A Quasi-strictly Non-volterra Quadratic Stochastic Operator

A. J. M. Hardin[1] · U. A. Rozikov[2]

Abstract

We consider a four-parameter family of non-Volterra operators defined on the two-dimensional simplex and show that, with one exception, each such operator has a unique fixed point. Depending on the parameters, we establish the type of this fixed point. We study the set of ω-limiting points for each trajectory and show that this set can be a single point or can contain a 2-periodic trajectory.

Keywords Quadratic stochastic operator · Simplex · Trajectory · Volterra and non-Volterra operators

Mathematics Subject Classification Primary 37N25; Secondary 92D10

1 Introduction

Quadratic stochastic operators (QSOs) frequently arise in many models of mathematical genetics, namely, in the theory of heredity (see e.g. [5,8] for motivations and results related to QSOs). Here we shall investigate a family of QSOs defined on the two-dimensional simplex. Let us give some definitions first. The $(m-1)$-dimensional simplex is defined by

$$S^{m-1} = \left\{ x = (x_1, x_2, \ldots, x_m) \in R^m : \text{ for any } i \ x_i \geq 0, \text{ and } \sum_{i=1}^{m} x_i = 1 \right\}.$$

$$(1.1)$$

✉ U. A. Rozikov
rozikovu@yandex.ru

A. J. M. Hardin
ahardin@ou.edu

[1] University of Oklahoma, 660 Parrington Oval, Norman, OK 73019, USA

[2] Institute of Mathematics, 81, Mirzo Ulug'bek str., Tashkent, Uzbekistan 100125

A QSO is the mapping $V : S^{m-1} \rightarrow S^{m-1}$ with

$$(Vx)_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j \qquad (1.2)$$

where

$$P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \sum_{k=1}^{m} P_{ij,k} = 1 \quad \text{for all } i, j, k. \qquad (1.3)$$

For a given $x^{(0)} \in S^{m-1}$ the trajectory $x^{(n)}$ of $x^{(0)}$ under the action of the QSO (1.2) is defined by $x^{(n+1)} = V(x^{(n)})$, where $n = 0, 1, 2, \ldots$.

One of the main problems in mathematical biology consists in the study of the asymptotic behavior of these trajectories.

Denote by $\omega(x^0)$ the set of ω-limiting points of trajectory $x^{(n)}$. Since S^{m-1} is a compact set and $\{x^{(n)}\} \subset S^{m-1}$ it follows that $\omega(x^0) \neq \emptyset$. It is clear that if $\omega(x^0)$ consists of a single point, then the trajectory converges and $\omega(x^0)$ is a fixed point of (1.2). The limit behavior of the trajectories of any QSO on one-dimensional space was fully studied by Lyubich [9]. However, the problem is still open even in the two-dimensional simplex.

Definition 1 [11] A quadratic stochastic operator is called Volterra if

$$P_{ij,k} = 0, \quad \text{for } k \notin \{i, j\}, \quad i, j, k = 1, \ldots, m, \qquad (1.4)$$

and it is called strictly non-Volterra if

$$P_{ij,k} = 0, \quad \text{for } k \in \{i, j\}, \quad i, j, k = 1, \ldots, m \qquad (1.5)$$

In [4,12] a Volterra operator of a bisexual population was investigated. However, in the non-Volterra case, many questions remain open and there seems to be no general theory available [1,6,10,11,13,14]. In [11] the conception of strictly non-Volterra QSOs was introduced, and it was proved that an arbitrary strictly non-Volterra quadratic stochastic operator on the two-dimensional simplex has a unique fixed point, which is not attracting.

Remark 1 A strictly non-Volterra operator exists only if $m \geq 3$. In [11] the case $m = 3$ was studied. In the present paper we will study the dynamics of the case $m = 3$ for a non-Volterra QSO which has a strictly non-Volterra x_1' and x_2'.

In this paper we consider a non-Volterra QSO (which we call quasi-strictly non-Volterra) defined on the two-dimensional simplex which has the form

$$V : \begin{cases} x_1' = \alpha x_2^2 + c x_3^2 + 2x_2 x_3, \\ x_2' = a x_1^2 + d x_3^2 + 2x_1 x_3, \\ x_3' = b x_1^2 + \beta x_2^2 + e x_3^2 + 2x_1 x_2, \end{cases} \qquad (1.6)$$

where $\alpha, \beta, a, b, c, d, e \geq 0$ and

$$a + b = 1, \quad \alpha + \beta = 1, \quad c + d + e = 1. \tag{1.7}$$

The paper is organized as follows. In Sect. 2 we study fixed points of (1.6), where we show that for $e \neq 1$, the QSO (1.6) has a unique fixed point. In the case $e = 1$, we show that the operator has two fixed points. In Sect. 3 we find conditions on parameters under which a fixed point is a repelling, attracting, or saddle point. In the last section we describe the ω-limit set of this non-Volterra QSO on S^2.

2 Fixed Point of Operator

Definition 2 A point $x \in S^{m-1}$ is called a fixed point of a QSO V if $V(x) = x$.

In our case the fixed point is a solution of the following system

$$\begin{cases} x_1 = \alpha x_2^2 + c x_3^2 + 2 x_2 x_3, \\ x_2 = a x_1^2 + d x_3^2 + 2 x_1 x_3, \\ x_3 = b x_1^2 + \beta x_2^2 + e x_3^2 + 2 x_1 x_2. \end{cases} \tag{2.1}$$

Theorem 1 *The non-Volterra QSO* (1.6) *has a unique fixed point* $x^* = (x_1^*, x_2^*, x_3^*) \in S^2$ *in all cases, except when* $e = 1$. *In the case* $e = 1$, *there are two fixed points of the system, one of which is* $(0, 0, 1)$.

Proof We shall consider all possible cases on a and α.

(1) Let $\alpha \neq 0$, $a \neq 0$. Substituting $x_1 = 1 - x_2 - x_3$ into the first equation of (2.1) gives

$$x_2 = \frac{-2x_3 - 1 + \sqrt{4(1 - \alpha c)x_3^2 + 4(1 - \alpha)x_3 + 1 + 4\alpha}}{2\alpha} \geq 0, \tag{2.2}$$

where $x_3 \in [0, \frac{\sqrt{1+4c}-1}{2c}]$ if $c \neq 0$ and $x_3 \in [0, 1]$ if $c = 0$. Similarly, the second equation in (2.1) gives

$$x_1 = \frac{-2x_3 - 1 + \sqrt{4(1 - ad)x_3^2 + 4(1 - a)x_3 + 1 + 4a}}{2a} \geq 0, \tag{2.3}$$

where $x_3 \in [0, \frac{\sqrt{1+4d}-1}{2d}]$ if $d \neq 0$ and $x_3 \in [0, 1]$ if $d = 0$.

