilon_1&\\&&r\epsilon_2\end{pmatrix}\begin{pmatrix}\frac{1}{r^2}&\\&*\end{pmatrix}$ $\sqcup\begin{pmatrix}r&&\\&&r\epsilon_1\\&r\epsilon_2&\end{pmatrix}\begin{pmatrix}\frac{1}{r^2}&\\&*\end{pmatrix}$
7	$\begin{pmatrix}0&1&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$2(p-1)p$	$\begin{pmatrix}r&&\\&s&\\ *&&*\end{pmatrix}\begin{pmatrix}\frac{1}{rs}&\\ *&*\end{pmatrix}$
8	$\begin{pmatrix}0&1&1\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$(p-1)^2p^2$	$\begin{pmatrix}r&&\\&s&\\&&s\end{pmatrix}\begin{pmatrix}\frac{1}{rs}&\\ *&*\end{pmatrix}$ $\sqcup\begin{pmatrix}r&&\\&&s\\&s&\end{pmatrix}\begin{pmatrix}\frac{1}{rs}&\\ *&*\end{pmatrix}$
9	$\begin{pmatrix}0&1&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$2(p-1)^2p$	$\begin{pmatrix}r&&\\&\frac{s^2}{r}&\\ *&&s\end{pmatrix}\begin{pmatrix}\frac{1}{s^2}&\\ *&*\end{pmatrix}$

	Orbit representative	*Orbit size*	*Stabilizer*
10	$\begin{pmatrix}0&1&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}1&0&0\\&0&0\\&&0\end{pmatrix}$	$2(p-1)^2p$	$\begin{pmatrix}r&&\\&s&\\*&&*\end{pmatrix}\begin{pmatrix}\frac{1}{rs}&\\&\frac{1}{r^2}\end{pmatrix}$
11	$\begin{pmatrix}0&1&1\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}1&0&0\\&0&0\\&&0\end{pmatrix}$	$(p-1)^3p^2$	$\begin{pmatrix}r&&\\&s&\\&&s\end{pmatrix}\begin{pmatrix}\frac{1}{rs}&\\*&\frac{1}{r^2}\end{pmatrix}$ $\sqcup\begin{pmatrix}r&&\\&&s\\&s&\end{pmatrix}\begin{pmatrix}\frac{1}{rs}&\\*&\frac{1}{r^2}\end{pmatrix}$
12	$\begin{pmatrix}0&1&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}1&0&0\\&0&0\\&&0\end{pmatrix}$	$2(p-1)^3p$	$\begin{pmatrix}r&&\\&\frac{s^2}{r}&\\*&&s\end{pmatrix}\begin{pmatrix}\frac{1}{s^2}&\\*&\frac{1}{r^2}\end{pmatrix}$
13	$\begin{pmatrix}0&0&0\\&1&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$2(p-1)$	$\begin{pmatrix}*&&\\*&s&\\*&&*\end{pmatrix}\begin{pmatrix}\frac{1}{s^2}&\\*&*\end{pmatrix}$
14	$\begin{pmatrix}0&0&0\\&1&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$\frac{(p-1)^2}{2}$	$\begin{pmatrix}*&&\\*&s&\\*&&\pm s\end{pmatrix}\begin{pmatrix}\frac{1}{s^2}&\\*&*\end{pmatrix}$ $\sqcup\begin{pmatrix}*&&\\*&&s\\*&\pm s&\end{pmatrix}\begin{pmatrix}\frac{1}{s^2}&\\*&*\end{pmatrix}$
15	$\begin{pmatrix}0&0&0\\&1&0\\&&\ell\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$\frac{(p-1)^2}{2}$	$\begin{pmatrix}*&&\\*&s&\\*&&\pm s\end{pmatrix}\begin{pmatrix}\frac{1}{s^2}&\\*&*\end{pmatrix}$ $\sqcup\begin{pmatrix}*&&\\*&&\pm s\\*&s\ell&\end{pmatrix}\begin{pmatrix}\frac{1}{s^2\ell}&\\*&*\end{pmatrix}$
16	$\begin{pmatrix}0&0&0\\&1&0\\&&0\end{pmatrix}\begin{pmatrix}1&0&0\\&0&0\\&&0\end{pmatrix}$	$2(p-1)^2$	$\begin{pmatrix}r&&\\*&s&\\*&&*\end{pmatrix}\begin{pmatrix}\frac{1}{s^2}&\\*&\frac{1}{r^2}\end{pmatrix}$
17	$\begin{pmatrix}0&0&0\\&1&0\\&&1\end{pmatrix}\begin{pmatrix}1&0&0\\&0&0\\&&0\end{pmatrix}$	$\frac{(p-1)^3}{2}$	$\begin{pmatrix}r&&\\*&s&\\*&&\pm s\end{pmatrix}\begin{pmatrix}\frac{1}{s^2}&\\*&\frac{1}{r^2}\end{pmatrix}$ $\sqcup\begin{pmatrix}r&&\\*&&s\\*&\pm s&\end{pmatrix}\begin{pmatrix}\frac{1}{s^2}&\\*&\frac{1}{r^2}\end{pmatrix}$
18	$\begin{pmatrix}0&0&0\\&1&0\\&&\ell\end{pmatrix}\begin{pmatrix}1&0&0\\&0&0\\&&0\end{pmatrix}$	$\frac{(p-1)^3}{2}$	$\begin{pmatrix}r&&\\*&s&\\*&&\pm s\end{pmatrix}\begin{pmatrix}\frac{1}{s^2}&\\&\frac{1}{r^2}\end{pmatrix}$ $\sqcup\begin{pmatrix}r&&\\*&&\pm s\\*&s\ell&\end{pmatrix}\begin{pmatrix}\frac{1}{s^2\ell}&\\*&\frac{1}{r^2}\end{pmatrix}$
19	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}1&0&0\\&0&0\\&&0\end{pmatrix}$	$p-1$	$\begin{pmatrix}r&&\\*&*&\\*&&*\end{pmatrix}\begin{pmatrix}*&\\&\frac{1}{r^2}\end{pmatrix}$ $\sqcup\begin{pmatrix}r&&\\*&&*\\*&*&\end{pmatrix}\begin{pmatrix}*&\\*&\frac{1}{r^2}\end{pmatrix}$
20	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	1	$G_x\times\mathrm{GL}_1$

(4.58)

Proof. To verify correctness, it suffices to check that the orbits of the same size are inequivalent, and that the stabilizers are correct, since the count of elements exhausts V/V_x.

Verification that equal sized orbits are inequivalent The verification that 2. and 4. are inequivalent follows from the fact that u^2+2uv and $u^2-\ell v^2$ are inequivalent under the action of $\mathrm{GL}_2\times\mathrm{GL}_1$ on $\mathrm{Sym}^2(\mathbf{F}_p^2)$. The same is true of 3. and 6. since both orbits are invariant under swapping v and w. 9. and 10. are inequivalent since the second quadratic form contains a u^2 term. 14. and 15. are inequivalent because when acting on v^2+w^2, the ratio between the coefficients on v^2 and w^2 changes by a square. The same holds for 17. and 18.

Calculation of stabilizers The fact that each of the listed stabilizers does in fact stabilize the form is checked in **OD11_stabilizers.nb**. The following proofs guarantee that the actual stabilizing group is no larger.

Assume throughout that the acting group is given in coordinates as

$$G_x\times\mathrm{GL}_1=\begin{pmatrix} r & & \\ a & s & \\ b & & t\end{pmatrix}\begin{pmatrix} x & \\ y & z\end{pmatrix}\sqcup\begin{pmatrix} r & & \\ a & & s \\ b & t & \end{pmatrix}\begin{pmatrix} x & \\ y & z\end{pmatrix}=G_1\sqcup G_2. \tag{4.59}$$

Recall that

$$V_x=\left(\begin{matrix} 0 & 0 & 0\\ & 0 & *\\ & & 0\end{matrix}\right)\left(\begin{matrix} 0 & * & *\\ & * & *\\ & & *\end{matrix}\right). \tag{4.60}$$

(1.) The pair of forms is $(u^2,0)$. The uv and uw coefficients in the first form force $a=b=0$. The u^2 coefficients imply $x=\frac{1}{r^2}$ and $y=0$.

(2.) The pair of forms is $(u^2+2uv,0)$. By considering the u^2 coefficient of the second form, $y=0$. Consider the action only on the first form where $u(u+2v) \bmod vw$ must be preserved. The action either preserves or exchanges the two lines $u=0$ and $u+2v=0$, which obtains the stabilizer claimed.

(3.) The pair of forms is $(u^2+2uv+2uw,0)$. The u^2 coefficient in the second quadratic form forces $y=0$. Depending on the choice of c, there are two ways of factoring $u^2+2uv+2uw+cvw$ into a pair of linear factors. If $c=0$ the factorization is $u(u+2v+2w)$. If $c\neq 0$ then both linear factors contain a u term, one contains a v term and the other contains a w term. Up to scaling the two forms, this obtains $(u+2v)(u+2w)=u^2+2uv+2uw+4vw$. To preserve $u(u+2v+2w) \bmod vw$, the GL_3 action can first either swap v and w, or not, then map $u=0$ to any of the four lines $u=0$, $u+2v+2w=0$, $u+2v=0$ or $u+2w=0$, and map $u+2v+2w=0$ to the other line in the corresponding factorization. This obtains the eight parts of the stabilizer given.

(4.) The pair of forms is $(u^2 - \ell v^2, 0)$. By considering the u^2 coefficient of the second form, $y = 0$. By considering the uv and uw coefficients of the first form, $a = b = 0$. By considering the w^2 coefficient of the first form, the acting group is contained in G_1. By considering the u^2 and v^2 coefficients of the first form, $s = \pm r$ and $x = \frac{1}{r^2}$.

(5.) The pair of forms is $(u^2 + 2uw - \ell v^2, 0)$. By considering the u^2 coefficient of the second form, $y = 0$, so it suffices to consider the action on the first form. First consider the possibility that the action is from G_2. This maps $u(u + 2w) - \ell v^2 \mapsto (ru+av+bw)(ru+av+bw+2sv)-t^2\ell w^2 \bmod vw$. To obtain that the uv coefficient is 0, $2s + a = -a$, but this makes the v^2 coefficient $-a^2$ which is not the negative of a non-square. Hence the acting matrix is from G_1. This maps $u(u+2w)-\ell v^2 \mapsto (ru+av+bw)(ru+av+bw+2tw)-s^2\ell v^2 \bmod vw$. By considering the uv coordinate, $a = 0$. It follows that the mapping either fixes $u = 0$ and $u + 2w = 0$ or exchanges them. These two possibilities obtain the two parts of the stabilizers given.

(6.) The pair of forms is $(u^2 - \ell v^2 - \ell w^2, 0)$. By considering the u^2 coefficient in the second form, $y = 0$, so it is sufficient to consider the action on the first form. By considering the uv and uw coefficients, $a = b = 0$. Either exchanging $v = 0$ and $w = 0$ or not leads to the stabilizer claimed.

(7.) The pair of forms is $(2uv, 0)$. Since there is no u^2 coefficient in either form, y and z are arbitrary, and it suffices to consider stabilization of the first form. To preserve $2uv \bmod vw$, note that $u \mapsto ru + av + bw$ with $r \neq 0$, and hence the uw coefficient forces $v = 0$ to be preserved. The v^2 coefficient forces $a = 0$. Now b is arbitrary, which obtains the stabilizer given.

(8.) The pair of forms is $(2uv + 2uw, 0)$. Since there is no u^2 coefficient in either form, y and z are arbitrary, and it suffices to consider stabilization of the first form. To preserve $2u(v + w) \bmod vw$, note that the only way to factor $2uv + 2uw + cvw$ into a pair of linear forms is with $c = 0$. Hence the lines $u = 0$ and $v + w = 0$ are preserved. Note that v and w may be preserved or exchanged, which leads to the stabilizer given.

(9.) The pair of forms is $(2uv + w^2, 0)$. Since there is no u^2 coefficient in either form, y and z are arbitrary, and it suffices to consider stabilization of the first form. Since there is no uw coefficient, v must be mapped to a scalar times itself. The same is then true of w. Considering the v^2 coefficient fixes $a = 0$, while b is arbitrary.

(10.) The pair of forms is $(2uv, u^2)$. This is the same as 7, except that the u^2 term in the second form fixes z.

(11.) The pair of forms is $(2uv + 2uw, u^2)$. This is the same as 8, except that the u^2 term in the second form fixes z.

(12.) The pair of forms is $(2uv + w^2, u^2)$. This is the same as 9, except that the u^2 term in the second form fixes z.

For the remainder of the forms, r, a, b are arbitrary since the first form does not depend on u.

(13.) The pair of forms is $(v^2, 0)$. The only constraints are that $v = 0$ is preserved, and that x is fixed.

(14.) The pair of forms is $(v^2+w^2,0)$. $v=0$ and $w=0$ may either be fixed or exchanged, the scaling of both variables has the same square.
(15.) The pair of forms is $(v^2+\ell w^2,0)$. This is the same as 14, except that if v and w are exchanged, a further scaling is necessary to preserve the ratio of the v^2 and w^2 coefficients.
(16.) The pair of forms is (v^2,u^2). This is the same as 13, except that the u^2 term in the second form fixes z.
(17.) The pair of forms is (v^2+w^2,u^2). This is the same as 14, except that the u^2 term in the second form fixes z.
(18.) The pair of forms is $(v^2+\ell w^2,u^2)$. This is the same as 15, except that the u^2 term in the second form fixes z.
(19.) The pair of forms is $(0,u^2)$. The only constraint is that the u^2 term in the second form fixes z.
(20.) The pair of forms is $(0,0)$. No constraint. □

4.2.4. Case of $\mathscr{O}_{D1^2}$

The acting group is

$$G_x \times \mathrm{GL}_1 = \begin{pmatrix} * & * & \\ * & * & \\ * & * & * \end{pmatrix} \begin{pmatrix} * & \\ * & * \end{pmatrix}. \tag{4.61}$$

The action is on the space V/V_x,

$$V_x = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & * \end{pmatrix} \begin{pmatrix} 0 & 0 & * \\ & 0 & * \\ & & * \end{pmatrix}. \tag{4.62}$$

Lemma 19. *Treat* $\mathrm{Sym}^2(\mathbf{F}_p^3)$ *as ternary quadratic forms in variables* u,v,w. *The group* $\begin{pmatrix} * & * & \\ * & * & \\ * & * & * \end{pmatrix} \times \mathrm{GL}_1$ *acts on* $\mathrm{Sym}^2(\mathbf{F}_p^3)/(w^2)$, *with the first factor acting by linear change of variable, and the* GL_1 *factor acting by multiplying the form by a scalar. Under this action,* $\mathrm{Sym}^2(\mathbf{F}_p^3)/(w^2)$ *has six orbits, with representatives*

a. $\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$

b. $\begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$

c. $\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$

d. $\begin{pmatrix} 1 & 0 & 0 \\ & -\ell & 0 \\ & & 0 \end{pmatrix}.$

Proof. The change of variable preserves (w^2), so the usual action of $\mathrm{GL}_3 \times \mathrm{GL}_1$ on $\mathrm{Sym}^2(\mathbf{F}_p^3)$ descends to the quotient. Identify the quotient by (uw, vw, w^2) with $\mathrm{Sym}^2(\mathbf{F}_p^2)$ acted on by $\mathrm{GL}_2(\mathbf{F}_p) \times \mathrm{GL}_1(\mathbf{F}_p)$. By Lemma 1, this action has four orbits with representatives

$$\begin{pmatrix} 0 & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ & -\ell \end{pmatrix}. \tag{4.63}$$

Extend these orbits to the quotient by (w^2) by fibering uw and vw over each $\mathrm{Sym}^2(\mathbf{F}_p^2)$ orbit. Above 0, uw and vw are equivalent. This obtains the orbits listed a. Above u^2, translating u by w can eliminate a uw term, so that the possibilities are u^2 and $u^2 + vw$, which are inequivalent, obtaining the b. orbits. Above uv, translating u and v by w can eliminate uw and vw terms, so the only c. orbit is uv. The same applies to $u^2 - \ell v^2$, so this is the only orbit in d. $\square$

The orbits of V/V_x under $G_x \times \mathrm{GL}_1$ are obtained by fibering the second quadratic form in u, v, treated as $\mathrm{Sym}^2(\mathbf{F}_p^2)$ over the orbits listed in Lemma 19, as described in the lemma that follows.

Lemma 20. *In the case of $\mathscr{O}_{D1^2}$, the action of $G_x \times \mathrm{GL}_1$ on V/V_x has 23 orbits, listed with a representative, size and stabilizer in the following table.*

Type a.	*Orbit representative*	*Orbit size*	*Stabilizer*
1.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	1	$G_x \times \mathrm{GL}_1$
2.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)(p+1)$	$\begin{pmatrix} s & * & \\ & * & \\ * & * & * \end{pmatrix} \begin{pmatrix} * & \\ * & \frac{1}{s^2} \end{pmatrix}$
3.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$\frac{(p-1)p(p+1)}{2}$	$\begin{pmatrix} s & & \\ & t & \\ * & * & * \end{pmatrix} \begin{pmatrix} * & \\ * & \frac{1}{st} \end{pmatrix} \sqcup \begin{pmatrix} & s & \\ t & & \\ * & * & * \end{pmatrix} \begin{pmatrix} * & \\ * & \frac{1}{st} \end{pmatrix}$
4.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & -\ell & 0 \\ & & 0 \end{pmatrix}$	$\frac{(p-1)^2 p}{2}$	$\begin{pmatrix} c & e & \\ \pm e\ell & \pm c & \\ * & * & * \end{pmatrix} \begin{pmatrix} * & \\ * & \frac{1}{c^2 - e^2\ell} \end{pmatrix}$
5.	$\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)^2(p+1)$	$\begin{pmatrix} s & * & \\ & * & \\ * & * & t \end{pmatrix} \begin{pmatrix} \frac{1}{st} & \\ * & \frac{1}{s^2} \end{pmatrix}$

Type a.	*Orbit representative*	*Orbit size*	*Stabilizer*
6.	$\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)^2p(p+1)$	$\begin{pmatrix} r & & \\ & s & \\ * & * & t \end{pmatrix} \begin{pmatrix} \frac{1}{rt} & \\ * & \frac{1}{s^2} \end{pmatrix}$
7.	$\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)^2p(p+1)$	$\begin{pmatrix} r & & \\ & s & \\ * & * & t \end{pmatrix} \begin{pmatrix} \frac{1}{rt} & \\ * & \frac{1}{rs} \end{pmatrix}$
8.	$\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}$	$\frac{(p-1)^3p(p+1)}{2}$	$\begin{pmatrix} s & & \\ & s & \\ * & * & t \end{pmatrix} \begin{pmatrix} \frac{1}{st} & \\ * & \frac{1}{s^2} \end{pmatrix}$ $\sqcup \begin{pmatrix} -s & 2s & \\ & s & \\ * & * & t \end{pmatrix} \begin{pmatrix} \frac{-1}{st} & \\ * & \frac{1}{s^2} \end{pmatrix}$
9.	$\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & -\ell & 0 \\ & & 0 \end{pmatrix}$	$\frac{(p-1)^3p(p+1)}{2}$	$\begin{pmatrix} s & & \\ & \pm s & \\ * & * & t \end{pmatrix} \begin{pmatrix} \frac{1}{st} & \\ * & \frac{1}{s^2} \end{pmatrix}$
10.	$\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)(p+1)$	$\begin{pmatrix} s & * & \\ & * & \\ * & * & t \end{pmatrix} \begin{pmatrix} \frac{1}{st} & \\ * & * \end{pmatrix}$

(4.64)

Type b.	*Orbit representative*	*Orbit size*	*Stabilizer*
11.	$\begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)p^2(p+1)$	$\begin{pmatrix} s & * & \\ & * & \\ & * & * \end{pmatrix} \begin{pmatrix} \frac{1}{s^2} & \\ & * \end{pmatrix}$
12.	$\begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)^2p^2(p+1)$	$\begin{pmatrix} s & a & \\ & t & \\ & * & * \end{pmatrix} \begin{pmatrix} \frac{1}{s^2} & \\ -\frac{2a}{s^2t} & \frac{1}{st} \end{pmatrix}$
13.	$\begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)^2p^3(p+1)$	$\begin{pmatrix} s & & \\ & t & \\ & * & * \end{pmatrix} \begin{pmatrix} \frac{1}{s^2} & \\ & \frac{1}{t^2} \end{pmatrix}$
14.	$\begin{pmatrix} 1 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)^2p^2(p+1)$	$\begin{pmatrix} s & -\frac{at}{s} & \\ & t & \\ a & * & \frac{s^2}{t} \end{pmatrix} \begin{pmatrix} \frac{1}{s^2} & \\ & * \end{pmatrix}$
15.	$\begin{pmatrix} 1 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)^3p^2(p+1)$	$\begin{pmatrix} s & -\frac{at}{s} & \\ & t & \\ a & * & \frac{s^2}{t} \end{pmatrix} \begin{pmatrix} \frac{1}{s^2} & \\ \frac{2a}{s^3} & \frac{1}{st} \end{pmatrix}$
16.	$\begin{pmatrix} 1 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)^3p^3(p+1)$	$\begin{pmatrix} s & & \\ & t & \\ & * & \frac{s^2}{t} \end{pmatrix} \begin{pmatrix} \frac{1}{s^2} & \\ & \frac{1}{t^2} \end{pmatrix}$

(4.65)

Type c.	*Orbit representative*	*Orbit size*	*Stabilizer*
17.	$\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$\frac{(p-1)p^4(p+1)}{2}$	$\begin{pmatrix} s & & \\ & t & \\ & & * \end{pmatrix} \begin{pmatrix} \frac{1}{st} & \\ & * \end{pmatrix} \sqcup \begin{pmatrix} & s & \\ t & & \\ & & * \end{pmatrix} \begin{pmatrix} \frac{1}{st} & \\ & * \end{pmatrix}$
18.	$\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(p-1)^2p^4(p+1)$	$\begin{pmatrix} s & & \\ & t & \\ & & * \end{pmatrix} \begin{pmatrix} \frac{1}{st} & \\ & \frac{1}{s^2} \end{pmatrix}$
19.	$\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}$	$\frac{(p-1)^3p^4(p+1)}{4}$	$\begin{pmatrix} s & & \\ & \pm s & \\ & & * \end{pmatrix} \begin{pmatrix} \frac{\pm 1}{s^2} & \\ & \frac{1}{s^2} \end{pmatrix} \sqcup \begin{pmatrix} & s & \\ \pm s & & \\ & & * \end{pmatrix} \begin{pmatrix} \frac{\pm 1}{s^2} & \\ & \frac{1}{s^2} \end{pmatrix}$
20.	$\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & \ell & 0 \\ & & 0 \end{pmatrix}$	$\frac{(p-1)^3p^4(p+1)}{4}$	$\begin{pmatrix} s & & \\ & \pm s & \\ & & * \end{pmatrix} \begin{pmatrix} \frac{\pm 1}{s^2} & \\ & \frac{1}{s^2} \end{pmatrix} \sqcup \begin{pmatrix} & \pm s & \\ s\ell & & \\ & & * \end{pmatrix} \begin{pmatrix} \frac{\pm 1}{s^2\ell} & \\ & \frac{1}{s^2\ell} \end{pmatrix}$

(4.66)

Type d.	*Orbit representative*	*Orbit size*	*Stabilizer*
21.	$\begin{pmatrix} 1 & 0 & 0 \\ & -\ell & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$\frac{(p-1)^2p^4}{2}$	$\begin{pmatrix} c & e & \\ \pm e\ell & \pm c & \\ & & * \end{pmatrix} \begin{pmatrix} \frac{1}{c^2-e^2\ell} & \\ & * \end{pmatrix}$
22.	$\begin{pmatrix} 1 & 0 & 0 \\ & -\ell & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}$	$\frac{(p-1)^3p^4(p+1)}{4}$	$\begin{pmatrix} s & & \\ & \pm s & \\ & & * \end{pmatrix} \begin{pmatrix} \frac{1}{s^2} & \\ & \frac{1}{s^2} \end{pmatrix} \sqcup \begin{pmatrix} & \pm s & \\ s\ell & & \\ & & * \end{pmatrix} \begin{pmatrix} \frac{-1}{s^2\ell} & \\ \frac{1}{s^2\ell^2} & \frac{1}{s^2\ell} \end{pmatrix}$
23*a*.	$\begin{pmatrix} 1 & 0 & 0 \\ & -\ell & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$, $(-1 = \square)$	$\frac{(p-1)^3p^4(p+1)}{4}$	$\begin{pmatrix} s & & \\ & \pm s & \\ & & * \end{pmatrix} \begin{pmatrix} \frac{1}{s^2} & \\ & \pm\frac{1}{s^2} \end{pmatrix} \sqcup \begin{pmatrix} & s & \\ \pm s\ell & & \\ & & * \end{pmatrix} \begin{pmatrix} \frac{-1}{s^2\ell} & \\ & \pm\frac{1}{s^2\ell} \end{pmatrix}$
23*b*.	$\begin{pmatrix} 1 & 0 & 0 \\ & -\ell & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & k & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}$, $1 - 4\ell k^2 \neq \square, (-1 \neq \square)$	$\frac{(p-1)^3p^4(p+1)}{4}$	$\star$

(4.67)

A description of the stabilizer $\star$ *is given in the proof.*

Proof. To classify the orbits of type a., when the first form is 0, the action on the second form is the same as $\mathrm{GL}_2 \times \mathrm{GL}_1$ acting on $\mathrm{Sym}^2(\mathbf{F}_p^2)$ which has the four orbits claimed. When the first form is uw, since $w = 0$ is preserved by the group action, modulo w, $u = 0$ is preserved, so that the action becomes $\begin{pmatrix} * & * \\ & * \end{pmatrix} \times \mathrm{GL}_1$ acting on $\mathrm{Sym}^2(\mathbf{F}_p^2)$. By Lemma 2, this action has six orbits, which gives the six orbits claimed.

To classify the orbits of type b., note that, by subtracting a multiple of the first form from the second, the u^2 term in the second form may be eliminated, so that it takes the form $c_1uv + c_2v^2$. When the first form is u^2, to stabilize the first quadratic form, u must be replaced by a multiple of itself. Thus the action on the second quadratic form is by $\begin{pmatrix} * & * \\ & * \end{pmatrix} \times \mathrm{GL}_1$, and uv is inequivalent with v^2 since the factor u cannot be eliminated. Meanwhile, if $c_2 \neq 0$ then the uv term may be eliminated by replacing v with $v + au$ for an appropriate a. Thus there are three inequivalent orbits with representatives $0, uv$ and v^2 under this action. When the first form is $u^2 + 2vw$, the first form may be stabilized by adjusting u by w and v by u simultaneously to keep the uw term 0. Thus the action on the second quadratic form (which does not depend on w) is the same, so there are three orbits in this case, also.

To classify the orbits of type c., by subtracting a multiple of the first form from the second it may be assumed that the second form is of type $c_1u^2 + c_2v^2$. To stabilize the first form, modulo w the lines $u = 0$ and $v = 0$ are either fixed or exchanged. If only one of c_1, c_2 is non-zero, it may be assumed that $c_1 \neq 0$ by exchanging u, v. Since the lines are preserved or exchanged, this is inequivalent to the orbits with both c_1 and $c_2 \neq 0$, and the orbits where c_1 and c_2 are related by a square or non-square are inequivalent, since the scaling in individual coordinates is quadratic.

To classify the orbits of type d., by subtracting a multiple of the first form from the second it may be assumed that the second form is of the type $c_1uv + c_2v^2$. To preserve the first quadratic form, the change of variables in u, v takes the form $\begin{pmatrix} c & e \\ \pm e\ell & \pm c \end{pmatrix}$. This maps v^2 in the second form to

$$(eu \pm cv)^2 = e^2u^2 \pm 2ceuv + c^2v^2 \equiv \pm 2ceuv + (c^2 + e^2\ell)v^2 \bmod u^2 - \ell v^2 \tag{4.68}$$

after subtracting a multiple of the first form to clear the u^2 term. If $-1 \neq \square$ modulo p then $c^2 + e^2\ell \not\equiv 0$ so uv and v^2 are inequivalent. Otherwise, v^2 and $2kuv + v^2$ are inequivalent since $ce \equiv k(c^2 + e^2\ell)$ leads to an irreducible quadratic. These are all of the inequivalent orbits, since testing a non-zero form as equivalent to v^2 leads to a quadratic equation which is either soluble (equivalent) or not (inequivalent).

Note that, since the orbits have already been classified, it suffices to check that each claimed stabilizer does in fact fix the form, since the orbits exactly cover the space. This verification is performed in **OD12_stabilizers.nb**, except for in the case 23b. In the case of 23 b., the stabilizer is obtained as follows. To preserve the top quadratic form, the stabilizer takes the form

$$\begin{pmatrix} c & e & \\ \pm e\ell & \pm c & \\ 0 & 0 & * \end{pmatrix} \begin{pmatrix} \frac{1}{c^2-e^2\ell} & \\ \frac{s}{c^2-e^2\ell} & t \end{pmatrix}. \tag{4.69}$$

The GL_3 part maps

$$v(v+2ku) \mapsto (eu \pm cv)((2ck+e)u \pm (2e\ell k + c)v). \tag{4.70}$$

Subtracting down to eliminate the coefficient on u^2, which determines s, this becomes

$$\pm \left(2c^2k + 2ce + 2e^2\ell k\right) uv + (c^2 + 4ce\ell k + e^2\ell)v^2. \tag{4.71}$$

To satisfy the stabilizer condition it is necessary and sufficient that the ratio of the uv and v^2 coefficients satisfies

$$\frac{\pm \left(2c^2k + 2ce + 2e^2\ell k\right)}{c^2 + 4ce\ell k + e^2\ell} = 2k. \tag{4.72}$$

For the $+$ condition to be satisfied, it follows that

$$(1 - 4\ell k^2)ce = 0, \tag{4.73}$$

which implies that $ce = 0$. Both $c = 0$ and $e = 0$ lead to solutions fixing t. The minus case obtains

$$-(c^2k + ce + e^2\ell k) = c^2k + 4ce\ell k^2 + e^2\ell k. \tag{4.74}$$

This leads to a reducible quadratic equation in c and e with two non-zero projective solutions. □

5. Evaluation of exponential sums

The following table lists those modulo p orbits which contain maximal and non-maximal elements, and representatives modulo p^2 for all maximal, or all non-maximal orbits. The classification of maximal and non-maximal $\bmod p^2$ orbits appears in Lemmas 12, 13, 14, and 15. By Lemma 5, in order to evaluate the Fourier transform of the maximal set at frequencies not divisible by p it suffices to consider only those $\bmod p^2$ orbits listed here.

mod p orbit	mod p^2 orbit representative	Justification
$\mathscr{O}_{1^2 11}$	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	This is the only non-maximal orbit
$\mathscr{O}_{1^2 2}$	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & -\ell \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	This is the only non-maximal orbit
$\mathscr{O}_{2^2}$	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & -\ell & 0 \\ & & 0 \end{pmatrix}$	This is the only non-maximal orbit
$\mathscr{O}_{1^4}$	$\begin{pmatrix} \epsilon p & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 1 & 0 \\ & & 0 \end{pmatrix},$ $\epsilon \in \mathbf{F}_p^\times / \{x^4 : x \in \mathbf{F}_p^\times\}$	These are all of the maximal orbits
$\mathscr{O}_{1^3 1}$	$\begin{pmatrix} \epsilon p & 0 & 0 \\ & 0 & \frac{1}{2} \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 1 & 0 \\ & & 0 \end{pmatrix},$ $\epsilon \in \mathbf{F}_p^\times / \{x^3 : x \in \mathbf{F}_p^\times\}$	These are all of the maximal orbits
$\mathscr{O}_{1^2 1^2}$	$\begin{pmatrix} p & 0 & 0 \\ & p & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix},$ $\begin{pmatrix} \ell p & 0 & 0 \\ & p & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix},$ $\begin{pmatrix} \ell p & 0 & 0 \\ & \ell p & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	These are all of the maximal orbits

(5.1)

The size of the stabilizer in $G(\mathbb{Z}/p^2\mathbb{Z})$ of each orbit listed above is given in the following table. These were obtained by multiplying the size of the stabilizer of the corresponding orbit in the G_x action on V/V_x by $p^{\dim G - \dim V_x} = p^{13 - \dim V_x}$, see Lemma 11.

mod p orbit	mod p^2 orbit representative	Stabilizer size
$\mathscr{O}_{1^2 11}$	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$2(p-1)^2 p^2$
$\mathscr{O}_{1^2 2}$	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & -\ell \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$2(p-1)^2 p^2$
$\mathscr{O}_{2^2}$	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & -\ell & 0 \\ & & 0 \end{pmatrix}$	$2(p-1)^2 p^3 (p+1)$
$\mathscr{O}_{1^4}$	$\begin{pmatrix} \epsilon p & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 1 & 0 \\ & & 0 \end{pmatrix},$ $\epsilon \in \mathbf{F}_p^\times / \{x^4 : x \in \mathbf{F}_p^\times\}$	$(p-1)p^4 \#\{\epsilon^4 = 1\}$
$\mathscr{O}_{1^3 1}$	$\begin{pmatrix} \epsilon p & 0 & 0 \\ & 0 & \frac{1}{2} \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 1 & 0 \\ & & 0 \end{pmatrix},$ $\epsilon \in \mathbf{F}_p^\times / \{x^3 : x \in \mathbf{F}_p^\times\}$	$(p-1)p^3 \#\{\epsilon^3 = 1\}$
$\mathscr{O}_{1^2 1^2}$	$\begin{pmatrix} p & 0 & 0 \\ & p & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$8(p-1)p^3$
$\mathscr{O}_{1^2 1^2}$	$\begin{pmatrix} \ell p & 0 & 0 \\ & \ell p & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$8(p-1)p^3$
$\mathscr{O}_{1^2 1^2}$	$\begin{pmatrix} \ell p & 0 & 0 \\ & p & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$4(p-1)p^3$

(5.2)

In evaluating the exponential sums write the time domain variable $x = x_0 + px_1$ and frequency variable $\xi = \xi_0 + p\xi_1$ where $\|x_0\|_\infty, \|\xi_0\|_\infty \leq \frac{p}{2}$. In this section it is useful to recall the formulas (see Lemma 9)

$$S_{p^2}(x,\xi) = p^{13}\mathscr{S}(x,\xi), \tag{5.3}$$
$$\mathscr{S}(x,\xi) = \sum_{g \in G^t_{x,\xi}} e_p\left([x_1, g\cdot\xi_0] + [x_0, g\cdot \xi_1]\right)$$

and (see the beginning of Section 3)

$$\Sigma_{p^2}(x,\xi) = \frac{p^{13}\mathscr{S}(x,\xi)}{\left|\mathrm{Stab}_{G(\mathbb{Z}/p^2\mathbb{Z})}(x)\right|} \tag{5.4}$$
$$\mathscr{M}(x,\xi) = \sum_{\substack{\mathscr{O}_{x'} \bmod p^2 : \mathscr{O}_{x'} \subset \mathscr{O}_x \bmod p \\ \text{maximal}}} \Sigma_{p^2}(x',\xi)$$
$$= -\sum_{\substack{\mathscr{O}_{x'} \bmod p^2 : \mathscr{O}_{x'} \subset \mathscr{O}_x \bmod p \\ \text{non-maximal}}} \Sigma_{p^2}(x',\xi).$$

The following lemma justifies classifying the $G_x \times \mathrm{GL}_1$ action on V/V_x in the case of ξ.

