

APPROXIMATING POINTWISE PRODUCTS OF LAPLACIAN EIGENFUNCTIONS

JIANFENG LU, CHRISTOPHER D. SOGGE, AND STEFAN STEINERBERGER

ABSTRACT. We consider Laplacian eigenfunctions on a d -dimensional bounded domain M (or a d -dimensional compact manifold M) with Dirichlet conditions. These operators give rise to a sequence of eigenfunctions $(e_\ell)_{\ell \in \mathbb{N}}$. We study the subspace of all pointwise products

$$A_n = \text{span} \{e_i(x)e_j(x) : 1 \leq i, j \leq n\} \subseteq L^2(M).$$

Clearly, that vector space has dimension $\dim(A_n) = n(n+1)/2$. We prove that products $e_i e_j$ of eigenfunctions are simple in a certain sense: for any $\varepsilon > 0$, there exists a low-dimensional vector space B_n that almost contains all products. More precisely, denoting the orthogonal projection $\Pi_{B_n} : L^2(M) \rightarrow B_n$, we have

$$\forall 1 \leq i, j \leq n \quad \|e_i e_j - \Pi_{B_n}(e_i e_j)\|_{L^2} \leq \varepsilon$$

and the size of the space $\dim(B_n)$ is relatively small: for every $\delta > 0$,

$$\dim(B_n) \lesssim_{M, \delta} \varepsilon^{-\delta} n^{1+\delta}.$$

We obtain the same sort of bounds for products of arbitrary length, as well for approximation in H^{-1} norm. Pointwise products of eigenfunctions are low-rank. This has implications, among other things, for the validity of fast algorithms in electronic structure computations.

1. INTRODUCTION

Let (M, g) be a compact Riemannian manifold of dimension $d \geq 2$. Let $\{e_j\}$ be an orthonormal basis of eigenfunctions with frequencies λ_j arranged so that $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$. Thus,

$$-\Delta_g e_j = \lambda_j^2 e_j, \quad \langle e_j, e_k \rangle = \int_M e_j \overline{e_k} dV_g = \delta_j^k.$$

Here Δ_g denotes the Laplace-Beltrami operator and dV_g is the volume element associated with the metric g on M . For simplicity, we are only considering eigenfunctions of Laplace-Beltrami operators but, using the same proof, all of our results extend to eigenfunctions of second order elliptic operators which are self-adjoint with respect to a smooth density on M . As we shall see in the final section we can also handle Dirichlet eigenfunctions in domains.

We ask a very simple question: what can be said about the function $e_i(x)e_j(x)$? Clearly, by L^2 -orthogonality, the function $e_i(x)e_j(x)$ has mean value 0 if $i \neq j$ but what else can be said about its spectral resolution, for example, the size of

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$\langle e_i e_j, e_k \rangle$? There are very few results overall; some results have been obtained in the presence of additional structure assumptions on Ω connected to number theory (see Bernstein & Reznikoff [2], Krötz & Stanton [12] and Sarnak [15]). Already the simpler question of understanding L^2 -size of the product is highly nontrivial: a seminal result of Burq-Gérard-Tzvetkov [8] states

$$\|e_\mu e_\lambda\|_{L^2} \lesssim \min(\lambda^{1/4}, \mu^{1/4}) \|e_\lambda\|_{L^2} \|e_\mu\|_{L^2}$$

on compact two-dimensional manifolds without boundary (this has been extended to higher dimensions [4, 7]). A recent result of the third author [23] (see also [9]) shows that one would generically (i.e., on typical manifolds in the presence of quantum chaos) expect $e_i(x)e_j(x)$ to be mainly supported at eigenfunctions having their eigenvalue close to $\max\{\lambda_i, \lambda_j\}$ and that deviation from this phenomenon, as in the case of Fourier series on $\mathbb{T}$ for example, requires eigenfunctions to be strongly correlated at the wavelength in a precise sense.