Now we may substitute (2.2) and (2.3) into $1 = x_1 + x_2 + x_3$, allowing $x = x_3$ and $f(x) = x$. This gives the function

$$f(x) = \frac{\alpha\sqrt{4(1 - ad)x^2 + 4(1 - a)x + 1 + 4a}}{2(\alpha + a - \alpha a)}$$

$$+ \frac{a\sqrt{4(1-\alpha c)x^2 + 4(1-\alpha)x + 1 + 4\alpha} - \alpha - a - 2\alpha a}{2(\alpha + a - \alpha a)}.$$

Now we define

$$g(x) = 4(1-ad)x^2 + 4(1-a)x + 1 + 4a > 0,$$
$$h(x) = 4(1-\alpha c)x^2 + 4(1-\alpha)x + 1 + 4\alpha > 0.$$

Thus, we have

$$g'(x) = 8(1-ad)x + 4(1-a) \geq 0, \quad h'(x) = 8(1-\alpha c)x + 4(1-\alpha) \geq 0,$$
$$g''(x) = 8(1-ad) \geq 0, \qquad\qquad h''(x) = 8(1-\alpha c)x \geq 0.$$

Then,

$$f(x) = \frac{\alpha\sqrt{g(x)}}{2(\alpha + a - \alpha a)} + \frac{a\sqrt{h(x)} - \alpha - a - 2\alpha a}{2(\alpha + a - \alpha a)}.$$

Differentiating $f(x)$ gives

$$f'(x) = \frac{\alpha g'(x)}{4(\alpha + a - \alpha a)\sqrt{g(x)}} + \frac{a h'(x)}{4(\alpha + a - \alpha a)\sqrt{h(x)}} \geq 0,$$

$$f''(x) = \frac{\alpha}{4(\alpha + a - \alpha a)} \frac{g(x)g''(x) - \frac{1}{2}g'(x)^2}{\sqrt{g(x)^3}}$$

$$+ \frac{a}{4(\alpha + a - \alpha a)} \frac{h(x)h''(x) - \frac{1}{2}h'(x)^2}{\sqrt{h(x)^3}} \geq 0.$$

These inequalities follow from (1.7) as well as the fact that $\alpha + a - \alpha a > 0$. Additionally, substituting the values for $g(x)$, $h(x)$, and their derivatives into the inequalities

$$g(x)g''(x) - \tfrac{1}{2}g'(x)^2 \geq 0, \; h(x)h''(x) - \tfrac{1}{2}h'(x)^2 \geq 0,$$

gives

$$(4(1-ad)x^2 + 4(1-a)x + 1 + 4a)(1-ad) - (2(1-ad)x + (1-a))^2 \geq 0,$$
$$(4(1-\alpha c)x^2 + 4(1-\alpha)x + 1 + 4\alpha)(1-\alpha c) - (2(1-\alpha c)x + (1-\alpha))^2 \geq 0.$$

This can be reduced to

$$\alpha(-6 + a + d + 4ad) \leq 0, \; a(-6 + \alpha + c + 4\alpha c) \leq 0.$$

Thus the inequalities $g(x)g''(x) - \frac{1}{2}g'(x)^2 \geq 0$ and $h(x)h''(x) - \frac{1}{2}h'(x)^2 \geq 0$ can be demonstrated to be true.

Additionally,

$$f(0) = \frac{\alpha(\sqrt{1+4a} - 1 - a) + a(\sqrt{1+4\alpha} - 1 - \alpha)}{2(\alpha + a - \alpha a)} > 0,$$

$$f(1) = \frac{\alpha\sqrt{9 - ad} + a\sqrt{9 - \alpha c} - \alpha - a - 2\alpha a}{2(\alpha + a - \alpha a)} \leq 1,$$

which follows from the above inequalities in addition to the fact that

$$\sqrt{9 - ad} \leq 3, \ \sqrt{9 - \alpha c} \leq 3.$$

Additionally, both $\sqrt{9 - ad}$ and $\sqrt{9 - \alpha c}$ can only be simultaneously equal to 3 when $c = d = 0$ (and thus when $e = 1$). This means that $f(1) = 1$ when $e = 1$ and $f(1) < 1$ otherwise.

Thus the function is increasing and convex in $[0, 1]$. Therefore, for $e \in [0, 1)$ the system has a unique fixed point, since $f(x)$ will only intersect the line x at one point in the domain $[0, 1]$. When $e = 1$ the system has a fixed point $(0, 0, 1)$. It can also be directly shown that when $e = 1$, $f(\frac{9}{10}) < \frac{9}{10}$. This demonstrates that the function $f(x)$ must cross the line x prior to reaching the value $f(1) = 1$. This proves that there are two fixed points when $e = 1$.

(2) Let $\alpha \neq 0$, $a = 0$ can be handled similarly. Substituting $x_1 = 1 - x_2 - x_3$ into the first equation of (2.1) gives

$$x_2 = \frac{-2x_3 - 1 + \sqrt{4(1 - \alpha c)x_3^2 + 4(1 - \alpha)x_3 + 1 + 4\alpha}}{2\alpha} \geq 0, \qquad (2.4)$$

where $x_3 \in [0, \frac{\sqrt{1+4c}-1}{2c}]$ if $c \neq 0$ and $x_3 \in [0, 1]$ if $c = 0$. The second equation in (2.1) gives

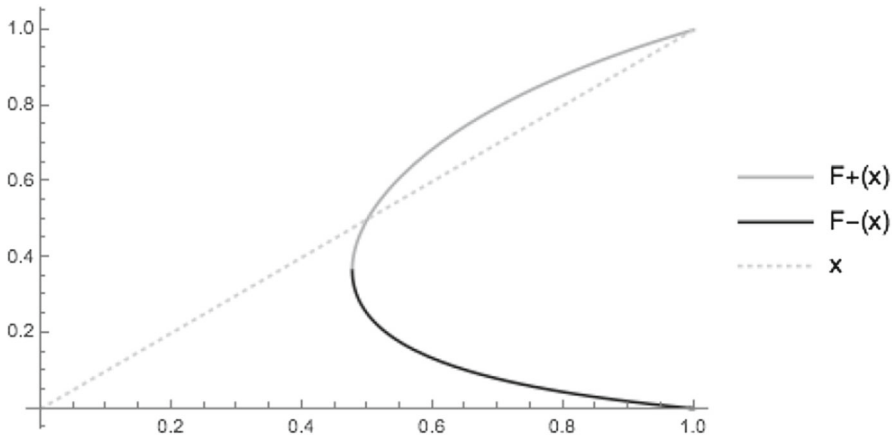$$x_1 = \frac{1 - x_3 - dx_3^2}{2x_3 + 1} \geq 0, \qquad (2.5)$$

where $x_3 \in [0, \frac{\sqrt{1+4d}-1}{2d}]$ if $d \neq 0$ and $x_3 \in [0, 1]$ if $d = 0$. This restriction ensures that x_1 is positive (Figs. 1, 2).