Lemma 21. *Let $x \in \mathscr{O}_{1^211}, \mathscr{O}_{1^22}, \mathscr{O}_{2^2}, \mathscr{O}_{1^4}, \mathscr{O}_{1^31}$ or $\mathscr{O}_{1^21^2} \bmod p$ and write $\xi = \xi_0 + p\xi_1$ satisfying $G_{x,\xi} \neq \emptyset$. Let $\lambda \in \mathbf{F}_p^\times$. Then*

$$\mathscr{M}(x, \xi_0 + p\xi_1) = \mathscr{M}(x, \xi_0 + p\lambda\xi_1). \tag{5.5}$$

Proof. Let $x = x_0 + px_1$ be a $\bmod p^2$ orbit representative from (5.1). Note that, by its definition the action set $G_{x,\xi}$ is invariant by multiplication by a scalar matrix in GL_2. By Lemma 9,

$$\mathscr{S}(x, \xi_0 + p\lambda\xi_1) = \sum_{g \in G^t_{x,\xi}} e_p\left([x_0, \lambda g\cdot\xi_1] + [x_1, g\cdot\xi_0]\right) \tag{5.6}$$
$$= \sum_{g \in G^t_{x,\xi}} e_p\left([x_0, g\cdot\xi_1] + \left[\lambda^{-1}x_1, g\cdot\xi_0\right]\right)$$
$$= \mathscr{S}(x_0 + p\lambda^{-1}x_1, \xi_0 + p\xi_1).$$

In the case of $\mathscr{O}_{1^211}, \mathscr{O}_{1^22}$ and $\mathscr{O}_{2^2}$, $x_1 = 0$, so

$$\mathscr{M}(x,\xi_0 + p\lambda\xi_1) = -\frac{p^{13}\mathscr{S}(x,\xi_0+p\lambda\xi_1)}{\left|\mathrm{Stab}_{G(\mathbb{Z}/p^2\mathbb{Z})}(x)\right|} = -\frac{p^{13}\mathscr{S}(x,\xi_0+p\xi_1)}{\left|\mathrm{Stab}_{G(\mathbb{Z}/p^2\mathbb{Z})}(x)\right|} = \mathscr{M}(x,\xi_0+p\xi_1). \tag{5.7}$$

In the case of $\mathscr{O}_{1^4}$, $\mathscr{O}_{1^31}$ and $\mathscr{O}_{1^21^2}$, multiplying x_1 by λ^{-1} permutes the maximal $\bmod p^2$ orbits above the $\bmod p$ orbit, while leaving the stabilizer size unchanged, see (5.2). Since $\mathscr{M}(x,\xi)$ is a sum of $\mathscr{S}(x,\xi)$ weighted by p^{13} times the inverse of the stabilizer, summed over these orbits, it follows that $\mathscr{M}(x,\xi_0+p\lambda\xi_1)=\mathscr{M}(x,\xi_0+p\xi_1)$ in this case, also. □

5.1. The exponential sums pair $(\mathscr{O}_{1^211}, \mathscr{O}_{D1^2})$

In this case,

$$x_0=\begin{pmatrix}0&0&0\\&1&0\\&&-1\end{pmatrix}\begin{pmatrix}0&0&\frac{1}{2}\\&0&0\\&&0\end{pmatrix},\qquad \xi_0=\begin{pmatrix}1&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\tag{5.8}$$

and $x_1=0$. The action set is

$$G^t_{x,\xi}=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\&a&b\\&c&d\end{pmatrix}\begin{pmatrix}b_{11}&b_{12}\\&b_{22}\end{pmatrix}.\tag{5.9}$$

The stabilizer has size

$$\left|\mathrm{Stab}_{G(\mathbb{Z}/p^2\mathbb{Z})}(x)\right|=2(p-1)^2p^2.\tag{5.10}$$

The GL_2 condition on $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ may be fibered as follows

$$\begin{aligned}\sum_{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{GL}_2}&=\sum_{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathbf{F}_p^4}-\sum_{\substack{\begin{pmatrix}b\lambda&b\\d\lambda&d\end{pmatrix}\\ \begin{pmatrix}b\\d\end{pmatrix}\in\mathbf{F}_p^2,\lambda\in\mathbf{F}_p^\times}}-\sum_{\substack{\begin{pmatrix}a&0\\c&0\end{pmatrix}\\ \begin{pmatrix}a\\c\end{pmatrix}\in\mathbf{F}_p^2}}-\sum_{\substack{\begin{pmatrix}0&b\\0&d\end{pmatrix}\\ \begin{pmatrix}b\\d\end{pmatrix}\in\mathbf{F}_p^2}}+p\sum_{\begin{pmatrix}0&0\\0&0\end{pmatrix}}\\&=\Sigma_1-\Sigma_2-\Sigma_3-\Sigma_4+\Sigma_5.\end{aligned}\tag{5.11}$$

In the summations that follow, unless otherwise indicated summations over matrices are constrained by the fact that the pairs of matrices are contained in $\mathrm{GL}_3\times\mathrm{GL}_2$. If $\Sigma_1-\Sigma_5$ is indicated then $a_{11},b_{11},b_{22}\in\mathbf{F}_p^\times$, $a_{12},a_{13},b_{12}\in\mathbf{F}_p$ and the remaining variables range according to the fibration above.

Representatives and exponential sum pairings are given in the table below. Since this is the single non-maximal orbit,

$$\mathscr{M}(x,\xi)=-\frac{p^{11}\mathscr{S}(x,\xi)}{2(p-1)^2}.\tag{5.12}$$

This table is generated in **exponential_sums_O1211_OD12.nb**.

Orbit	ξ_1	$[x_0, g\cdot\xi_1]$	$\mathscr{S}(x,\xi)$	$\mathscr{M}(x,\xi)$
1.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	0	$(p-1)^5p^4(p+1)$	$-\frac{(p-1)^3p^{15}(p+1)}{2}$
2.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$b_{11}(b^2-d^2)$	$(p-1)^5p^4$	$-\frac{(p-1)^3p^{15}}{2}$
3.	$\begin{pmatrix}0&0&0\\&0&\frac{1}{2}\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$b_{11}(ab-cd)$	$-2(p-1)^4p^4$	$(p-1)^2p^{15}$
4.	$\begin{pmatrix}0&0&0\\&-\ell&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$b_{11}(-a^2\ell+b^2+c^2\ell-d^2)$	0	0
5.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&1\\&0&0\\&&0\end{pmatrix}$	$b_{11}(b^2-d^2)+b_{22}a_{11}d$	$-(p-1)^3p^4(2p-1)$	$\frac{(p-1)p^{15}(2p-1)}{2}$
6.	$\begin{pmatrix}0&0&0\\&1&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&1\\&0&0\\&&0\end{pmatrix}$	$b_{11}(a^2-c^2)+b_{22}a_{11}d$	$(p-1)^3p^4$	$-\frac{(p-1)p^{15}}{2}$
7.	$\begin{pmatrix}0&0&0\\&0&\frac{1}{2}\\&&0\end{pmatrix}\begin{pmatrix}0&0&1\\&0&0\\&&0\end{pmatrix}$	$b_{11}(ab-cd)+b_{22}a_{11}d$	$2(p-1)^3p^4$	$-(p-1)p^{15}$
8.	$\begin{pmatrix}0&0&0\\&1&1\\&&0\end{pmatrix}\begin{pmatrix}0&0&1\\&0&0\\&&0\end{pmatrix}$	$b_{11}(a(a+2b)-c(c+2d))$ $+b_{22}a_{11}d$	$-2(p-1)^2p^4$	p^{15}
9.	$\begin{pmatrix}0&0&0\\&-\ell&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&1\\&0&0\\&&0\end{pmatrix}$	$b_{11}(-a^2\ell+b^2+c^2\ell-d^2)$ $+b_{22}a_{11}d$	0	0
10.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&1\\&0&0\\&&0\end{pmatrix}$	$b_{22}a_{11}d$	$-(p-1)^4p^4$	$\frac{(p-1)^2p^{15}}{2}$
11.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}$	$b_{12}(b^2-d^2)+b_{22}a_{13}d$	0	0
12.	$\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}$	$b_{11}(2ab-2cd)+b_{12}(b^2-d^2)$ $+b_{22}a_{13}d$	0	0
13.	$\begin{pmatrix}0&0&0\\&1&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}$	$b_{11}(a^2-c^2)+b_{12}(b^2-d^2)$ $+b_{22}a_{13}d$	0	0
14.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&1&0\\&0&0\\&&1\end{pmatrix}$	$b_{12}(b^2-d^2)$ $+b_{22}(a_{11}c+a_{13}d)$	0	0
15.	$\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}\begin{pmatrix}0&1&0\\&0&0\\&&1\end{pmatrix}$	$b_{11}(2ab-2cd)+b_{12}(b^2-d^2)$ $+b_{22}(a_{11}c+a_{13}d)$	0	0
16.	$\begin{pmatrix}0&0&0\\&1&0\\&&0\end{pmatrix}\begin{pmatrix}0&1&0\\&0&0\\&&1\end{pmatrix}$	$b_{11}(a^2-c^2)+b_{12}(b^2-d^2)$ $+b_{22}(a_{11}c+a_{13}d)$	0	0
17.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}$	$b_{12}(2ab-2cd)+b_{22}(a_{12}d+a_{13}c)$	0	0
18.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}$	$b_{11}(b^2-d^2)+b_{12}(2ab-2cd)$ $+b_{22}(a_{12}d+a_{13}c)$	0	0
19.	$\begin{pmatrix}0&0&0\\&1&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}$	$b_{11}(a^2+b^2-c^2-d^2)$ $+b_{12}(2ab-2cd)$ $+b_{22}(a_{12}d+a_{13}c)$	0	0
20.	$\begin{pmatrix}0&0&0\\&\ell&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}$	$b_{11}(a^2\ell+b^2-c^2\ell-d^2)$ $+b_{12}(2ab-2cd)$ $+b_{22}(a_{12}d+a_{13}c)$	0	0
21.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&-\ell&0\\&&1\end{pmatrix}$	$b_{12}(-a^2\ell+b^2+c^2\ell-d^2)$ $+b_{22}(-a_{12}c\ell+a_{13}d)$	0	0
22.	$\begin{pmatrix}0&0&0\\&1&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&-\ell&0\\&&1\end{pmatrix}$	$b_{11}(a^2-c^2)$ $+b_{12}(-a^2\ell+b^2+c^2\ell-d^2)$ $+b_{22}(-a_{12}c\ell+a_{13}d)$	0	0
23a.	$\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&-\ell&0\\&&1\end{pmatrix}$	$b_{11}(2ab-2cd)$ $+b_{12}(-a^2\ell+b^2+c^2\ell-d^2)$ $+b_{22}(-a_{12}c\ell+a_{13}d)$	0	0
23b.	$\begin{pmatrix}0&0&0\\&1&k\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&-\ell&0\\&&1\end{pmatrix}$	$b_{11}(a^2+2abk-c^2-2cdk)$ $+b_{12}(-a^2\ell+b^2+c^2\ell-d^2)$ $+b_{22}(-a_{12}c\ell+a_{13}d)$	0	0

1. In this case $[x_0, g\cdot\xi_1] = 0$, so

$$\mathscr{S}(x,\xi) = |G_{x,\xi}| = (p-1)^5 p^4 (p+1). \tag{5.13}$$

2. In this case

$$[x_0, g\cdot\xi_1] = b_{11}(b^2 - d^2). \tag{5.14}$$

The fibration gives

$$\Sigma_1 = \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(b^2-d^2)).$$

Substitute $b := b+d$, $d := b-d$ to obtain

$$\Sigma_1 = (p-1)^2 p^5 \sum_{b_{11}\in\mathbf{F}_p^\times, b,d\in\mathbf{F}_p} e_p(b_{11}bd).$$

The sum over d vanishes unless $b=0$, so $\Sigma_1 = (p-1)^3p^6$. The remaining summations may be performed similarly,

$$\begin{aligned}
\Sigma_2 &= \sum_{\begin{pmatrix} * & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(b^2-d^2)) \\
&= (p-1)^3 p^3 \sum_{b_{11}\in\mathbf{F}_p^\times, b,d\in\mathbf{F}_p} e_p(b_{11}bd) \\
&= (p-1)^4 p^4 \\
\Sigma_3 &= \sum_{\begin{pmatrix} * & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} 1 = (p-1)^3 p^5 \\
\Sigma_4 &= \sum_{\begin{pmatrix} * & * & * \\ & 0 & b \\ & 0 & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(b^2-d^2)) \\
&= (p-1)^2 p^3 \sum_{b_{11}\in\mathbf{F}_p^\times, b,d\in\mathbf{F}_p} e_p(b_{11}bd) \\
&= (p-1)^3 p^4
\end{aligned}$$

$$\Sigma_5 = p \sum_{\begin{pmatrix} * & * & * \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} 1 = (p-1)^3 p^4.$$

Thus

$$\mathscr{S}(x,\xi) = \Sigma_1 - \Sigma_2 - \Sigma_3 - \Sigma_4 + \Sigma_5 = (p-1)^5 p^4. \tag{5.15}$$

3. In this case

$$[x_0, g \cdot \xi_1] = b_{11}(ab - cd). \tag{5.16}$$

The fibration gives

$$\Sigma_1 = \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(ab - cd))$$

Summation in b, d vanishes unless $a = c = 0$. Thus $\Sigma_1 = (p-1)^3 p^5$.

$$\Sigma_2 = \sum_{\begin{pmatrix} * & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(b^2 - d^2)\lambda)$$

Substitute $b := b + d$, $d := b - d$ so $\Sigma_2 = (p-1)^4 p^4$.

$$\Sigma_3 = \sum_{\begin{pmatrix} * & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} 1 = (p-1)^3 p^5$$

$$\Sigma_4 = \Sigma_3 = (p-1)^3 p^5$$
$$\Sigma_5 = (p-1)^3 p^4.$$

Thus

$$\mathscr{S}(x,\xi) = (p-1)^3 p^4 [p - (p-1) - 2p + 1] = -2(p-1)^4 p^4. \tag{5.17}$$

4. In this case

$$[x_0, g \cdot \xi_1] = b_{11}(-a^2\ell + b^2 + c^2\ell - d^2) = b_{11}(-(a^2 - c^2)\ell + b^2 - d^2). \tag{5.18}$$

The fibration gives

$$\begin{aligned}
\Sigma_1 &= \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(-(a^2-c^2)\ell + (b^2-d^2))) \\
&= (p-1)^2 p^3 \sum_{a,b,c,d \in \mathbf{F}_p, b_{11} \in \mathbf{F}_p^\times} e_p(b_{11}(-ac\ell + bd)) \\
&= (p-1)^3 p^5 \\
\Sigma_2 &= \sum_{\begin{pmatrix} * & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(b^2-d^2)(-\lambda^2\ell + 1))
\end{aligned}$$

Since $-\lambda^2\ell + 1 \neq 0$ this factor may be eliminated by a change of variable in the b_{11} summation. Substitute $b := b + d$, $d := b - d$ to obtain $\Sigma_2 = (p-1)^4 p^4$. Similarly,

$$\begin{aligned}
\Sigma_3 &= \sum_{\begin{pmatrix} * & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(-b_{11}(a^2-c^2)\ell) = (p-1)^3 p^4 \\
\Sigma_4 &= \sum_{\begin{pmatrix} * & * & * \\ & 0 & b \\ & 0 & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(b^2-d^2)) = (p-1)^3 p^4 \\
\Sigma_5 &= (p-1)^3 p^4.
\end{aligned}$$

Thus

$$\mathscr{S}(x,\xi) = (p-1)^3 p^4 (p - (p-1) - 1) = 0. \tag{5.19}$$

5. In this case

$$[x_0, g \cdot \xi_1] = b_{11}(b^2 - d^2) + b_{22} a_{11} d. \tag{5.20}$$

Summing in b_{22} gives -1 if $d \neq 0$ and $p - 1$ if $d = 0$. Thus,

$$\begin{aligned}
\mathscr{S}(x,\xi) &= -\frac{1}{p-1} \sum_{g \in G^t_{x,\xi}} e_p(b_{11}(b^2-d^2)) + \frac{p}{p-1} \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}b^2) \qquad (5.21) \\
&= -\frac{2.}{p-1} - (p-1)^3 p^5
\end{aligned}$$

$$= -(p-1)^4 p^4 - (p-1)^3 p^5$$
$$= -(p-1)^3 p^4 (2p-1).$$

6. In this case

$$[x_0, g \cdot \xi_1] = b_{11}(a^2 - c^2) + b_{22} a_{11} d. \tag{5.22}$$

Sum over b_{22} as before to obtain

$$\begin{aligned}\mathscr{S}(x,\xi) = &-\frac{1}{p-1} \sum_{g \in G^t_{x,\xi}} e_p(b_{11}(a^2 - c^2)) \\ &+ \frac{p}{p-1} \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & \end{pmatrix} \begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a^2 - c^2)).\end{aligned} \tag{5.23}$$

The $c = 0$ term in the second sum may be added without altering the sum, since

$$\sum_{b_{11} \in \mathbf{F}_p^\times, a \in \mathbf{F}_p} e_p(b_{11} a^2) = 0. \tag{5.24}$$

Thus, substituting $a + c$ and $a - c$ for a and c as before,

$$\begin{aligned}\mathscr{S}(x,\xi) &= -\frac{2.}{p-1} + (p-1)^3 p^5 \\ &= -(p-1)^4 p^4 + (p-1)^3 p^5 \\ &= (p-1)^3 p^4.\end{aligned} \tag{5.25}$$

7. In this case

$$[x_0, g \cdot \xi_1] = b_{11}(ab - cd) + b_{22} a_{11} d. \tag{5.26}$$

Sum in b_{22} to find

$$\mathscr{S}(x,\xi) = -\frac{1}{p-1} \sum_{g \in G^t_{x,\xi}} e_p(b_{11}(ab - cd)) + \frac{p}{p-1} \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & 0 \end{pmatrix} \begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11} ab)$$

$$= -\frac{3.}{p-1} = 2(p-1)^3 p^4.$$

8. In this case

$$[x_0, g \cdot \xi_1] = b_{11}(a(a+2b) - c(c+2d)) + b_{22}a_{11}d. \tag{5.27}$$

Sum in b_{22} to find

$$\begin{aligned}\mathscr{S}(x,\xi) = &-\frac{1}{p-1}\sum_{g \in G^t_{x,\xi}} e_p(b_{11}(a(a+2b) - c(c+2d))) \\ &+ \frac{p}{p-1} \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a(a+2b) - c^2))\end{aligned}$$

The first sum is the sum from 3. In the second sum, replace $a := a+b$ to obtain

$$\mathscr{S}(x,\xi) = -\frac{3.}{p-1} + (p-1)p^4 \sum_{b_{11}, b, c \in \mathbf{F}_p^\times, a \in \mathbf{F}_p} e_p(b_{11}(a^2 - b^2 - c^2))$$

The sum may be evaluated in terms of the Gauss sum

$$\tau = \sum_{n \in \mathbf{F}_p} e_p\left(n^2\right). \tag{5.28}$$

Here and below, $\left(\frac{\cdot}{p}\right)$ indicates the Legendre symbol. When $b_{11} \neq 0$,

$$\sum_{n \in \mathbf{F}_p} e_p\left(b_{11}n^2\right) = \tau \left(\frac{b_{11}}{p}\right). \tag{5.29}$$

Also,

$$\sum_{b_{11} \in \mathbf{F}_p^\times} \left(\frac{b_{11}}{p}\right) = 0. \tag{5.30}$$

The Gauss sum satisfies

$$\tau^2 = \left(\frac{-1}{p}\right) p. \tag{5.31}$$

With these facts in hand,

$$\begin{aligned}\mathscr{S}(x,\xi) &= 2(p-1)^3p^4 + (p-1)p^4 \sum_{b_{11}\in\mathbf{F}_p^\times} \tau\left(\frac{b_{11}}{p}\right)\left(\tau\left(\frac{-b_{11}}{p}\right)-1\right)^2\\ &= 2(p-1)^3p^4 - 2(p-1)^2p^5\\ &= -2(p-1)^2p^4.\end{aligned}$$

9. In this case

$$[x_0, g\cdot\xi_1] = b_{11}(-(a^2-c^2)\ell + (b^2-d^2)) + b_{22}a_{11}d. \tag{5.32}$$

Sum in b_{22} to find

$$\begin{aligned}\mathscr{S}(x,\xi) &= -\frac{4.}{p-1} + \frac{p}{p-1} \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & 0\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(-a^2\ell + b^2 + c^2\ell))\\ &= (p-1)p^4 \sum_{b_{11}\in\mathbf{F}_p^\times} \tau\left(\frac{-b_{11}\ell}{p}\right)\left(\tau\left(\frac{b_{11}}{p}\right)-1\right)\left(\tau\left(\frac{b_{11}\ell}{p}\right)-1\right)\\ &= 0.\end{aligned}$$

10. In this case

$$[x_0, g\cdot\xi_1] = b_{22}a_{11}d. \tag{5.33}$$

Summing in b_{22},

$$\begin{aligned}\mathscr{S}(x,\xi) &= -\frac{|G_{x,\xi}|}{p-1} + p\#\begin{pmatrix} * & * & * \\ & * & * \\ & * & \end{pmatrix}\begin{pmatrix} * & * \\ & 1\end{pmatrix}\\ &= -(p-1)^4p^4(p+1) + (p-1)^4p^5\\ &= -(p-1)^4p^4.\end{aligned}$$

11.-23. These sums vanish on summing in a_{12}, a_{13} and b_{12}.

5.2. The exponential sums pair $(\mathscr{O}_{1^22}, \mathscr{O}_{D1^2})$

In this case,

$$x_0 = \begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & -\ell \end{pmatrix}\begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \qquad \xi_0 = \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \tag{5.34}$$

and $x_1 = 0$. The action set is

$$G_{x,\xi}^t = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & a & b \\ & c & d \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix}, \tag{5.35}$$

which is fibered as before. The stabilizer has size $\left|\mathrm{Stab}_{G(\mathbb{Z}/p^2\mathbb{Z})}(x)\right| = 2(p-1)^2p^2$.

Representatives and exponential sum pairings are given in the table below. Since this is the single non-maximal orbit,

$$\mathscr{M}(x,\xi) = -\frac{p^{11}\mathscr{S}(x,\xi)}{2(p-1)^2}. \tag{5.36}$$

See **exponential_sums_O122_OD12.nb**.

Orbit	ξ_1	$[x_0, g\cdot\xi_1]$	$\mathscr{S}(x,\xi)$	$\mathscr{M}(x,\xi)$
1.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	0	$(p-1)^5p^4(p+1)$	$-\frac{(p-1)^3p^{15}(p+1)}{2}$
2.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(b^2-d^2\ell)$	$-(p-1)^4p^4(p+1)$	$\frac{(p-1)^2p^{15}(p+1)}{2}$
3.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & \frac{1}{2} \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(ab-cd\ell)$	0	0
4.	$\begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(-a^2\ell+b^2+c^2\ell^2-d^2\ell)$	$2(p-1)^3p^4(p+1)$	$-(p-1)p^{15}(p+1)$
5.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(b^2-d^2\ell)+b_{22}a_{11}d$	$(p-1)^3p^4$	$-\frac{(p-1)p^{15}}{2}$
6.	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a^2-c^2\ell)+b_{22}a_{11}d$	$(p-1)^3p^4$	$-\frac{(p-1)p^{15}}{2}$
7.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & \frac{1}{2} \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(ab-cd\ell)+b_{22}a_{11}d$	0	0
8.	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 1 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a(a+2b)-c(c+2d)\ell)$ $+b_{22}a_{11}d$	0	0
9.	$\begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(-a^2\ell+b^2+c^2\ell^2-d^2\ell)$ $+b_{22}a_{11}d$	$-2(p-1)^2p^4$	p^{15}
10.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{22}a_{11}d$	$-(p-1)^4p^4$	$\frac{(p-1)^2p^{15}}{2}$
11.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$	$b_{12}(b^2-d^2\ell)+b_{22}a_{13}d$	0	0
12.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$	$b_{11}(2ab-2cd\ell)$ $+b_{12}(b^2-d^2\ell)+b_{22}a_{13}d$	0	0
13.	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$	$b_{11}(a^2-c^2\ell)$ $+b_{12}(b^2-d^2\ell)+b_{22}a_{13}d$	0	0
14.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$	$b_{12}(b^2-d^2\ell)$ $+b_{22}(a_{11}c+a_{13}d)$	0	0

Orbit	ξ_1	$[x_0, g \cdot \xi_1]$	$\mathscr{S}(x,\xi)$	$\mathscr{M}(x,\xi)$
15.	$\begin{pmatrix}0&0&0\\ &0&1\\ &&0\end{pmatrix}\begin{pmatrix}0&1&0\\ &0&0\\ &&1\end{pmatrix}$	$b_{11}(2ab-2cd\ell)+b_{12}(b^2-d^2\ell)$ $+b_{22}(a_{11}c+a_{13}d)$	0	0
16.	$\begin{pmatrix}0&0&0\\ &1&0\\ &&0\end{pmatrix}\begin{pmatrix}0&1&0\\ &0&0\\ &&1\end{pmatrix}$	$b_{11}(a^2-c^2\ell)+b_{12}(b^2-d^2\ell)$ $+b_{22}(a_{11}c+a_{13}d)$	0	0
17.	$\begin{pmatrix}0&0&0\\ &0&0\\ &&0\end{pmatrix}\begin{pmatrix}0&0&0\\ &0&1\\ &&0\end{pmatrix}$	$b_{12}(2ab-2cd\ell)$ $+b_{22}(a_{12}d+a_{13}c)$	0	0
18.	$\begin{pmatrix}0&0&0\\ &0&0\\ &&1\end{pmatrix}\begin{pmatrix}0&0&0\\ &0&1\\ &&0\end{pmatrix}$	$b_{11}(b^2-d^2\ell)+b_{12}(2ab-2cd\ell)$ $+b_{22}(a_{12}d+a_{13}c)$	0	0
19.	$\begin{pmatrix}0&0&0\\ &1&0\\ &&1\end{pmatrix}\begin{pmatrix}0&0&0\\ &0&1\\ &&0\end{pmatrix}$	$b_{11}(a^2+b^2-c^2\ell-d^2\ell)$ $+b_{12}(2ab-2cd\ell)+b_{22}(a_{12}d+a_{13}c)$	0	0
20.	$\begin{pmatrix}0&0&0\\ &\ell&0\\ &&1\end{pmatrix}\begin{pmatrix}0&0&0\\ &0&1\\ &&0\end{pmatrix}$	$b_{11}(a^2\ell+b^2-c^2\ell^2-d^2\ell)$ $+b_{12}(2ab-2cd\ell)+b_{22}(a_{12}d+a_{13}c)$	0	0
21.	$\begin{pmatrix}0&0&0\\ &0&0\\ &&0\end{pmatrix}\begin{pmatrix}0&0&0\\ &-\ell&0\\ &&1\end{pmatrix}$	$b_{12}(-a^2\ell+b^2+c^2\ell^2-d^2\ell)$ $+b_{22}(-a_{12}c\ell+a_{13}d)$	0	0
22.	$\begin{pmatrix}0&0&0\\ &1&0\\ &&0\end{pmatrix}\begin{pmatrix}0&0&0\\ &-\ell&0\\ &&1\end{pmatrix}$	$b_{11}(a^2-c^2\ell)$ $+b_{12}(-a^2\ell+b^2+c^2\ell^2-d^2\ell)$ $+b_{22}(-a_{12}c\ell+a_{13}d)$	0	0
23a.	$\begin{pmatrix}0&0&0\\ &0&1\\ &&0\end{pmatrix}\begin{pmatrix}0&0&0\\ &-\ell&0\\ &&1\end{pmatrix}$	$b_{11}(2ab-2cd\ell)$ $+b_{12}(-a^2\ell+b^2+c^2\ell^2-d^2\ell)$ $+b_{22}(-a_{12}c\ell+a_{13}d)$	0	0
23b.	$\begin{pmatrix}0&0&0\\ &1&k\\ &&0\end{pmatrix}\begin{pmatrix}0&0&0\\ &-\ell&0\\ &&1\end{pmatrix}$	$b_{11}(a^2+2abk-c^2\ell-2cd\ell k)$ $+b_{12}(-a^2\ell+b^2+c^2\ell^2-d^2\ell)$ $+b_{22}(-a_{12}c\ell+a_{13}d)$	0	0

The exponential sums are as follows.

1. In this case $[x_0, g \cdot \xi_1] = 0$ so that

$$\mathscr{S}(x,\xi) = |G_{x,\xi}| = (p-1)^5 p^4 (p+1). \tag{5.37}$$

2. In this case

$$[x_0, g \cdot \xi_1] = b_{11}(b^2 - d^2\ell). \tag{5.38}$$

Since b, d are not 0 simultaneously, $b^2 - d^2\ell \neq 0$, and the sum over b_{11} is -1, so

$$\mathscr{S}(x,\xi) = -\frac{|G_{x,\xi}|}{p-1} = -(p-1)^4 p^4 (p+1). \tag{5.39}$$

3. In this case

$$[x_0, g\cdot\xi_1] = b_{11}(ab - cd\ell). \tag{5.40}$$

The fibration gives

$$\Sigma_1 = \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(ab-cd\ell)) = (p-1)^3p^5$$

$$\Sigma_2 = \sum_{\begin{pmatrix} * & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(b^2-d^2\ell)\lambda)$$

Summation in b and d gives $\tau^2\left(\frac{-\ell}{p}\right) = -p$, independent of $b_{11}\lambda$. Thus $\Sigma_2 = -(p-1)^4p^4$.

$$\Sigma_3 = \sum_{\begin{pmatrix} * & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} 1 = (p-1)^3p^5$$

$$\Sigma_4 = \Sigma_3 = (p-1)^3p^5$$

$$\Sigma_5 = (p-1)^3p^4$$

Thus

$$\mathscr{S}(x,\xi) = 0. \tag{5.41}$$

4. In this case

$$[x_0, g\cdot\xi_1] = b_{11}(-a^2\ell + b^2 + c^2\ell^2 - d^2\ell). \tag{5.42}$$

The fibration gives

$$\Sigma_1 = \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(-a^2\ell + b^2 + c^2\ell^2 - d^2\ell))$$

Expanding in Gauss sums, summation in a, b, c, d gives p^2 independent of b_{11}, so $\Sigma_1 = (p-1)^3 p^5$.

$$\Sigma_2 = \sum_{\begin{pmatrix} * & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(b^2 - d^2\ell)(-\lambda^2\ell + 1))$$

Since $-\lambda^2\ell + 1 \neq 0$, summation in b^2 and d^2 obtains $-p$ as in Σ_2 of 3., so that $\Sigma_2 = -(p-1)^4 p^4$. Similarly,

$$\Sigma_3 = \sum_{\begin{pmatrix} * & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(-a^2\ell + c^2\ell^2)) = -(p-1)^3 p^4$$

$$\Sigma_4 = \sum_{\begin{pmatrix} * & * & * \\ & 0 & b \\ & 0 & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(b^2 - d^2\ell)) = -(p-1)^3 p^4$$

$$\Sigma_5 = (p-1)^3 p^4.$$

Thus

$$\mathscr{S}(x,\xi) = 2(p-1)^3 p^4 (p+1). \tag{5.43}$$

5. In this case

$$[x_0, g \cdot \xi_1] = b_{11}(b^2 - d^2\ell) + b_{22}a_{11}d. \tag{5.44}$$

The sum over b_{11} is -1 since $b^2 - d^2\ell \neq 0$. Sum in b_{22} to find

$$\begin{aligned} \mathscr{S}(x,\xi) &= \frac{|G_{x,\xi}|}{(p-1)^2} - \frac{p}{(p-1)^2} \# \begin{pmatrix} * & * & * \\ & * & * \\ & * & \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix} \\ &= (p-1)^3 p^4 (p+1) - (p-1)^3 p^5 \\ &= (p-1)^3 p^4. \end{aligned}$$

6. In this case

$$[x_0, g \cdot \xi_1] = b_{11}(a^2 - c^2\ell) + b_{22}a_{11}d. \tag{5.45}$$

As for 5., $\mathscr{S}(x,\xi) = (p-1)^3 p^4$.