In this paper, we ask the question on the numerical rank of the space spanned by the pointwise products of eigenfunctions

$$A_n = \text{span} \{e_i(x)e_j(x) : 1 \leq i, j \leq n\}.$$

This is a natural quantity for measuring the complexity of the products but also motivated by the density fitting approximation to the electron repulsion integral in the quantum chemistry literature. Given a set of eigenfunctions, the four-center two-electron repulsion integral

$$(ij|kl) = \int_{\Omega \times \Omega} \frac{e_i(x)e_j(x)e_k(y)e_l(y)}{|x-y|} dx dy$$

is a central quantity in electronic structure theories. If we are working with the first n eigenfunctions $(e_i)_{1 \leq i \leq n}$, then one has to evaluate $\mathcal{O}(n^4)$ integrals.

It has been empirically observed in the literature (see e.g., [14] by the first author and Lexing Ying) that the space A_n can in practice be very well approximated by another vector space B_n with $\dim(B_n) \sim c \cdot n$, often referred as density fitting in quantum chemistry literature. This then drastically reduces the number of integrals in need of evaluation to $\mathcal{O}(n^2)$ and can be used for fast algorithms for electronic structure calculations as in e.g., [13]. Our result is inspired by the empirical success of density fitting and gives a mathematical justification.

2. MAIN RESULTS

Motivated by the above, for $1 \leq i \leq j \leq n$, we are interested in estimating products of eigenfunctions $e_i e_j$ by finite linear combinations of eigenfunctions. With this in mind, let for $\nu \in \mathbb{N}$

$$(1) \quad E_\nu f = \sum_{k=0}^{\nu} \langle f, e_k \rangle e_k$$

denote the projection of $f \in L^2(M)$ onto the space, B_ν , spanned by $\{e_k\}_{k=0}^{\nu}$, and

$$(2) \quad \mathcal{R}_\nu f = f - E_\nu f$$

denote the “remainder term” for this projection. Thus,

$$(3) \quad \|f - E_\nu f\|_{L^2(M)}^2 = \|\mathcal{R}_\nu f\|_2^2 = \sum_{k > \nu} |\langle f, e_k \rangle|^2.$$

Our first result says that, if, as above, $i, j \leq n$ then the “eigenproduct” $e_i e_j$ can be well approximated by elements of B_ν if ν is not much larger than n .

Theorem 1. *Fix (M, g) as above. Then there is a $\sigma = \sigma_d$ so that given $\kappa \in \mathbb{N}$ there is a uniform constant C_κ such that if $i, j \leq n < \nu$ and $n \in \mathbb{N}$ we have*

$$(4) \quad \|\mathcal{R}_\nu(e_i e_j)\|_{L^2(M)} \leq C_\kappa n^\sigma (n/\nu)^\kappa.$$

Furthermore, there is a fixed σ_∞ depending only on d so that

$$(5) \quad \|\mathcal{R}_\nu(e_i e_j)\|_{L^\infty(M)} \leq C_\kappa n^{\sigma_\infty} (n/\nu)^\kappa.$$

Thus, given $\delta > 0$ there is a constant $C_{M,\delta}$ so that if $\varepsilon \in (0, 1)$ we have for $n \gg 1$

$$(6) \quad \|\mathcal{R}_\nu(e_i e_j)\|_{L^\infty(M)} < \varepsilon, \quad \text{if } \nu = \lceil C_{M,\delta} n^{1+\delta} \varepsilon^{-\delta} \rceil.$$

Note that the results in Theorem 1 trivially hold on the torus $\mathbb{T}^d \simeq (-\pi, \pi]^d$, since, in this case, $e_i e_j$ must be a trigonometric polynomial of degree $\lambda_i + \lambda_j \leq 2\lambda_n$ and $\mathcal{R}_\nu$ annihilates trigonometric polynomials of degree λ_ν . Thus, $\mathcal{R}_\nu(e_i e_j) = 0$ in this case if ν is larger than a fixed multiple of n . The result is also trivial on the round sphere for similar reasons since the product of a spherical harmonic of degree i with one of degree j is a linear combination of ones of degree $\leq i + j$ whose expansion involves the Clebsch-Gordan coefficients.