Directly substituting (2.4) and (2.5) into $1 = x_1 + x_2 + x_3$, allowing $x = x_3$, gives

$$1 = x + \frac{1 - x - dx^2}{2x + 1} + \frac{-2x - 1 + \sqrt{4(1 - \alpha c)x^2 + 4(1 - \alpha)x + 1 + 4\alpha}}{2\alpha}.$$

Solving for the xs present in the first two terms of the above equation [but not any of the xs in the (2.4) term], and allowing $F_\pm(x) = x$, gives the functions

$$F_\pm(x) = \frac{1 + 2\alpha + 2x - \sqrt{h(x)} \pm \sqrt{q(x)}}{2\alpha(2 - d)},$$

Fig. 1 $\alpha = .0001, e = 1$
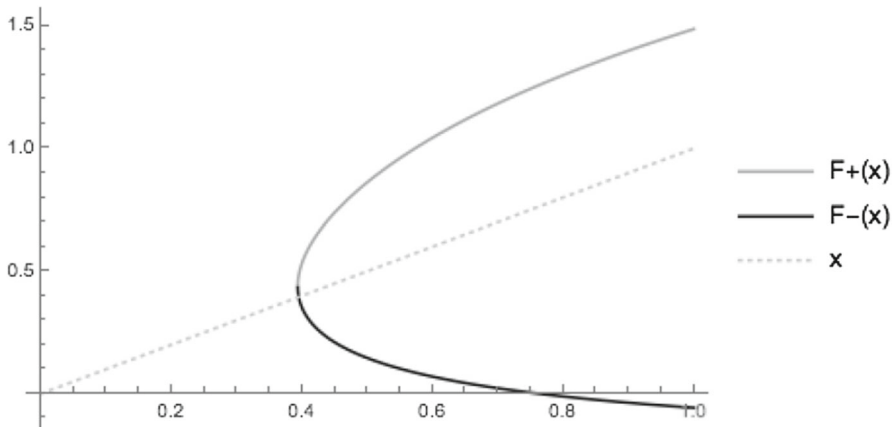
Fig. 2 $\alpha = .0001, e = 0.18$

where

$$h(x) = 4(1 - \alpha c)x^2 + 4(1 - \alpha)x + 1 + 4\alpha \geq 0,$$
$$q(x) = 2\alpha(2 - d)(1 - \sqrt{h(x)} + 2x) + (1 - \sqrt{h(x)} + 2x + 2\alpha)^2.$$

Two examples of the function $F_{\pm}$ graphed against the line x are given below. ☐

Lemma 1 $F_{\pm}(x) = x$ *at a unique point when* $e \in [0, 1)$ *and at two points—one of which is* $F_+(1) = 1$—*when* $e = 1$.

Proof We note that when $q(x) = 0$,

$$x_q = \frac{1}{2c}\left[-5 + d + \sqrt{12 - 8d + d^2}\right.$$
$$\left. + (37 + 2d^2 - 10\sqrt{12 - 8d + d^2} + 2d(-9 + \sqrt{12 - 8d + d^2})\right)$$

$$- 2c(2 - d - \sqrt{12 - 8d + d^2} + \alpha(14 + d^2 - 4\sqrt{12 - 8d + d^2}$$
$$+ d(-8 + \sqrt{12 - 8d + d^2})))^{\frac{1}{2}}\Big].$$

The curve $F_\pm$ is imaginary when $x < x_q$. When $x = x_q$, $F_+(x) = F_-(x)$, and when $x > x_q$, $F_+(x) > F_-(x)$.

Additionally, at $x = 1$ we have

$$F_+(1) = \frac{3 + 2\alpha - \sqrt{h(1)} + \sqrt{q(1)}}{2\alpha(2 - d)} \geq 1,$$

where

$$h(1) = 9 - 4\alpha c \geq 0,$$
$$q(1) = 2\alpha(2 - d)(3 - \sqrt{h(1)}) + (3 - \sqrt{h(1)} + 2\alpha)^2.$$

It can be readily shown that in the case that $e = 1$, $F_+(1) = 1$. When $e \in [0, 1)$, we would like to show that $F_+(1) > 1$. This can be reduced to showing that $\sqrt{q(1)} + 2\alpha d + 3 > \sqrt{h(1)} + 2\alpha$. It can be easily shown that $\sqrt{q(1)} > 2\alpha$. Additionally, when $d = 0$, $3 > \sqrt{h(1)}$. When $d \neq 0$, $3 \geq \sqrt{h(1)}$, but $2\alpha d > 0$. The culmination of the above facts proves the inequality $\sqrt{q(1)} + 2\alpha d + 3 > \sqrt{h(1)} + 2\alpha$ to be always true.

Thus, $F_+(1) > 1$ when $e < 1$. It can be similarly shown that $F_-(1) < 1$. Analysis of the derivations of $F_\pm$ shows that the following inequalities are true.

$$F'_+(x) = \frac{1}{2\alpha(2 - d)}\left(2 - \frac{h'(x)}{2\sqrt{h(x)}} + \frac{q'(x)}{2\sqrt{q(x)}}\right) \geq 0,$$

$$F'_-(x) = \frac{1}{2\alpha(2 - d)}\left(2 - \frac{h'(x)}{2\sqrt{h(x)}} - \frac{q'(x)}{2\sqrt{q(x)}}\right) \leq 0,$$

$$F''_+(x) = \frac{1}{2\alpha(2 - d)}\left(-\frac{h(x)h''(x) - \frac{1}{2}(h'(x))^2}{\sqrt{(h(x))^3}} + \frac{q(x)q''(x) - \frac{1}{2}(q'(x))^2}{\sqrt{(q(x))^3}}\right) \leq 0,$$

for all $x > x_q$.

The above demonstrates that $F_\pm(x)$ must intersect the line x at least once on $[0, 1)$. Because $F_+(x)$ is increasing and concave, and $F_-(x)$ is decreasing, $F_\pm(x)$ and x will only intersect once on $[0, 1)$. Additionally, $F_+(1) = 1$ only when $e = 1$. Thus the lemma is proven. □

Therefore, a direct extension of the above lemma shows that when $e = 1$, the system (1.6) has two fixed points—one of which is $(0, 0, 1)$. And when $e < 1$, it has a unique fixed point.

(3) The case $\alpha = 0$, $a \neq 0$ can be handled analogously to the previous case.