7. In this case

$$[x_0, g \cdot \xi_1] = b_{11}(ab - cd\ell) + b_{22}a_{11}d. \tag{5.46}$$

Sum in b_{22} to find

$$\mathscr{S}(x,\xi) = -\frac{1}{p-1}\sum_{g\in G^t_{x,\xi}} e_p(b_{11}(ab-cd\ell)) + \frac{p}{p-1}\sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & 0\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}ab)$$

The first sum is the same as for 3., which is 0, while in the second sum, $b_{11}b \neq 0$ so that the sum over a, which now runs over $\mathbf{F}_p$, is 0. Thus $\mathscr{S}(x,\xi) = 0$.

8. In this case

$$[x_0, g \cdot \xi_1] = b_{11}(a(a+2b) - c(c+2d)\ell) + b_{22}a_{11}d. \tag{5.47}$$

Sum in b_{22} to find

$$\begin{aligned}\mathscr{S}(x,\xi) = &-\frac{1}{p-1}\sum_{g\in G^t_{x,\xi}} e_p(b_{11}(a(a+2b) - c(c+2d)\ell)) \\ &+ \frac{p}{p-1}\sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a(a+2b) - c^2\ell))\end{aligned}$$

After making a column operation on GL_2, the first sum is equal to the sum from 3., which is 0. The second sum may be evaluated by replacing $a := a + b$, which is permissible since a ranges in $\mathbf{F}_p$, so that $a(a + 2b)$ becomes $a^2 - b^2$. This obtains

$$\begin{aligned}\mathscr{S}(x,\xi) &= (p-1)p^4 \sum_{b_{11},b,c\in\mathbf{F}_p^\times, a\in\mathbf{F}_p} e_p(b_{11}(a^2 - b^2 - c^2\ell)) \\ &= (p-1)p^4 \sum_{b_{11}\in\mathbf{F}_p^\times} \tau\left(\frac{b_{11}}{p}\right)\left(\tau\left(\frac{-b_{11}}{p}\right) - 1\right)\left(\tau\left(\frac{-b_{11}\ell}{p}\right) - 1\right) \\ &= 0.\end{aligned}$$

9. In this case

$$[x_0, g \cdot \xi_1] = b_{11}(-a^2\ell + b^2 + c^2\ell^2 - d^2\ell) + b_{22}a_{11}d. \tag{5.48}$$

Sum in b_{22} to find

$$\begin{aligned}
\mathscr{S}(x,\xi) &= -\frac{4.}{p-1} + \frac{p}{p-1} \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & & c \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(-a^2\ell + b^2 + c^2\ell^2)) \\
&= -2(p-1)^2p^4(p+1) + (p-1)p^4 \sum_{b_{11}\in\mathbf{F}_p^\times} \tau\left(\frac{-b_{11}\ell}{p}\right)\left(\tau\left(\frac{b_{11}}{p}\right) - 1\right)^2 \\
&= -2(p-1)^2p^4(p+1) + 2(p-1)^2p^5 \\
&= -2(p-1)^2p^4.
\end{aligned}$$

10. In this case

$$[x_0, g\cdot\xi_1] = b_{22}a_{11}d. \tag{5.49}$$

Summing in b_{22},

$$\begin{aligned}
\mathscr{S}(x,\xi) &= -\frac{|G_{x,\xi}|}{p-1} + p\#\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}\begin{pmatrix} * & * \\ & 1 \end{pmatrix} \\
&= -(p-1)^4p^4(p+1) + (p-1)^4p^5 \\
&= -(p-1)^4p^4.
\end{aligned}$$

11.-23. These sums vanish on summing in a_{12}, a_{13} and b_{12}.

5.3. The exponential sums pair $(\mathscr{O}_{2^2}, \mathscr{O}_{D11})$

For this pair, the standard representatives are

$$x_0 = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ & -\ell & 0 \\ & & 0 \end{pmatrix}, \qquad \xi_0 = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}. \tag{5.50}$$

The acting set is

$$G^t_{x,\xi} = \left\{\begin{pmatrix} a & c\lambda\ell & a_{13} \\ c & a\lambda & a_{23} \\ & & a_{33} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix}\right\}. \tag{5.51}$$

The range of summation is $(a,c) \in \mathbf{F}_p^2 \setminus \{(0,0)\}$, $\lambda, a_{33}, b_{11}, b_{22} \in \mathbf{F}_p^\times$, and $a_{13}, a_{23}, b_{12} \in \mathbf{F}_p$. Thus $|G_{x,\xi}| = (p-1)^5p^3(p+1)$. The stabilizer has size

$$\left|\mathrm{Stab}_{G(\mathbb{Z}/p^2\mathbb{Z})}(x)\right| = 2(p-1)^2p^3(p+1). \tag{5.52}$$

Representatives and exponential sum pairings are given in the table below. Since this is the single non-maximal orbit,

$$\mathscr{M}(x,\xi) = -\frac{p^{10}\mathscr{S}(x,\xi)}{2(p-1)^2(p+1)}. \tag{5.53}$$

See **exponential_sums_O22_OD11.nb**.

Orbit	ξ_1	$[x_0, g\cdot\xi_1]$	$\mathscr{S}(x,\xi)$	$\mathscr{M}(x,\xi)$
1.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}$	$b_{12}a_{33}^2 + b_{22}(a_{13}^2 - a_{23}^2\ell)$	0	0
2.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&1\\&&1\end{pmatrix}$	$b_{12}a_{33}^2+$ $b_{22}(a_{13}^2 - a_{23}^2\ell - 2aa_{23}\lambda\ell + 2a_{13}c\lambda\ell)$	0	0
3.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&1\\&0&1\\&&1\end{pmatrix}$	$b_{12}a_{33}^2+$ $b_{22}(2aa_{13} + a_{13}^2 - a_{23}^2\ell - 2a_{23}c\ell$ $-2aa_{23}\lambda\ell + 2a_{13}c\lambda\ell)$	0	0
4.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&-\ell&0\\&&1\end{pmatrix}$	$b_{12}a_{33}^2+$ $b_{22}(a_{13}^2 - a_{23}^2\ell + a^2\lambda^2\ell^2 - c^2\lambda^2\ell^3)$	0	0
5.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&1\\&-\ell&0\\&&1\end{pmatrix}$	$b_{12}a_{33}^2+$ $b_{22}(2aa_{13} + a_{13}^2 - a_{23}^2\ell - 2a_{23}c\ell$ $+a^2\lambda^2\ell^2 - c^2\lambda^2\ell^3)$	0	0
6.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}-\ell&0&0\\&-\ell&0\\&&1\end{pmatrix}$	$b_{12}a_{33}^2+$ $b_{22}(a_{13}^2 - a^2\ell - a_{23}^2\ell + c^2\ell^2$ $+a^2\lambda^2\ell^2 - c^2\lambda^2\ell^3)$	0	0
7.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}$	$b_{22}(-2aa_{23}\lambda\ell + 2a_{13}c\lambda\ell)$	0	0
8.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&1\\&0&1\\&&0\end{pmatrix}$	$b_{22}(a_{13}(2a + 2c\lambda\ell) - a_{23}(2a\lambda\ell + 2c\ell))$	0	0
9.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}1&0&0\\&0&1\\&&0\end{pmatrix}$	$b_{22}(a^2 - c^2\ell - 2aa_{23}\lambda\ell + 2a_{13}c\lambda\ell)$	0	0
10.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}$	$b_{11}a_{33}^2 + b_{22}(-2aa_{23}\lambda\ell + 2a_{13}c\lambda\ell)$	0	0
11.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&1\\&0&1\\&&0\end{pmatrix}$	$b_{11}a_{33}^2+$ $b_{22}(a_{13}(2a + 2c\lambda\ell) - a_{23}(2a\lambda\ell + 2c\ell))$	0	0
12.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}1&0&0\\&0&1\\&&0\end{pmatrix}$	$b_{11}a_{33}^2+$ $b_{22}(a^2 - c^2\ell - 2aa_{23}\lambda\ell + 2a_{13}c\lambda\ell)$	0	0
13.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&1&0\\&&0\end{pmatrix}$	$b_{22}(-a^2 + c^2\ell)\lambda^2\ell$	$-(p-1)^4p^3(p+1)$	$\frac{(p-1)^2p^{13}}{2}$
14.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}1&0&0\\&1&0\\&&0\end{pmatrix}$	$b_{22}(a^2 - c^2\ell)(1 - \lambda^2\ell)$	$-(p-1)^4p^3(p+1)$	$\frac{(p-1)^2p^{13}}{2}$
15.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}\ell&0&0\\&1&0\\&&0\end{pmatrix}$	$b_{22}(a^2 - c^2\ell)(1 - \lambda^2)\ell$	$(p-1)^3p^3(p+1)^2$	$-\frac{(p-1)p^{13}(p+1)}{2}$
16.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&1&0\\&&0\end{pmatrix}$	$b_{11}a_{33}^2 + b_{22}(-a^2 + c^2\ell)\lambda^2\ell$	$(p-1)^3p^3(p+1)$	$-\frac{(p-1)p^{13}}{2}$
17.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}1&0&0\\&1&0\\&&0\end{pmatrix}$	$b_{11}a_{33}^2 + b_{22}(a^2 - c^2\ell)(1 - \lambda^2\ell)$	$(p-1)^3p^3(p+1)$	$-\frac{(p-1)p^{13}}{2}$
18.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}\ell&0&0\\&1&0\\&&0\end{pmatrix}$	$b_{11}a_{33}^2 + b_{22}(a^2 - c^2\ell)(1 - \lambda^2)\ell$	$-(p-1)^2p^3(p+1)^2$	$\frac{p^{13}(p+1)}{2}$
19.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$b_{11}a_{33}^2$	$-(p-1)^4p^3(p+1)$	$\frac{(p-1)^2p^{13}}{2}$
20.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	0	$(p-1)^5p^3(p+1)$	$-\frac{(p-1)^3p^{13}}{2}$

The exponential sums are as follows.

1.-12. These sums vanish. For sums 1.-6., sum in b_{12} to force $a_{33} = 0$. For sums 7.-12. sum in a_{13} and a_{23} to force $a = c = 0$. In either case, this makes the acting matrix singular.

13. In this case

$$[x_0, g \cdot \xi_1] = b_{22}(-a^2 + c^2\ell)\lambda^2\ell. \tag{5.54}$$

Since $(-a^2 + c^2\ell) \neq 0$, sum in b_{22} to obtain

$$\mathscr{S}(x, \xi) = -\frac{|G_{x,\xi}|}{p-1} = -(p-1)^4 p^3 (p+1). \tag{5.55}$$

14. In this case

$$[x_0, g \cdot \xi_1] = b_{22}(a^2 - c^2\ell)(1 - \lambda^2\ell). \tag{5.56}$$

Since $(a^2 - c^2\ell)(1 - \lambda^2\ell) \neq 0$, sum in b_{22} to obtain as for 13,

$$\mathscr{S}(x, \xi) = -(p-1)^4 p^3 (p+1). \tag{5.57}$$

15. In this case

$$[x_0, g \cdot \xi_1] = b_{22}(a^2 - c^2\ell)(1 - \lambda^2)\ell. \tag{5.58}$$

Changing variables in b_{22},

$$\begin{aligned}\mathscr{S}(x, \xi) &= \sum_{\begin{pmatrix} a & c\lambda\ell & a_{13} \\ c & a\lambda & a_{23} \\ & & a_{33} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix}} e_p(b_{22}(a^2 - c^2\ell)(1 - \lambda^2)\ell) \\ &= (p-1)^3 p^3 (p+1) \sum_{b_{22}, \lambda \in (\mathbb{Z}/p\mathbb{Z})^\times} e_p(b_{22}(1 - \lambda^2)) \\ &= (p-1)^3 p^3 (p+1)^2.\end{aligned}$$

16. In this case

$$[x_0, g \cdot \xi_1] = b_{11}a_{33}^2 + b_{22}(-a^2 + c^2\ell)\lambda^2\ell. \tag{5.59}$$

Sum in b_{11} to obtain $\mathscr{S}(x, \xi) = \frac{-13.}{p-1} = (p-1)^3 p^3 (p+1)$.

17. In this case

$$[x_0, g \cdot \xi_1] = b_{11}a_{33}^2 + b_{22}(a^2 - c^2\ell)(1 - \lambda^2\ell). \tag{5.60}$$

Sum in b_{11} to obtain $\mathscr{S}(x,\xi) = \frac{-14.}{p-1} = (p-1)^3p^3(p+1)$.

18. In this case

$$[x_0, g \cdot \xi_1] = b_{11}a_{33}^2 + b_{22}(a^2 - c^2\ell)(1 - \lambda^2)\ell. \tag{5.61}$$

Sum in b_{11} to obtain $\mathscr{S}(x,\xi) = \frac{-15.}{p-1} = -(p-1)^2p^3(p+1)^2$.

19. In this case

$$[x_0, g \cdot \xi_1] = b_{11}a_{33}^2. \tag{5.62}$$

Sum in b_{11} to obtain $\mathscr{S}(x,\xi) = \frac{-|G_{x,\xi}|}{p-1} = -(p-1)^4p^3(p+1)$.

20. In this case

$$[x_0, g \cdot \xi_1] = 0 \tag{5.63}$$

so $\mathscr{S}(x,\xi) = |G_{x,\xi}| = (p-1)^5p^3(p+1)$.

5.4. The exponential sums pair $(\mathscr{O}_{2^2}, \mathscr{O}_{D2})$

Take standard representatives

$$x_0 = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ & -\ell & 0 \\ & & 0 \end{pmatrix}, \qquad \xi_0 = \begin{pmatrix} \ell & \beta & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \tag{5.64}$$

with $\ell u^2 + 2\beta uv + v^2$ irreducible. The acting set is ($g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$)

$$G_{x,\xi}^t = \left\{ g = \begin{pmatrix} a & b & a_{13} \\ c & d & a_{23} \\ & & a_{33} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix}, [u^2 - \ell v^2, g \cdot (\ell u^2 + 2\beta uv + v^2)] = 0 \right\}.$$
(5.65)

As determined in Appendix B, $|G_{x,\xi}| = (p-1)^4p^3(p+1)^2$. The stabilizer has size

$$\left|\mathrm{Stab}_{G(\mathbb{Z}/p^2\mathbb{Z})}(x)\right| = 2(p-1)^2p^3(p+1). \tag{5.66}$$

Representatives and exponential sum pairings are given in the table below. Since this is the single non-maximal orbit,

$$\mathscr{M}(x,\xi) = -\frac{p^{10}\mathscr{S}(x,\xi)}{2(p-1)^2(p+1)}. \tag{5.67}$$

See **exponential_sums_O22_OD2.nb**.

Orbit	ξ_1	$[x_0, g\cdot\xi_1]$	$\mathscr{S}(x,\xi)$	$\mathscr{M}(x,\xi)$
1.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}$	$b_{12}a_{33}^2+b_{22}(a_{13}^2-a_{23}^2\ell)$	0	0
2.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&1\\&&1\end{pmatrix}$	$b_{12}a_{33}^2$ $+b_{22}(a_{13}^2+2a_{13}b-a_{23}^2-2a_{23}d\ell)$	0	0
3.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&1&0\\&0&0\\&&1\end{pmatrix}$	$b_{12}a_{33}^2$ $+b_{22}(a_{13}^2+2ab-a_{23}^2\ell-2cd\ell)$	0	0
4.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}$	$2b_{22}(a_{13}b-a_{23}d\ell)$	0	0
5.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}$	$b_{11}a_{33}^2+2b_{22}(a_{13}b-a_{23}d\ell)$	0	0
6.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&1&0\\&&0\end{pmatrix}$	$b_{22}(b^2-d^2\ell)$	$-(p-1)^3p^3(p+1)^2$	$\frac{(p-1)p^{13}(p+1)}{2}$
7.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&1&0\\&&0\end{pmatrix}$	$b_{11}a_{33}^2+b_{22}(b^2-d^2\ell)$	$(p-1)^2p^3(p+1)^2$	$-\frac{p^{13}(p+1)}{2}$
8a.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&1&0\\&0&0\\&&0\end{pmatrix}$	$2b_{22}(ab-cd\ell)$	$(p-1)^4p^3(p+1)$	$-\frac{(p-1)^2p^{13}}{2}$
8b.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&k&0\\&1&0\\&&0\end{pmatrix}$	$b_{22}(b^2+2abk-d^2\ell-2cd\ell k)$	$(p-1)^4p^3(p+1)$	$-\frac{(p-1)^2p^{13}}{2}$
9a.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&1&0\\&0&0\\&&0\end{pmatrix}$	$b_{11}a_{33}^2+b_{22}(2ab-2cd\ell)$	$-(p-1)^3p^3(p+1)$	$\frac{(p-1)p^{13}}{2}$
9b.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&k&0\\&1&0\\&&0\end{pmatrix}$	$b_{11}a_{33}^2+$ $b_{22}(b^2+2abk-d^2\ell-2cd\ell k)$	$-(p-1)^3p^3(p+1)$	$\frac{(p-1)p^{13}}{2}$
10	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$b_{11}a_{33}^2$	$-(p-1)^3p^3(p+1)^2$	$\frac{(p-1)p^{13}(p+1)}{2}$
11	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	0	$(p-1)^4p^3(p+1)^2$	$-\frac{(p-1)^2p^{13}(p+1)}{2}$

The exponential sums are as follows.

1.-5. These sums vanish on summing in b_{12}, a_{13} and a_{23}.

6. In this case,

$$[x_0, g\cdot\xi_1] = b_{22}(b^2-d^2\ell). \tag{5.68}$$

Since $b^2-d^2\ell\neq 0$, summing in b_{22} gives -1, so

$$\mathscr{S}(x,\xi_1) = \frac{-|G_{x,\xi_1}|}{p-1} = -(p-1)^3p^3(p+1)^2. \tag{5.69}$$

7. In this case,

$$[x_0, g\cdot\xi_1] = b_{11}a_{33}^2 + b_{22}(b^2-d^2\ell). \tag{5.70}$$

Summing in b_{11} and b_{22} each give -1, so

$$\mathscr{S}(x,\xi) = \frac{|G_{x,\xi}|}{(p-1)^2} = (p-1)^2 p^3 (p+1)^2. \tag{5.71}$$

8a. $p \equiv 1 \bmod 4$. In this case,

$$[x_0, g \cdot \xi_1] = 2b_{22}(ab - cd\ell). \tag{5.72}$$

By summing in b_{22},

$$\begin{aligned}\mathscr{S}(x,\xi) &= -\frac{|G_{x,\xi}|}{p-1} \\ &\quad + \frac{p}{p-1}\#\left\{g \in G_{x,\xi}^t : [u^2-\ell v^2, g\cdot uv] = 0, [u^2 - \ell v^2, g\cdot(\ell u^2+v^2)] = 0\right\} \\ &= -\frac{|G_{x,\xi}|}{p-1} \\ &\quad + \frac{p}{p-1}\#\left\{g \in G_{x,\xi} : [g\cdot(u^2-\ell v^2), uv] = 0, [g\cdot(u^2-\ell v^2), \ell u^2 + v^2] = 0\right\}.\end{aligned}$$

The conditions on g are equivalent to g acting on $u^2 - \ell v^2$ by a scalar. Thus

$$\begin{aligned}\mathscr{S}(x,\xi) &= -\frac{|G_{x,\xi}|}{p-1} + \frac{p}{p-1}\#\left\{\begin{pmatrix} c & e\ell & * \\ \pm e & \pm c & * \\ & & * \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}\right\} \\ &= -(p-1)^3p^3(p+1)^2 + 2(p-1)^3p^4(p+1) \\ &= (p-1)^4p^3(p+1).\end{aligned}$$

9a. $p \equiv 1 \bmod 4$. In this case,

$$[x_0, g\cdot\xi_1] = b_{11}a_{33}^2 + b_{22}(2ab - 2cd\ell). \tag{5.73}$$

The sum in b_{11} is -1, so

$$\mathscr{S}(x,\xi) = -\frac{8a.}{p-1} = -(p-1)^3p^3(p+1). \tag{5.74}$$

10. In this case,

$$[x_0, g\cdot\xi_1] = b_{11}a_{33}^2. \tag{5.75}$$

The sum in b_{11} is -1, so

$$\mathscr{S}(x,\xi) = -\frac{|G_{x,\xi}|}{p-1} = -(p-1)^3p^3(p+1)^2. \tag{5.76}$$

11. $\xi_1 = 0$. In this case $\mathscr{S}(x,\xi) = |G_{x,\xi}| = (p-1)^4p^3(p+1)^2$.
The relation $9b. = -\frac{8b.}{p-1}$ holds as in the a. case. Since the sum of $e_p([x,\xi])$ over ξ in a full mod p orbit vanishes by Lemma 5, $8a. = 8b.$ and $9a. = 9b.$

5.5. The exponential sums pair $(\mathscr{O}_{1^21^2}, \mathscr{O}_{D1^2})$

For this pair,

$$x_0 = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \qquad \xi_0 = \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \tag{5.77}$$

and

$$x_1 = \begin{pmatrix} x & 0 & 0 \\ & y & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \tag{5.78}$$

with x, y equal to either 1 or ℓ. The acting set is

$$G^t_{x,\xi} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & a & b \\ & c & d \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix} \sqcup \begin{pmatrix} & a & b \\ a_{21} & a_{22} & a_{23} \\ & c & d \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix}. \tag{5.79}$$

Write this as $G^t_{x,\xi,1} \sqcup G^t_{x,\xi,2}$. This obtains

$$[x_1, g \cdot \xi_0] = b_{11}(xa_{11}^2 + ya_{21}^2). \tag{5.80}$$

Note that

$$G_{x,\xi,1} = G_{x,\xi,2} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.81}$$

Since x_0 is invariant under $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ while in x_1 this exchanges x and y, the exponential sums may be written by setting

$$x_1' = \begin{pmatrix} \epsilon & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \tag{5.82}$$

$[x_1', g \cdot \xi_0] = b_{11}a_{11}^2\epsilon$ and

$$\mathscr{S}(x, \xi) = \sum_{\epsilon = x,y} \sum_{g \in G^t_{x,\xi,1}} e_p\left([x_1', g \cdot \xi_0] + [x_0, g \cdot \xi_1]\right). \tag{5.83}$$

The sum over $G^t_{x,\xi,1}$ is fibered as for the pairs $(\mathscr{O}_{1^211}, \mathscr{O}_{D1^2})$ and $(\mathscr{O}_{1^22}, \mathscr{O}_{D1^2})$.

The stabilizer has size

$$\left|\mathrm{Stab}_{G(\mathbb{Z}/p^2\mathbb{Z})}(x)\right| = \begin{cases} 8(p-1)p^3 & x = y \\ 4(p-1)p^3 & x \neq y \end{cases}. \tag{5.84}$$

Representatives and exponential sum pairings are given in the table below. Summing over the three maximal orbits

$$\mathscr{M}(x,\xi) = \sum_{(x,y)\in\{(1,1),(1,\ell),(\ell,\ell)\}} \frac{2^{\delta(x\neq y)}p^{10}\mathscr{S}(x,\xi)}{8(p-1)}. \tag{5.85}$$

See **exponential_sums_O1212_OD12.nb**.

Orbit	ξ_1	$[x_0, g\cdot\xi_1] + [x_1', g\cdot\xi_0]$	$\mathscr{S}(x,\xi)$	$\mathscr{M}(x,\xi)$
1.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$b_{11}a_{11}^2\epsilon$	$-2(p-1)^4p^4(p+1)$	$-(p-1)^3p^{14}(p+1)$
2.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$b_{11}(a_{11}^2\epsilon+d^2)$	$(p-1)^3p^6\left(\left(\frac{-x}{p}\right)+\left(\frac{-y}{p}\right)\right)$ $+2(p-1)^3p^4$	$(p-1)^2p^{14}$
3.	$\begin{pmatrix}0&0&0\\&0&\frac{1}{2}\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$b_{11}(a_{11}^2\epsilon+cd)$	$-2(p-1)^4p^4$	$-(p-1)^3p^{14}$
4.	$\begin{pmatrix}0&0&0\\&-\ell&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$b_{11}(a_{11}^2\epsilon-c^2\ell+d^2)$	$2(p-1)^3p^4(p+1)$	$(p-1)^2p^{14}(p+1)$
5.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&1\\&0&0\\&&0\end{pmatrix}$	$b_{11}(a_{11}^2\epsilon+d^2)+b_{22}a_{11}b$	$2(p-1)^3p^4$	$(p-1)^2p^{14}$
6.	$\begin{pmatrix}0&0&0\\&1&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&1\\&0&0\\&&0\end{pmatrix}$	$b_{11}(a_{11}^2\epsilon+c^2)+b_{22}a_{11}b$	$-2(p-1)^2p^4$ $-(p-1)^2p^5\left(\left(\frac{-x}{p}\right)+\left(\frac{-y}{p}\right)\right)$	$-(p-1)p^{14}$
7.	$\begin{pmatrix}0&0&0\\&0&\frac{1}{2}\\&&0\end{pmatrix}\begin{pmatrix}0&0&1\\&0&0\\&&0\end{pmatrix}$	$b_{11}(a_{11}^2\epsilon+cd)$ $+b_{22}a_{11}b$	$2(p-1)^3p^4$	$(p-1)^2p^{14}$
8.	$\begin{pmatrix}0&0&0\\&1&1\\&&0\end{pmatrix}\begin{pmatrix}0&0&1\\&0&0\\&&0\end{pmatrix}$	$b_{11}(a_{11}^2\epsilon+c(c+2d))$ $+b_{22}a_{11}b$	$-2(p-1)^2p^4$ $-(p-1)^2p^5\left(\left(\frac{-x}{p}\right)+\left(\frac{-y}{p}\right)\right)$	$-(p-1)p^{14}$
9.	$\begin{pmatrix}0&0&0\\&-\ell&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&1\\&0&0\\&&0\end{pmatrix}$	$b_{11}(a_{11}^2\epsilon-c^2\ell+d^2)$ $+b_{22}a_{11}b$	$-2(p-1)^2p^4$ $+(p-1)^2p^5\left(\left(\frac{x}{p}\right)+\left(\frac{y}{p}\right)\right)$	$-(p-1)p^{14}$
10.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&1\\&0&0\\&&0\end{pmatrix}$	$b_{11}a_{11}^2\epsilon+b_{22}a_{11}b$	$2(p-1)^3p^4$	$(p-1)^2p^{14}$
11.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}$	$b_{11}a_{11}^2\epsilon+b_{12}d^2+b_{22}a_{13}b$	0	0
12.	$\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}$	$b_{11}(a_{11}^2\epsilon+2cd)+b_{12}d^2$ $+b_{22}a_{13}b$	0	0
13.	$\begin{pmatrix}0&0&0\\&1&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}$	$b_{11}(a_{11}^2\epsilon+c^2)+b_{12}d^2$ $+b_{22}a_{13}b$	0	0
14.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&1&0\\&0&0\\&&1\end{pmatrix}$	$b_{11}a_{11}^2\epsilon+b_{12}d^2$ $+b_{22}(aa_{11}+a_{13}b)$	0	0
15.	$\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}\begin{pmatrix}0&1&0\\&0&0\\&&1\end{pmatrix}$	$b_{11}(a_{11}^2\epsilon+2cd)+b_{12}d^2$ $+b_{22}(a_{11}a+a_{13}b)$	0	0
16.	$\begin{pmatrix}0&0&0\\&1&0\\&&0\end{pmatrix}\begin{pmatrix}0&1&0\\&0&0\\&&1\end{pmatrix}$	$b_{11}(a_{11}^2\epsilon+c^2)+b_{12}d^2$ $+b_{22}(a_{11}a+a_{13}b)$	0	0
17.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}$	$b_{11}a_{11}^2\epsilon+2b_{12}cd$ $+b_{22}(a_{12}b+a_{13}a)$	0	0
18.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}$	$b_{11}(a_{11}^2\epsilon+d^2)+2b_{12}cd$ $+b_{22}(a_{12}b+a_{13}a)$	0	0
19.	$\begin{pmatrix}0&0&0\\&1&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}$	$b_{11}(a_{11}^2\epsilon+c^2+d^2)$ $+2b_{12}cd+b_{22}(a_{12}b+a_{13}a)$	0	0
20.	$\begin{pmatrix}0&0&0\\&\ell&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}$	$b_{11}(a_{11}^2\epsilon+c^2\ell+d^2)$ $+2b_{12}cd+b_{22}(a_{12}b+a_{13}a)$	0	0

Orbit	ξ_1	$[x_0, g\cdot\xi_1]$	$\mathscr{S}(x,\xi)$	$\mathscr{M}(x,\xi)$
21.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}$	$b_{11}a_{11}^2\epsilon + b_{12}(-c^2\ell + d^2)$ $+b_{22}(-a_{12}a\ell + a_{13}b)$	0	0
22.	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + c^2) + b_{12}(-c^2\ell + d^2)$ $+b_{22}(-a_{12}a\ell + a_{13}b)$	0	0
23a.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + 2cd)$ $+b_{12}(-c^2\ell + d^2)$ $+b_{22}(-a_{12}a\ell + a_{13}b)$	0	0
23b.	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & k \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + c^2 + 2cdk)$ $+b_{12}(-c^2\ell + d^2)$ $+b_{22}(-a_{12}a\ell + a_{13}b)$	0	0

The exponential sums are as follows.