On a general manifold there are no simple representations for eigenproducts and, in particular, it seems rare that the product of two eigenfunctions of frequency $\leq \lambda_n$ or less can be expressed by linear combinations of eigenfunctions of frequency λ_ν or less with ν being a fixed multiple of n . On the other hand, our result says that, in an approximate sense, one has a weaker version if one is willing to replace $\nu \approx n$ by $\nu \approx n^{1+\delta}$ for any $\delta > 0$. As we shall see, the proof of Theorem 1 will also allow us to show that the same results hold for eigenproducts of any fixed length.

Theorem 2. *Fix $\ell = 2, 3, \dots$. Then there is a $\sigma_{d,\ell} < \infty$ so that if $j_1, \dots, j_\ell \leq n < \nu$, with $n, \nu \in \mathbb{N}$, we have*

$$(7) \quad \|\mathcal{R}_\nu(e_{j_1} \cdots e_{j_\ell})\|_{L^\infty(M)} \leq C_{M,\kappa,\ell} n^{\sigma_{d,\ell}} (n/\nu)^\kappa, \quad \kappa = 1, 2, \dots,$$

with the uniform constant $C_{M,\kappa,\ell}$ depending only on (M, g) , κ and ℓ . Thus, given $\delta > 0$ we have for $\varepsilon \in (0, 1)$

$$(8) \quad \|\mathcal{R}_\nu(e_{j_1} \cdots e_{j_\ell})\|_{L^\infty(M)} < \varepsilon, \quad \text{if } \nu = \lceil C_{M,\ell,\delta} n^{1+\delta} \varepsilon^{-\delta} \rceil.$$

To prove these results we shall use a classical Sobolev embedding; for products with more terms, we use a variation that comes from a combination of Leibniz's rule together with estimates established by the second author [18].

Lemma 3. *For $\sigma \in \mathbb{R}$, let $\|f\|_{H^\sigma(M)} = \|(I - \Delta_g)^{\sigma/2} f\|_{L^2(M)}$ denote the norm for the Sobolev space of order σ on M . Then*

$$(9) \quad \|f\|_{L^\infty(M)} \lesssim \|f\|_{H^\sigma(M)}, \quad \text{if } \sigma > d/2.$$

Also, if $\ell, n, \mu \in \mathbb{N}$ and if $1 \leq j_1, \dots, j_\ell \leq n$,

$$(10) \quad \left\| \prod_{k=1}^{\ell} e_{j_k} \right\|_{H^\mu(M)} \leq C_{\ell,\mu,M} \lambda_n^{\mu+\ell\sigma(2\ell,d)},$$

if, for $p > 2$ we set

$$(11) \quad \sigma(p, d) = \max\left\{ \frac{d-1}{2} \left(\frac{1}{2} - \frac{1}{p} \right), d \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{1}{2} \right\}.$$

These bounds arise naturally from estimates established by the second author [18] saying that if $p > 2$ then for $j \geq 1$ we have

$$(12) \quad \|e_j\|_{L^p(M)} \lesssim \lambda_j^{\sigma(p,d)},$$

with $\sigma(p, d)$ being as in (11).

3. PROOFS

3.1. Proof of Theorem 1 and Theorem 2.

Proof. To prove the L^2 -estimate (4) we note that if $\mu \in \mathbb{N}$, we have

$$\begin{aligned} \|\mathcal{R}_\nu h\|_{L^2}^2 &= \sum_{k>\nu} |\langle h, e_k \rangle|^2 \leq \lambda_\nu^{-2\mu} \sum_{k>\nu} \lambda_k^{2\mu} |\langle h, e_k \rangle|^2 \\ &\leq \lambda_\nu^{-2\mu} \|h\|_{H^\mu}^2. \end{aligned}$$

If we take $h = e_i e_j$ and use this along with (10) we conclude that

$$\|\mathcal{R}_\nu(e_i e_j)\|_{L^2} \lesssim \lambda_\nu^{-\mu} \lambda_n^{2\sigma(4,d)+\mu}.$$

Since, by the Weyl formula, $n \approx \lambda_n^d$ and $\nu \approx \lambda_\nu^d$, this inequality implies that

$$\|\mathcal{R}_\nu(e_i e_j)\|_{L^2(M)} \leq C (n/\nu)^{\mu/d} n^{\frac{2}{d}\sigma(4,d)}.$$