(4) Let $\alpha = a = 0$. The system is therefore

$$\begin{cases} x_1 = cx_3^2 + 2x_2x_3 \\ x_2 = dx_3^2 + 2x_1x_3 \\ x_3 = (x_1 + x_2)^2 + ex_3^2 = (1 - x_3)^2 + ex_3^2 \end{cases} \tag{2.6}$$

The proof of the fourth case will follow directly from the following lemma.

Lemma 2 *If $e \in [0, 1)$, the system has a unique fixed point, (x_1^*, x_2^*, x_3^*), where*

$$x_1^* = \frac{7 + 24d - 2e - 19de - 5e^2 + 2e^2d + (3 - 11d + 4de + 3d^2)\sqrt{5 - 4e}}{2(1 + e)(-13 + 6\sqrt{5 - 4e} + 6e + e^2)},$$

$$x_2^* = 1 - \frac{7 + 24d - 2e - 19de - 5e^2 + 2e^2d + (3 - 11d + 4de + 3d^2)\sqrt{5 - 4e}}{2(1 + e)(-13 + 6\sqrt{5 - 4e} + 6e + e^2)}$$

$$\quad - \frac{3 - \sqrt{5 - 4e}}{2(1 + e)},$$

$$x_3^* = \frac{3 - \sqrt{5 - 4e}}{2(1 + e)}.$$

However, when $e = 1$ the system has two fixed points: $(0, 0, 1)$ *and* $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$.

Proof Examine each possible case.

- Let $e \in [0, 1)$. Solving the third equation of (2.6) for x_3 gives $x_3^* = \frac{3 \pm \sqrt{T}}{2(1+e)} > 0$ where $T = 5 - 4e > 0$.

  Assume for the purpose of contradiction that $\frac{3 + \sqrt{T}}{2(1+e)} \leq 1$. This can be reduced to $\sqrt{T} \leq 2e - 1$. If $e \leq \frac{1}{2}$, then $2e - 1 \leq 0$ and $\sqrt{T} \geq 0$, so $\frac{3+\sqrt{T}}{2(1+e)} \not\leq 1$. Additionally, if $e > \frac{1}{2}$ then $0 < \sqrt{T} \leq 2e - 1$ which can be reduced to $1 \leq e$ which is not true for $e \in [0, 1)$. Thus $\frac{3+\sqrt{T}}{2(1+e)} \not\leq 1$ for any $e \in [0, 1)$ and is not a fixed point of $x_3' = (1 - x_3)^2 + ex_3^2$.

  It can be proved that $\frac{3-\sqrt{T}}{2(1+e)} \leq 1$ because it can be reduced to $\sqrt{T} \geq 1 - 2e$. As it was shown previously that , $\sqrt{T} \leq 1 - 2e$ is false for all $e \in [0, 1)$, it must be that $\sqrt{T} > 1 - 2e$. Therefore,

$$x_3^* = \frac{3 - \sqrt{5 - 4e}}{2(1 + e)},$$

  is a unique fixed point of the system.
  Substituting x_3^* into the first two equations of (2.6) gives $x_1 = cx_3^{*2} + 2x_2x_3^*$ and $x_2 = dx_3^{*2} + 2x_1x_3^*$. Substituting this value of x_2 into x_1 and reducing gives

$$x_1^* = \frac{(3x_3^* - 1)(c + 2dx_3^*)}{5 + e - 12x_3^*},$$

which can be written as

$$x_1^* = \frac{7 + 25d - 2e - 19de - 5e^2 + 2e^2d + (3 - 11d + 4de + 3d^2)\sqrt{5 - 4e}}{2(1 + e)(-13 + 6\sqrt{5 - 4e} + 6e + e^2)}.$$

Additionally, we know that $x_2^* = 1 - x_1^* - x_3^*$ which yields

$$x_2^* = 1 - \frac{7 + 25d - 2e - 19de - 5e^2 + 2e^2d + (3 - 11d + 4de + 3d^2)\sqrt{5 - 4e}}{2(1 + e)(-13 + 6\sqrt{5 - 4e} + 6e + e^2)}$$

$$- \frac{3 - \sqrt{5 - 4e}}{2(1 + e)}.$$

Thus, (x_1^*, x_2^*, x_3^*) is a unique fixed point of the system.

- Let $e = 1$. The system is therefore

$$\begin{cases} x_1 = 2x_2x_3, \\ x_2 = 2x_1x_3, \\ x_3 = (1 - x_3)^2 + x_3^2. \end{cases} \tag{2.7}$$

Additionally, we know that $x_3^* = \frac{3 \pm \sqrt{(5-4e)}}{2(1+e)}$; therefore, $x_3^* = \frac{1}{2}$ or 1. When $x_3^* = \frac{1}{2}$, it follows from (2.7) that $x_1 = x_2$, and it follows from $x_1 + x_2 + x_3 = 1$ that $x_1 + x_2 = \frac{1}{2}$. Thus, $x_1^* = x_2^* = \frac{1}{4}$ and the point $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) \in S^2$ is a fixed point of the system. When $x_3^* = 1$, it follows from $x_1 + x_2 + x_3 = 1$ that $x_1^* = x_2^* = 0$. Thus, the point $(0, 0, 1) \in S^2$ is a second fixed point of the system. □

By the above cases, all possible values for the system are considered and the theorem is proved.

3 The Type of the Fixed Point

Definition 3 [2] A fixed point x^* of the operator V is called hyperbolic if its Jacobian J at x^* has no eigenvalues on the unit circle.

Definition 4 [2] A hyperbolic fixed point x^* is called:

(i) attracting if all the eigenvalues of the Jacobian $J(x^*)$ are < 1 in absolute value;
(ii) repelling if all the eigenvalues of the Jacobian $J(x^*)$ are > 1 in absolute value;
(iii) a saddle otherwise.

To find the type of a fixed point we use $x_3 = 1 - x_1 - x_2$ to rewrite QSO (1.6) as follows:

$$V : \begin{cases} x_1' = c - 2cx_1 + cx_1^2 + 2(c - 1)x_1x_2 + 2(1 - c)x_2 + (\alpha + c - 2)x_2^2, \\ x_2' = d - 2dx_2 + dx_2^2 + 2(d - 1)x_1x_2 + 2(1 - d)x_1 + (a + d - 2)x_1^2, \end{cases}$$

where $(x_1, x_2) \in \{(x, y) : x, y \geq 0, \ 0 \leq x + y \leq 1\}$ and x_1, x_2, are the first two coordinates of a point lying in the two-dimensional simplex.