1. In this case

$$[x_0, g\cdot\xi_1] + [x_1', g\cdot\xi_0] = b_{11}a_{11}^2\epsilon. \tag{5.86}$$

The sum in b_{11} gives -1, so

$$\mathscr{S}(x,\xi) = -\frac{|G_{x,\xi}|}{p-1} = -2(p-1)^4p^4(p+1). \tag{5.87}$$

2. In this case

$$[x_0, g\cdot\xi_1] + [x_1', g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon + d^2). \tag{5.88}$$

The following sums are evaluated by first summing the quadratic terms to obtain Gauss sums.

$$\begin{aligned}\Sigma_1 &= \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & b \\ & c & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + d^2)) \\ &= (p-1)p^6 \sum_{\epsilon=x,y}\sum_{b_{11}\in\mathbf{F}_p^\times} \left(\tau\left(\frac{b_{11}\epsilon}{p}\right) - 1\right)\tau\left(\frac{b_{11}}{p}\right) \\ &= (p-1)^2p^7\left(\left(\frac{-x}{p}\right) + \left(\frac{-y}{p}\right)\right)\end{aligned}$$

$$\Sigma_2 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + d^2))$$

$$= (p-1)^2 p^4 \sum_{\epsilon=x,y} \sum_{b_{11}\in \mathbf{F}_p^\times} \left(\tau\left(\frac{b_{11}\epsilon}{p}\right) - 1\right)\tau\left(\frac{b_{11}}{p}\right)$$

$$= (p-1)^3 p^5 \left(\left(\frac{-x}{p}\right) + \left(\frac{-y}{p}\right)\right)$$

$$\Sigma_3 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon)$$

$$= -2(p-1)^2 p^5$$

$$\Sigma_4 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & 0 & b \\ & 0 & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + d^2))$$

$$= (p-1) p^4 \sum_{\epsilon=x,y} \sum_{b_{11}\in \mathbf{F}_p^\times} \left(\tau\left(\frac{b_{11}\epsilon}{p}\right) - 1\right)\tau\left(\frac{b_{11}}{p}\right)$$

$$= (p-1)^2 p^5 \left(\left(\frac{-x}{p}\right) + \left(\frac{-y}{p}\right)\right)$$

$$\Sigma_5 = p\sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon)$$

$$= -2(p-1)^2 p^4.$$

Thus

$$\mathscr{S}(x,\xi) = (p-1)^3 p^6 \left(\left(\frac{-x}{p}\right) + \left(\frac{-y}{p}\right)\right) + 2(p-1)^3 p^4. \tag{5.89}$$

3. In this case

$$[x_0, g\cdot\xi_1] + [x_1', g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon + cd). \tag{5.90}$$

$$\Sigma_1 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & b \\ & c & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + cd))$$

First sum in d to force $c = 0$. The sum in b_{11} is now -1. Thus $\Sigma_1 = -2(p-1)^2 p^6$.

$$\Sigma_2 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix} \begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2 \epsilon + d^2 \lambda))$$

This sum vanishes by summing over d and λ.

$$\begin{aligned} \Sigma_3 &= \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix} \begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11} a_{11}^2 \epsilon) \\ &= -2(p-1)^2 p^5 \\ \Sigma_4 &= \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & 0 & b \\ & 0 & d \end{pmatrix} \begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11} a_{11}^2 \epsilon) \\ &= -2(p-1)^2 p^5 \\ \Sigma_5 &= p \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix} \begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11} a_{11}^2 \epsilon) = -2(p-1)^2 p^4. \end{aligned}$$

Thus

$$\mathscr{S}(x, \xi) = -2(p-1)^4 p^4. \tag{5.91}$$

4. In this case

$$[x_0, g \cdot \xi_1] + [x_1', g \cdot \xi_0] = b_{11}(a_{11}^2 \epsilon - c^2 \ell + d^2). \tag{5.92}$$

$$\begin{aligned} \Sigma_1 &= \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & b \\ & c & d \end{pmatrix} \begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2 \epsilon - c^2 \ell + d^2)) \\ &= (p-1)p^5 \sum_{\epsilon=x,y} \sum_{b_{11} \in \mathbf{F}_p^\times} \left(\tau \left(\frac{b_{11}\epsilon}{p} \right) - 1 \right) \tau^2 \left(\frac{-\ell}{p} \right) \\ &= 2(p-1)^2 p^6 \end{aligned}$$

$$\Sigma_2 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + d^2(-\lambda^2\ell + 1)))$$

Set apart the $d = 0$ term, which evaluates to $-2(p-1)^3p^4$. When $d \neq 0$, replace $\lambda := d\lambda$ to obtain

$$\begin{aligned}\Sigma_2 &= -2(p-1)^3p^4 + (p-1)p^4 \sum_{\epsilon=x,y} \sum_{b_{11},a_{11},\lambda,d\in\mathbf{F}_p^\times} e_p(b_{11}(a_{11}^2\epsilon - \lambda^2\ell + d^2)) \\ &= -2(p-1)^3p^4 \\ &+ (p-1)p^4 \sum_{\epsilon=x,y} \sum_{b_{11}\in\mathbf{F}_p^\times} \left(\tau\left(\frac{b_{11}\epsilon}{p}\right) - 1\right)\left(\tau\left(-\frac{b_{11}\ell}{p}\right) - 1\right)\left(\tau\left(\frac{b_{11}}{p}\right) - 1\right)\end{aligned}$$

The sum over b_{11} vanishes unless zero or two factors of $\left(\frac{b_{11}}{p}\right)$ are chosen when the product is expanded. Using $\left(\frac{\ell}{p}\right) = -1$ obtains

$$\begin{aligned}\Sigma_2 &= -2(p-1)^3p^4 - 2(p-1)^2p^4 + 2(p-1)^2p^5 \\ &\quad + (p-1)^2p^5\left(\left(\frac{x}{p}\right) - \left(\frac{-x}{p}\right) + \left(\frac{y}{p}\right) - \left(\frac{-y}{p}\right)\right) \\ &= (p-1)^2p^5\left(\left(\frac{x}{p}\right) - \left(\frac{-x}{p}\right) + \left(\frac{y}{p}\right) - \left(\frac{-y}{p}\right)\right) \\ \Sigma_3 &= \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon - c^2\ell)) \\ &= (p-1)p^4 \sum_{\epsilon=x,y} \sum_{b_{11}\in\mathbf{F}_p^\times} \left(\tau\left(\frac{b_{11}\epsilon}{p}\right) - 1\right)\tau\left(\frac{-b_{11}\ell}{p}\right) \\ &= -(p-1)^2p^5\left(\left(\frac{x}{p}\right) + \left(\frac{y}{p}\right)\right) \\ \Sigma_4 &= \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & 0 & b \\ & 0 & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + d^2)) \\ &= (p-1)^2p^5\left(\left(\frac{-x}{p}\right) + \left(\frac{-y}{p}\right)\right)\end{aligned}$$

$$\Sigma_5 = p \sum_{\epsilon = x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix} \begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11} a_{11}^2 \epsilon) = -2(p-1)^2 p^4.$$

Thus

$$\mathscr{S}(x, \xi) = 2(p-1)^3 p^4 (p+1). \tag{5.93}$$

5. In this case

$$[x_0, g \cdot \xi_1] + [x_1', g \cdot \xi_0] = b_{11}(a_{11}^2 \epsilon + d^2) + b_{22} a_{11} b. \tag{5.94}$$

In Σ_1, Σ_2 and Σ_4 below, sum in b to find that the sum vanishes. In the remaining sums, sum in b_{11}. This obtains

$$\Sigma_1 = \sum_{\epsilon = x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & b \\ & c & d \end{pmatrix} \begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2 \epsilon + d^2) + b_{22} a_{11} b) = 0$$

$$\Sigma_2 = \sum_{\epsilon = x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix} \begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2 \epsilon + d^2) + b_{22} a_{11} b) = 0$$

$$\Sigma_3 = \sum_{\epsilon = x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix} \begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11} a_{11}^2 \epsilon) = -2(p-1)^2 p^5$$

$$\Sigma_4 = \sum_{\epsilon = x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & 0 & b \\ & 0 & d \end{pmatrix} \begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2 \epsilon + d^2) + b_{22} a_{11} b) = 0$$

$$\Sigma_5 = p \sum_{\epsilon = x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix} \begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11} a_{11}^2 \epsilon) = -2(p-1)^2 p^4.$$

Thus

$$\mathscr{S}(x, \xi) = 2(p-1)^3 p^4. \tag{5.95}$$

6. In this case

$$[x_0, g \cdot \xi_1] + [x_1', g \cdot \xi_0] = b_{11}(a_{11}^2 \epsilon + c^2) + b_{22} a_{11} b. \tag{5.96}$$

Here in Σ_1, Σ_2 and Σ_4, sum in b to conclude that the sums vanish. In Σ_3, sum first in a_{11} and c. In Σ_5, sum in b_{11}. This obtains

$$\Sigma_1 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & b \\ & c & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + c^2) + b_{22}a_{11}b) = 0$$

$$\Sigma_2 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + d^2\lambda^2) + b_{22}a_{11}b) = 0$$

$$\begin{aligned}\Sigma_3 &= \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + c^2)) \\ &= (p-1)p^4 \sum_{\epsilon=x,y} \sum_{b_{11}\in\mathbf{F}_p^\times} \tau\left(\frac{b_{11}}{p}\right)\left(\tau\left(\frac{b_{11}\epsilon}{p}\right) - 1\right) \\ &= (p-1)^2p^5\left(\left(\frac{-x}{p}\right) + \left(\frac{-y}{p}\right)\right)\end{aligned}$$

$$\Sigma_4 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & 0 & b \\ & 0 & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon + b_{22}a_{11}b) = 0$$

$$\Sigma_5 = p\sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -2(p-1)^2p^4.$$

Thus

$$\mathscr{S}(x,\xi) = -2(p-1)^2p^4 - (p-1)^2p^5\left(\left(\frac{-x}{p}\right) + \left(\frac{-y}{p}\right)\right). \tag{5.97}$$

7. In this case

$$[x_0, g\cdot\xi_1] + [x_1', g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon + cd) + b_{22}a_{11}b. \tag{5.98}$$

As in 5. and 6., sum in b to show that $\Sigma_1, \Sigma_2, \Sigma_4 = 0$. In the remaining sums, sum in b_{11}. This obtains,

$$\Sigma_1 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & b \\ & c & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + cd) + b_{22}a_{11}b) = 0$$

$$\Sigma_2 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + d^2\lambda) + b_{22}a_{11}b) = 0$$

$$\Sigma_3 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -2(p-1)^2p^5$$

$$\Sigma_4 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & 0 & b \\ & 0 & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon + b_{22}a_{11}b) = 0$$

$$\Sigma_5 = p\sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -2(p-1)^2p^4.$$

Thus

$$\mathscr{S}(x,\xi) = 2(p-1)^3p^4. \tag{5.99}$$

8. In this case

$$[x_0, g\cdot\xi_1] + [x_1', g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon + c(c+2d)) + b_{22}a_{11}b. \tag{5.100}$$

Sum in b to show that Σ_1, Σ_2 and Σ_4 vanish. In Σ_3, sum first in a_{11} and c, while in Σ_5 sum in b_{11} first. This obtains,

$$\Sigma_1 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & b \\ & c & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + c(c+2d)) + b_{22}a_{11}b) = 0$$

$$\Sigma_2 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + d^2\lambda(\lambda+2)) + b_{22}a_{11}b) = 0$$

$$\Sigma_3 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + c^2))$$

$$= (p-1)p^4 \sum_{\epsilon=x,y} \sum_{b_{11}\in\mathbf{F}_p^\times} \tau\left(\frac{b_{11}}{p}\right)\left(\tau\left(\frac{b_{11}\epsilon}{p}\right)-1\right)$$

$$= (p-1)^2p^5\left(\left(\frac{-x}{p}\right)+\left(\frac{-y}{p}\right)\right)$$

$$\Sigma_4 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & 0 & b \\ & 0 & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon + b_{22}a_{11}b) = 0$$

$$\Sigma_5 = p\sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -2(p-1)^2p^4.$$

Thus

$$\mathscr{S}(x,\xi) = -2(p-1)^2p^4 - (p-1)^2p^5\left(\left(\frac{-x}{p}\right)+\left(\frac{-y}{p}\right)\right). \tag{5.101}$$

9. In this case

$$[x_0, g\cdot\xi_1] + [x_1', g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon - c^2\ell + d^2) + b_{22}a_{11}b. \tag{5.102}$$

Sum in b to show that Σ_1, Σ_2 and Σ_4 vanish. In Σ_3 sum in a_{11} and c first, while in Σ_5 sum in b_{11} first. This obtains,

$$\Sigma_1 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & b \\ & c & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon - c^2\ell + d^2) + b_{22}a_{11}b) = 0$$

$$\Sigma_2 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + d^2(-\ell\lambda^2+1)) + b_{22}a_{11}b) = 0$$

$$\Sigma_3 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon - c^2\ell))$$

$$= (p-1)p^4 \sum_{\epsilon=x,y} \sum_{b_{11}\in\mathbf{F}_p^\times} \tau\left(\frac{-b_{11}\ell}{p}\right)\left(\tau\left(\frac{b_{11}\epsilon}{p}\right)-1\right)$$

$$= -(p-1)^2p^5\left(\left(\frac{x}{p}\right)+\left(\frac{y}{p}\right)\right)$$

$$\Sigma_4 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & 0 & b \\ & 0 & d\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + d^2) + b_{22}a_{11}b) = 0$$

$$\Sigma_5 = p\sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & 0 & 0 \\ & 0 & 0\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -2(p-1)^2p^4.$$

Thus

$$\mathscr{S}(x,\xi) = -2(p-1)^2p^4 + (p-1)^2p^5\left(\left(\frac{x}{p}\right)+\left(\frac{y}{p}\right)\right). \tag{5.103}$$

10. In this case

$$[x_0, g\cdot\xi_1] + [x_1', g\cdot\xi_0] = b_{11}a_{11}^2\epsilon + b_{22}a_{11}b. \tag{5.104}$$

Sum in b in Σ_1, Σ_2 and Σ_4 to show that these sums vanish. In Σ_3 and Σ_5 sum in b_{11} first. This obtains

$$\Sigma_1 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & b \\ & c & d\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon + b_{22}a_{11}b) = 0$$

$$\Sigma_2 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & b\lambda & b \\ & d\lambda & d\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon + b_{22}a_{11}b) = 0$$

$$\Sigma_3 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & 0 \\ & c & 0\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -2(p-1)^2p^5$$

$$\Sigma_4 = \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & 0 & b \\ & 0 & d\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon + b_{22}a_{11}b) = 0$$

$$\Sigma_5 = p \sum_{\epsilon=x,y} \sum_{\begin{pmatrix} a_{11} & * & * \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -2(p-1)^2p^4.$$

Thus

$$\mathscr{S}(x,\xi) = 2(p-1)^3p^4. \tag{5.105}$$

11.-23. These sums vanish on summing in a_{12}, a_{13} and b_{12}.

5.6. The exponential sums pair $(\mathscr{O}_{1^21^2}, \mathscr{O}_{D11})$

The standard representatives in this case are

$$x_0 = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2} & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \qquad \xi_0 = \begin{pmatrix} 1 & 0 & 0 \\ & -1 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \tag{5.106}$$

and

$$x_1 = \begin{pmatrix} x & 0 & 0 \\ & y & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}. \tag{5.107}$$

The acting set is the subset

$$G_{x,\xi}^t \subset \begin{pmatrix} * & * & a_{13} \\ * & * & a_{23} \\ & & a_{33} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix} \tag{5.108}$$

in which the GL_3 factor acts by

$$\begin{aligned} u + v &\mapsto \alpha u + \beta v \\ u - v &\mapsto \lambda(\alpha u - \beta v) \end{aligned}$$

thus

$$(u+v)(u-v) \mapsto \alpha^2\lambda u^2 - \beta^2\lambda v^2. \tag{5.109}$$

Thus

$$[x_1, g\cdot\xi_0] = b_{11}(\alpha^2 x - \beta^2 y)\lambda. \tag{5.110}$$

The stabilizer has size

$$\left|\mathrm{Stab}_{G(\mathbb{Z}/p^2\mathbb{Z})}(x)\right| = \begin{cases} 8(p-1)p^3 & x = y \\ 4(p-1)p^3 & x \neq y \end{cases}. \tag{5.111}$$

Representatives and exponential sum pairings are given in the table below. Summing over the three maximal orbits

$$\mathscr{M}(x,\xi) = \sum_{(x,y)\in\{(1,1),(1,\ell),(\ell,\ell)\}} \frac{2^{\delta(x\neq y)}p^{10}\mathscr{S}(x,\xi)}{8(p-1)}. \tag{5.112}$$

See **exponential_sums_O1212_OD11.nb**.

Orbit	ξ_1	$[x_0, g\cdot\xi_1]+[x_1, g\cdot\xi_0]$	$\mathscr{S}(x,\xi)$	$\mathscr{M}(x,\xi)$
1.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}$	$b_{11}(\alpha^2x-\beta^2y)\lambda+b_{12}a_{33}^2$ $+b_{22}a_{13}a_{23}$	0	0
2.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&1\\&0&-1\\&&1\end{pmatrix}$	$b_{11}(\alpha^2x-\beta^2y)\lambda+b_{12}a_{33}^2$ $+b_{22}(a_{13}a_{23}-a_{13}\beta\lambda+a_{23}\alpha\lambda)$	0	0
3.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&2\\&0&0\\&&1\end{pmatrix}$	$b_{11}(\alpha^2x-\beta^2y)\lambda+b_{12}a_{33}^2$ $+b_{22}(a_{23}\alpha(1+\lambda)$ $+a_{13}\beta(1-\lambda)+a_{13}a_{23})$	0	0
4.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}-\ell&\ell&0\\&-\ell&0\\&&1\end{pmatrix}$	$b_{11}(\alpha^2x-\beta^2y)\lambda+b_{12}a_{33}^2$ $+b_{22}(a_{13}a_{23}+\alpha\beta\lambda^2\ell)$	0	0
5.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}-\ell&\ell&1\\&-\ell&1\\&&1\end{pmatrix}$	$b_{11}(\alpha^2x-\beta^2y)\lambda+b_{12}a_{33}^2$ $+b_{22}(a_{13}a_{23}+a_{13}\beta$ $+a_{23}\alpha+\alpha\beta\lambda^2\ell)$	0	0
6.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}-2\ell&0&0\\&-2\ell&0\\&&1\end{pmatrix}$	$b_{11}(\alpha^2x-\beta^2y)\lambda+b_{12}a_{33}^2$ $+b_{22}(\alpha\beta(\lambda^2-1)\ell+a_{13}a_{23})$	0	0
7.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&1\\&0&-1\\&&0\end{pmatrix}$	$b_{11}(\alpha^2x-\beta^2y)\lambda$ $+b_{22}(a_{23}\alpha-a_{13}\beta)\lambda$	0	0
8.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&2\\&0&0\\&&0\end{pmatrix}$	$b_{11}(\alpha^2x-\beta^2y)\lambda$ $+b_{22}(a_{13}\beta(1-\lambda)$ $+a_{23}\alpha(1+\lambda))$	0	0
9.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}1&1&1\\&1&-1\\&&0\end{pmatrix}$	$b_{11}(\alpha^2x-\beta^2y)\lambda$ $+b_{22}(-a_{13}\beta\lambda+a_{23}\alpha\lambda+\alpha\beta)$	0	0
10.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}1&1&1\\&1&-1\\&&0\end{pmatrix}$	$b_{11}(a_{33}^2+\alpha^2\lambda x-\beta^2\lambda y)$ $+b_{22}(-a_{13}\beta\lambda+a_{23}\alpha\lambda+\alpha\beta)$	0	0
11.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&2\\&0&0\\&&0\end{pmatrix}$	$b_{11}(a_{33}^2+\alpha^2\lambda x-\beta^2\lambda y)$ $+b_{22}(a_{13}\beta(1-\lambda)$ $+a_{23}\alpha(1+\lambda))$	0	0
12.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}1&1&1\\&1&-1\\&&0\end{pmatrix}$	$b_{11}(a_{33}^2+\alpha^2\lambda x-\beta^2\lambda y)$ $+b_{22}(-a_{13}\beta\lambda+a_{23}\alpha\lambda+\alpha\beta)$	0	0
13.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}1&-1&0\\&1&0\\&&0\end{pmatrix}$	$b_{11}(\alpha^2x-\beta^2y)\lambda-b_{22}\alpha\beta\lambda^2$	$-(p-1)^3p^3$ $-(p-1)^3p^4\left(\frac{xy}{p}\right)$	$-\frac{(p-1)^2p^{13}}{2}$
14.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}2&0&0\\&2&0\\&&0\end{pmatrix}$	$b_{11}(\alpha^2x-\beta^2y)\lambda$ $+b_{22}\alpha\beta(1-\lambda^2)$	$(p-1)^2p^3(p+1)$ $(p-1)^2p^4(p+1)\left(\frac{xy}{p}\right)$	$\frac{(p-1)p^{13}(p+1)}{2}$
15.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}1+\ell&-1+\ell&0\\&1+\ell&0\\&&0\end{pmatrix}$	$b_{11}(\alpha^2x-\beta^2y)\lambda$ $+b_{22}\alpha\beta(\ell-\lambda^2)$	$-(p-1)^3p^3$ $-(p-1)^3p^4\left(\frac{xy}{p}\right)$	$-\frac{(p-1)^2p^{13}}{2}$

Orbit	ξ_1	$[x_0, g\cdot\xi_1]+[x_1, g\cdot\xi_0]$	$\mathscr{S}(x,\xi)$	$\mathscr{M}(x,\xi)$
16.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{33}^2+\alpha^2\lambda x-\beta^2\lambda y)$ $-b_{22}\alpha\beta\lambda^2$	$(p-1)^2p^3$ $+(p-1)^2p^4\left(\frac{xy}{p}\right)$	$\frac{(p-1)p^{13}}{2}$
17.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ & 2 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{33}^2+\alpha^2\lambda x-\beta^2\lambda y)$ $+b_{22}\alpha\beta(1-\lambda^2)$	$-(p-1)p^3(p+1)$ $-(p-1)p^4(p+1)\left(\frac{xy}{p}\right)$ $-(p-1)p^5$ $\times\left(\left(\frac{x}{p}\right)+\left(\frac{-x}{p}\right)\right.$ $\left.+\left(\frac{y}{p}\right)+\left(\frac{-y}{p}\right)\right)$	$-\frac{p^{13}(p+1)}{2}$
18.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1+\ell & -1+\ell & 0 \\ & 1+\ell & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{33}^2+\alpha^2\lambda x-\beta^2\lambda y)$ $+b_{22}\alpha\beta(\ell-\lambda^2)$	$(p-1)^2p^3$ $+(p-1)^2p^4\left(\frac{xy}{p}\right)$	$\frac{(p-1)p^{13}}{2}$
19.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{33}^2+\alpha^2\lambda x-\beta^2\lambda y)$	$-(p-1)^3p^3$ $-(p-1)^3p^4\left(\frac{xy}{p}\right)$	$\frac{-(p-1)^2p^{13}}{2}$
20.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(\alpha^2 x-\beta^2 y)\lambda$	$(p-1)^4p^3$ $+(p-1)^4p^4\left(\frac{xy}{p}\right)$	$\frac{(p-1)^3p^{13}}{2}$

The orbital exponential sums are as follows.

1.-12. These sums vanish on summing over a_{13}, a_{23}, b_{12}.

13. Here

$$[x_0, g\cdot\xi_1]+[x_1, g\cdot\xi_0] = b_{11}(\alpha^2 x-\beta^2 y)\lambda - b_{22}\alpha\beta\lambda^2. \tag{5.113}$$

The sum over b_{22} is -1 since $\alpha,\beta,\lambda \neq 0$. Hence,

$$\begin{aligned}\mathscr{S}(x,\xi) &= \sum_{\begin{pmatrix} \alpha,\beta,\lambda & * \\ & * \\ & * \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22}\end{pmatrix}} e_p(b_{11}(\alpha^2 x-\beta^2 y)\lambda - b_{22}\alpha\beta\lambda^2) \\ &= -(p-1)^2p^3\sum_{\alpha,\beta,b_{11}\in\mathbf{F}_p^\times} e_p(b_{11}(\alpha^2 x-\beta^2 y)) \\ &= -(p-1)^2p^3\sum_{b_{11}\in\mathbf{F}_p^\times}\left(\tau\left(\frac{b_{11}x}{p}\right)-1\right)\left(\tau\left(\frac{-b_{11}y}{p}\right)-1\right) \\ &= -(p-1)^3p^3-(p-1)^3p^4\left(\frac{xy}{p}\right).\end{aligned}$$

14. Here

$$[x_0, g\cdot\xi_1]+[x_1, g\cdot\xi_0] = b_{11}(\alpha^2 x-\beta^2 y)\lambda + b_{22}\alpha\beta(1-\lambda^2). \tag{5.114}$$

Thus,

$$\mathscr{S}(x,\xi) = \sum_{\begin{pmatrix} \alpha,\beta,\lambda & * \\ & * \\ & * \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22}\end{pmatrix}} e_p(b_{11}(\alpha^2 x-\beta^2 y)\lambda + b_{22}\alpha\beta(1-\lambda^2)).$$

Sum in b_{22} and split in to $\lambda = \pm 1$ and otherwise to obtain

$$\begin{aligned}\mathscr{S}(x,\xi) &= 2(p-1)^2p^3 \sum_{\alpha,\beta,b_{11}\in\mathbf{F}_p^\times} e_p(b_{11}(\alpha^2 x - \beta^2 y)) \\ &\quad - (p-3)(p-1)p^3 \sum_{\alpha,\beta,b_{11}\in\mathbf{F}_p^\times} e_p(b_{11}(\alpha^2 x - \beta^2 y)) \\ &= (p-1)p^3(p+1) \sum_{b_{11}\in\mathbf{F}_p^\times} \left(\tau\left(\frac{b_{11}x}{p}\right) - 1\right)\left(\tau\left(\frac{-b_{11}y}{p}\right) - 1\right) \\ &= (p-1)^2p^4(p+1)\left(\frac{xy}{p}\right) + (p-1)^2p^3(p+1).\end{aligned}$$

15. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}(\alpha^2 x - \beta^2 y)\lambda + b_{22}\alpha\beta(\ell - \lambda^2). \tag{5.115}$$

Since $\ell - \lambda^2 \neq 0$, the sum in b_{22} is -1. Hence, as in 13.,

$$\begin{aligned}\mathscr{S}(x,\xi) &= \sum_{\begin{pmatrix}\alpha,\beta,\lambda & * \\ & * \\ & *\end{pmatrix}\begin{pmatrix}b_{11} & * \\ & b_{22}\end{pmatrix}} e_p(b_{11}(\alpha^2 x - \beta^2 y)\lambda + b_{22}\alpha\beta(\ell - \lambda^2)) \\ &= -(p-1)^2p^3 \sum_{\alpha,\beta,b_{11}\in\mathbf{F}_p^\times} e_p(b_{11}(\alpha^2 x - \beta^2 y)) \\ &= -(p-1)^3p^3 - (p-1)^3p^4\left(\frac{xy}{p}\right).\end{aligned}$$

16. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}(a_{33}^2 + \alpha^2\lambda x - \beta^2\lambda y) - b_{22}\alpha\beta\lambda^2. \tag{5.116}$$

The sum in b_{22} is -1. Then replace $b_{11} := b_{11}\lambda$ and sum in λ to obtain

$$\begin{aligned}\mathscr{S}(x,\xi) &= \sum_{\begin{pmatrix}\alpha,\beta,\lambda & * \\ & * \\ & *\end{pmatrix}\begin{pmatrix}b_{11} & * \\ & b_{22}\end{pmatrix}} e_p(b_{11}(a_{33}^2 + \alpha^2\lambda x - \beta^2\lambda y) - b_{22}\alpha\beta\lambda^2) \\ &= (p-1)p^3 \sum_{\alpha,\beta,b_{11}\in\mathbf{F}_p^\times} e_p(b_{11}(\alpha^2 x - \beta^2 y)) \\ &= (p-1)^2p^3 + (p-1)^2p^4\left(\frac{xy}{p}\right).\end{aligned}$$

17. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}(\alpha^2\lambda x - \beta^2\lambda y + a_{33}^2) - b_{22}\alpha\beta(\lambda^2-1). \tag{5.117}$$

Sum in b_{22} which obtains -1 unless $\lambda=\pm1$. Thus

$$\begin{aligned}\mathscr{S}(x,\xi) &= \sum_{\begin{pmatrix}\alpha,\beta,\lambda & * \\ & * \\ & *\end{pmatrix}\begin{pmatrix}b_{11} & * \\ & b_{22}\end{pmatrix}} e_p(b_{11}(a_{33}^2+\alpha^2\lambda x-\beta^2\lambda y)+b_{22}\alpha\beta(1-\lambda^2)) \\ &= -\frac{1}{p-1}\sum_{\begin{pmatrix}\alpha,\beta,\lambda & * \\ & * \\ & *\end{pmatrix}\begin{pmatrix}b_{11} & * \\ & b_{22}\end{pmatrix}} e_p(b_{11}(a_{33}^2+\alpha^2\lambda x-\beta^2\lambda y)) \\ &\quad + \frac{p}{p-1}\sum_{\lambda=\pm}\sum_{\begin{pmatrix}\alpha,\beta & * \\ & * \\ & *\end{pmatrix}\begin{pmatrix}b_{11} & * \\ & b_{22}\end{pmatrix}} e_p(b_{11}(a_{33}^2+\alpha^2\lambda x-\beta^2\lambda y))\end{aligned}$$

In the first sum, substitute $b_{11} := b_{11}\lambda$, then sum in λ. In the second sum, sum in α, β and a_{33} to obtain Gauss sums. Hence,

$$\begin{aligned}\mathscr{S}(x,\xi) &= (p-1)p^3\sum_{\alpha,\beta,b_{11}\in\mathbf{F}_p^\times} e_p(b_{11}(\alpha^2 x-\beta^2 y)) \\ &\quad + p^4\sum_{\lambda=\pm}\sum_{b_{11}\in\mathbf{F}_p^\times}\left(\tau\left(\frac{b_{11}\lambda x}{p}\right)-1\right)\left(\tau\left(\frac{-b_{11}\lambda y}{p}\right)-1\right)\left(\tau\left(\frac{b_{11}}{p}\right)-1\right) \\ &= (p-1)^2p^4\left(\frac{xy}{p}\right)+(p-1)^2p^3-2(p-1)p^4-2(p-1)p^5\left(\frac{xy}{p}\right) \\ &\quad -(p-1)p^5\left(\left(\frac{x}{p}\right)+\left(\frac{-x}{p}\right)+\left(\frac{y}{p}\right)+\left(\frac{-y}{p}\right)\right) \\ &= -(p-1)p^4(p+1)\left(\frac{xy}{p}\right)-(p-1)p^3(p+1) \\ &\quad -(p-1)p^5\left(\left(\frac{x}{p}\right)+\left(\frac{-x}{p}\right)+\left(\frac{y}{p}\right)+\left(\frac{-y}{p}\right)\right).\end{aligned}$$

18. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}(a_{33}^2+\alpha^2\lambda x-\beta^2\lambda y)+b_{22}\alpha\beta(\ell-\lambda^2). \tag{5.118}$$

The sum in b_{22} is -1. Then replace $b_{11} := b_{11}\lambda$ and sum in λ. This obtains

$$\begin{aligned}\mathscr{S}(x,\xi) &= \sum_{\begin{pmatrix}\alpha,\beta,\lambda & * \\ & * \\ & *\end{pmatrix}\begin{pmatrix}b_{11} & * \\ & b_{22}\end{pmatrix}} e_p(b_{11}(a_{33}^2+\alpha^2\lambda x-\beta^2\lambda y)+b_{22}\alpha\beta(\ell-\lambda^2)) \\ &= (p-1)p^3\sum_{\alpha,\beta,b_{11}\in\mathbf{F}_p^\times} e_p(b_{11}(\alpha^2x-\beta^2y)) \\ &= (p-1)^2p^3+(p-1)^2p^4\left(\frac{xy}{p}\right).\end{aligned}$$

19. Here

$$[x_0, g\cdot\xi_1]+[x_1, g\cdot\xi_0] = b_{11}(a_{33}^2+\alpha^2\lambda x-\beta^2\lambda y). \tag{5.119}$$

This is the same as 18 without the sum over b_{22}. Hence,

$$\begin{aligned}\mathscr{S}(x,\xi) &= \sum_{\begin{pmatrix}\alpha,\beta,\lambda & * \\ & * \\ & *\end{pmatrix}\begin{pmatrix}b_{11} & * \\ & b_{22}\end{pmatrix}} e_p(b_{11}(a_{33}^2+\alpha^2\lambda x-\beta^2\lambda y)) \\ &= -(p-1)^2p^3\sum_{\alpha,\beta,b_{11}\in\mathbf{F}_p^\times} e_p(b_{11}(\alpha^2x-\beta^2y)) \\ &= -(p-1)^3p^3-(p-1)^3p^4\left(\frac{xy}{p}\right).\end{aligned}$$

20. Here

$$[x_0, g\cdot\xi_1]+[x_1, g\cdot\xi_0] = b_{11}(\alpha^2x-\beta^2y)\lambda. \tag{5.120}$$

Replace $b_{11} := b_{11}\lambda$. Hence,

$$\begin{aligned}\mathscr{S}(x,\xi) &= \sum_{\begin{pmatrix}\alpha,\beta,\lambda & * \\ & * \\ & *\end{pmatrix}\begin{pmatrix}b_{11} & * \\ & b_{22}\end{pmatrix}} e_p(b_{11}(\alpha^2x-\beta^2y)\lambda) \\ &= (p-1)^4p^3+(p-1)^4p^4\left(\frac{xy}{p}\right).\end{aligned}$$

5.7. The exponential sums pair $(\mathscr{O}_{1^21^2}, \mathscr{O}_{D2})$

Standard representatives are

$$x_0 = \left(\begin{pmatrix}0 & 0 & 0 \\ & 0 & 0 \\ & & 1\end{pmatrix}\begin{pmatrix}0 & \frac{1}{2} & 0 \\ & 0 & 0 \\ & & 0\end{pmatrix}\right), \qquad \xi_0 = \left(\begin{pmatrix}-\ell & 0 & 0 \\ & 1 & 0 \\ & & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0 \\ & 0 & 0 \\ & & 0\end{pmatrix}\right), \tag{5.121}$$

and

$$x_1 = \begin{pmatrix} x & 0 & 0 \\ & y & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}. \tag{5.122}$$

The acting set is

$$G^t_{x,\xi} = \left\{ \begin{pmatrix} a & c\ell & a_{13} \\ c\lambda & a\lambda & a_{23} \\ & & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix} \right\}. \tag{5.123}$$

The range of summation is $(a, c) \in \mathbf{F}_p^2 \setminus \{(0, 0)\}$, $\lambda, a_{33}, b_{11}, b_{22} \in \mathbf{F}_p^\times$, and $a_{13}, a_{23}, b_{12} \in \mathbf{F}_p$.

The GL_3 action maps

$$\begin{aligned} -\ell u^2 + v^2 \mapsto & -\ell(au + c\lambda v)^2 + (c\ell u + a\lambda v)^2 \\ & = (-a^2\ell + c^2\ell^2)u^2 + (a^2\lambda^2 - c^2\lambda^2\ell)v^2 \bmod uv. \end{aligned}$$

Thus

$$[x_1, g \cdot \xi_0] = b_{11}(a^2 - c^2\ell)(-\ell x + \lambda^2 y). \tag{5.124}$$

The stabilizer has size

$$\left|\mathrm{Stab}_{G(\mathbb{Z}/p^2\mathbb{Z})}(x)\right| = \begin{cases} 8(p-1)p^3 & x = y \\ 4(p-1)p^3 & x \neq y \end{cases}. \tag{5.125}$$

Representatives and exponential sum pairings are given in the table below. Summing over the three maximal orbits

$$\mathscr{M}(x, \xi) = \sum_{(x,y) \in \{(1,1),(1,\ell),(\ell,\ell)\}} \frac{2^{\delta(x \neq y)} p^{10} \mathscr{S}(x, \xi)}{8(p-1)}. \tag{5.126}$$

See **exponential_sums_O1212_OD2.nb**.