As $\mu \in \mathbb{N}$ is arbitrary, this of course yields (4) with σ there being $\frac{2}{d}\sigma(4, d)$. To prove the sup-norm bounds assume that $\mu \in \mathbb{N}$ is larger than $(d+1)/2$. Then, by (9) and the above argument we have

$$\begin{aligned} \|\mathcal{R}_\nu(e_i e_j)\|_\infty^2 &\lesssim \|\mathcal{R}_\nu(e_i e_j)\|_{H^{(d+1)/2}}^2 \\ &\leq \lambda_\nu^{d+1-2\mu} \|e_i e_j\|_{H^\mu}^2 \\ &\lesssim \lambda_\nu^{d+1-2\mu} \lambda_n^{4\sigma(4,d)+2\mu}. \end{aligned}$$

From this, we obtain

$$\begin{aligned} \|\mathcal{R}_\nu(e_i e_j)\|_\infty &\lesssim \lambda_n^{2\sigma(4,d)+(d+1)/2} (\lambda_n/\lambda_\nu)^{\mu-(d+1)/2} \\ &\approx n^{\frac{2}{d}\sigma(4,d)+\frac{d+1}{2d}} (n/\nu)^{[\mu-(d+1)/2]/d}, \end{aligned}$$

which yields (5) with

$$\sigma_\infty = \frac{2}{d}\sigma(4, d) + \frac{d+1}{2d}.$$

Also, (8) is a trivial consequence of (7). This argument clearly also gives the approximation bounds (7) for eigenproducts of length ℓ . Indeed by (9), if $\mu \in \mathbb{N}$ is larger than $(d+1)/2$,

$$\|\mathcal{R}_\nu(e_{j_1} \cdots e_{j_\ell})\|_\infty \leq \lambda_\nu^{(d+1)/2-\mu} \left\| \prod_{k=1}^\ell e_{j_k} \right\|_{H^\mu} \lesssim \lambda_\nu^{(d+1)/2-\mu} \lambda_n^{\mu+\ell\sigma(2\ell,d)},$$

which yields (7) with

$$\sigma_{d,\ell} = \frac{\ell}{d}\sigma(2\ell, d) + \frac{d+1}{2d}.$$

□

3.2. Proof of Lemma 3.

Proof. To prove (10) we first recall some basic facts about Sobolev spaces on manifolds. See [19] for more details. First, if $1 = \sum_{j=1}^N \varphi_j$ is a fixed smooth partition of unity with

$$\text{supp } \varphi_j \subseteq \Omega_j,$$

where $\Omega_j \subset M$ is a coordinate patch, we have for fixed $\mu \in \mathbb{N}$

$$\|f\|_{H^\mu(M)} \approx \sum_{j=1}^N \sum_{|\alpha| \leq \mu} \|\partial^\alpha(\varphi_j f)\|_{L^2(\mathbb{R}^n)}.$$

Here, the L^2 -norms are taken with respect to our local coordinates. $\|\prod_{k=1}^\ell e_{j_k}\|_{H^\mu(M)}$ is dominated by a finite sum of terms of the form

$$\|\partial^\alpha(\varphi \cdot \prod_{k=1}^\ell e_{j_k})\|_{L^2},$$

where $\varphi = \varphi_j$ for some $j = 1, \dots, N$ and $|\alpha| \leq \mu$. By Leibniz's rule, we can thus dominate the left side of (10) by a finite sum of terms of the form

$$\|\prod_{k=1}^\ell L_k e_{j_k}\|_{L^2(M)},$$

where $L_k : C^\infty(M) \rightarrow C^\infty(M)$ are differential operators with smooth coefficients of order m_k with

$$(13) \quad m_1 + \dots + m_\ell \leq \mu.$$

As a result, by Hölder's inequality, $\|\prod_{k=1}^\ell e_{j_k}\|_{H^\mu(M)}$ is majorized by a finite sum of terms of the form