The Jacobian, $J(x^*)$, has the representation

$$
\begin{pmatrix}
-2c(1 - x_1^* - x_2^*) - 2x_2^* & 2(\alpha - 1)x_2^* + 2(1 - c)(1 - x_1^* - x_2^*) \\
2(a - 1)x_1^* + 2(1 - d)(1 - x_1^* - x_2^*) & -2d(1 - x_1^* - x_2^*) - 2x_1^*
\end{pmatrix}.
$$
(3.1)

The Jacobian (3.1) has the eigenvalues $\lambda_{1,2} = ex_3^* - 1 \pm \sqrt{D(a, \alpha, c, e)}$, where

$$
\begin{aligned}
D \equiv D(a, \alpha, c, e) &= (ex_3^* - 1)^2 + 4ex_3^{*2} \\
&+ 4[(b\beta - 1)x_1^*x_2^* + (a(1 - c) - 1)x_1^*x_3^* + (\alpha(1 - d) - 1)x_2^*x_3^*]. \quad (3.2)
\end{aligned}
$$

The classification of these eigenvalues is as follows:

$$
\begin{cases}
\text{If } D < -1 + (1 - ex_3^*)^2, & \text{the fixed point is repelling;} \\
\text{If } D = -1 + (1 - ex_3^*)^2, & \text{the fixed point is nonhyperbolic;} \\
\text{If } -1 + (1 - ex_3^*)^2 < D < 0, & \text{the fixed point is attracting;} \\
\text{If } D = 0 \text{ and } ex_3^* = 0, & \text{the fixed point is nonhyperbolic;} \\
\text{If } D = 0 \text{ and } ex_3^* > 0, & \text{the fixed point is attracting;} \\
\text{If } 0 < D < e^2 x_3^{*2}, & \text{the fixed point is attracting;} \\
\text{If } D = e^2 x_3^{*2}, & \text{the fixed point is nonhyperbolic;} \\
\text{If } e^2 x_3^{*2} < D < (2 - ex_3^*)^2, & \text{the fixed point is a saddle point;} \\
\text{If } D = (2 - ex_3^*)^2, & \text{the fixed point is nonhyperbolic;} \\
\text{If } (2 - ex_3^*)^2 < D, & \text{the fixed point is repelling.}
\end{cases}
$$
(3.3)

In [11] it was proven that strictly non-Volterra QSOs with $m = 3$ have a unique fixed point and that the type of the hyperbolic fixed point can never be attracting. However, in the system (1.6), the introduction of the parameter e has caused an attracting fixed point to become possible, as evidenced by the following example.

Example When $c = \beta = \frac{5}{8}, d = b = 0, \alpha = e = \frac{3}{8}$, and $a = 1$ the system (1.6) can be written

$$
V : \begin{cases}
x_1' = \frac{3}{8}x_2^2 + \frac{5}{8}x_3^2 + 2x_2x_3, \\
x_2' = x_1^2 + 2x_1x_3, \\
x_3' = \frac{5}{8}x_2^2 + \frac{3}{8}x_3^2 + 2x_1x_2.
\end{cases}
$$
(3.4)
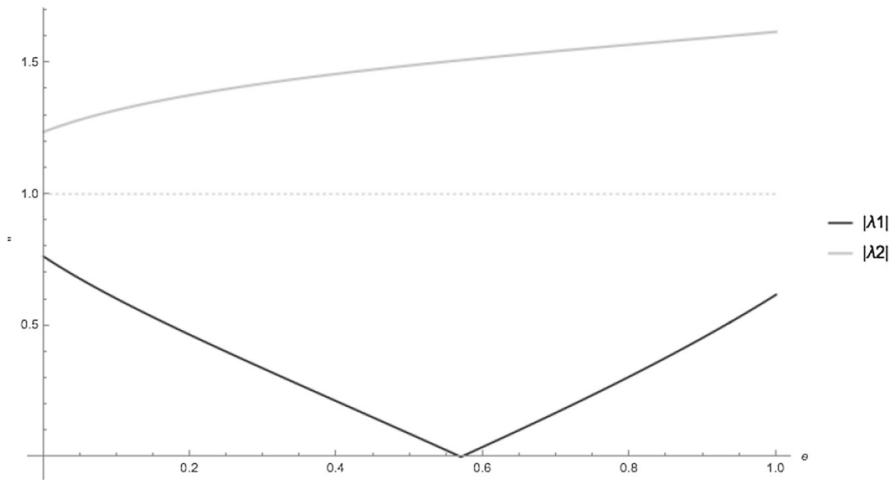
Fig. 3 Effect of e on eigenvalues

The fixed point of this system is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Substituting these values into the eigenvalues of the Jacobian gives $\lambda_{1,2} = -\frac{7}{8} \pm \frac{1}{8}\sqrt{\frac{13}{3}} i$. Therefore, $|\lambda_{1,2}| = \sqrt{\frac{5}{6}} < 1$, and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is attracting.

In the case where $\alpha = a = 0$ (i.e. system (2.6)), the eigenvalues of the Jacobian can be written

$$\lambda_{1,2} = ex_3^* - 1 \pm \sqrt{1 + 2(e - 2)x_3^* + (4 + e^2)x_3^{*2}},$$

where $x_3^* = \frac{3 - \sqrt{5 - 4e}}{2(1+3)}$. It can be proven that $|\lambda_1| < 1$ and $|\lambda_2| > 1$ for all e. The inequality $|\lambda_1| < 1$ can be proven from the facts that

$$0 \le 1 + 2(e - 2)x_3^* + (4 + e^2)x_3^{*2},$$

$$0 < ex_3^* + \sqrt{1 + 2(e - 2)x_3^* + (4 + e^2)x_3^{*2}},$$

$$x_3^{*2} + ex_3^* < \frac{3}{4} + x_3^*.$$

The last inequality, $x_3^{*2} + ex_3^* < \frac{3}{4} + x_3^*$, follows from the fact that $x_3^{*2} \le x_3^*$ and from substituting the value of x_3^* into the inequality $ex_3^* < \frac{3}{4}$, which gives $3e < 3 + 2e\sqrt{5 - 4e}$. Additionally, $|\lambda_2| > 1$ can be proven from the fact that $ex_3^* - 1 - \sqrt{1 + 2(e - 2)x_3^* + (4 + e^2)x_3^{*2}} < -1$. This can be reduced to the quadratic $1 + (2e - 4)x_3^* + 4x_3^{*2}$, which is always positive. This means that the fixed point x_3^* is a saddle point for all $e \in [0, 1)$. This can be seen clearly in the graph (Fig. 3).

In the case that $a = \alpha = 0$ and $e = 1$, the fixed point $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ is a saddle point and $(0, 0, 1)$ is a repeller.