Orbit	ξ_1	$[x_0, g\cdot\xi_1]+[x_1, g\cdot\xi_0]$	$\mathscr{S}(x,\xi)$	$\mathscr{M}(x,\xi)$
1.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}$	$-b_{11}(a^2-c^2\ell)(\ell x-\lambda^2 y)$ $+b_{12}a_{33}^2+b_{22}a_{13}a_{23}$	0	0
2.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&1\\&&1\end{pmatrix}$	$-b_{11}(a^2-c^2\ell)(\ell x-\lambda^2 y)$ $+b_{12}a_{33}^2$ $+b_{22}(a_{13}a_{23}+a_{13}a\lambda+a_{23}c\ell)$	0	0
3.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&1&0\\&0&0\\&&1\end{pmatrix}$	$-b_{11}(a^2-c^2\ell)(\ell x-\lambda^2 y)$ $+b_{12}a_{33}^2$ $+b_{22}(a_{13}a_{23}+a^2\lambda+c^2\lambda\ell)$	0	0
4.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}$	$-b_{11}(a^2-c^2\ell)(\ell x-\lambda^2 y)$ $+b_{22}(a_{13}a\lambda+a_{23}c\ell)$	0	0
5.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&0&1\\&&0\end{pmatrix}$	$-b_{11}(a^2-c^2\ell)(\ell x-\lambda^2 y)$ $+b_{11}a_{33}^2+b_{22}(a_{13}a\lambda+a_{23}c\ell)$	0	0
6.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&1&0\\&&0\end{pmatrix}$	$-b_{11}(a^2-c^2\ell)(\ell x-\lambda^2 y)$ $+b_{22}ac\lambda\ell$	$(p-1)^3p^3$ $-(p-1)^3p^4\left(\frac{xy}{p}\right)$	$\frac{(p-1)^2p^{13}}{2}$
7.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&1&0\\&&0\end{pmatrix}$	$-b_{11}(a^2-c^2\ell)(\ell x-\lambda^2 y)$ $+b_{11}a_{33}^2+b_{22}ac\lambda\ell$	$-(p-1)^2p^3$ $+(p-1)^2p^4\left(\frac{xy}{p}\right)$ $+(p-1)^2p^5\left(\left(\frac{x}{p}\right)-\left(\frac{-x}{p}\right)\right.$ $\left.+\left(\frac{y}{p}\right)-\left(\frac{-y}{p}\right)\right)$	$-\frac{(p-1)p^{13}}{2}$
8a.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&1&0\\&0&0\\&&0\end{pmatrix}$	$-b_{11}(a^2-c^2\ell)(\ell x-\lambda^2 y)$ $+b_{22}(a^2\lambda+c^2\lambda\ell)$	$-(p-1)^2p^3(p+1)$ $+(p-1)^2p^4(p+1)\left(\frac{xy}{p}\right)$	$-\frac{(p-1)p^{13}(p+1)}{2}$
8b.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&k&0\\&1&0\\&&0\end{pmatrix}$	$-b_{11}(a^2-c^2\ell)(\ell x-\lambda^2 y)$ $+b_{22}(ac\lambda\ell+a^2\lambda k+c^2\lambda\ell k)$	$-(p-1)^2p^3(p+1)$ $+(p-1)^2p^4(p+1)\left(\frac{xy}{p}\right)$	$-\frac{(p-1)p^{13}(p+1)}{2}$
9a.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&1&0\\&0&0\\&&0\end{pmatrix}$	$-b_{11}(a^2-c^2\ell)(\ell x-\lambda^2 y)$ $+b_{11}a_{33}^2+b_{22}(a^2\lambda+c^2\lambda\ell)$	$(p-1)p^3(p+1)$ $-(p-1)p^4(p+1)\left(\frac{xy}{p}\right)$	$\frac{p^{13}(p+1)}{2}$
9b.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&k&0\\&1&0\\&&0\end{pmatrix}$	$-b_{11}(a^2-c^2\ell)(\ell x-\lambda^2 y)$ $+b_{11}a_{33}^2$ $+b_{22}(ac\lambda\ell+a^2\lambda k+c^2\lambda\ell k)$	$(p-1)p^3(p+1)$ $-(p-1)p^4(p+1)\left(\frac{xy}{p}\right)$	$\frac{p^{13}(p+1)}{2}$
10.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$-b_{11}(a^2-c^2\ell)(\ell x-\lambda^2 y)$ $+b_{11}a_{33}^2$	$-(p-1)^2p^3(p+1)$ $+(p-1)^2p^4(p+1)\left(\frac{xy}{p}\right)$	$-\frac{(p-1)p^{13}(p+1)}{2}$
11.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$-b_{11}(a^2-c^2\ell)(\ell x-\lambda^2 y)$	$(p-1)^3p^3(p+1)$ $-(p-1)^3p^4(p+1)\left(\frac{xy}{p}\right)$	$\frac{(p-1)^2p^{13}(p+1)}{2}$

1.-5. These sums vanish on summing in b_{12}, a_{13} and a_{23}.

6. Here

$$[x_0, g\cdot\xi_1]+[x_1, g\cdot\xi_0] = -b_{11}(a^2-c^2\ell)(\ell x-\lambda^2 y)+b_{22}ac\lambda\ell. \tag{5.127}$$

Note that $a^2-c^2\ell\neq 0$. Thus summation over b_{11} obtains -1 if $\ell x-\lambda^2 y\neq 0$ and $p-1$ otherwise. Thus,

$$\begin{aligned}\mathscr{S}(x,\xi) &= \sum_{\begin{pmatrix}a & c\ell & *\\ c\lambda & a\lambda & *\\ & & *\end{pmatrix}\begin{pmatrix}b_{11} & *\\ & b_{22}\end{pmatrix}} e_p(b_{11}(a^2-c^2\ell)(-\ell x+\lambda^2 y)+b_{22}ac\lambda\ell)\\ &= -(p-1)p^3\sum_{(a,c)\neq(0,0),\lambda,b_{22}\in\mathbf{F}_p^\times} e_p(b_{22}ac\lambda\ell)\end{aligned}$$

$$+ (p-1)p^4 \sum_{\substack{\lambda^2 \equiv \ell x y^{-1} \\ (a,c)\neq(0,0), b_{22}\in \mathbf{F}_p^\times}} e_p(b_{22}ac\lambda\ell).$$

Since the number of solutions to $\ell x - \lambda^2 y = 0$ is $1 - \left(\frac{xy}{p}\right)$, and since the sum over a is 0 unless $c = 0$, in which case it is $p-1$, it follows that

$$\begin{aligned}\mathscr{S}(x,\xi) &= -(p-1)^4p^3 + (p-1)^3p^4\left(1 - \left(\frac{xy}{p}\right)\right) \\ &= (p-1)^3p^3 - (p-1)^3p^4\left(\frac{xy}{p}\right).\end{aligned}$$

7. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = -b_{11}(a^2 - c^2\ell)(\ell x - \lambda^2 y) + b_{11}a_{33}^2 + b_{22}ac\lambda\ell. \tag{5.128}$$

Thus

$$\mathscr{S}(x,\xi) = \sum_{\begin{pmatrix} a & c\ell & * \\ c\lambda & a\lambda & * \\ & & * \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22}\end{pmatrix}} e_p(b_{11}((a^2 - c^2\ell)(-\ell x + \lambda^2 y) + a_{33}^2) + b_{22}ac\lambda\ell)$$

Sum in b_{22} to obtain

$$\begin{aligned}\mathscr{S}(x,\xi) = &-\frac{1}{p-1}\sum_{\begin{pmatrix} a & c\ell & * \\ c\lambda & a\lambda & * \\ & & * \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22}\end{pmatrix}} e_p(b_{11}((a^2 - c^2\ell)(-\ell x + \lambda^2 y) + a_{33}^2)) \\ &+ \frac{p}{p-1}\sum_{\begin{pmatrix} a & c\ell & * \\ c\lambda & a\lambda & * \\ & & * \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22}\end{pmatrix}, ac=0} e_p(b_{11}((a^2 - c^2\ell)(-\ell x + \lambda^2 y) + a_{33}^2)) \\ = &-\frac{1}{p-1}\Sigma_1 + \frac{p}{p-1}\Sigma_2.\end{aligned}$$

Here, setting apart the case $-\ell x + \lambda^2 y = 0$, and adding and subtracting the $(a,c) = (0,0)$ term,

$$\Sigma_1 = (p-1)^2p^3(p+1)\sum_{\substack{\lambda^2\equiv \ell x y^{-1} \\ b_{11}, a_{33}\in \mathbf{F}_p^\times}} e_p(b_{11}a_{33}^2)$$

$$+ (p-1)p^3 \sum_{\substack{\lambda^2 \not\equiv \ell x y^{-1} \\ b_{11}, a_{33} \in \mathbf{F}_p^\times \\ a, c \in \mathbf{F}_p}} e_p(b_{11}((a^2 - c^2\ell)(-\ell x + \lambda^2 y) + a_{33}^2))$$

$$- (p-1)p^3 \sum_{\substack{\lambda^2 \not\equiv \ell x y^{-1} \\ b_{11}, a_{33} \in \mathbf{F}_p^\times}} e_p(b_{11} a_{33}^2)$$

In the first and third sums, summation over b_{11} gives -1, while in the second sum, summation over a, c gives $-p$ via the Gauss sums, followed by summation over b_{11} which gives -1 again once the term involving a and c has been removed. This obtains

$$\begin{aligned} &= -(p-1)^3 p^3 (p+1) \left(1 - \left(\frac{xy}{p}\right)\right) + (p-1)^2 p^4 \left(p - 2 + \left(\frac{xy}{p}\right)\right) \\ &\quad + (p-1)^2 p^3 \left(p - 2 + \left(\frac{xy}{p}\right)\right) \\ &= -(p-1)^2 p^3 (p+1) + (p-1)^2 p^4 (p+1) \left(\frac{xy}{p}\right). \end{aligned}$$

To evaluate Σ_2, set apart the $c = 0$ and $a = 0$ terms to obtain

$$\begin{aligned} \Sigma_2 = (p-1)p^3 &\sum_{\lambda, a, b_{11}, a_{33} \in \mathbf{F}_p^\times} e_p(b_{11}(-a^2 \ell x + a^2 \lambda^2 y + a_{33}^2)) \\ &+ (p-1)p^3 \sum_{\lambda, c, b_{11}, a_{33} \in \mathbf{F}_p^\times} e_p(b_{11}(c^2 \ell^2 x - c^2 \lambda^2 \ell y + a_{33}^2)). \end{aligned}$$

In these sums, replace $\lambda := \lambda a$ and $\lambda := \lambda c$, then sum the squared variables to obtain Gauss sums. Thus,

$$\begin{aligned} \Sigma_2 &= (p-1)p^3 \sum_{b_{11} \in \mathbf{F}_p^\times} \left(\tau\left(\frac{-b_{11}\ell x}{p}\right) - 1\right) \left(\tau\left(\frac{b_{11} y}{p}\right) - 1\right) \left(\tau\left(\frac{b_{11}}{p}\right) - 1\right) \\ &\quad + (p-1)p^3 \sum_{b_{11} \in \mathbf{F}_p^\times} \left(\tau\left(\frac{-b_{11}\ell y}{p}\right) - 1\right) \left(\tau\left(\frac{b_{11} x}{p}\right) - 1\right) \left(\tau\left(\frac{b_{11}}{p}\right) - 1\right) \\ &= -2(p-1)^2 p^3 + (p-1)^2 p^4 \left\{2\left(\frac{xy}{p}\right) + \left(\frac{x}{p}\right) - \left(\frac{-x}{p}\right) + \left(\frac{y}{p}\right) - \left(\frac{-y}{p}\right)\right\} \end{aligned}$$

Hence

$$\begin{aligned} \mathscr{S}(x, \xi) = -(p-1)^2 p^3 &+ (p-1)^2 p^4 \left(\frac{xy}{p}\right) \\ &+ (p-1)p^5 \left\{\left(\frac{x}{p}\right) - \left(\frac{-x}{p}\right) + \left(\frac{y}{p}\right) - \left(\frac{-y}{p}\right)\right\}. \end{aligned}$$

8a. $p \equiv 1 \bmod 4$. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = -b_{11}(a^2 - c^2\ell)(\ell x - \lambda^2 y) + b_{22}(a^2\lambda + c^2\lambda\ell). \qquad (5.129)$$

Here $a^2\lambda + c^2\lambda\ell \neq 0$ so the sum in b_{22} is -1. Since the sum in b_{11} is -1 if $\ell x - \lambda^2 y \neq 0$ and $p-1$ otherwise, it follows that

$$\begin{aligned}
\mathscr{S}(x,\xi) &= \sum_{\begin{pmatrix} a & c\ell & * \\ c\lambda & a\lambda & * \\ & & * \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22} \end{pmatrix}} e_p(b_{11}(a^2 - c^2\ell)(-\ell x + \lambda^2 y) + b_{22}(a^2\lambda + c^2\lambda\ell)) \\
&= -(p-1)p^3 \sum_{(a,c)\neq(0,0),\lambda,b_{22}\in\mathbf{F}_p^\times} e_p(b_{22}(a^2\lambda + c^2\lambda\ell)) \\
&\quad + (p-1)p^4 \sum_{\substack{\lambda^2 \equiv \ell x y^{-1} \\ (a,c)\neq(0,0),b_{22}\in\mathbf{F}_p^\times}} e_p(b_{22}(a^2\lambda + c^2\lambda\ell)) \\
&= (p-1)^3p^3(p+1) - (p-1)^2p^4(p+1)\left(1 - \left(\frac{xy}{p}\right)\right) \\
&= -(p-1)^2p^3(p+1) + (p-1)^2p^4(p+1)\left(\frac{xy}{p}\right).
\end{aligned}$$

8b. $p \equiv 3 \bmod 4$, $\ell - 4k^2 = \square$. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = -b_{11}(a^2 - c^2\ell)(\ell x - \lambda^2 y) + b_{22}((a^2\lambda + c^2\lambda\ell)k + ac\lambda\ell). \quad (5.130)$$

Note that as a quadratic form in a, c, the discriminant is $\lambda^2\ell(\ell - 4k^2) \neq \square$ so that this form is irreducible. Thus the sum in b_{22} is -1 as in 8a, which reduces to that case,

$$\mathscr{S}(x,\xi) = -(p-1)^2p^3(p+1) + (p-1)^2p^4(p+1)\left(\frac{xy}{p}\right).$$

9a. $p \equiv 1 \bmod 4$. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = -b_{11}(a^2 - c^2\ell)(\ell x - \lambda^2 y) + b_{11}a_{33}^2 + b_{22}(a^2\lambda + c^2\lambda\ell). \quad (5.131)$$

Since $a^2\lambda + c^2\lambda\ell \neq 0$, summing in b_{22} obtains -1, so

$$\begin{aligned}
\mathscr{S}(x,\xi) &= -\frac{1}{p-1} \sum_{\begin{pmatrix} a & c\ell & * \\ c\lambda & a\lambda & * \\ & & * \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22} \end{pmatrix}} e_p(b_{11}((a^2 - c^2\ell)(-\ell x + \lambda^2 y) + a_{33}^2)) \\
&= (p-1)p^3(p+1) - (p-1)p^4(p+1)\left(\frac{xy}{p}\right),
\end{aligned}$$

see Σ_1 of 7.

9b. Here

$$[x_0, g\cdot\xi_1]+[x_1, g\cdot\xi_0]=-b_{11}(a^2-c^2\ell)(\ell x-\lambda^2 y)+b_{11}a_{33}^2+b_{22}(ac\lambda\ell+a^2\lambda k+c^2\lambda\ell k). \tag{5.132}$$

This is equal to 9a., see the calculation in 8b.

10. Here

$$[x_0, g\cdot\xi_1]+[x_1, g\cdot\xi_0]=-b_{11}(a^2-c^2\ell)(\ell x-\lambda^2 y)+b_{11}a_{33}^2. \tag{5.133}$$

This is Σ_1 of 7., so

$$\begin{aligned}\mathscr{S}(x,\xi) &= \sum_{\begin{pmatrix} a & c\ell & * \\ c\lambda & a\lambda & * \\ & & * \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22}\end{pmatrix}} e_p(b_{11}((a^2-c^2\ell)(-\ell x+\lambda^2 y)+a_{33}^2)) \\ &= -(p-1)^2p^3(p+1)+(p-1)^2p^4(p+1)\left(\frac{xy}{p}\right).\end{aligned}$$

11. Here

$$[x_0, g\cdot\xi_1]+[x_1, g\cdot\xi_0]=-b_{11}(a^2-c^2\ell)(\ell x-\lambda^2 y). \tag{5.134}$$

Thus, splitting on $\ell x-\lambda^2 y=0$ or not,

$$\begin{aligned}\mathscr{S}(x,\xi) &= \sum_{\begin{pmatrix} a & c\ell & * \\ c\lambda & a\lambda & * \\ & & * \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22}\end{pmatrix}} e_p(b_{11}(a^2-c^2\ell)(-\ell x+\lambda^2 y)) \\ &= -(p-1)^4p^3(p+1)+(p-1)^3p^4(p+1)\left(1-\left(\frac{xy}{p}\right)\right) \\ &= (p-1)^3p^3(p+1)-(p-1)^3p^4(p+1)\left(\frac{xy}{p}\right).\end{aligned}$$

5.8. *The exponential sums pair* $(\mathscr{O}_{1^31}, \mathscr{O}_{D1^2})$

Standard representatives are

$$x_0=\begin{pmatrix} 0 & 0 & 0 \\ & 0 & \frac{1}{2} \\ & & 0\end{pmatrix}\begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 1 & 0 \\ & & 0\end{pmatrix}, \qquad \xi_0=\begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0\end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0\end{pmatrix}, \tag{5.135}$$

and

$$x_1 = \begin{pmatrix} \epsilon & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \qquad \epsilon \in \mathbf{F}_p^\times / \{x^3 : x \in \mathbf{F}_p^\times\}. \tag{5.136}$$

The acting set is

$$G_{x,\xi}^t = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & a & b \\ & c & d \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix}. \tag{5.137}$$

Thus

$$[x_1, g \cdot \xi_0] = b_{11} a_{11}^2 \epsilon. \tag{5.138}$$

GL_2 is fibered as for the pairs $(\mathscr{O}_{1^211}, \mathscr{O}_{D1^2})$, $(\mathscr{O}_{1^22}, \mathscr{O}_{D1^2})$ and $(\mathscr{O}_{1^21^2}, \mathscr{O}_{D1^2})$.

The stabilizer has size

$$\left|\mathrm{Stab}_{G(\mathbb{Z}/p^2\mathbb{Z})}(x)\right| = (p-1)p^3 \#\{\epsilon^3 = 1\}. \tag{5.139}$$

Representatives and exponential sum pairings are given in the table below. Summing over the maximal orbits,

$$\mathscr{M}(x,\xi) = \sum_{\epsilon \in \mathbf{F}_p^\times / \{x^3 : x \in \mathbf{F}_p^\times\}} \frac{p^{10} \mathscr{S}(x,\xi)}{(p-1)\#\{\epsilon^3 = 1\}}. \tag{5.140}$$

See **exponential_sums_O131_OD12.nb**.

Orbit	ξ_1	$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0]$	$\mathscr{S}(x,\xi)$	$\mathscr{M}(x,\xi)$
1.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}a_{11}^2\epsilon$	$-(p-1)^4p^4(p+1)$	$-(p-1)^3p^{14}(p+1)$
2.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + bd)$	$-(p-1)^4p^4$	$-(p-1)^3p^{14}$
3.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & \frac{1}{2} \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}\left(a_{11}^2\epsilon + \frac{1}{2}(ad+bc)\right)$	$2(p-1)^3p^4$	$2(p-1)^2p^{14}$
4.	$\begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon - ac\ell + bd)$	0	0
5.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + bd) + b_{22}a_{11}d$	$-(p-1)^4p^4$	$-(p-1)^3p^{14}$
6.	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + ac) + b_{22}a_{11}d$	$(p-1)^3p^4$	$(p-1)^2p^{14}$
7.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & \frac{1}{2} \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}\left(a_{11}^2\epsilon + \frac{1}{2}(ad+bc)\right) + b_{22}a_{11}d$	$(p-1)^2p^4(p-2)$	$(p-1)p^{14}(p-2)$
8.	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 1 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + ac + ad + bc)$ $+b_{22}a_{11}d$	$-2(p-1)^2p^4$	$-2(p-1)p^{14}$
9.	$\begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon - ac\ell + bd) + b_{22}a_{11}d$	0	0
10.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}a_{11}^2\epsilon + b_{22}a_{11}d$	$(p-1)^3p^4$	$(p-1)^2p^{14}$
11.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$	$b_{11}a_{11}^2\epsilon + b_{12}bd$ $+b_{22}(a_{13}d + b^2)$	$(p-1)^3p^4$	$(p-1)^2p^{14}$

Orbit	ξ_1	$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0]$	$\mathscr{S}(x,\xi)$	$\mathscr{M}(x,\xi)$
12.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + ad + bc) + b_{12}bd$ $+b_{22}(a_{13}d + b^2)$	$-(p-1)^2p^4$	$-(p-1)p^{14}$
13.	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + ac) + b_{12}bd$ $+b_{22}(a_{13}d + b^2)$	0	0
14.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$	$b_{11}a_{11}^2\epsilon + b_{12}bd$ $+b_{22}(a_{11}c + a_{13}d + b^2)$	$-(p-1)^2p^4$	$-(p-1)p^{14}$
15.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + ad + bc) + b_{12}bd$ $+b_{22}(a_{11}c + a_{13}d + b^2)$	$-(p-1)^2p^4(p+1)$ $+(p-1)p^6$ $\times\#\{x \in \mathbf{F}_p^\times : x^3 \equiv \epsilon\}$	p^{14}
16.	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + ac) + b_{12}bd$ $+b_{22}(a_{11}c + a_{13}d + b^2)$	0	0
17.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$	$b_{11}a_{11}^2\epsilon + b_{12}(ad + bc)$ $+b_{22}(a_{12}d + a_{13}c + 2ab)$	0	0
18.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + bd) + b_{12}(ad + bc)$ $+b_{22}(a_{12}d + a_{13}c + 2ab)$	0	0
19.	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + ac + bd) + b_{12}(ad + bc)$ $+b_{22}(a_{12}d + a_{13}c + 2ab)$	0	0
20.	$\begin{pmatrix} 0 & 0 & 0 \\ & \ell & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + ac\ell + bd) + b_{12}(ad + bc)$ $+b_{22}(a_{12}d + a_{13}c + 2ab)$	0	0
21.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}$	$b_{11}a_{11}^2\epsilon + b_{12}(-ac\ell + bd)$ $+b_{22}(-a_{12}c\ell + a_{13}d - a^2\ell + b^2)$	0	0
22.	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + ac) + b_{12}(-ac\ell + bd)$ $+b_{22}(-a_{12}c\ell + a_{13}d - a^2\ell + b^2)$	0	0
23a.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + ad + bc) + b_{12}(-ac\ell + bd)$ $+b_{22}(-a_{12}c\ell + a_{13}d - a^2\ell + b^2)$	0	0
23b.	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & k \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + ac + adk + bck)$ $+b_{12}(-ac\ell + bd)$ $+b_{22}(-a_{12}c\ell + a_{13}d - a^2\ell + b^2)$	0	0

The exponential sums are as follows.

1. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}a_{11}^2\epsilon. \tag{5.141}$$

Thus

$$\sum_{\begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -\frac{|G_{x,\xi}|}{p-1} = -(p-1)^4p^4(p+1). \tag{5.142}$$

2. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon + bd). \tag{5.143}$$

$$\Sigma_1 = \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + bd)) = -(p-1)^2p^6$$

$$\Sigma_2 = \sum_{\begin{pmatrix} * & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + bd)) = -(p-1)^3p^4$$

$$\Sigma_3 = \sum_{\begin{pmatrix} * & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -(p-1)^2p^5$$

$$\Sigma_4 = \sum_{\begin{pmatrix} * & * & * \\ & 0 & b \\ & 0 & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + bd)) = -(p-1)^2p^4$$

$$\Sigma_5 = p\sum_{\begin{pmatrix} * & * & * \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -(p-1)^2p^4.$$

Thus

$$\mathscr{S}(x,\xi) = \Sigma_1 - \Sigma_2 - \Sigma_3 - \Sigma_4 + \Sigma_5 = -(p-1)^4p^4. \tag{5.144}$$

3. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}\left(a_{11}^2\epsilon + \frac{1}{2}(ad+bc)\right). \tag{5.145}$$

$$\begin{aligned}\Sigma_1 &= \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p\left(b_{11}\left(a_{11}^2\epsilon + \frac{1}{2}(ad+bc)\right)\right) \\ &= -(p-1)^2p^5 \\ \Sigma_2 &= \sum_{\begin{pmatrix} * & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + bd\lambda)) \\ &= -(p-1)^3p^4\end{aligned}$$

$$\Sigma_3 = \sum_{\begin{pmatrix} * & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -(p-1)^2p^5$$

$$\Sigma_4 = \sum_{\begin{pmatrix} * & * & * \\ & 0 & b \\ & 0 & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -(p-1)^2p^5$$

$$\Sigma_5 = p \sum_{\begin{pmatrix} * & * & * \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -(p-1)^2p^4.$$

Thus

$$\mathscr{S}(x,\xi) = 2(p-1)^3p^4. \tag{5.146}$$

4. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon - ac\ell + bd). \tag{5.147}$$

$$\Sigma_1 = \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon - ac\ell + bd)) = -(p-1)^2p^5$$

$$\Sigma_2 = \sum_{\begin{pmatrix} * & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + bd(-\lambda^2\ell + 1))) = -(p-1)^3p^4$$

$$\Sigma_3 = \sum_{\begin{pmatrix} * & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon - ac\ell)) = -(p-1)^2p^4$$

$$\Sigma_4 = \sum_{\begin{pmatrix} * & * & * \\ & 0 & b \\ & 0 & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + bd)) = -(p-1)^2p^4$$

$$\Sigma_5 = p \sum_{\begin{pmatrix} * & * & * \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -(p-1)^2p^4.$$

Thus

$$\mathscr{S}(x,\xi) = 0.$$

5. Here

$$[x_0, g \cdot \xi_1] + [x_1, g \cdot \xi_0] = b_{11}(a_{11}^2\epsilon + bd) + b_{22}a_{11}d. \tag{5.148}$$

In Σ_1, Σ_2 and Σ_4, summation in b vanishes unless $d = 0$. Thus,

$$\Sigma_1 = \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + bd) + b_{22}a_{11}d) = -(p-1)^2p^6$$

$$\Sigma_2 = \sum_{\begin{pmatrix} * & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22} \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + bd) + b_{22}a_{11}d) = -(p-1)^3p^4$$

$$\Sigma_3 = \sum_{\begin{pmatrix} * & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -(p-1)^2p^5$$

$$\Sigma_4 = \sum_{\begin{pmatrix} * & * & * \\ & 0 & b \\ & 0 & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22} \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + bd) + b_{22}a_{11}d) = -(p-1)^2p^4$$

$$\Sigma_5 = p \sum_{\begin{pmatrix} * & * & * \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -(p-1)^2p^4.$$

Thus

$$\mathscr{S}(x,\xi) = -(p-1)^4p^4. \tag{5.149}$$

6. Here

$$[x_0, g \cdot \xi_1] + [x_1, g \cdot \xi_0] = b_{11}(a_{11}^2\epsilon + ac) + b_{22}a_{11}d. \tag{5.150}$$

$$\Sigma_1 = \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22} \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + ac) + b_{22}a_{11}d) = 0$$

$$\Sigma_2 = \sum_{\begin{pmatrix} * & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22} \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + bd\lambda^2) + b_{22}a_{11}d) = -(p-1)^3p^4$$

$$\Sigma_3 = \sum_{\begin{pmatrix} * & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + ac)) = -(p-1)^2p^4$$

$$\Sigma_4 = \sum_{\begin{pmatrix} * & * & * \\ & 0 & b \\ & 0 & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22} \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon + b_{22}a_{11}d) = 0$$

$$\Sigma_5 = p \sum_{\begin{pmatrix} * & * & * \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -(p-1)^2p^4.$$

Thus

$$\mathscr{S}(x,\xi) = (p-1)^3p^4. \tag{5.151}$$

7. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}\left(a_{11}^2\epsilon + \frac{1}{2}(ad+bc)\right) + b_{22}a_{11}d. \tag{5.152}$$

Here

$$\Sigma_1 = \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22} \end{pmatrix}} e_p\left(b_{11}\left(a_{11}^2\epsilon + \frac{1}{2}(ad+bc)\right) + b_{22}a_{11}d\right) = -(p-1)^2p^5$$

$$\Sigma_2 = \sum_{\begin{pmatrix} * & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22} \end{pmatrix}} e_p(b_{11}\left(a_{11}^2\epsilon + bd\lambda\right) + b_{22}a_{11}d) = -(p-1)^3p^4$$

$$\Sigma_3 = \sum_{\begin{pmatrix} * & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -(p-1)^2p^5$$

$$\Sigma_4 = \sum_{\begin{pmatrix} * & * & * \\ & 0 & b \\ & 0 & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22} \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon + b_{22}a_{11}d) = 0$$

$$\Sigma_5 = p \sum_{\begin{pmatrix} * & * & * \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -(p-1)^2p^4.$$

Thus

$$\mathscr{S}(x,\xi) = (p-1)^2p^4(p-2). \tag{5.153}$$

8. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon + ac + ad + bc) + b_{22}a_{11}d. \tag{5.154}$$

Thus (in Σ_2, summation in d vanishes if $\lambda = -2$)

$$\Sigma_1 = \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & b \\ & c & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22} \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + ac + ad + bc) + b_{22}a_{11}d)$$
$$= -(p-1)^2p^5$$

$$\Sigma_2 = \sum_{\begin{pmatrix} * & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22} \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + bd(\lambda^2+2\lambda)) + b_{22}a_{11}d)$$
$$= -(p-2)p^4(p-1)^2$$

$$\Sigma_3 = \sum_{\begin{pmatrix} * & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + ac)) = -(p-1)^2p^4$$

$$\Sigma_4 = \sum_{\begin{pmatrix} * & * & * \\ & 0 & b \\ & 0 & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22} \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon + b_{22}a_{11}d) = 0$$

$$\Sigma_5 = p \sum_{\begin{pmatrix} * & * & * \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -(p-1)^2p^4.$$

Thus

$$\mathscr{S}(x,\xi) = -2(p-1)^2p^4.$$

9. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon - ac\ell + bd) + b_{22}a_{11}d. \tag{5.155}$$

Thus

$$\Sigma_1 = \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & b \\ & c & d\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22}\end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon - ac\ell + bd) + b_{22}a_{11}d) = -(p-1)^2p^5$$

$$\Sigma_2 = \sum_{\begin{pmatrix} * & * & * \\ & b\lambda & b \\ & d\lambda & d\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22}\end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + bd(-\lambda^2\ell + 1)) + b_{22}a_{11}d) = -(p-1)^3p^4$$

$$\Sigma_3 = \sum_{\begin{pmatrix} * & * & * \\ & a & 0 \\ & c & 0\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & *\end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon - ac\ell)) = -(p-1)^2p^4$$

$$\Sigma_4 = \sum_{\begin{pmatrix} * & * & * \\ & 0 & b \\ & 0 & d\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22}\end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + bd) + b_{22}a_{11}d) = -(p-1)^2p^4$$

$$\Sigma_5 = p\sum_{\begin{pmatrix} * & * & * \\ & 0 & 0 \\ & 0 & 0\end{pmatrix}\begin{pmatrix} * & * \\ & *\end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -(p-1)^2p^4.$$

Thus $\mathscr{S}(x,\xi) = 0$.

10. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}a_{11}^2\epsilon + b_{22}a_{11}d. \tag{5.156}$$

Thus

$$\Sigma_1 = \sum_{\begin{pmatrix} a_{11} & * & * \\ & a & b \\ & c & d\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22}\end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon + b_{22}a_{11}d) = 0$$

$$\Sigma_2 = \sum_{\begin{pmatrix} * & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22} \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon + b_{22}a_{11}d) = 0$$

$$\Sigma_3 = \sum_{\begin{pmatrix} * & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -(p-1)^2p^5$$

$$\Sigma_4 = \sum_{\begin{pmatrix} * & * & * \\ & 0 & b \\ & 0 & d \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & b_{22} \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon + b_{22}a_{11}d) = 0$$

$$\Sigma_5 = p \sum_{\begin{pmatrix} * & * & * \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -(p-1)^2p^4.$$

Thus

$$\mathscr{S}(x,\xi) = (p-1)^3p^4. \tag{5.157}$$

11. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}a_{11}^2\epsilon + b_{12}bd + b_{22}(a_{13}d + b^2). \tag{5.158}$$

Summing in a_{13} forces $d = 0$. Thus, summing in b_{11} and b_{22},

$$\begin{aligned}\mathscr{S}(x,\xi) &= \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon + b_{22}b^2) \\ &= (p-1)^3p^4.\end{aligned}$$

12. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon + ad + bc) + b_{12}bd + b_{22}(a_{13}d + b^2). \tag{5.159}$$

Summing in a_{13} forces $d = 0$. Thus, summing in b_{22}, b, then b_{11},

$$\mathscr{S}(x,\xi) = \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + bc) + b_{22}b^2)$$

$$= -(p-1)^2 p^4.$$

13. Here

$$[x_0, g \cdot \xi_1] + [x_1, g \cdot \xi_0] = b_{11}(a_{11}^2 \epsilon + ac) + b_{12}bd + b_{22}(a_{13}d + b^2). \tag{5.160}$$

Sum in a_{13} to force $d = 0$. Now sum in a to force $c = 0$. Thus the sum vanishes.