$$(14) \quad \prod_{k=1}^\ell \|L_k e_{j_k}\|_{L^{2\ell}(M)},$$

where the L_k are as above. Since L_k is a differential operator of order m_k , for any $1 < p < \infty$, standard L^p elliptic regularity estimates give

$$\|L_k h\|_{L^p(M)} \lesssim \|(I - \Delta_g)^{m_k/2} h\|_{L^p(M)}.$$

Thus, since $(I - \Delta_g)^{m_k/2} e_{j_k} = (1 + \lambda_{j_k}^2)^{m_k/2} e_{j_k}$ and $0 < \lambda_1 \leq \lambda_{j_k} \leq \lambda_n$

$$\|L_k e_{j_k}\|_{L^{2\ell}(M)} \lesssim \lambda_n^{m_k} \|e_{j_k}\|_{L^{2\ell}(M)} \lesssim \lambda_n^{m_k + \sigma(2\ell, d)},$$

using (12) in the last inequality. Since (14) has ℓ factors and (13) is valid we obtain (10) from this, which finishes the proof of Lemma 3. $\square$

3.3. Some Remarks. We just showed that, if $\sigma(p, d)$ is as in (11) then if $n, \nu \in \mathbb{N}$ and $j_1, \dots, j_\ell \leq n$ and $n < \nu$, then we have

$$(15) \quad \|\mathcal{R}_\nu(\prod_{k=1}^\ell e_{j_k})\|_{L^2(M)} \leq C_\kappa \lambda_n^{\ell\sigma(2\ell, d)} (\lambda_n/\lambda_\nu)^\kappa \approx n^{\ell\sigma(2\ell, d)/d} (n/\nu)^{\kappa/d},$$

for each $\kappa = 1, 2, 3, \dots$. Since $\|\mathcal{R}_\nu\|_{L^2 \rightarrow L^2} = 1$, by Hölder's inequality and (12) we clearly have

$$(16) \quad \left\| \mathcal{R}_\nu \left(\prod_{k=1}^{\ell} e_{j_k} \right) \right\|_{L^2(M)} \leq C \lambda_n^{\ell \sigma(2l, d)}, \quad \text{if } \nu \leq n.$$

Thus, if $\kappa \in \mathbb{N}$ is fixed, we have the uniform bounds

$$(17) \quad \left\| \mathcal{R}_\nu \left(\prod_{k=1}^{\ell} e_{j_k} \right) \right\|_{L^2(M)} \leq C_\kappa \lambda_n^{\ell \sigma(2l, d)} (1 + \lambda_\nu / \lambda_n)^{-\kappa}$$

$$(18) \quad \approx n^{\ell \sigma(2l, d)/d} (1 + \nu/n)^{-\kappa/d}.$$

Inequality (17) is saturated on the sphere S^d for ν smaller than a fixed multiple of n if $j_1 = \dots = j_\ell = n$. Specifically, if $\ell \geq 3$ or if $\ell = 2$ and $d \geq 3$ one can check that zonal functions saturate the bound. For the remaining case where $\ell = d = 2$, the highest weight spherical harmonics saturate the bounds if $\lambda_\nu + 1 \leq 2\lambda_n$. On the other hand, as we pointed out before, the left side of (17) is zero in this case if $\lambda_\nu + 1 > 2\lambda_n$.

The proof of Theorems 1–2 shows that if $1 \leq j_k \leq n < \nu$ and $\kappa \in \mathbb{N}$, then

$$\left\| \mathcal{R}_\nu \left(\prod_{k=1}^{\ell} e_{j_k} \right) \right\|_{L^2(M)} \leq C_\kappa (\lambda_\nu / \lambda_n)^{-\kappa} \prod_{k=1}^{\ell} \|e_{j_k}\|_{L^{2\ell}(M)}.$$