Additionally, for all cases where $e = 1$ and $(0, 0, 1)$ is a second fixed point of the system, it can be seen that $\lambda_{1,2} = \pm 2$. Thus the second fixed point that occurs when $e = 1$ is always a repeller.

Remark 2 A non-Volterra QSO with $m = 3$ (1.6) also has a unique fixed point in all cases except $e = 1$ for which there are two fixed points, one of which-(0,0,1)-is always a repeller. Additionally, the fixed point of a non-Volterra QSO may be attracting.

4 The ω-Limit Set

In this section we shall describe the ω-limit set of trajectories under certain parameter restrictions. Let $x^0 = (x_1^0, x_2^0, x_3^0) \in S^2$ be the initial point and let $\{x^{(n)}, n = 0, 1, 2, \ldots\}$ be the trajectory of x^0 under the action of the operator (1.6); that is,

$$x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}) = V(x^{(n-1)}), \qquad n = 1, 2, \ldots \quad x^{(0)} = x^0.$$

For simplicity we shall examine the case in which $a = \alpha = 0$; therefore the operator can be written as (2.6), i.e.,

$$\begin{cases} x_1' = cx_3^2 + 2x_2 x_3, \\ x_2' = dx_3^2 + 2x_1 x_3, \\ x_3' = (1 - x_3)^2 + ex_3^2, \end{cases} \tag{4.1}$$

which demonstrates that the trajectory of the third coordinate $\{x_3^{(n)}\}$ is defined by the dynamical system of

$$\varphi(x) = (1 - x)^2 + ex^2.$$

4.1 Case $e = 1$

In this case operator has the form (denoted by V_1)

$$V_1 : \begin{cases} x_1' = 2x_2 x_3, \\ x_2' = 2x_1 x_3, \\ x_3' = (1 - x_3)^2 + x_3^2. \end{cases} \tag{4.2}$$

This operator has been studied in [7]: It is easy to see that

$$\mathrm{Fix}(V_1) = \{(1/4, 1/4, 1/2), (0, 0, 1)\}.$$

For any $(x, y, 1/2)$ with $x + y = 1/2$, we have

$$V_1^2(x, y, 1/2) = V_1(y, x, 1/2) = (x, y, 1/2).$$

In fact the set of all 2-periodic points is

$$\text{Per}_2(V_1) = \{(x, y, 1/2) \in S^2 : x + y = 1/2\}.$$

Moreover it is easy to see that $V_1(x, y, 0) = (0, 0, 1)$, for any $(x, y, 0) \in S^2$.
For $\theta \in [0, \infty)$, denote

$$M_\theta = \begin{cases} \{(x, y, z) \in S^2 : xy = 0\}, & \text{if } \theta = 0, \\ \{(x, y, z) \in S^2 : x = \theta y \text{ or } x = \frac{1}{\theta}y\}, & \text{if } \theta \in (0, \infty). \end{cases}$$

Theorem 2 [7] *If $e = 1$, then*

1. *For any initial point (x_1^0, x_2^0, x_3^0), with $x_3^0 = 0$ or $x_3^0 = 1$ we have*

$$\lim_{n \to \infty} V_1^n(x_1^0, x_2^0, x_3^0) = (0, 0, 1).$$

2. *For any initial point $x^0 = (x_1^0, x_2^0, x_3^0)$, with $x_3^0 \neq 0$ and $x_3^0 \neq 1$, there exists $\theta \in [0, +\infty)$, such that $x^0 \in M_\theta$. Moreover,*

$$\lim_{n \to \infty} V_1^n(x_1^0, x_2^0, x_3^0) = \begin{cases} \left(\frac{\theta}{2(\theta+1)}, \frac{1}{2(\theta+1)}, \frac{1}{2} \right), & n = 2k, k = 1, 2, 3, \dots \\ \left(\frac{1}{2(\theta+1)}, \frac{\theta}{2(\theta+1)}, \frac{1}{2} \right), & n = 2k+1, k = 1, 2, 3, \dots \end{cases}$$

4.2 Case $e \in [0, 1)$

Now we consider the operator (4.1) for all $e \neq 1$. By the above given results we know that this operator has a unique fixed point:

$$\text{Fix}(V) = \{x^* = (x_1^*, x_2^*, x_3^*)\},$$

which is never attractive.

Let us describe periodic points of the operator. By (4.1) the sequence $x^{(n+1)} = V^n(x^0)$ has the form

$$\begin{cases} x_1^{(n+1)} = c(x_3^{(n)})^2 + 2x_2^{(n)}x_3^{(n)}, \\ x_2^{(n+1)} = d(x_3^{(n)})^2 + 2x_1^{(n)}x_3^{(n)}, \\ x_3^{(n+1)} = \varphi(x_3^{(n)}). \end{cases} \quad (4.3)$$

Lemma 3 *If $1/4 \leq e < 1$ then the operator (4.1) does not have any n-periodic point, $n \geq 2$, different from the fixed point x^*.*

Proof First we give analysis of the equation $\varphi(\varphi(x)) = x$, (for existence of 2-periodic points) which has solutions $x_3^*, \overline{x}_3 = \frac{1+\sqrt{1-4e}}{2(1+e)}$, and $\overline{\overline{x}}_3 = \frac{1-\sqrt{1-4e}}{2(1+e)}$, where $\varphi(\overline{x}_3) = \overline{\overline{x}}_3$ and vice-versa. These numbers exist if and only iff $e < 1/4$, and when $e = 1/4$ the

three numbers coincide. Thus for $1/4 \le e < 1$ there are no 2-periodic points of φ. By Sharkovskii's theorem [2] we have that $\varphi^n(x) = x$ does not have solution $x \ne x_3^*$ for all $n \ge 2$. Thus, $x_3^{(n)} = x_3$ has unique solution $x_3 = x_3^*$ for any $n \ge 1$. Using this fact we reduce the equation $V^n(x) = x$ (with $x = (x_1, x_2, x_3)$) of n-periodic points to the equation $L^n(\hat{x}) = \hat{x}$ (with $\hat{x} = (x_1, x_2)$), where L is the linear operator given by

$$L : \begin{cases} x' = c(x_3^*)^2 + 2x_3^* y, \\ y' = d(x_3^*)^2 + 2x_3^* x. \end{cases}$$

It is then easy to see that this linear operator has a unique fixed point (x_1^*, x_2^*) (the first two coordinates of the fixed point x^*). This fixed point is attractive and therefore by the known theorem of liner dynamical systems (see Chapter 3 of [3]), we see that all trajectories of the linear operator tend to the fixed point. Therefore this linear operator has no periodic points except x^*. □

Lemma 4 *If $0 \le e < 1/4$, then the operator* (4.1) *has 2-periodic points, $(\overline{x}_1, \overline{x}_3, \overline{x}_3)$ and $(\overline{\overline{x}}_1, \overline{\overline{x}}_2, \overline{\overline{x}}_3)$—different from the fixed point x^*—which are described explicitly below. Moreover the operator does not have any n-periodic point for all $n \ge 3$.*

The following proof will rely on the concept of topological conjugacy.