14. Here

$$[x_0, g \cdot \xi_1] + [x_1, g \cdot \xi_0] = b_{11}a_{11}^2 \epsilon + b_{12}bd + b_{22}(a_{11}c + a_{13}d + b^2). \tag{5.161}$$

Sum in a_{13} to force $d = 0$. Hence, summing in b_{11}, c and then b_{22},

$$\mathscr{S}(x, \xi) = \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2 \epsilon + b_{22}(a_{11}c + b^2)) = -(p-1)^2 p^4. \tag{5.162}$$

15. Here

$$[x_0, g \cdot \xi_1] + [x_1, g \cdot \xi_0] = b_{11}(a_{11}^2 \epsilon + ad + bc) + b_{12}bd + b_{22}(a_{11}c + a_{13}d + b^2). \tag{5.163}$$

Sum in a_{13} to force $d = 0$. Thus

$$\mathscr{S}(x, \xi) = \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2 \epsilon + bc) + b_{22}(a_{11}c + b^2)).$$

Sum in b_{22}; when $b^2 \neq -a_{11}c$ the sum is -1, otherwise $p - 1$. Hence,

$$\begin{aligned}\mathscr{S}(x, \xi) = &-\frac{1}{p-1} \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2 \epsilon + b_{11}bc) \\ &+ \frac{p}{p-1} \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & \end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}, b^2 = -a_{11}c} e_p(b_{11}a_{11}^2 \epsilon + b_{11}bc)\end{aligned}$$

In the first sum, sum first in b then in b_{11} obtaining -1 in both sums. In the second sum, replace $c = -\frac{b^2}{a_{11}}$, then replace $b_{11} := \frac{b_{11}}{a_{11}}$ and sum in b_{11} to obtain

$$\begin{aligned}\mathscr{S}(x,\xi) &= -(p-1)^2p^4 + \frac{p}{p-1}\sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & & c\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}, b^2=-a_{11}c} e_p(b_{11}(\epsilon a_{11}^3 - b^3)) \\ &= -(p-1)^2p^4 - (p-1)^2p^5 + \#\{(a_{11},b)\in(\mathbf{F}_p^\times)^2 : \epsilon a_{11}^3 \equiv b^3\}p^6 \\ &= -(p-1)^2p^4(p+1) + (p-1)p^6\#\{x\in\mathbf{F}_p^\times : x^3\equiv\epsilon\}.\end{aligned}$$

16. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon + ac) + b_{12}bd + b_{22}(a_{11}c + a_{13}d + b^2). \tag{5.164}$$

Sum in a_{13} to force $d=0$. Sum in a to force $c=0$. Thus the sum vanishes.
17.-23. Summing in a_{12}, a_{13} forces $c=d=0$ so these sums vanish.

5.9. The exponential sums pair $(\mathscr{O}_{1^31}, \mathscr{O}_{Cs})$

Standard representatives are

$$x_0 = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & \frac12 \\ & & 0\end{pmatrix}\begin{pmatrix} 0 & 0 & \frac12 \\ & 1 & 0 \\ & & 0\end{pmatrix}, \qquad \xi_0 = \begin{pmatrix} 0 & -1 & 0 \\ & 0 & 0 \\ & & 0\end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0\end{pmatrix} \tag{5.165}$$

and

$$x_1 = \begin{pmatrix} \epsilon & 0 & 0 \\ & 0 & 0 \\ & & 0\end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0\end{pmatrix}, \epsilon\in\mathbf{F}_p^\times/\{x^3 : x\in\mathbf{F}_p^\times\}. \tag{5.166}$$

The acting set is

$$G_{x,\xi}^t = \begin{pmatrix} a & a_{12} & a_{13} \\ & b & a_{23} \\ & & a_{33}\end{pmatrix}\begin{pmatrix} c & b_{12} \\ & \frac{bc}{a}\end{pmatrix}, \tag{5.167}$$

and

$$[x_1, g\cdot\xi_0] = (-2aca_{12} + b_{12}a^2)\epsilon. \tag{5.168}$$

The stabilizer has size

$$\left|\mathrm{Stab}_{G(\mathbb{Z}/p^2\mathbb{Z})}(x)\right| = (p-1)p^3\#\{\epsilon^3 = 1\}. \tag{5.169}$$

Representatives and exponential sum pairings are given in the table below. Summing over the maximal orbits,

$$\mathscr{M}(x,\xi) = \sum_{\epsilon \in \mathbf{F}_p^\times/\{x^3 : x \in \mathbf{F}_p^\times\}} \frac{p^{10}\mathscr{S}(x,\xi)}{(p-1)\#\{\epsilon^3 = 1\}}. \tag{5.170}$$

See **exponential_sums_O131_OCs.nb**.

Orbit	ξ_1	$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0]$	$\mathscr{S}(x,\xi)$	$\mathscr{M}(x,\xi)$
1.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$	$b_{12}a_{23}a_{33} + \frac{bc}{a}(a_{23}^2 + a_{13}a_{33})$ $+(-2aca_{12} + b_{12}a^2)\epsilon$	0	0
2.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}$	$\frac{b^3c}{a} + (-2aca_{12} + b_{12}a^2)\epsilon$	0	0
3.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ & 1 & 0 \\ & & 0 \end{pmatrix}$	$bc\left(\frac{b^2}{a} + a_{33}\right) + (-2aca_{12} + b_{12}a^2)\epsilon$	0	0
4.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}$	$c\left(\frac{b^3}{a} + a_{23}a_{33}\right) + (-2aca_{12} + b_{12}a^2)\epsilon$	0	0
5.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & \frac{1}{2} \\ & & 1 \end{pmatrix}$	$\frac{bc}{a}\left(ba_{23} + \frac{a_{12}a_{33}}{2} + a_{13}a_{33} + a_{23}^2\right)$ $+b_{12}\left(a_{23}a_{33} + \frac{a_{33}b}{2}\right)$ $+(-2aca_{12} + b_{12}a^2)\epsilon$	0	0
6.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$	$\frac{bc}{a}(2ba_{23} + a_{12}a_{33}) + bb_{12}a_{33}$ $+(-2aca_{12} + b_{12}a^2)\epsilon$	0	0
7.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$	$\frac{bc}{a}(2ba_{23} + a_{12}a_{33}) + bb_{12}a_{33} + ca_{23}a_{33}$ $+(-2aca_{12} + b_{12}a^2)\epsilon$	0	0
8.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}$	$\frac{bc}{a}(-b^2\ell + a_{13}a_{33} + a_{23}^2) + b_{12}a_{23}a_{33}$ $+(-2aca_{12} + b_{12}a^2)\epsilon$	0	0
9.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$bca_{33} + (-2aca_{12} + b_{12}a^2)\epsilon$	0	0
10.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$ca_{23}a_{33} + (-2aca_{12} + b_{12}a^2)\epsilon$	0	0
11.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(-2aca_{12} + b_{12}a^2)\epsilon$	0	0

Each of these sums vanishes on summing in the indicated variables.

1.-4. a_{12}

5. a_{13}

6. a_{23}

7. a_{12}, b_{12}. The dependence on these variables is $a_{12}\left(\frac{bc}{a}a_{33} - 2ac\epsilon\right)$ and $b_{12}\left(ba_{33} + a^2\epsilon\right)$. The corresponding linear forms are independent since $p > 3$ is assumed.

8. a_{13}

9.-11. a_{12}

5.10. The exponential sums pair $(\mathscr{O}_{1^4}, \mathscr{O}_{D1^2})$

Standard representatives

$$x_0 = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 1 & 0 \\ & & 0 \end{pmatrix}, \qquad \xi_0 = \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \tag{5.171}$$

and

$$x_1 = \begin{pmatrix} \epsilon & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \epsilon \in \mathbf{F}_p^\times / \{x^4 : x \in \mathbf{F}_p^\times\}. \tag{5.172}$$

The acting set is

$$G_{x,\xi}^t = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & a & b \\ & c & d \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix}, \tag{5.173}$$

which is fibered as for the other exponential sums involving $\mathscr{O}_{D1^2}$, and

$$[x_1, g \cdot \xi_0] = b_{11} a_{11}^2 \epsilon. \tag{5.174}$$

The stabilizer has size

$$\left|\mathrm{Stab}_{G(\mathbb{Z}/p^2\mathbb{Z})}(x)\right| = (p-1)p^4 \#\{\epsilon^4 = 1\}. \tag{5.175}$$

Representatives and exponential sum pairings are given in the table below. Summing over the maximal orbits,

$$\mathscr{M}(x, \xi) = \sum_{\epsilon \in \mathbf{F}_p^\times / \{x^4 : x \in \mathbf{F}_p^\times\}} \frac{p^9 \mathscr{S}(x, \xi)}{(p-1)\#\{\epsilon^4 = 1\}}. \tag{5.176}$$

See **exponential_sums_O14_OD12.nb**.

Orbit	ξ_1	$[x_0, g \cdot \xi_1] + [x_1, g \cdot \xi_0]$	$\mathscr{S}(x, \xi)$	$\mathscr{M}(x, \xi)$
1.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11} a_{11}^2 \epsilon$	$-(p-1)^4 p^4 (p+1)$	$-(p-1)^3 p^{13} (p+1)$
2.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2 \epsilon + d^2)$	$(p-1)^3 p^4$ $+ \left(\frac{-\epsilon}{p}\right) (p-1)^3 p^6$	$(p-1)^2 p^{13}$
3.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & \frac{1}{2} \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2 \epsilon + cd)$	$-(p-1)^4 p^4$	$-(p-1)^3 p^{13}$
4.	$\begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2 \epsilon - c^2 \ell + d^2)$	$(p-1)^3 p^4 (p+1)$	$(p-1)^2 p^{13} (p+1)$
5.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2 \epsilon + d^2) + b_{22} a_{11} d$	$-(p-1)^2 p^4 (p^2 - p + 1)$ $-(p-1)^2 p^6 \left(\frac{-\epsilon}{p}\right)$	$-(p-1) p^{13} (p^2 - p + 1)$

Orbit	ξ_1	$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0]$	$\mathscr{S}(x,\xi)$	$\mathscr{M}(x,\xi)$
6.	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + c^2) + b_{22}a_{11}d$	$(p-1)^3p^4$	$(p-1)^2p^{13}$
7.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & \frac{1}{2} \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + cd) + b_{22}a_{11}d$	$-(p-1)^4p^4$	$-(p-1)^3p^{13}$
8.	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 1 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + c^2 + 2cd) + b_{22}a_{11}d$	$-(p-1)^4p^4$ $\left(1+\left(\frac{-\epsilon}{p}\right)\right)(p-1)^2p^6$	$(p-1)p^{13}(2p-1)$
9.	$\begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon - c^2\ell + d^2) + b_{22}a_{11}d$	$-(p-1)^2p^4$ $-(p-1)^2p^6\left(\frac{\epsilon}{p}\right)$	$-(p-1)p^{13}$
10.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$b_{11}a_{11}^2\epsilon + b_{22}a_{11}d$	$(p-1)^3p^4$	$(p-1)^2p^{13}$
11.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$	$b_{11}a_{11}^2\epsilon + b_{12}d^2$ $+b_{22}(a_{13}d + b^2)$	$(p-1)^3p^4$	$(p-1)^2p^{13}$
12.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + 2cd) + b_{12}d^2$ $+b_{22}(a_{13}d + b^2)$	$(p-1)^3p^4$	$(p-1)^2p^{13}$
13.	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + c^2) + b_{12}d^2$ $+b_{22}(a_{13}d + b^2)$	$-(p-1)^2p^4$ $-(p-1)^2p^5\left(\frac{-\epsilon}{p}\right)$	$-(p-1)p^{13}$
14.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$	$b_{11}a_{11}^2\epsilon + b_{12}d^2$ $+b_{22}(a_{11}c + a_{13}d + b^2)$	$-(p-1)^2p^4$	$-(p-1)p^{13}$
15.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + 2cd) + b_{12}d^2$ $+b_{22}(a_{11}c + a_{13}d + b^2)$	$-(p-1)^2p^4$	$-(p-1)p^{13}$
16.	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + c^2) + b_{12}d^2$ $+b_{22}(a_{11}c + a_{13}d + b^2)$	$-(p-1)^2p^4(p+1)$ $-(p-1)^2p^5\left(\frac{-\epsilon}{p}\right)$ $+(p-1)p^6$ $\times\#\{x : x^4 \equiv -\epsilon \bmod p\}$	p^{13}
17.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$	$b_{11}a_{11}^2\epsilon + 2b_{12}cd$ $+b_{22}(a_{12}d + a_{13}c + 2ab)$	0	0
18.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + d^2) + 2b_{12}cd$ $+b_{22}(a_{12}d + a_{13}c + 2ab)$	0	0
19.	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + c^2 + d^2) + 2b_{12}cd$ $+b_{22}(a_{12}d + a_{13}c + 2ab)$	0	0
20.	$\begin{pmatrix} 0 & 0 & 0 \\ & \ell & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + c^2\ell + d^2) + 2b_{12}cd$ $+b_{22}(a_{12}d + a_{13}c + 2ab)$	0	0
21.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}$	$b_{11}a_{11}^2\epsilon + b_{12}(-c^2\ell + d^2)$ $+b_{22}(-a_{12}c\ell + a_{13}d - a^2\ell + b^2)$	0	0
22.	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + c^2) + b_{12}(-c^2\ell + d^2)$ $+b_{22}(-a_{12}c\ell + a_{13}d - a^2\ell + b^2)$	0	0
23a.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + 2cd)$ $+b_{12}(-c^2\ell + d^2)$ $+b_{22}(-a_{12}c\ell + a_{13}d - a^2\ell + b^2)$	0	0
23b.	$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & k \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}$	$b_{11}(a_{11}^2\epsilon + c(c + 2dk))$ $+b_{12}(-c^2\ell + d^2)$ $+b_{22}(-a_{12}c\ell + a_{13}d - a^2\ell + b^2)$	0	0

The exponential sums are as follows.

1. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}a_{11}^2\epsilon. \tag{5.177}$$

Thus, summing in b_{11},

$$\mathscr{S}(x,\xi) = \sum_{g\in G_{x,\xi}^t} e_p(b_{11}a_{11}^2\epsilon) = -\frac{|G_{x,\xi}|}{p-1} = -(p-1)^4p^4(p+1). \tag{5.178}$$

2. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon + d^2). \tag{5.179}$$

Thus

$$\begin{aligned}
\Sigma_1 &= \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & d\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + d^2)) \\
&= (p-1)p^6 \sum_{b_{11}\in\mathbf{F}_p^\times}\left(\tau\left(\frac{b_{11}\epsilon}{p}\right) - 1\right)\tau\left(\frac{b_{11}}{p}\right) \\
&= (p-1)^2p^7\left(\frac{-\epsilon}{p}\right) \\
\Sigma_2 &= \sum_{\begin{pmatrix} * & * & * \\ & b\lambda & b \\ & d\lambda & d\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + d^2)) \\
&= (p-1)^2p^4 \sum_{b_{11}\in\mathbf{F}_p^\times}\left(\tau\left(\frac{b_{11}\epsilon}{p}\right) - 1\right)\tau\left(\frac{b_{11}}{p}\right) = (p-1)^3p^5\left(\frac{-\epsilon}{p}\right) \\
\Sigma_3 &= \sum_{\begin{pmatrix} * & * & * \\ & a & 0 \\ & c & 0\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) \\
&= -(p-1)^2p^5 \\
\Sigma_4 &= \sum_{\begin{pmatrix} * & * & * \\ & 0 & b \\ & 0 & d\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + d^2))
\end{aligned}$$

$$= (p-1)p^4 \sum_{b_{11} \in \mathbf{F}_p^\times} \left(\tau\left(\frac{b_{11}\epsilon}{p}\right) - 1\right) \tau\left(\frac{b_{11}}{p}\right) = (p-1)^2 p^5 \left(\frac{-\epsilon}{p}\right)$$

$$\Sigma_5 = p \sum_{\begin{pmatrix} * & * & * \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix} \begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11} a_{11}^2 \epsilon) = -(p-1)^2 p^4.$$

Thus

$$\mathscr{S}(x, \xi) = (p-1)^3 p^4 + (p-1)^3 p^6 \left(\frac{-\epsilon}{p}\right). \tag{5.180}$$

3. Here

$$[x_0, g \cdot \xi_1] + [x_1, g \cdot \xi_0] = b_{11}(a_{11}^2 \epsilon + cd). \tag{5.181}$$

$$\Sigma_1 = \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & d \end{pmatrix} \begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2 \epsilon + cd))$$

$$= -(p-1)^2 p^6$$

$$\Sigma_2 = \sum_{\begin{pmatrix} * & * & * \\ & b\lambda & b \\ & d\lambda & d \end{pmatrix} \begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2 \epsilon + d^2 \lambda)) = 0$$

$$\Sigma_3 = \sum_{\begin{pmatrix} * & * & * \\ & a & 0 \\ & c & 0 \end{pmatrix} \begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11} a_{11}^2 \epsilon) = -(p-1)^2 p^5$$

$$\Sigma_4 = \sum_{\begin{pmatrix} * & * & * \\ & 0 & b \\ & 0 & d \end{pmatrix} \begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11} a_{11}^2 \epsilon) = -(p-1)^2 p^5$$

$$\Sigma_5 = p \sum_{\begin{pmatrix} * & * & * \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix} \begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11} a_{11}^2 \epsilon) = -(p-1)^2 p^4.$$

Thus

$$\mathscr{S}(x, \xi) = -(p-1)^4 p^4. \tag{5.182}$$

4. Here

$$[x_1, g\cdot\xi_0] + [x_0, g\cdot\xi_1] = b_{11}(a_{11}^2\epsilon - c^2\ell + d^2). \tag{5.183}$$

$$\begin{aligned}
\Sigma_1 &= \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & d\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon - c^2\ell + d^2)) \\
&= (p-1)p^5 \sum_{b_{11}\in\mathbf{F}_p^\times} \left(\tau\left(\frac{b_{11}\epsilon}{p}\right) - 1\right)\tau^2\left(\frac{-\ell}{p}\right) \\
&= (p-1)^2p^6 \\
\Sigma_2 &= \sum_{\begin{pmatrix} * & * & * \\ & b\lambda & b \\ & d\lambda & d\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + d^2(1-\lambda^2\ell)))
\end{aligned}$$

Split off the $d = 0$ term. In the $d \neq 0$ terms replace $\lambda := d\lambda$. This obtains

$$\begin{aligned}
\Sigma_2 &= (p-1)^2p^4 \sum_{a_{11},b_{11}\in\mathbf{F}_p^\times} e_p(b_{11}a_{11}^2\epsilon) + (p-1)p^4 \sum_{\lambda,a_{11},b_{11},d\in\mathbf{F}_p^\times} e_p(b_{11}(a_{11}^2\epsilon + d^2 - \lambda^2\ell)) \\
&= -(p-1)^3p^4 + (p-1)p^4 \sum_{b_{11}\in\mathbf{F}_p^\times}\left(\tau\left(\frac{b_{11}\epsilon}{p}\right) - 1\right)\left(\tau\left(\frac{b_{11}}{p}\right) - 1\right) \\
&\quad\times\left(\tau\left(\frac{-b_{11}\ell}{p}\right) - 1\right) \\
&= -(p-1)^3p^4 - (p-1)^2p^4 - (p-1)^2p^5\left\{\left(\frac{-\epsilon}{p}\right) - \left(\frac{\epsilon}{p}\right) - 1\right\} \\
&= -(p-1)^2p^5\left\{\left(\frac{-\epsilon}{p}\right) - \left(\frac{\epsilon}{p}\right)\right\} \\
\Sigma_3 &= \sum_{\begin{pmatrix} * & * & * \\ & a & 0 \\ & c & 0\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon - c^2\ell)) \\
&= (p-1)p^4 \sum_{b_{11}\in\mathbf{F}_p^\times}\left(\tau\left(\frac{b_{11}\epsilon}{p}\right) - 1\right)\tau\left(\frac{-b_{11}\ell}{p}\right) \\
&= -(p-1)^2p^5\left(\frac{\epsilon}{p}\right)
\end{aligned}$$

$$\begin{aligned}\Sigma_4 &= \sum_{\begin{pmatrix} * & * & * \\ & 0 & b \\ & 0 & d\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + d^2)) \\ &= (p-1)^2 p^5 \left(\frac{-\epsilon}{p}\right) \\ \Sigma_5 &= p \sum_{\begin{pmatrix} * & * & * \\ & 0 & 0 \\ & 0 & 0\end{pmatrix}\begin{pmatrix} b_{11} & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) = -(p-1)^2p^4.\end{aligned}$$

Thus

$$\mathscr{S}(x,\xi) = (p-1)^3p^4(p+1). \tag{5.184}$$

5. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon + d^2) + b_{22}a_{11}d. \tag{5.185}$$

Summing in b_{22} obtains -1 if $d \neq 0$, $p-1$ if $d = 0$, so that

$$\begin{aligned}\mathscr{S}(x,\xi) &= -\frac{2.}{p-1} + \frac{p}{p-1}\sum_{\begin{pmatrix} * & * & * \\ & * & * \\ & * & \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) \\ &= -(p-1)^2p^4 - (p-1)^2p^6\left(\frac{-\epsilon}{p}\right) - (p-1)^3p^5 \\ &= -(p-1)^2p^4(p^2-p+1) - (p-1)^2p^6\left(\frac{-\epsilon}{p}\right).\end{aligned}$$

6. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon + c^2) + b_{22}a_{11}d. \tag{5.186}$$

Summing in b_{22} obtains -1 if $d \neq 0$ and $p-1$ if $d = 0$, so that,

$$\begin{aligned}\mathscr{S}(x,\xi) &= -\frac{2.}{p-1} + \frac{p}{p-1}\sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + c^2)) \\ &= -\frac{2.}{p-1} + (p-1)p^5\sum_{b_{11}\in\mathbf{F}_p^\times}\left(\tau\left(\frac{b_{11}\epsilon}{p}\right) - 1\right)\left(\tau\left(\frac{b_{11}}{p}\right) - 1\right)\end{aligned}$$

$$= -\frac{2.}{p-1} + (p-1)^2 p^5 + (p-1)^2 p^6 \left(\frac{-\epsilon}{p}\right)$$
$$= (p-1)^3 p^4.$$

7. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon + cd) + b_{22}a_{11}d. \tag{5.187}$$

Thus, summing in b_{22} as above,

$$\begin{aligned}\mathscr{S}(x,\xi) &= -\frac{3.}{p-1} + \frac{p}{p-1} \sum_{\begin{pmatrix} * & * & * \\ & * & * \\ & * & \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) \\ &= -\frac{3.}{p-1} - (p-1)^3 p^5 \\ &= -(p-1)^4 p^4.\end{aligned}$$

8. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon + c^2 + 2cd) + b_{22}a_{11}d \tag{5.188}$$

Sum in b_{22} to find

$$\begin{aligned}\mathscr{S}(x,\xi) &= -\frac{3.}{p-1} + \frac{p}{p-1} \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + c^2)) \\ &= -\frac{3.}{p-1} + (p-1)p^5 \sum_{b_{11}\in\mathbf{F}_p^\times} \left(\tau\left(\frac{b_{11}\epsilon}{p}\right) - 1\right)\left(\tau\left(\frac{b_{11}}{p}\right) - 1\right) \\ &= (p-1)^3 p^4 + (p-1)^2 p^5 + (p-1)^2 p^6 \left(\frac{-\epsilon}{p}\right) \\ &= -(p-1)^4 p^4 + \left(1 + \left(\frac{-\epsilon}{p}\right)\right)(p-1)^2 p^6.\end{aligned}$$

9. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon - c^2\ell + d^2) + b_{22}a_{11}d. \tag{5.189}$$

Sum in b_{22} to find

$$\begin{aligned}\mathscr{S}(x,\xi) &= -\frac{4.}{p-1} + \frac{p}{p-1} \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon - c^2\ell)) \\ &= -(p-1)^2p^4(p+1) \\ &\quad + (p-1)p^5 \sum_{b_{11}\in\mathbf{F}_p^\times} \left(\tau\left(\frac{b_{11}\epsilon}{p}\right) - 1\right)\left(\tau\left(\frac{-b_{11}\ell}{p}\right) - 1\right) \\ &= -(p-1)^2p^4(p+1) + (p-1)^2p^5 - (p-1)^2p^6\left(\frac{\epsilon}{p}\right) \\ &= -(p-1)^2p^4 - (p-1)^2p^6\left(\frac{\epsilon}{p}\right).\end{aligned}$$

10. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}a_{11}^2\epsilon + b_{22}a_{11}d. \tag{5.190}$$

Sum in b_{22} to find

$$\begin{aligned}\mathscr{S}(x,\xi) &= -\frac{1.}{p-1} + \frac{p}{p-1} \sum_{\begin{pmatrix} * & * & * \\ & * & * \\ & * & \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon) \\ &= (p-1)^3p^4(p+1) - (p-1)^3p^5 = (p-1)^3p^4.\end{aligned}$$

In sums 11-16 sum over b_{12} to force $d = 0$.

11. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}a_{11}^2\epsilon + b_{12}d^2 + b_{22}(a_{13}d + b^2). \tag{5.191}$$

Thus

$$\mathscr{S}(x,\xi) = \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon + b_{22}b^2) = (p-1)^3p^4, \tag{5.192}$$

since summation in b_{11} and b_{22} each give -1.

12. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon + 2cd) + b_{12}d^2 + b_{22}(a_{13}d + b^2). \tag{5.193}$$

Thus

$$\mathscr{S}(x,\xi) = \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon + b_{22}b^2) = (p-1)^3p^4, \tag{5.194}$$

since summation in b_{11} and b_{22} each give -1.

13. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon + c^2) + b_{12}d^2 + b_{22}(a_{13}d + b^2). \tag{5.195}$$

Now sum in b_{22}, obtaining -1, and sum in a_{11} and c to obtain Gauss sums, to find

$$\begin{aligned}\mathscr{S}(x,\xi) &= \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + c^2) + b_{22}b^2) \\ &= -(p-1)p^4 \sum_{b_{11}\in\mathbf{F}_p^\times} \left(\tau\left(\frac{b_{11}\epsilon}{p}\right) - 1\right)\left(\tau\left(\frac{b_{11}}{p}\right) - 1\right) \\ &= -(p-1)^2p^4 - (p-1)^2p^5\left(\frac{-\epsilon}{p}\right).\end{aligned}$$

14. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}a_{11}^2\epsilon + b_{12}d^2 + b_{22}(a_{11}c + a_{13}d + b^2). \tag{5.196}$$

Now sum in b_{11}, obtaining -1, then sum in a_{11}, obtaining -1, and finally sum in b_{22}, obtaining -1, to obtain

$$\begin{aligned}\mathscr{S}(x,\xi) &= \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & c & \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}} e_p(b_{11}a_{11}^2\epsilon + b_{22}(b^2 + a_{11}c)) \\ &= -(p-1)^2p^4.\end{aligned}$$

15. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{11}(a_{11}^2\epsilon + 2cd) + b_{12}d^2 + b_{22}(a_{11}c + a_{13}d + b^2). \tag{5.197}$$

After setting $d = 0$ this is the same sum as in 14., so

$$\mathscr{S}(x,\xi) = 14. = -(p-1)^2p^4. \tag{5.198}$$

16. Here

$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{22}(b^2 + a_{11}c + a_{13}d) + b_{12}d^2 + b_{11}(c^2 + a_{11}^2\epsilon). \tag{5.199}$$

Thus

$$\begin{aligned}\mathscr{S}(x,\xi) &= \sum_{\begin{pmatrix} * & * & * \\ & a & b \\ & & c\end{pmatrix}\begin{pmatrix} * & * \\ & *\end{pmatrix}} e_p(b_{11}(a_{11}^2\epsilon + c^2) + b_{22}(b^2 + a_{11}c)) \\ &= p^4 \sum_{b_{11}, b_{22}, a_{11}, b, c \in \mathbf{F}_p^\times} e_p(b_{11}(a_{11}^2\epsilon + c^2) + b_{22}(b^2 + a_{11}c)).\end{aligned}$$

Summing in b_{22} obtains -1 if $b^2 + a_{11}c \neq 0$ and $p-1$ otherwise. Splitting the sum according to this and substituting $b_{11} := \frac{b_{11}}{c^2}$ in the second sum obtains

$$\begin{aligned}\mathscr{S}(x,\xi) = -p^4 &\sum_{b_{11}, a_{11}, b, c \in \mathbf{F}_p^\times} e_p(b_{11}(a_{11}^2\epsilon + c^2)) \\ &+ p^5 \sum_{b_{11}, b, c\in\mathbf{F}_p^\times} e_p(b_{11}(\epsilon b^4 + c^4)) \\ = -(p-1)p^4 &\sum_{b_{11}\in\mathbf{F}_p^\times}\left(\tau\left(\frac{b_{11}\epsilon}{p}\right) - 1\right)\left(\tau\left(\frac{b_{11}}{p}\right)-1\right) \\ &- (p-1)^2p^5 + (p-1)p^6\#\{x : x^4 \equiv -\epsilon \bmod p\} \\ = -(p-1)^2p^4(p+1) &- (p-1)^2p^5\left(\frac{-\epsilon}{p}\right) + (p-1)p^6\#\{x : x^4\equiv -\epsilon \bmod p\}.\end{aligned}$$

17.-23. These sums vanish since summing in a_{12} and a_{13} forces $c = d = 0$.

5.11. The exponential sums pair $(\mathscr{O}_{1^4}, \mathscr{O}_{D11})$

Standard representatives are

$$x_0 = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1\end{pmatrix}\begin{pmatrix} 0 & 0 & \frac12 \\ & 1 & 0 \\ & & 0\end{pmatrix}, \qquad \xi_0 = \begin{pmatrix} 0 & \frac12 & 0 \\ & 0 & 0 \\ & & 0\end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0\end{pmatrix}, \tag{5.200}$$

and

$$x_1 = \begin{pmatrix} \epsilon & 0 & 0 \\ & 0 & 0 \\ & & 0\end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0\end{pmatrix}, \epsilon \in \mathbf{F}_p^\times/\{x^4 : x \in \mathbf{F}_p^\times\}. \tag{5.201}$$

The acting set is

$$G^t_{x,\xi} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix} \sqcup \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & & a_{23} \\ & & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix}. \tag{5.202}$$

Here

$$[x_1, g \cdot \xi_0] = b_{11}a_{11}a_{12}\epsilon. \tag{5.203}$$

The stabilizer has size

$$\left|\mathrm{Stab}_{G(\mathbb{Z}/p^2\mathbb{Z})}(x)\right| = (p-1)p^4\#\{\epsilon^4 = 1\}. \tag{5.204}$$

Representatives and exponential sum pairings are given in the table below. Summing over the maximal orbits,

$$\mathscr{M}(x,\xi) = \sum_{\epsilon \in \mathbf{F}_p^\times/\{x^4 : x \in \mathbf{F}_p^\times\}} \frac{p^9 \mathscr{S}(x,\xi)}{(p-1)\#\{\epsilon^4 = 1\}}. \tag{5.205}$$

See **exponential_sums_O14_OD11.nb**.