We then used the universal L^p -bounds (12) of one of us to control the right side and prove our results. Substantially improved eigenfunction estimates would of course lead to better bounds for the L^2 -approximation of products of eigenfunctions. For instance, the “random wave model” of Berry [3] predicts that eigenfunctions on Riemannian manifolds with chaotic geodesic flow should have $\|e_\lambda\|_{L^p} = O(1)$ for $p < \infty$. (See e.g., [24] for more details.) If this optimistic conjecture were valid we would have, for any fixed $\ell = 2, 3, \dots$,

$$\left\| \mathcal{R}_\nu \left(\prod_{k=1}^{\ell} e_{j_k} \right) \right\|_2 < \varepsilon \quad \text{if } \nu > C_\delta n \varepsilon^{-\delta} \text{ and } 1 \leq j_k \leq n,$$

with $\delta > 0$ being arbitrary. We should note, though, that there has been much work on obtaining improved L^p -estimates for eigenfunctions over the last forty years and only logarithmic improvements over the universal bounds (12) have been obtained assuming, say, that one has negative curvature. See, e.g., [1, 5, 6, 11, 20, 21, 22].

We also would like to point out that the argument that we have employed to obtain the approximation bounds in Theorem 1 yield higher order Sobolev estimates as well. Indeed, if $\ell = 2, 3, \dots$ and $\mu \in \mathbb{N}$ are fixed, the proof of (6) shows that if, $1 \leq j_k \leq n$, $k = 1, \dots, \ell$,

$$(19) \quad \left\| \mathcal{R}_\nu \left(\prod_{k=1}^{\ell} e_{j_k} \right) \right\|_{H^\mu} < \varepsilon \quad \text{for } \varepsilon \in (0, 1) \quad \text{if } \nu = O(n^{1+\delta} \varepsilon^{-\delta}),$$

where $\delta > 0$ can be chosen to be as small as we like. This in turn allows us to control the C^m norms of $\mathcal{R}_\nu(e_{j_1} \cdots e_{j_\ell})$ for any $m \in \mathbb{N}$ for such ν .

4. APPROXIMATION IN H^{-1}

Motivated by the discussion of the four-center two-electron repulsion integral, which is a somewhat better behaved quantity due to the smoothing effects of the potential, it is also natural to look for an approximation result in a function space that captures the smoothing effect of the potential. Since a multiple of $|x - y|^{-1}$ is the fundamental solution of the Laplacian in $\mathbb{R}^3$, the appropriate physically relevant problems involve the Sobolev space H^{-1} equipped with the norm defined by

$$(20) \quad \|f\|_{H^{-1}}^2 = \|(I - \Delta_g)^{-1/2} f\|_{L^2}^2 = \sum (1 + \lambda_k^2)^{-1} |\langle f, e_k \rangle|^2.$$

Our H^{-1} approximation result then is the following.

Theorem 4. *Let (M, g) be a compact Riemannian manifold of dimension $2 \leq d \leq 4$. Then if $1 \leq i, j \leq n < \nu$ and $\varepsilon \in (0, 1)$ we have*

$$(21) \quad \|\mathcal{R}_\nu(e_i e_j)\|_{H^{-1}} < \varepsilon, \quad \text{if } \nu = O(n^{\mu(d)} \varepsilon^{-d}),$$

where

$$(22) \quad \mu(d) = \begin{cases} 1/4, & \text{if } d = 2 \\ 1/2, & \text{if } d = 3 \\ 1, & \text{if } d = 4. \end{cases}$$

Thus, our results only involve sublinear growth of ν in terms of n for $d \leq 3$ and linear growth for $d = 4$, which matches up nicely with the trivial cases of $\mathbb{T}^d$ and S^d for these dimensions. We only stated things in dimensions $d \leq 4$ since in higher dimensions we cannot get improved H^{-1} bounds compared to the L^2 bounds in Theorem 1.