Definition 5 Let $f : A \to A$ and $g : B \to B$ be two maps. f and g are called topologically conjugate if there exists a homeomorphism $h : A \to B$ such that, $h \circ f = g \circ h$.

Additionally, it is known (as shown in [2]) that mappings which are topologically conjugate are completely equivalent in terms of their dynamics. In particular, h gives a one-to-one correspondence between periodic points of f and g.

Proof It can be seen from $|\varphi'(x_3^*)| = |1 - \sqrt{5 - 4e}|$ that if $e \in [0, \frac{1}{4})$, then x_3^* is a repelling fixed point of $\varphi(x)$. As mentioned in the proof of the previous lemma, if $e < \frac{1}{4}$ then the function $\varphi(\varphi(x))$ has fixed points $x_3^*, \overline{x}_3 = \frac{1+\sqrt{1-4e}}{2(1+e)}$, and $\overline{\overline{x}}_3 = \frac{1-\sqrt{1-4e}}{2(1+e)}$, i.e., $\varphi^2(\overline{x}_3) = \overline{x}_3$, and $\varphi^2(\overline{\overline{x}}_3) = \overline{\overline{x}}_3$. By substituting $x_3 = \overline{x}_3, \overline{\overline{x}}_3$ in the first and second equations of the system $V^2(x) = x$ and solving it with respect to x_1 and x_2, we get

$$
\begin{aligned}
\overline{x}_1 &= \frac{2de + (1 - e - 2e^2)c + (2de - c(1 + e))\sqrt{1 - 4e}}{2(e - 1)^2(e + 1)}, \\
\overline{\overline{x}}_1 &= \frac{2d + (1 - e - 2e^2)c + (-2d + c(e + 1))\sqrt{1 - 4e}}{2(e - 1)^2(e + 1)}, \\
\overline{x}_2 &= \frac{2ce + (1 - e - 2e^2)d + (2ce - d(1 + e))\sqrt{1 - 4e}}{2(e - 1)^2(e + 1)}, \\
\overline{\overline{x}}_2 &= \frac{2c + (1 - e - 2e^2)d + (-2c + d(e + 1))\sqrt{1 - 4e}}{2(e - 1)^2(e + 1)}.
\end{aligned}
\tag{4.4}
$$

Now we show that the operator has no n-periodic point if $n \ge 3$. It is easy to see that for each solution $x = \tilde{x}_3$ of $\varphi^n(x) = x$, one gets a unique $(\tilde{x}_1, \tilde{x}_2)$ from $V^n(x) = x$.

Therefore the number of periodic points of V is equal to the number of periodic points of φ. Now we show that φ does not have n-periodic points for any $n \geq 3$. Taking $h(x) = ax + b$ one can see that our function φ is topologically conjugate to the logistic map $\xi(x) = \mu x(1 - x)$ with $\mu = 1 + \sqrt{5 - 4e}$. For $e \in [0, \frac{1}{4})$ we have $\mu \in (3, 1 + \sqrt{5}]$. For the logistic map the following is known (see [15]).

If μ between 3 and $1 + \sqrt{5} \approx 3.236$, then ξ has one 2-periodic orbit and all trajectories (except when started at the fixed point) will approach this 2-periodic orbit.

From this fact, by the conjugacy argument, it follows that φ, and thus V, do not have n-periodic point for any $n \geq 3$. $\qquad\square$

Remark 3 The conjugacy argument mentioned in the proof of Lemma 4 can be also used to give an alternative proof of Lemma 3. In this case $\mu = 1 + \sqrt{5 - 4e} \in (2, 3]$ and (see [15]) the function ξ has no periodic points (except fixed points). All trajectories will converge to the non-zero fixed point.

Theorem 3 *Let* $e \in [0, 1)$.

1. *If* $e < \frac{1}{4}$ *then there exists an open set* $\mathcal{U} \subset S^2$ *such that* $\overline{x}, \overline{\overline{x}} \in \mathcal{U}$ *and for any* $x^0 = (x_1^0, x_2^0, x_3^0) \in \mathcal{U}$ *we have*

$$\lim_{n \to \infty} V^n(x^0) = \begin{cases} \overline{x}, & \text{if } x_3^0 \neq x_3^* \text{ and } n = 2k, \\ x^*, & \text{if } x_3^0 = x_3^*, \\ \overline{\overline{x}}, & \text{if } x_3^0 \neq x_3^* \text{ and } n = 2k + 1, \end{cases} \qquad (4.5)$$

   where $x^* = (x_1^*, x_2^*, x_3^*)$ *is fixed point and* $\overline{x} = (\overline{x}_1, \overline{x}_3, \overline{x}_3)$, $\overline{\overline{x}} = (\overline{\overline{x}}_1, \overline{\overline{x}}_2, \overline{\overline{x}}_3)$ *are periodic points described above.*
2. *If* $e \geq \frac{1}{4}$ *then there exists an open set* $U \subset S^2$ *such that* $x^* \in U$ *and for any* $x^0 = (x_1^0, x_2^0, x_3^0) \in U$ *we have*

$$\lim_{n \to \infty} V^n(x_1^0, x_2^0, x_3^0) = (x_1^*, x_2^*, x_3^*).$$

Proof (1) For $e \in [0, \frac{1}{4})$, it can be seen from $|\varphi'(x_3^*)| = |1 - \sqrt{5 - 4e}|$ that x_3^* is a repelling fixed point of $\varphi(x)$. Additionally, when $e < \frac{1}{4}$ the fixed points $\overline{x}_3$ and $\overline{\overline{x}}_3$ of function $g(x) = \varphi(\varphi(x))$ are attracting, which follow from $|g'(\overline{x}_3)| < 1$ and $|g'(\overline{\overline{x}}_3)| < 1$. Define the operator $W : [0, 1]^2 \to [0, 1]^2$ by the first and the last coordinate of the operator V:

$$W : \begin{cases} x_1' = cx_3^2 + 2x_3(1 - x_1 - x_3), \\ x_3' = \varphi(x_3). \end{cases} \qquad (4.6)$$

Now using the Jacobian of the operator $W(W(x))$ one can see that the 2-periodic orbit $\{\overline{x}, \overline{\overline{x}}\}$ is a unique attracting orbit, and the fixed point x^* is a saddle point of V. The operator V has the following invariant sets:

$$\gamma = \{(x_1, x_2, x_3) \in S^2 : x_3 = x_3^*\},$$

$$\Gamma = \{(x_1, x_2, x_3) \in S^2 : x_3 = \overline{x}_3 \text{ or } \overline{\overline{x}}_3\}.$$