Orbit	ξ_1	$[x_0, g \cdot \xi_1] + [x_1, g \cdot x_0]$	$\mathscr{S}(x,\xi)$	$\mathscr{M}(x,\xi)$
1.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$	$b_{11}a_{11}a_{12}\epsilon + b_{12}a_{33}^2 + b_{22}(a_{13}a_{33} + a_{23}^2)$	0	0
2.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 1 \end{pmatrix}$	$b_{11}a_{11}a_{12}\epsilon + b_{12}a_{33}^2$ $+b_{22}(a_{12}a_{33} + a_{13}a_{33} + 2a_{22}a_{23} + a_{23}^2)$	0	0
3.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 1 \\ & & 1 \end{pmatrix}$	$b_{11}a_{11}a_{12}\epsilon + b_{12}a_{33}^2$ $+b_{22}(a_{23}(2a_{21} + 2a_{22} + a_{23}) + a_{33}(a_{11} + a_{12} + a_{13}))$	0	0
4.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}$	$b_{11}a_{11}a_{12}\epsilon + b_{12}a_{33}^2 + b_{22}(a_{13}a_{33} - a_{22}^2\ell + a_{23}^2)$	0	0
5.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}$	$b_{11}a_{11}a_{12}\epsilon + b_{12}a_{33}^2$ $+b_{22}(a_{11}a_{33} + a_{13}a_{33} + 2a_{21}a_{23} - a_{22}^2\ell + a_{23}^2)$	0	0
6.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} -\ell & 0 & 0 \\ & -\ell & 0 \\ & & 1 \end{pmatrix}$	$b_{11}a_{11}a_{12}\epsilon + b_{12}a_{33}^2$ $+b_{22}(a_{13}a_{33} - (a_{21}^2 + a_{22}^2)\ell + a_{23}^2)$	0	0
7.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$	$b_{11}a_{11}a_{12}\epsilon + b_{22}(a_{12}a_{33} + 2a_{22}a_{23})$	0	0
8.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$	$b_{11}a_{11}a_{12}\epsilon$ $+b_{22}(2a_{23}(a_{21} + a_{22}) + a_{33}(a_{11} + a_{12}))$	0	0
9.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$	$b_{11}a_{11}a_{12}\epsilon + b_{22}(a_{12}a_{33} + a_{21}^2 + 2a_{22}a_{23})$	0	0
10.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}a_{12}\epsilon + a_{33}^2) + b_{22}(a_{12}a_{33} + 2a_{22}a_{23})$	0	0
11.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$	$b_{11}(a_{11}a_{12}\epsilon + a_{33}^2)$ $+b_{22}(2a_{23}(a_{21} + a_{22}) + a_{33}(a_{11} + a_{12})$	0	0

Orbit	ξ_1	$[x_0, g\cdot\xi_1]+[x_1, g\cdot x_0]$	$\mathscr{S}(x,\xi)$	$\mathscr{M}(x,\xi)$
12.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}1&0&0\\&0&1\\&&0\end{pmatrix}$	$b_{11}(a_{11}a_{12}\epsilon+a_{33}^2)+b_{22}(a_{12}a_{33}+a_{21}^2+2a_{22}a_{23})$	0	0
13.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&1&0\\&&0\end{pmatrix}$	$b_{11}a_{11}a_{12}\epsilon+b_{22}a_{22}^2$	0	0
14.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}1&0&0\\&1&0\\&&0\end{pmatrix}$	$b_{11}a_{11}a_{12}\epsilon+b_{22}(a_{21}^2+a_{22}^2)$	0	0
15.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}\ell&0&0\\&1&0\\&&0\end{pmatrix}$	$b_{11}a_{11}a_{12}\epsilon+b_{22}(a_{21}^2\ell+a_{22}^2)$	0	0
16.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&1&0\\&&0\end{pmatrix}$	$b_{11}(a_{11}a_{12}\epsilon+a_{33}^2)+b_{22}a_{22}^2$	0	0
17.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}1&0&0\\&1&0\\&&0\end{pmatrix}$	$b_{11}(a_{11}a_{12}\epsilon+a_{33}^2)+b_{22}(a_{21}^2+a_{22}^2)$	0	0
18.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}\ell&0&0\\&1&0\\&&0\end{pmatrix}$	$b_{11}(a_{11}a_{12}\epsilon+a_{33}^2)+b_{22}(a_{21}^2\ell+a_{22}^2)$	0	0
19.	$\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$b_{11}(a_{11}a_{12}\epsilon+a_{33}^2)$	0	0
20.	$\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}$	$b_{11}a_{11}a_{12}\epsilon$	0	0

All of the orbital exponential sums vanish. This is checked as follows.

1.-6. These sums vanish on summing over b_{12}.

7.-12. These sums vanish on summing over a_{23} and a_{11}. The summation in a_{23} forces $a_{22}=0$ in each case, so that a_{11} ranges in $\mathbf{F}_p$.

13.-20. These sums vanish on summing over a_{11} or a_{12}.

5.12. The exponential sums pair $(\mathscr{O}_{1^4}, \mathscr{O}_{1^4})$

Standard representatives are

$$x_0=\begin{pmatrix}0&0&0\\&0&0\\&&1\end{pmatrix}\begin{pmatrix}0&0&\frac{1}{2}\\&1&0\\&&0\end{pmatrix},\qquad \xi_0=\begin{pmatrix}0&0&-1\\&1&0\\&&0\end{pmatrix}\begin{pmatrix}2&0&0\\&0&0\\&&0\end{pmatrix} \tag{5.206}$$

and

$$x_1=\begin{pmatrix}\epsilon&0&0\\&0&0\\&&0\end{pmatrix}\begin{pmatrix}0&0&0\\&0&0\\&&0\end{pmatrix}, \epsilon\in\mathbf{F}_p^\times/\{x^4 : x\in\mathbf{F}_p^\times\}. \tag{5.207}$$

The acting set is

$$G_{x,\xi}^t=\begin{pmatrix}a&a_{12}&a_{13}\\&b&a_{23}\\&&c\end{pmatrix}\begin{pmatrix}d&b_{12}\\&e\end{pmatrix}, b^2=ac, cd=ae. \tag{5.208}$$

Here

$$[x_1, g\cdot\xi_0] = (d(a_{12}^2 - 2aa_{13}) + 2b_{12}a^2)\epsilon. \tag{5.209}$$

The stabilizer has size

$$\left|\mathrm{Stab}_{G(\mathbb{Z}/p^2\mathbb{Z})}(x)\right| = (p-1)p^4\#\{\epsilon^4 = 1\}. \tag{5.210}$$

Representatives and exponential sum pairings are given in the table below. Summing over the maximal orbits,

$$\mathscr{M}(x,\xi) = \sum_{\epsilon\in\mathbf{F}_p^\times/\{x^4 : x\in\mathbf{F}_p^\times\}} \frac{p^9\mathscr{S}(x,\xi)}{(p-1)\#\{\epsilon^4=1\}}. \tag{5.211}$$

See **exponential_sums_O14_O14.nb**.

Orbit	ξ_1	$[x_0, g\cdot\xi_1] + [x_1, g\cdot x_0]$	$\mathscr{S}(x,\xi)$	$\mathscr{M}(x,\xi)$
1.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$	$b_{12}c^2 + e(a_{13}c + a_{23}^2) + (d(a_{12}^2 - 2aa_{13}) + 2b_{12}a^2)\epsilon$	0	0
2.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$	$e(2a_{23}b + a_{12}c) + (d(a_{12}^2 - 2aa_{13}) + 2b_{12}a^2)\epsilon$	0	0
3.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ & 1 & 0 \\ & & 0 \end{pmatrix}$	$c^2d + e(-ac + b^2) + (d(a_{12}^2 - 2aa_{13}) + 2b_{12}a^2)\epsilon$	0	0
4.	$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$	$(d(a_{12}^2 - 2aa_{13}) + 2b_{12}a^2)\epsilon$	0	0

All of the orbital exponential sums vanish, as is checked below.

1. $\xi_1 = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$. Here

$$[x_0, g\cdot\xi_1] = b_{12}c^2 + e(a_{13}c + a_{23}^2)$$
$$[x_0, g\cdot\xi_1] + [x_1, g\cdot\xi_0] = b_{12}c^2 + e(a_{13}c + a_{23}^2) + (d(a_{12}^2 - 2aa_{13}) + 2b_{12}a^2)\epsilon.$$

Sum in b_{12} to find $2a^2\epsilon + c^2 = 0$ or $2a^2\epsilon + \left(\frac{ae}{d}\right)^2 = 0$. Sum in a_{13} to find $2ad\epsilon = ce$ or $2ad\epsilon = \frac{ae^2}{d}$. These two conditions are inconsistent, as the first implies $2d^2\epsilon = -e^2$ and the second implies $2d^2\epsilon = e^2$, so the sum vanishes.
2. $\xi_1 = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$. The sum vanishes on summing over b_{12}.
3. $\xi_1 = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ & 1 & 0 \\ & & 0 \end{pmatrix}$. The sum vanishes on summing over b_{12}.
4. $\xi_1 = 0$. The sum vanishes on summing over b_{12}.

6. Summation

Combining the previous sections proves the following theorem.

Theorem 2. *The Fourier transform of the maximal set is supported on the mod p orbits $\mathscr{O}_0, \mathscr{O}_{D1^2}, \mathscr{O}_{D11}$ and $\mathscr{O}_{D2}$. It is given explicitly in the following tables.*

1. *Case* $\mathscr{O}_0$, $\xi = p\xi_0$.

Orbit	$p^{-12}\widehat{\mathbf{1}_{\max}}(p\xi_0)$	*Orbit size*
$\mathscr{O}_0$	$(p-1)^4p(p+1)^2(p^5+2p^4+4p^3+4p^2+3p+1)$	1
$\mathscr{O}_{D1^2}$	$-(p-1)^3p(p+1)^4$	$(p-1)(p+1)(p^2+p+1)$
$\mathscr{O}_{D11}$	$-(p-1)^3p(2p^3+6p^2+4p+1)$	$(p-1)p(p+1)^2(p^2+p+1)/2$
$\mathscr{O}_{D2}$	$(p-1)^2p(2p^2+3p+1)$	$(p-1)^2p(p+1)(p^2+p+1)/2$
$\mathscr{O}_{Dns}$	$(p-1)^2p(2p^2+3p+1)$	$(p-1)^2p^2(p+1)(p^2+p+1)$
$\mathscr{O}_{Cs}$	$-p^7+5p^5-3p^4-3p^3+p^2+p$	$(p-1)^2p(p+1)^2(p^2+p+1)$
$\mathscr{O}_{Cns}$	$(p-1)^2p(2p^2+3p+1)$	$(p-1)^2p^3(p+1)(p^2+p+1)$
$\mathscr{O}_{B11}$	$(p-1)^2p(2p^2+3p+1)$	$(p-1)^2p^2(p+1)^2(p^2+p+1)/2$
$\mathscr{O}_{B2}$	$(p-1)^2p(2p^2+3p+1)$	$(p-1)^3p^2(p+1)(p^2+p+1)/2$
$\mathscr{O}_{1^4}$	$p(p^3-3p^2+p+1)$	$(p-1)^3p^2(p+1)^2(p^2+p+1)$
$\mathscr{O}_{1^31}$	$p(p^3-3p^2+p+1)$	$(p-1)^3p^3(p+1)^2(p^2+p+1)$
$\mathscr{O}_{1^21^2}$	$(p-1)^2p(3p+1)$	$(p-1)^2p^4(p+1)^2(p^2+p+1)/2$
$\mathscr{O}_{2^2}$	$-(p-1)p(p+1)^2$	$(p-1)^3p^4(p+1)(p^2+p+1)/2$
$\mathscr{O}_{1^211}$	$p(p^3-3p^2+p+1)$	$(p-1)^3p^4(p+1)^2(p^2+p+1)/2$
$\mathscr{O}_{1^22}$	$p(p^3-3p^2+p+1)$	$(p-1)^3p^4(p+1)^2(p^2+p+1)/2$
$\mathscr{O}_{1111}$	$-p^3+p^2+p$	$(p-1)^4p^4(p+1)^2(p^2+p+1)/24$
$\mathscr{O}_{112}$	$-p^3+p^2+p$	$(p-1)^4p^4(p+1)^2(p^2+p+1)/4$
$\mathscr{O}_{22}$	$-p^3+p^2+p$	$(p-1)^4p^4(p+1)^2(p^2+p+1)/8$
$\mathscr{O}_{13}$	$-p^3+p^2+p$	$(p-1)^4p^4(p+1)^2(p^2+p+1)/3$
$\mathscr{O}_{4}$	$-p^3+p^2+p$	$(p-1)^4p^4(p+1)^2(p^2+p+1)/4$

(6.1)

2. *Case* $\mathscr{O}_{D1^2}$.

Orbit	$p^{-12}\widehat{\mathbf{1}_{\max}}(\xi)$	*Orbit size*
1.	$-(p-1)^3p(p+1)^3$	$(p-1)p^4(p+1)(p^2+p+1)$
2.	$(p-1)^2p(2p+1)$	$(p-1)^2p^4(p+1)^2(p^2+p+1)$
3.	$(p-1)^2p(2p+1)$	$(p-1)^2p^5(p+1)^2(p^2+p+1)/2$
4.	$-(p-1)p(p+1)$	$(p-1)^3p^5(p+1)(p^2+p+1)/2$
5.	$p(p^3-2p^2+1)$	$(p-1)^3p^4(p+1)^2(p^2+p+1)$
6.	$-(p-1)p(p+1)$	$(p-1)^3p^5(p+1)^2(p^2+p+1)$
7.	$-(p-1)p(p+1)$	$(p-1)^3p^5(p+1)^2(p^2+p+1)$
8.	p	$(p-1)^4p^5(p+1)^2(p^2+p+1)/2$
9.	p	$(p-1)^4p^5(p+1)^2(p^2+p+1)/2$
10.	$(p-1)^2p(p+1)^2$	$(p-1)^2p^4(p+1)^2(p^2+p+1)$
11.	$(p-1)^2p(p+1)$	$(p-1)^2p^6(p+1)^2(p^2+p+1)$
12.	$-(p-1)p$	$(p-1)^3p^6(p+1)^2(p^2+p+1)$
13.	$-(p-1)p$	$(p-1)^3p^7(p+1)^2(p^2+p+1)$
14.	$-(p-1)p(p+1)$	$(p-1)^3p^6(p+1)^2(p^2+p+1)$
15.	p	$(p-1)^4p^6(p+1)^2(p^2+p+1)$
16.	p	$(p-1)^4p^7(p+1)^2(p^2+p+1)$
17.	0	$(p-1)^2p^8(p+1)^2(p^2+p+1)/2$
18.	0	$(p-1)^3p^8(p+1)^2(p^2+p+1)$
19.	0	$(p-1)^4p^8(p+1)^2(p^2+p+1)/4$
20.	0	$(p-1)^4p^8(p+1)^2(p^2+p+1)/4$
21.	0	$(p-1)^3p^8(p+1)(p^2+p+1)/2$
22.	0	$(p-1)^4p^8(p+1)^2(p^2+p+1)/4$
23.	0	$(p-1)^4p^8(p+1)^2(p^2+p+1)/4$

(6.2)

3. *Case* $\mathscr{O}_{D11}$.

Orbit	$p^{-12}\widehat{\mathbf{1}_{\max}}(\xi)$	*Orbit size*
1.	0	$(p-1)^2p^{10}(p+1)^2(p^2+p+1)/2$
2.	0	$(p-1)^3p^{10}(p+1)^2(p^2+p+1)/2$
3.	0	$(p-1)^4p^{10}(p+1)^2(p^2+p+1)/8$
4.	0	$(p-1)^3p^{10}(p+1)^2(p^2+p+1)/2$
5.	0	$(p-1)^4p^{10}(p+1)^2(p^2+p+1)/4$
6.	0	$(p-1)^4p^{10}(p+1)^2(p^2+p+1)/8$
7.	0	$(p-1)^2p^8(p+1)^2(p^2+p+1)$
8.	0	$(p-1)^3p^9(p+1)^2(p^2+p+1)/2$
9.	0	$(p-1)^3p^8(p+1)^2(p^2+p+1)$
10.	0	$(p-1)^3p^8(p+1)^2(p^2+p+1)$
11.	0	$(p-1)^4p^9(p+1)^2(p^2+p+1)/2$
12.	0	$(p-1)^4p^8(p+1)^2(p^2+p+1)$
13.	0	$(p-1)^2p^7(p+1)^2(p^2+p+1)$
14.	$(p-1)p^2$	$(p-1)^3p^7(p+1)^2(p^2+p+1)/4$
15.	$-(p-1)p^2$	$(p-1)^3p^7(p+1)^2(p^2+p+1)/4$
16.	0	$(p-1)^3p^7(p+1)^2(p^2+p+1)$
17.	$-p^2$	$(p-1)^4p^7(p+1)^2(p^2+p+1)/4$
18.	p^2	$(p-1)^4p^7(p+1)^2(p^2+p+1)/4$
19.	0	$(p-1)^2p^7(p+1)^2(p^2+p+1)/2$
20.	0	$(p-1)p^7(p+1)^2(p^2+p+1)/2$

(6.3)

4. *Case* $\mathscr{O}_{D2}$.

Orbit	$p^{-12}\widehat{\mathbf{1}_{\max}}(\xi)$	*Orbit size*
1.	0	$(p-1)^3p^{10}(p+1)(p^2+p+1)/2$
2.	0	$(p-1)^4p^{10}(p+1)^2(p^2+p+1)/4$
3.	0	$(p-1)^4p^{10}(p+1)^2(p^2+p+1)/4$
4.	0	$(p-1)^3p^9(p+1)^2(p^2+p+1)/2$
5.	0	$(p-1)^4p^9(p+1)^2(p^2+p+1)/2$
6.	$(p-1)p^2$	$(p-1)^3p^7(p+1)^2(p^2+p+1)/4$
7.	$-p^2$	$(p-1)^4p^7(p+1)^2(p^2+p+1)/4$
8.	$-(p-1)p^2$	$(p-1)^3p^7(p+1)^2(p^2+p+1)/4$
9.	p^2	$(p-1)^4p^7(p+1)^2(p^2+p+1)/4$
10.	0	$(p-1)^3p^7(p+1)(p^2+p+1)/2$
11.	0	$(p-1)^2p^7(p+1)(p^2+p+1)/2$

(6.4)

Proof. Since $\mathbf{1}_{\max}$ is $G(\mathbb{Z}/p^2\mathbb{Z})$-invariant, it is the sum of the indicator functions of some $G(\mathbb{Z}/p^2\mathbb{Z})$ orbits on $V(\mathbb{Z}/p^2\mathbb{Z})$.

When $\xi \in \mathscr{O}_0$ so that $\xi = p\xi_1$,

$$\widehat{\mathbf{1}_{\max}}(p\xi_1) = \sum_{x \in V(\mathbb{Z}/p^2\mathbb{Z})} \mathbf{1}_{\max}(x) e_{p^2}([x, p\xi_1]) \tag{6.5}$$

$$= \sum_{\mathscr{O} \in G(\mathbb{Z}/p\mathbb{Z})\backslash V(\mathbb{Z}/p\mathbb{Z})} \sum_{x \in \mathscr{O} \bmod p} e_p([x, \xi_1]) \left(\sum_{x_1 \in V(\mathbb{Z}/p\mathbb{Z})} \mathbf{1}_{\max}(x + px_1) \right).$$

The density

$$\mu(\mathscr{O}) = p^{-12} \sum_{x_1 \in V(\mathbb{Z}/p\mathbb{Z})} \mathbf{1}_{\max}(x + px_1) \tag{6.6}$$

of maximal orbits above x depends only on the orbit $\mathscr{O}$, and is given in the following table.

Orbit	$\mathscr{O}_{1^4}$	$\mathscr{O}_{1^31}$	$\mathscr{O}_{2^2}$	$\mathscr{O}_{1^21^2}$	$\mathscr{O}_{1^22}$	$\mathscr{O}_{1^211}$	$\mathscr{O}_4$	$\mathscr{O}_{2^2}$	$\mathscr{O}_{13}$	$\mathscr{O}_{112}$	$\mathscr{O}_{1111}$
Density	$\frac{p-1}{p}$	$\frac{p-1}{p}$	$\frac{p^2-1}{p^2}$	$\left(\frac{p-1}{p}\right)^2$	$\frac{p-1}{p}$	$\frac{p-1}{p}$	1	1	1	1	1

(6.7)

The Fourier transform above $\mathscr{O}_0$ was thus calculated according to the formula

$$\widehat{\mathbf{1}_{\max}}(p\xi_1) = p^{12} \sum_{\mathscr{O} \in G(\mathbb{Z}/p\mathbb{Z})\backslash V(\mathbb{Z}/p\mathbb{Z})} \mu(\mathscr{O}) \sum_{x \in \mathscr{O} \bmod p} e_p([x, \xi_1]), \tag{6.8}$$

see **modp_sums.nb**. The orbital exponential sums $\sum_{x \in \mathscr{O}} e_p([x, \xi_1])$ are obtained as the columns of the matrix M in [10] p. 27, as a function of the orbits of ξ_1 under the $G(\mathbb{Z}/p\mathbb{Z})$ action. The orbit sizes of the mod p orbits were also determined in [10].

The orbit sizes of the orbits in V/V_x under the G_x action were obtained in Lemmas 19, 17, and 16. The orbit sizes of the orbits in $V(\mathbb{Z}/p^2\mathbb{Z})$ under $G(\mathbb{Z}/p^2\mathbb{Z})$ were obtained from the formula

$$\left|\mathscr{O}_\xi \bmod p^2\right| = \left|\mathscr{O}_{\xi_0 \bmod p}\right| \cdot p^{\dim V_\xi} \cdot \left|\mathscr{O}_{\xi_1} \subset V/V_\xi\right|, \tag{6.9}$$

from Lemma 11.

When $\xi \not\equiv 0 \bmod p$, it was verified in Lemma 5 that for $\mathscr{O} \in G(\mathbb{Z}/p\mathbb{Z})\backslash V(\mathbb{Z}/p\mathbb{Z})$ with density of maximal elements 0 or 1,

$$\sum_{\substack{x \in V(\mathbb{Z}/p^2\mathbb{Z}) \\ x \in \mathscr{O} \bmod p}} \mathbf{1}_{\max}(x) e_{p^2}([x, \xi]) = \mathscr{M}(x, \xi) = 0. \tag{6.10}$$

Hence, in the formula

$$\begin{aligned} \widehat{\mathbf{1}_{\max}}(\xi) &= \sum_{x \in V(\mathbb{Z}/p^2\mathbb{Z})} \mathbf{1}_{\max}(x) e_{p^2}([x, \xi]) \\ &= \sum_{\mathscr{O} \in G(\mathbb{Z}/p\mathbb{Z}) \backslash V(\mathbb{Z}/p\mathbb{Z})} \mathscr{M}(x, \xi) \end{aligned} \tag{6.11}$$

the sum may be restricted to $\mathscr{O} \in \{\mathscr{O}_{1^2 11}, \mathscr{O}_{1^2 2}, \mathscr{O}_{1^2 1^2}, \mathscr{O}_{2^2}, \mathscr{O}_{1^3 1}, \mathscr{O}_{1^4}\}$. This sum is further restricted, since by Lemma 9, $\mathscr{M}(x, \xi) = 0$ if $G_{x,\xi} = \emptyset$. This proves that $\widehat{\mathbf{1}_{\max}}(\xi) = 0$ unless $\xi \in \mathscr{O}_0, \mathscr{O}_{D1^2}, \mathscr{O}_{D11}, \mathscr{O}_{D2}, \mathscr{O}_{Cs}$ or $\mathscr{O}_{1^4}$. For each of those pairs $x, \xi \not\equiv 0 \bmod p$ such that $G_{x,\xi} \neq \emptyset$, $\mathscr{M}(x, \xi)$ was calculated. This demonstrated that in fact the Fourier transform vanishes over $\mathscr{O}_{Cs}$ and $\mathscr{O}_{1^4}$, also. The Fourier transform displayed is the result of adding the columns $\mathscr{M}(x, \xi)$ from the previous section, Sections 5.1, 5.2, 5.5, 5.8 and 5.10 for the Fourier transform above $\mathscr{O}_{D1^2}$, Sections 5.3 and 5.6 for the Fourier transform above $\mathscr{O}_{D11}$, and Sections 5.4 and 5.7 for the Fourier transform above $\mathscr{O}_{D2}$.

For the calculations in this proof, see **summation.nb**. □

Proof of Theorem 1. The Fourier transform of $\mathbf{1}_{\text{non-max}}$ is $p^{24}\delta_{\xi=0} - \widehat{\mathbf{1}_{\max}}(\xi)$. The formulas obtained in Theorem 1 are the result of combining the formulas for the Fourier transform with the size of the orbit of each frequency. These calculations are performed in **summation.nb**. □

Appendix A. Computation of annihilator subspaces

The annihilator subspaces for standard representatives are as follows. For the computations in this Appendix, see **annihilator_spaces.nb**.

(1) Case of $\mathscr{O}_{D1^2}$, $x = (0, w^2)$.

$$\left(I + p\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}\right)\left(\left(I + p\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}\right)\begin{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix}\end{pmatrix}\left(I + p\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33}\end{pmatrix}\right)\right)$$

$$= \begin{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0\end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1\end{pmatrix}\end{pmatrix} + p\begin{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & b_{12}\end{pmatrix} \\ \begin{pmatrix} 0 & 0 & a_{13} \\ & 0 & a_{23} \\ & & 2a_{33} + b_{22}\end{pmatrix}\end{pmatrix}.$$

Thus

$$V_x = \begin{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & *\end{pmatrix} \\ \begin{pmatrix} 0 & 0 & * \\ & 0 & * \\ & & *\end{pmatrix}\end{pmatrix}.$$

(2) Case of $\mathscr{O}_{D11}$, $x = (0, vw)$.

$$\left(I + p\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}\right)\left(\left(I + p\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}\right)\begin{matrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0\end{pmatrix}\end{matrix}\left(I + p\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33}\end{pmatrix}\right)\right)$$

$$= \begin{matrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0\end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ & 0 & \frac{1}{2} \\ & & 0\end{pmatrix}\end{matrix} + p\begin{matrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & \frac{b_{12}}{2} \\ & & 0\end{pmatrix} \\ \begin{pmatrix} 0 & \frac{a_{13}}{2} & \frac{a_{12}}{2} \\ & a_{23} & \frac{a_{22}+a_{33}+b_{22}}{2} \\ & & a_{32}\end{pmatrix}\end{matrix}.$$

Thus

$$V_x = \begin{matrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & * \\ & & 0\end{pmatrix} \\ \begin{pmatrix} 0 & * & * \\ & * & * \\ & & *\end{pmatrix}\end{matrix}.$$

(3) Case of $\mathscr{O}_{D2}$, $x = (0, v^2 - \ell w^2)$.

$$\left(I + p\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}\right)\left(\left(I + p\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}\right)\begin{matrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\ell\end{pmatrix}\end{matrix}\left(I + p\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33}\end{pmatrix}\right)\right)$$

$$= \begin{matrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0\end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & -\ell\end{pmatrix}\end{matrix} + p\begin{matrix}\begin{pmatrix} 0 & 0 & 0 \\ & b_{12} & 0 \\ & & -b_{12}\ell\end{pmatrix} \\ \begin{pmatrix} 0 & a_{12} & -a_{13}\ell \\ & 2a_{22}+b_{22} & -a_{23}\ell + a_{32} \\ & & -2a_{33}\ell - b_{22}\ell\end{pmatrix}\end{matrix}.$$

Thus

$$V_x = \begin{matrix}\begin{pmatrix} 0 & 0 & 0 \\ & z & 0 \\ & & -z\ell\end{pmatrix} \\ \begin{pmatrix} 0 & * & * \\ & * & * \\ & & *\end{pmatrix}\end{matrix}.$$

(4) Case of $\mathscr{O}_{Dns}$, $x = (0, u^2 - vw)$.

$$\left(I + p\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}\right)\left(\left(I + p\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}\right)\begin{matrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0\end{pmatrix}\end{matrix}\left(I + p\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33}\end{pmatrix}\right)\right)$$

$$= \begin{matrix}\begin{pmatrix}0 & 0 & 0\\ & 0 & 0\\ & & 0\end{pmatrix}\\ \begin{pmatrix}1 & 0 & 0\\ & 0 & -\frac{1}{2}\\ & & 0\end{pmatrix}\end{matrix} + p\begin{matrix}\begin{pmatrix}b_{12} & 0 & 0\\ & 0 & -\frac{b_{12}}{2}\\ & & 0\end{pmatrix}\\ \begin{pmatrix}2a_{11}+b_{22} & -\frac{a_{13}}{2}+a_{21} & -\frac{a_{12}}{2}+a_{31}\\ & -a_{23} & -\frac{a_{22}+a_{33}+b_{22}}{2}\\ & & -a_{32}\end{pmatrix}\end{matrix}.$$

Thus

$$V_x = \begin{matrix}\begin{pmatrix}z & 0 & 0\\ & 0 & -\frac{z}{2}\\ & & 0\end{pmatrix}\\ \begin{pmatrix}* & * & *\\ & * & *\\ & & *\end{pmatrix}\end{matrix}.$$

(5) Case of $\mathscr{O}_{Cs}$, $x = (w^2, vw)$.

$$\left(I + p\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\right)\left(\left(I + p\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\right)\begin{matrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\\ \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \frac{1}{2}\\ 0 & \frac{1}{2} & 0\end{pmatrix}\end{matrix}\left(I + p\begin{pmatrix}a_{11} & a_{21} & a_{31}\\ a_{12} & a_{22} & a_{32}\\ a_{13} & a_{23} & a_{33}\end{pmatrix}\right)\right)$$

$$= \begin{matrix}\begin{pmatrix}0 & 0 & 0\\ & 0 & 0\\ & & 1\end{pmatrix}\\ \begin{pmatrix}0 & 0 & 0\\ & 0 & \frac{1}{2}\\ & & 0\end{pmatrix}\end{matrix} + p\begin{matrix}\begin{pmatrix}0 & 0 & a_{13}\\ & 0 & a_{23}+\frac{b_{12}}{2}\\ & & 2a_{33}+b_{11}\end{pmatrix}\\ \begin{pmatrix}0 & \frac{a_{13}}{2} & \frac{a_{12}}{2}\\ & a_{23} & \frac{a_{22}+a_{33}+b_{22}}{2}\\ & & a_{32}+b_{21}\end{pmatrix}\end{matrix}.$$

Thus

$$V_x = \begin{matrix}\begin{pmatrix}0 & 0 & z\\ & 0 & *\\ & & *\end{pmatrix}\\ \begin{pmatrix}0 & \frac{z}{2} & *\\ & * & *\\ & & *\end{pmatrix}\end{matrix}.$$

(6) Case of $\mathscr{O}_{Cns}$, $x = (vw, uw)$.

$$\left(I + p\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\right)\left(\left(I + p\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\right)\begin{matrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \frac{1}{2}\\ 0 & \frac{1}{2} & 0\end{pmatrix}\\ \begin{pmatrix}0 & 0 & \frac{1}{2}\\ 0 & 0 & 0\\ \frac{1}{2} & 0 & 0\end{pmatrix}\end{matrix}\left(I + p\begin{pmatrix}a_{11} & a_{21} & a_{31}\\ a_{12} & a_{22} & a_{32}\\ a_{13} & a_{23} & a_{33}\end{pmatrix}\right)\right)$$

$$= \begin{matrix}\begin{pmatrix}0 & 0 & 0\\ & 0 & \frac{1}{2}\\ & & 0\end{pmatrix}\\ \begin{pmatrix}0 & 0 & \frac{1}{2}\\ & 0 & 0\\ & & 0\end{pmatrix}\end{matrix} + p\begin{matrix}\begin{pmatrix}0 & \frac{a_{13}}{2} & \frac{a_{12}+b_{12}}{2}\\ & a_{23} & \frac{a_{22}+a_{33}+b_{11}}{2}\\ & & a_{32}\end{pmatrix}\\ \begin{pmatrix}a_{13} & \frac{a_{23}}{2} & \frac{a_{11}+a_{33}+b_{22}}{2}\\ & 0 & \frac{a_{21}+b_{21}}{2}\\ & & a_{31}\end{pmatrix}\end{matrix}.$$

Thus

$$V_x = \begin{matrix} \begin{pmatrix} 0 & \frac{z}{2} & * \\ & y & * \\ & & * \end{pmatrix} \\ \begin{pmatrix} z & \frac{y}{2} & * \\ & 0 & * \\ & & * \end{pmatrix} \end{matrix}.$$

(7) Case of $\mathscr{O}_{B11}$, $x = (v^2, w^2)$.

$$\left(I + p\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right) \left(\left(I + p\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right) \begin{matrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix} \left(I + p\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}\right) \right)$$

$$= \begin{matrix} \begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \end{matrix} + p \begin{matrix} \begin{pmatrix} 0 & a_{12} & 0 \\ & 2a_{22} + b_{11} & a_{32} \\ & & b_{12} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & a_{13} \\ & b_{21} & a_{23} \\ & & 2a_{33} + b_{22} \end{pmatrix} \end{matrix}.$$

Thus

$$V_x = \begin{matrix} \begin{pmatrix} 0 & * & 0 \\ & * & * \\ & & * \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & * \\ & * & * \\ & & * \end{pmatrix} \end{matrix}.$$

(8) Case of $\mathscr{O}_{B2}$, $x = (vw, v^2 + \ell w^2)$.

$$\left(I + p\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right) \left(\left(I + p\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right) \begin{matrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \ell \end{pmatrix} \end{matrix} \left(I + p\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}\right) \right)$$

$$= \begin{matrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & \frac{1}{2} \\ & & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & \ell \end{pmatrix} \end{matrix} + p \begin{matrix} \begin{pmatrix} 0 & \frac{a_{13}}{2} & \frac{a_{12}}{2} \\ & a_{23} + b_{12} & \frac{a_{22}+a_{33}+b_{11}}{2} \\ & & a_{32} + b_{12}\ell \end{pmatrix} \\ \begin{pmatrix} 0 & a_{12} & a_{13}\ell \\ & 2a_{22} + b_{22} & a_{23}\ell + a_{32} + \frac{b_{21}}{2} \\ & & 2a_{33}\ell + b_{22}\ell \end{pmatrix} \end{matrix}.$$

Thus

$$V_x = \begin{matrix} \begin{pmatrix} 0 & \frac{z}{2} & \frac{y}{2} \\ & * & * \\ & & * \end{pmatrix} \\ \begin{pmatrix} 0 & y & z\ell \\ & * & * \\ & & * \end{pmatrix} \end{matrix}.$$

(9) Case of $\mathscr{O}_{1^4}$, $(w^2, uw + v^2)$.