Note that if $x^0 \in \gamma$ then $\lim_{n \to \infty} x_3^{(n)} = x_3^*$. If $x^0 \in \Gamma$ then $\lim_{n \to \infty} x_3^{(n)} = \overline{x}_3$ when n is even and $\lim_{n \to \infty} x_3^{(n)} = \overline{\overline{x}}_3$ when n is odd, in these cases the trajectory for x_1 can be written as

$$x_1^{(n+2)} = c\overline{\overline{x}}_2^2 + 2(d\overline{x}_3^2 + 2x_1^{(n)}\overline{x}_3)\overline{\overline{x}}_3 \text{ for even } n,$$

and

$$x_1^{(n+2)} = c\overline{x}_2^2 + 2(d\overline{\overline{x}}_3^2 + 2x_1^{(n)}\overline{\overline{x}}_3)\overline{x}_3 \text{ for odd } n.$$

The existence of the limit (4.5) follows from general theory of dynamical systems (see [2]) and the uniqueness of the attracting 2-periodic points.

(2) Next we shall consider when $e = \frac{1}{4}$. It can be seen from $|\varphi'(x_3^*)| = |1 - \sqrt{5 - 4e}|$ that when $e = \frac{1}{4}$, x_3^* is nonhyperbolic. It can be shown that the quadratic function $\varphi(x) - \frac{2}{5}$ has roots at $\frac{2}{5}$ and $\frac{6}{5}$. Additionally, $\varphi(x) - \frac{2}{5}$ is concave for all x. Therefore,

$$\begin{cases} \varphi(x) - \frac{2}{5} > 0 \Rightarrow \varphi(x) > \frac{2}{5}, & x < \frac{2}{5}, \\ \varphi(x) - \frac{2}{5} < 0 \Rightarrow \varphi(x) < \frac{2}{5}, & x > \frac{2}{5}. \end{cases}$$

which demonstrates that $x_3^{(n)}$ oscillates between $[0, \frac{2}{5})$ and $(\frac{2}{5}, 1]$. Moreover, it can be demonstrated that

$$\begin{cases} x < \varphi(\varphi(x)), & x < \frac{2}{5}, \\ x > \varphi(\varphi(x)), & x > \frac{2}{5}. \end{cases}$$

Thus, $\lim x_3^{(n)} = \frac{2}{5}$.

It can be seen from $|\varphi'(x_3^*)| = |1 - \sqrt{5 - 4e}|$ that when $e \in (\frac{1}{4}, 1)$, x_3^* is an attracting fixed point of $\varphi(x)$.

Therefore, $x_3^{(n)}$ will converge to x_3^* when $e \geq \frac{1}{4}$. On the invariant line γ a trajectory of this operator is as follows:

$$x^{(n)} = (x_1^{(n)}, 1 - x_1^{(n)} - x_3^*, x_3^*),$$

where $x_1^{(n)}$ satisfies the equality

$$x_1^{(n+1)} = x_3^*(2 - (2 - c)x_3^* - 2x_1^{(n)}). \tag{4.7}$$

It follows from (4.7) that $\lim_{n \to \infty} x_1^{(n)} = x_1^*$. Therefore, when $\lim_{n \to \infty} x_3^{(n)} = x_3^*$, then $\lim_{n \to \infty} V(x^0) = (x_1^*, x_2^*, x_3^*)$. $\square$

5 Conclusion

As was discussed in the introduction, there does not exist a general theory for non-Volterra quadratic stochastic operators. This paper represents an additional step towards a more comprehensive understanding of this family of operators. A complete understanding of non-Volterra QSOs would not only be a significant advance in the field of mathematical genetics and dynamical systems, but it would also answer questions about the modeling of populations that have complex genetic structures for certain traits. Further research in this area could include a more complete description of the ω-limit set of the operator studied here, as well as an investigation into a more general theory for non-Volterra QSOs.

References

1. Blath, J., Jamilov, U.U., Scheutzow, M.: (G, μ)-quadratic stochastic operators. J. Differ. Equ. Appl. **20**(8), 1258–1267 (2014)
2. Devaney, R.L.: An Introduction to Chaotic Dynamical Systems: Studies in Nonlinearity. Westview Press, Boulder (2003)
3. Galor, O.: Discrete Dynamical Systems. Springer, Berlin (2007)
4. Ganikhodjaev, N.N., Jamilov, U.U., Mukhitdinov, R.T.: Non-ergodic quadratic operators of bisexual population. Ukr. Math. J. **65**(6), 1152–1160 (2013)
5. Ganikhodzhaev, R.N., Mukhamedov, F.M., Rozikov, U.A.: Quadratic stochastic operators and processes: results and open problems. Inf. Dimens. Anal. Quantum Prob. Relat. Top. **14**(2), 279–335 (2011)
6. Jamilov, U.U., Ladra, M., Mukhitdinov, R.T.: On the equiprobable strictly non-Volterra quadratic stochastic operators. Qual. Theory Dyn. Syst. **16**(3), 645–655 (2017)
7. Khamraev, A. Yu.: On dynamics of a quasi-strongly non Volterra quadratic stochastic operator. Preprint (2018)
8. Lyubich, Y.I.: Mathematical Structures in Population Genetics. Biomathematics, vol. 22. Springer, Berlin, Heidelberg (1992)
9. Lyubich, YI.: Iterations of quadratic maps. Math. Econ. Func. Anal. 109–138, M. Nauka (1974) (**Russian**)
10. Rozikov, U.A., Jamilov, U.U.: F-quadratic stochastic operators. Math. Notes **83**(4), 554–559 (2008)
11. Rozikov, U.A., Jamilov, U.U.: The dynamics of strictly non-Volterra quadratic stochastic operators on the two dimensional simplex. Sb. Math. **200**(9), 1339–1351 (2009)
12. Rozikov, U.A., Jamilov, U.U.: Volterra quadratic stochastic of a two-sex population. Ukr. Math. J. **63**(7), 1136–1153 (2011)
13. Rozikov, U.A., Zada, A.: On dynamics of l-Volterra quadratic stochastic operators. Int. J. Biomath. **3**(2), 143–159 (2010)
14. Rozikov, U.A., Zada, A.: On a class of separable quadratic stochastic operators. Lobachevskii J. Math. **32**(4), 385–394 (2011)
15. Sharkovskii, A.N., Kolyada, S.F., Sivak, A.G., Fedorenko, V.V.: Dynamics of One-Dimensional Mappings. Naukova Dumka, Kiev (1989). (Russian)