$$\left(I + p\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}\right)\left(\left(I + p\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}\right)\begin{array}{c}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix} \\ \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0\end{pmatrix}\end{array}\left(I + p\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33}\end{pmatrix}\right)\right)$$

$$= \begin{array}{c}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1\end{pmatrix} \\ \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 1 & 0 \\ & & 0\end{pmatrix}\end{array} + p\begin{array}{c}\begin{pmatrix} 0 & 0 & a_{13} + \frac{b_{12}}{2} \\ & b_{12} & a_{23} \\ & & 2a_{33} + b_{11}\end{pmatrix} \\ \begin{pmatrix} a_{13} & a_{12} + \frac{a_{23}}{2} & \frac{a_{11}+a_{33}+b_{22}}{2} \\ & 2a_{22} + b_{22} & \frac{a_{21}}{2} + a_{32} \\ & & a_{31} + b_{21}\end{pmatrix}\end{array}.$$

Thus

$$V_x = \begin{array}{c}\begin{pmatrix} 0 & 0 & y + \frac{z}{2} \\ & z & * \\ & & *\end{pmatrix} \\ \begin{pmatrix} y & * & * \\ & * & * \\ & & *\end{pmatrix}\end{array}.$$

(10) Case of $\mathscr{O}_{1^3 1}$, $(vw, uw + v^2)$.

$$\left(I + p\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}\right)\left(\left(I + p\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}\right)\begin{array}{c}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0\end{pmatrix} \\ \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0\end{pmatrix}\end{array}\left(I + p\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33}\end{pmatrix}\right)\right)$$

$$= \begin{array}{c}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & \frac{1}{2} \\ & & 0\end{pmatrix} \\ \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 1 & 0 \\ & & 0\end{pmatrix}\end{array} + p\begin{array}{c}\begin{pmatrix} 0 & \frac{a_{13}}{2} & \frac{a_{12}+b_{12}}{2} \\ & a_{23} + b_{12} & \frac{a_{22}+a_{33}+b_{11}}{2} \\ & & a_{32}\end{pmatrix} \\ \begin{pmatrix} a_{13} & a_{12} + \frac{a_{23}}{2} & \frac{a_{11}+a_{33}+b_{22}}{2} \\ & 2a_{22} + b_{22} & a_{32} + \frac{a_{21}+b_{21}}{2} \\ & & a_{31}\end{pmatrix}\end{array}.$$

Thus

$$V_x = \begin{array}{c}\begin{pmatrix} 0 & \frac{z}{2} & * \\ & * & * \\ & & *\end{pmatrix} \\ \begin{pmatrix} z & * & * \\ & * & * \\ & & *\end{pmatrix}\end{array}.$$

(11) Case of $\mathscr{O}_{1^21^2}$, (w^2, uv).

$$\left(I + p\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}\right)\left(\left(I + p\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}\right)\begin{matrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}\end{matrix}\left(I + p\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33}\end{pmatrix}\right)\right)$$

$$= \begin{matrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0\end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ & 0 & 0 \\ & & 0\end{pmatrix}\end{matrix} + p\begin{matrix}\begin{pmatrix} 0 & \frac{b_{12}}{2} & a_{13} \\ & 0 & a_{23} \\ & & 2a_{33} + b_{11}\end{pmatrix} \\ \begin{pmatrix} a_{12} & \frac{a_{11}+a_{22}+b_{22}}{2} & \frac{a_{32}}{2} \\ & a_{21} & \frac{a_{31}}{2} \\ & & b_{21}\end{pmatrix}\end{matrix}.$$

Thus

$$V_x = \begin{matrix}\begin{pmatrix} 0 & * & * \\ & 0 & * \\ & & *\end{pmatrix} \\ \begin{pmatrix} * & * & * \\ & * & * \\ & & *\end{pmatrix}\end{matrix}.$$

(12) Case of $\mathscr{O}_{2^2}$, $(w^2, u^2 - \ell v^2)$.

$$\left(I + p\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}\right)\left(\left(I + p\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}\right)\begin{matrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\ell & 0 \\ 0 & 0 & 0\end{pmatrix}\end{matrix}\left(I + p\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33}\end{pmatrix}\right)\right)$$

$$= \begin{matrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1\end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ & -\ell & 0 \\ & & 0\end{pmatrix}\end{matrix} + p\begin{matrix}\begin{pmatrix} b_{12} & 0 & a_{13} \\ & -b_{12}\ell & a_{23} \\ & & 2a_{33} + b_{11}\end{pmatrix} \\ \begin{pmatrix} 2a_{11} + b_{22} & -a_{12}\ell + a_{21} & a_{31} \\ & -2a_{22}\ell - b_{22}\ell & -a_{32}\ell \\ & & b_{21}\end{pmatrix}\end{matrix}.$$

Thus

$$V_x = \begin{matrix}\begin{pmatrix} z & 0 & * \\ & -z\ell & * \\ & & *\end{pmatrix} \\ \begin{pmatrix} * & * & * \\ & * & * \\ & & *\end{pmatrix}\end{matrix}.$$

(13) Case of $\mathscr{O}_{1^211}$, $(v^2 - w^2, uw)$.

$$\left(I + p\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}\right)\left(\left(I + p\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}\right)\begin{matrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1\end{pmatrix} \\ \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0\end{pmatrix}\end{matrix}\left(I + p\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33}\end{pmatrix}\right)\right)$$

$$
= \begin{matrix} \begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 0 & 0 \\ & & 0 \end{pmatrix} \end{matrix} + p \begin{matrix} \begin{pmatrix} 0 & a_{12} & -a_{13} + \frac{b_{12}}{2} \\ & 2a_{22} + b_{11} & -a_{23} + a_{32} \\ & & -2a_{33} - b_{11} \end{pmatrix} \\ \begin{pmatrix} a_{13} & \frac{a_{23}}{2} & \frac{a_{11}+a_{33}+b_{22}}{2} \\ & b_{21} & \frac{a_{21}}{2} \\ & & a_{31} - b_{21} \end{pmatrix} \end{matrix} .
$$

Thus

$$
V_x = \begin{matrix} \begin{pmatrix} 0 & * & * \\ & * & * \\ & & * \end{pmatrix} \\ \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \end{matrix} .
$$

(14) Case of $\mathscr{O}_{1^22}$, $(v^2 - \ell w^2, uw)$.

$$
\left(I + p \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right) \left(\left(I + p \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right) \begin{matrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\ell \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \end{matrix} \left(I + p \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}\right) \right)
$$

$$
= \begin{matrix} \begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & -\ell \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 0 & 0 \\ & & 0 \end{pmatrix} \end{matrix} + p \begin{matrix} \begin{pmatrix} 0 & a_{12} & -a_{13}\ell + \frac{b_{12}}{2} \\ & 2a_{22} + b_{11} & -a_{23}\ell + a_{32} \\ & & -2a_{33}\ell - b_{11}\ell \end{pmatrix} \\ \begin{pmatrix} a_{13} & \frac{a_{23}}{2} & \frac{a_{11}+a_{33}+b_{22}}{2} \\ & b_{21} & \frac{a_{21}}{2} \\ & & a_{31} - b_{21}\ell \end{pmatrix} \end{matrix} .
$$

Thus

$$
V_x = \begin{matrix} \begin{pmatrix} 0 & * & * \\ & * & * \\ & & * \end{pmatrix} \\ \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \end{matrix} .
$$

Appendix B. Determination of action sets

For each of the pairings listed in Lemma 8 the action set $G_{x,\xi}$ is calculated. The calculations in this Appendix are performed in **action_sets.nb**.

1. Case $(\mathscr{O}_{1^211}, \mathscr{O}_{D1^2})$: For standard representatives

$$
x = \begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \tag{B.1}
$$

and

$$V_x = \begin{pmatrix} 0 & * & * \\ & * & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}, \qquad V_\xi = \begin{pmatrix} * & * & * \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} * & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}. \tag{B.2}$$

Since the second form is eliminated by V_x, in the action on ξ, it may be assumed that the GL_2 action acts on the second quadratic form by an arbitrary scalar. The action on ξ is thus given by

$$\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix} \right) \cdot \xi = \begin{pmatrix} b_{11}a_{11}^2 & b_{11}a_{11}a_{21} & b_{11}a_{11}a_{31} \\ & b_{11}a_{21}^2 & b_{11}a_{21}a_{31} \\ & & b_{11}a_{31}^2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}.$$

This forces $a_{21} = a_{31} = 0$, so the action set is

$$G^t_{x,\xi} = \begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix} \begin{pmatrix} * & * \\ & * \end{pmatrix}, \qquad |G_{x,\xi}| = (p-1)^5 p^4 (p+1). \tag{B.3}$$

2. Case $(\mathscr{O}_{1^22}, \mathscr{O}_{D1^2})$: For standard representatives

$$x = \begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & -\ell \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \tag{B.4}$$

and

$$V_x = \begin{pmatrix} 0 & * & * \\ & * & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}, \qquad V_\xi = \begin{pmatrix} * & * & * \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} * & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \tag{B.5}$$

the action set is, as for 1.,

$$G^t_{x,\xi} = \begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix} \begin{pmatrix} * & * \\ & * \end{pmatrix}, \qquad |G_{x,\xi}| = (p-1)^5 p^4 (p+1). \tag{B.6}$$

3. Case $(\mathscr{O}_{2^2}, \mathscr{O}_{D11})$: For standard representatives

$$x = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & -\ell & 0 \\ & & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \tag{B.7}$$

and

$$V_x = \begin{pmatrix} z & 0 & * \\ & -z\ell & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}, \qquad V_\xi = \begin{pmatrix} * & * & * \\ & * & * \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & * & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}. \tag{B.8}$$

Since the second form is eliminated by V_x, in the action on ξ, it may be assumed that the GL_2 action acts on the second quadratic form by an arbitrary scalar.

The action on ξ is thus given by

$$\left(\begin{pmatrix} a & b & a_{13} \\ c & d & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix}\right) \cdot \xi = \begin{pmatrix} 2b_{11}ab & b_{11}(ad+bc) & b_{11}(aa_{32}+ba_{31}) \\ & 2b_{11}cd & b_{11}(ca_{32}+da_{31}) \\ & & 2b_{11}a_{31}a_{32} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}.$$

This forces $a_{31} = a_{32} = 0$ and $ab = cd\ell$, so that

$$G^t_{x,\xi} = \begin{pmatrix} a & b & * \\ c & d & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * \\ & * \end{pmatrix}, \qquad ab = cd\ell. \tag{B.9}$$

There are $(p-1)^3$ choices of a, b, c, d with $ab \neq 0$ and $2(p-1)^2$ choices with $ab = 0$, so $|G_{x,\xi}| = (p-1)^5 p^3 (p+1)$. The two parametrizations are equivalent since the cardinalities are the same.
The action set may be parametrized,

$$\begin{pmatrix} a & c\lambda\ell & a_{13} \\ c & a\lambda & a_{23} \\ & & a_{33} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix}, \qquad \lambda \in \mathbf{F}_p^\times, (a, c) \in \mathbf{F}_p^2 \setminus \{(0,0)\}. \tag{B.10}$$

4. Case $(\mathscr{O}_{2^2}, \mathscr{O}_{D2})$: For standard representatives

$$x = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & -\ell & 0 \\ & & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} \ell & \beta & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \tag{B.11}$$

with $\ell u^2 + 2\beta uv + v^2$ irreducible the annihilated spaces are

$$V_x = \begin{pmatrix} z & 0 & * \\ & -z\ell & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}, \qquad V_\xi = \begin{pmatrix} * & * & * \\ & * & * \\ & & 0 \end{pmatrix} \begin{pmatrix} z\ell & z\beta & 0 \\ & z & 0 \\ & & 0 \end{pmatrix}. \tag{B.12}$$

Since the second form is eliminated by V_x, in the action on ξ, it may be assumed that the GL_2 action acts on the second quadratic form by an arbitrary scalar.
The action on ξ is thus given by

$$\begin{aligned} &\left(\begin{pmatrix} a & b & a_{13} \\ c & d & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix}\right) \cdot \xi \\ &= \begin{pmatrix} b_{11}(a^2\ell + 2ab\beta + b^2) & b_{11}(ac\ell + ad\beta + bc\beta + bd) & b_{11}(aa_{31}\ell + aa_{32}\beta + ba_{31}\beta + ba_{32}) \\ & b_{11}(c^2\ell + 2cd\beta + d^2) & b_{11}(ca_{31}\ell + ca_{32}\beta + da_{31}\beta + da_{32}) \\ & & b_{11}(a_{31}^2\ell + 2a_{31}a_{32}\beta + a_{32}^2) \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}. \end{aligned}$$

Since the lower right entry in the first quadratic form is a quadratic irreducible on a_{31} and a_{32} which must vanish, this forces $a_{31} = a_{32} = 0$. The action set is thus

$$G^t_{x,\xi} = \begin{pmatrix} a & b & * \\ c & d & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * \\ & * \end{pmatrix}, \qquad \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\ell u^2 + 2\beta uv + v^2), u^2 - \ell v^2\right] = 0. \tag{B.13}$$

To count $G_{x,\xi}$, first note that the number of quadratic irreducibles in $\mathrm{Sym}^2(\mathbf{F}_p)$ is $\frac{(p-1)^2p}{2}$. The number of irreducibles which are orthogonal to $u^2 - \ell v^2$ is

$$\begin{aligned}&\#\{(\lambda,\beta) : \lambda(\ell u^2 + v^2) + 2\beta uv \text{ irreducible}\} \\ &= \#\{(\lambda,\beta) : 4\beta^2 - 4\lambda^2\ell \neq \square\}.\end{aligned} \tag{B.14}$$

As λ runs through $\mathbb{Z}/p\mathbb{Z}$, together $4\lambda^2\ell$ and $4\lambda^2$ run through $\mathbb{Z}/p\mathbb{Z}$ twice. Thus

$$\#\{(\lambda,\beta) : 4\beta^2 - 4\lambda^2\ell \neq \square\} + \#\{(\lambda,\beta) : 4\beta^2 - 4\lambda^2 \neq \square\} = (p-1)p. \tag{B.15}$$

Since

$$\#\{(\lambda,\beta) : 4\beta^2 - 4\lambda^2 \neq \square\} = \#\{(a,b) : ab \neq \square\} = \frac{(p-1)^2}{2}, \tag{B.16}$$

it follows that

$$\#\{(\lambda,\beta) : 4\beta^2 - 4\lambda^2\ell \neq \square\} = \frac{(p-1)(p+1)}{2}. \tag{B.17}$$

Since the quadratic irreducibles are a single $\mathrm{GL}_2 \times \mathrm{GL}_1$ orbit in $\mathrm{Sym}^2(\mathbf{F}_p)$, which are uniformly covered by the action of the group on a representative, it follows that

$$\begin{aligned}&\#\left\{g \in \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z}) : \left[g\cdot(\ell u^2 + 2\beta uv + v^2), u^2 - \ell v^2\right] = 0\right\} \\ &= \frac{\frac{(p-1)(p+1)}{2}}{\frac{(p-1)^2p}{2}}(p^2-p)(p^2-1) \\ &= (p-1)(p+1)^2.\end{aligned} \tag{B.18}$$

Thus $|G_{x,\xi}| = (p-1)^4p^3(p+1)^2$.

5. Case $(\mathscr{O}_{1^21^2}, \mathscr{O}_{D1^2})$: For standard representatives

$$x = \begin{pmatrix}0 & 0 & 0\\ & 0 & 0\\ & & 1\end{pmatrix}\begin{pmatrix}0 & \frac{1}{2} & 0\\ & 0 & 0\\ & & 0\end{pmatrix}, \qquad \xi = \begin{pmatrix}1 & 0 & 0\\ & 0 & 0\\ & & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ & 0 & 0\\ & & 0\end{pmatrix} \tag{B.19}$$

and

$$V_x = \begin{pmatrix}0 & * & *\\ & 0 & *\\ & & *\end{pmatrix}\begin{pmatrix}* & * & *\\ & * & *\\ & & *\end{pmatrix}, \qquad V_\xi = \begin{pmatrix}* & * & *\\ & 0 & 0\\ & & 0\end{pmatrix}\begin{pmatrix}* & 0 & 0\\ & 0 & 0\\ & & 0\end{pmatrix}. \tag{B.20}$$

Since the second form is eliminated by V_x, in the action on ξ, it may be assumed that the GL_2 action acts on the second quadratic form by an arbitrary scalar.

The action on ξ is thus given by

$$\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix}\right) \cdot \xi = \begin{pmatrix} b_{11}a_{11}^2 & b_{11}a_{11}a_{21} & b_{11}a_{11}a_{31} \\ & b_{11}a_{21}^2 & b_{11}a_{21}a_{31} \\ & & b_{11}a_{31}^2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}.$$

This forces $a_{31} = a_{11}a_{21} = 0$. The action set is

$$G_{x,\xi}^t = \begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix} \begin{pmatrix} * & * \\ & * \end{pmatrix} \sqcup \begin{pmatrix} & * & * \\ * & * & * \\ & * & * \end{pmatrix} \begin{pmatrix} * & * \\ & * \end{pmatrix}, \qquad |G_{x,\xi}| = 2(p-1)^5 p^4 (p+1). \tag{B.21}$$

6. Case $(\mathscr{O}_{1^21^2}, \mathscr{O}_{D11})$: For standard representatives

$$x = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 1 & 0 & 0 \\ & -1 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \tag{B.22}$$

and

$$V_x = \begin{pmatrix} 0 & * & * \\ & 0 & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}, \qquad V_\xi = \begin{pmatrix} * & * & * \\ & * & * \\ & & 0 \end{pmatrix} \begin{pmatrix} z & 0 & 0 \\ & -z & 0 \\ & & 0 \end{pmatrix}. \tag{B.23}$$

Since the second form is eliminated by V_x, in the action on ξ, it may be assumed that the GL_2 action acts on the second quadratic form by an arbitrary scalar. The action on ξ is thus given by

$$\begin{aligned} &\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix}\right) \cdot \xi \\ &= \begin{pmatrix} b_{11}(a_{11}^2 - a_{12}^2) & b_{11}(a_{11}a_{21} - a_{12}a_{22}) & b_{11}(a_{11}a_{31} - a_{12}a_{32}) \\ & b_{11}(a_{21}^2 - a_{22}^2) & b_{11}(a_{21}a_{31} - a_{22}a_{32}) \\ & & b_{11}(a_{31}^2 - a_{32}^2) \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}. \end{aligned}$$

Since the first two columns of the GL_3 matrix are linearly independent, vanishing of the w dependence in the first quadratic form forces $a_{31} = a_{32} = 0$. The action set maps $u^2 - v^2 = (u+v)(u-v)$ to a form $c_1u^2 + c_2v^2 = c_1(u+\gamma v)(u-\gamma v)$ and thus satisfies

$$G_{x,\xi}^t \subset \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * \\ & * \end{pmatrix}, \begin{matrix} u+v \mapsto \alpha u + \beta v \\ u - v \mapsto \lambda(\alpha u - \beta v) \end{matrix}, \alpha, \beta, \lambda \in (\mathbb{Z}/p\mathbb{Z})^\times. \tag{B.24}$$

The size is $|G_{x,\xi}| = (p-1)^6 p^3$.

7. Case $(\mathscr{O}_{1^21^2}, \mathscr{O}_{D2})$: For standard representatives

$$x = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} -\ell & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \tag{B.25}$$

and

$$V_x = \begin{pmatrix} 0 & * & * \\ & 0 & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}, \qquad V_\xi = \begin{pmatrix} * & * & * \\ & * & * \\ & & 0 \end{pmatrix} \begin{pmatrix} -z\ell & 0 & 0 \\ & z & 0 \\ & & 0 \end{pmatrix}. \tag{B.26}$$

Since the second form is eliminated by V_x, in the action on ξ, it may be assumed that the GL_2 action acts on the second quadratic form by an arbitrary scalar. The action on ξ is thus given by

$$\begin{aligned} &\left(\begin{pmatrix} a & b & a_{13} \\ c & d & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix} \right) \cdot \xi \\ &= \begin{pmatrix} b_{11}(-a^2\ell + b^2) & b_{11}(-ac\ell + bd) & b_{11}(-aa_{31}\ell + ba_{32}) \\ & b_{11}(-c^2\ell + d^2) & b_{11}(-ca_{31}\ell + da_{32}) \\ & & b_{11}(-a_{31}^2\ell + a_{32}^2) \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}. \end{aligned}$$

This forces $a_{31} = a_{32} = 0$. The action set is

$$G_{x,\xi}^t = \begin{pmatrix} a & b & * \\ c & d & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * \\ & * \end{pmatrix}, \qquad ac\ell = bd. \tag{B.27}$$

As for 3., $|G_{x,\xi}| = (p-1)^5 p^3 (p+1)$, and the action set has parametrization,

$$\begin{pmatrix} a & c\ell & * \\ c\lambda & a\lambda & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * \\ & * \end{pmatrix}, \qquad \lambda \in \mathbf{F}_p^\times, (a,c) \in \mathbf{F}_p^2 \setminus \{(0,0)\}. \tag{B.28}$$

8. Case $(\mathscr{O}_{1^31}, \mathscr{O}_{D1^2})$: For standard representatives

$$x = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & \frac{1}{2} \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 1 & 0 \\ & & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \tag{B.29}$$

and

$$V_x = \begin{pmatrix} 0 & \frac{z}{2} & * \\ & * & * \\ & & * \end{pmatrix} \begin{pmatrix} z & * & * \\ & * & * \\ & & * \end{pmatrix}, \qquad V_\xi = \begin{pmatrix} * & * & * \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} * & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}. \tag{B.30}$$

Under the action on ξ the two forms are proportional, and this can be orthogonal to V_x only if the second form is 0, and thus it may be assumed that the second form is acted on by a scalar under the GL_2 action.
The action on ξ is thus given by

$$\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix}\right) \cdot \xi = \begin{pmatrix} b_{11}a_{11}^2 & b_{11}a_{11}a_{21} & b_{11}a_{11}a_{31} \\ & b_{11}a_{21}^2 & b_{11}a_{21}a_{31} \\ & & b_{11}a_{31}^2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}.$$

The v^2 and w^2 coefficients force $a_{21} = a_{31} = 0$, so the action set is

$$G_{x,\xi}^t = \begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix} \begin{pmatrix} * & * \\ & * \end{pmatrix}, \qquad |G_{x,\xi}| = (p-1)^5 p^4 (p+1). \tag{B.31}$$

9. Case $(\mathscr{O}_{1^31}, \mathscr{O}_{Cs})$: For standard representatives

$$x = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & \frac{1}{2} \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 1 & 0 \\ & & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 0 & -1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \tag{B.32}$$

and

$$V_x = \begin{pmatrix} 0 & \frac{z}{2} & * \\ & * & * \\ & & * \end{pmatrix} \begin{pmatrix} z & * & * \\ & * & * \\ & & * \end{pmatrix}, \qquad V_\xi = \begin{pmatrix} * & * & * \\ & * & -z \\ & & 0 \end{pmatrix} \begin{pmatrix} * & * & z \\ & 0 & 0 \\ & & 0 \end{pmatrix} \tag{B.33}$$

Since V_x annihilates the dependence on w in both forms, the action on ξ has no w dependence in both forms, and thus this remains true of any linear combination of the forms. Thus the action by GL_2 can be inverted with this still the case, so that the GL_3 action must carry each form to a form independent of w. From the second form, u^2, it follows that u is mapped by the GL_3 action to a linear form not containing w. Now from the first form, $2uv$, it follows that v is mapped to a linear form not containing w.
The action on ξ is given by

$$\begin{aligned} &\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ & & a_{33} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right) \cdot \xi \\ &= \begin{pmatrix} -2b_{11}a_{11}a_{12} + b_{12}a_{11}^2 & b_{11}(-a_{11}a_{22} - a_{12}a_{21}) + b_{12}a_{11}a_{21} & 0 \\ & -2b_{11}a_{21}a_{22} + b_{12}a_{21}^2 & 0 \\ & & 0 \end{pmatrix} \\ &\begin{pmatrix} -2b_{21}a_{11}a_{12} + b_{22}a_{11}^2 & b_{21}(-a_{11}a_{22} - a_{12}a_{21}) + b_{22}a_{11}a_{21} & 0 \\ & -2b_{21}a_{21}a_{22} + b_{22}a_{21}^2 & 0 \\ & & 0 \end{pmatrix}. \end{aligned}$$

Considering the v^2 coefficients gives

$$a_{21}(-2b_{11}a_{22}+b_{12}a_{21})=0, \qquad a_{21}(-2b_{21}a_{22}+b_{22}a_{21})=0. \tag{B.34}$$

This forces $a_{21}=0$, since otherwise either $a_{21}=a_{22}=0$ or the matrix $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ is singular, both of which are impossible. It now follows from the uv coefficient in the second quadratic form that $-a_{11}a_{22}b_{21}=0$. Since $a_{21}=0$, the matrix on GL_3 would be singular if either $a_{11}=0$ or $a_{22}=0$, so that $b_{21}=0$.

The action has now been simplified to

$$\left(\begin{pmatrix} a & a_{12} & a_{13} \\ & b & a_{23} \\ & & a_{33} \end{pmatrix}, \begin{pmatrix} c & b_{12} \\ & d \end{pmatrix}\right)\cdot\xi = \begin{pmatrix} a^2b_{12}-2aa_{12}c & -abc & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} a^2d & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}.$$

Thus $abc=a^2d$ so that the action set is

$$G_{x,\xi}^t = \begin{pmatrix} a & * & * \\ & b & * \\ & & * \end{pmatrix}\begin{pmatrix} c & * \\ & \frac{bc}{a} \end{pmatrix}, \qquad |G_{x,\xi}|=(p-1)^4p^4. \tag{B.35}$$

10. Case $(\mathscr{O}_{1^4}, \mathscr{O}_{D1^2})$: For standard representatives

$$x = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 1 & 0 \\ & & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \tag{B.36}$$

and

$$V_x = \begin{pmatrix} 0 & 0 & y+\frac{z}{2} \\ & z & * \\ & & * \end{pmatrix}\begin{pmatrix} y & * & * \\ & * & * \\ & & * \end{pmatrix}, \qquad V_\xi = \begin{pmatrix} * & * & * \\ & 0 & 0 \\ & & 0 \end{pmatrix}\begin{pmatrix} * & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}. \tag{B.37}$$

Since the two forms in the action on ξ are linearly dependent, to be orthogonal to V_x it is necessary that the second form is 0, and thus it may be assumed that the second form is acted on by a scalar under the GL_2 action.

The action on ξ is thus given by

$$\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix}\right)\cdot\xi = \begin{pmatrix} b_{11}a_{11}^2 & b_{11}a_{11}a_{21} & b_{11}a_{11}a_{31} \\ & b_{11}a_{21}^2 & b_{11}a_{21}a_{31} \\ & & b_{11}a_{31}^2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}.$$

From the w^2 coefficient it follows that $a_{31}=0$. It now follows from the v^2 coefficient that $a_{21}^2=0$. The action set is

$$G_{x,\xi}^t = \begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}, \qquad |G_{x,\xi}|=(p-1)^5p^4(p+1). \tag{B.38}$$

11. Case $(\mathscr{O}_{1^4}, \mathscr{O}_{D11})$: For standard representatives

$$x = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 1 & 0 \\ & & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \tag{B.39}$$

and

$$V_x = \begin{pmatrix} 0 & 0 & y + \frac{z}{2} \\ & z & * \\ & & * \end{pmatrix} \begin{pmatrix} y & * & * \\ & * & * \\ & & * \end{pmatrix}, \qquad V_\xi = \begin{pmatrix} * & * & * \\ & * & * \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & * & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}. \tag{B.40}$$

Since the action on ξ results in two forms which are linearly dependent, to be orthogonal to V_x the second form is 0, and thus it may be assumed that the second form is acted on by a scalar under the GL_2 action.
The action on ξ is thus given by

$$\begin{aligned} &\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix} \right) \cdot \xi \\ &= \begin{pmatrix} 2b_{11}a_{11}a_{12} & b_{11}(a_{11}a_{22} + a_{12}a_{21}) & b_{11}(a_{11}a_{32} + a_{12}a_{31}) \\ & 2b_{11}a_{21}a_{22} & b_{11}(a_{21}a_{32} + a_{22}a_{31}) \\ & & 2b_{11}a_{31}a_{32} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}. \end{aligned}$$

It follows from w^2 coefficient that one of a_{31} and a_{32} is 0. Now considering the uw, vw coordinates, both a_{31} and a_{32} are zero, or else the GL_3 matrix would have a 0 column. It follows from the v^2 coefficient that $a_{21}a_{22} = 0$, either of which can hold. The action set is thus

$$G_{x,\xi}^t = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * \\ & * \end{pmatrix} \sqcup \begin{pmatrix} * & * & * \\ * & & * \\ & & * \end{pmatrix} \begin{pmatrix} * & * \\ & * \end{pmatrix}, \qquad |G_{x,\xi}| = 2(p-1)^5 p^4. \tag{B.41}$$

12. Case $(\mathscr{O}_{1^4}, \mathscr{O}_{1^4})$: For standard representatives

$$x = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ & 1 & 0 \\ & & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 0 & 0 & -1 \\ & 1 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \tag{B.42}$$

and

$$V_x = \begin{pmatrix} 0 & 0 & y + \frac{z}{2} \\ & z & * \\ & & * \end{pmatrix} \begin{pmatrix} y & * & * \\ & * & * \\ & & * \end{pmatrix}, \qquad V_\xi = \begin{pmatrix} * & * & * \\ & * & * \\ & & y \end{pmatrix} \begin{pmatrix} * & * & -z + y \\ & z & 0 \\ & & 0 \end{pmatrix}, \tag{B.43}$$

any combination $\lambda_1(v^2 - 2uw) + 2\lambda_2 u^2$ with $\lambda_1 \neq 0$ cannot be a double line, and hence $b_{21} = 0$. By considering the v^2 and w^2 coefficients in the second quadratic form, it follows that $a_{21} = a_{31} = 0$. The action on ξ is thus given by

$$\begin{aligned}&\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & a_{32} & a_{33}\end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ & b_{22}\end{pmatrix}\right) \cdot \xi \\ &= \begin{pmatrix} b_{11}(a_{12}^2 - 2a_{11}a_{13}) + 2b_{12}a_{11}^2 & b_{11}(-a_{11}a_{23} + a_{12}a_{22}) & b_{11}(-a_{11}a_{33} + a_{12}a_{32}) \\ & b_{11}a_{22}^2 & b_{11}a_{22}a_{32} \\ & & b_{11}a_{32}^2 \end{pmatrix} \\ &\quad \times \begin{pmatrix} 2b_{22}a_{11}^2 & 0 & 0 \\ & 0 & 0 \\ & & 0\end{pmatrix}.\end{aligned}$$

Considering the w^2 coefficient of the first quadratic form implies $a_{32} = 0$. The action thus reduces to

$$\begin{aligned}&\left(\begin{pmatrix} a & * & * \\ & b & * \\ & & c\end{pmatrix}, \begin{pmatrix} d & * \\ & e\end{pmatrix}\right) \cdot \xi \\ &= \begin{pmatrix} 2a^2b_{12} - 2aa_{13}d + a_{12}^2 d & -aa_{23}d + a_{12}bd & -acd \\ & b^2 d & 0 \\ & & 0\end{pmatrix} \begin{pmatrix} 2a^2 e & 0 & 0 \\ & 0 & 0 \\ & & 0\end{pmatrix}.\end{aligned} \tag{B.44}$$

The action set is thus

$$G_{x,\xi}^t = \begin{pmatrix} a & * & * \\ & b & * \\ & & c\end{pmatrix} \begin{pmatrix} d & * \\ & e\end{pmatrix}, \qquad b^2 = ac, b^2 d = a^2 e, \qquad |G_{x,\xi}| = (p-1)^3 p^4. \tag{B.45}$$

References

[1] Manjul Bhargava, Higher composition laws. II. On cubic analogues of Gauss composition, Ann. Math. (2) 159 (2) (2004) 865–886.
[2] Manjul Bhargava, Higher composition laws. III. The parametrization of quartic rings, Ann. Math. (2) 159 (3) (2004) 1329–1360.
[3] Manjul Bhargava, Higher composition laws. IV. The parametrization of quintic rings, Ann. Math. (2) 167 (1) (2008) 53–94.
[4] Wieb Bosma, John Cannon, Catherine Playoust, The Magma algebra system. I. The user language, J. Symb. Comput. 24 (1997) 235–265.
[5] Robert Hough, The shape of quartic fields, arXiv preprint, arXiv:1706.09987, 2017.
[6] Mikio Sato, Takuro Shintani, On zeta functions associated with prehomogeneous vector spaces, Ann. Math. (2) 100 (1974) 131–170.
[7] Takuro Shintani, On Dirichlet series whose coefficients are class numbers of integral binary cubic forms, J. Math. Soc. Jpn. 24 (1972) 132–188.
[8] Takashi Taniguchi, Frank Thorne, Orbital L-functions for the space of binary cubic forms, Can. J. Math. 65 (6) (2013) 1320–1383.
[9] Takashi Taniguchi, Frank Thorne, Secondary terms in counting functions for cubic fields, Duke Math. J. 162 (13) (2013) 2451–2508.

[10] Takashi Taniguchi, Frank Thorne, Orbital exponential sums for prehomogeneous vector space, arXiv:1607.07827v1, 2016.
[11] Wolfram Research, Inc., Mathematica, Version 11.3, 2018, Champaign, IL.
[12] D.J. Wright, A. Yukie, Prehomogeneous vector spaces and field extensions, Invent. Math. 110 (2) (1992) 283–314.
[13] A. Yukie, Shintani Zeta Functions, London Mathematical Society Lecture Note Series, vol. 183, Cambridge University Press, Cambridge, 1993.