Proof. Under the above hypothesis, we claim that

$$(23) \quad \|\mathcal{R}_\nu(e_i e_j)\|_{H^{-1}} \leq C \lambda_\nu^{-1} \lambda_n^{2\sigma(4,d)} \approx \nu^{-1/d} n^{2\sigma(4,d)/d},$$

where, as in (11),

$$(24) \quad \sigma(4, d) = \begin{cases} \frac{1}{8}, & \text{if } d = 2 \\ \frac{d-2}{4}, & \text{if } d \geq 3. \end{cases}$$

Since

$$\nu^{-1/d} n^{2\sigma(4,d)/d} < \varepsilon \iff \nu > \varepsilon^{-d} n^{\mu(d)},$$

where $\mu(d)$, as in (22), equals $2\sigma(4, d)$, we conclude that in order to obtain (21), we just need to prove (23). To prove this we use Hölder's inequality and (12) to get

$$\begin{aligned} \|\mathcal{R}_\nu(e_i e_j)\|_{H^{-1}}^2 &\leq \lambda_\nu^{-2} \sum_{k > \nu} |\langle e_i e_j, e_k \rangle|^2 \\ &\leq \lambda_\nu^{-2} \|e_i e_j\|_2^2 \leq \lambda_\nu^{-2} \|e_i\|_4^2 \|e_j\|_4^2 \lesssim \lambda_\nu^{-2} \lambda_n^{4\sigma(4,d)}, \end{aligned}$$

as desired. $\square$

5. RESULTS FOR MANIFOLDS WITH BOUNDARY AND DOMAINS

Let us conclude by extending our results to eigenfunctions of the Dirichlet Laplacian on M , where M is either a d -dimensional relatively compact Riemannian manifold or a bounded domain in $\mathbb{R}^d$ and where the boundary, ∂M , of M is smooth. Thus, we shall consider an L^2 -normalized basis of eigenfunctions $\{e_j\}_{j=1}^\infty$ satisfying

$$-\Delta_g e_j(x) = \lambda_j^2 e_j(x), \quad x \in M, \quad \text{and } e_j|_{\partial M} = 0.$$

As before, we shall assume that the frequencies of the eigenfunctions are arranged in increasing order, i.e., $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$.

If we then define $\mathcal{R}_\nu$ as in (2) we have the following.

Theorem 5. *Let $\ell = 2, 3, \dots$. Assume that $j_k \leq n$, $k = 1, \dots, \ell$. Then given $\delta > 0$ we have for $\varepsilon \in (0, 1)$*

$$(25) \quad \left\| \mathcal{R}_\nu \left(\prod_{k=1}^\ell e_{j_k} \right) \right\|_{L^\infty} < \varepsilon, \quad \text{if } \nu = O(n^{1+\delta} \varepsilon^{-\delta}).$$

Also, if $2 \leq d \leq 4$ and $\|f\|_{H^{-1}}$ is as in (20) and $i, j \leq n \in \mathbb{N}$

$$(26) \quad \left\| \mathcal{R}_\nu(e_i e_j) \right\|_{H^{-1}} < \varepsilon, \quad \text{if } \nu = O(n^{\mu(d)} \varepsilon^{-d}),$$

where $\mu(2) = 1/3$, $\mu(3) = 2/3$ and $\mu(4) = 1$.

To prove these results one uses the eigenfunction estimates in [17] which say that bounds of the form (12) are valid, but where the exponent $\sigma(p, d)$ may be larger than the one in (11) for a certain range of p depending on d . Our earlier argument only used the fact that this exponent was finite in order to get bounds of the form (25) and the same applies here. In fact by using such a bound along with elliptic regularity estimates (see [10, §9.6]), one obtains (25).

By the same argument we can also obtain bounds of the form (19) in this setting.

To prove (26) one, as before, only needs to use L^4 eigenfunction estimates. Specifically, by the arguments from the preceding section, we obtain (26) just by using the fact that, for Dirichlet eigenfunctions, we have (12) for $p = 4$ with

$$\sigma(4, d) = \begin{cases} 1/6, & \text{if } d = 2 \\ 1/3, & \text{if } d = 3 \\ 1/2, & \text{if } d = 4. \end{cases}$$

These bounds were obtained in [17].

We remark that if M has geodesically concave boundary, the results in [16] say that we have the more favorable bounds where, as in the boundaryless case, $\sigma(4, d)$ is given by (24) and so, in this case, we can recover the bounds in (21)–(22).

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(J. L.) DEPARTMENT OF MATHEMATICS, DEPARTMENT OF PHYSICS, AND DEPARTMENT OF CHEMISTRY, DUKE UNIVERSITY, BOX 90320, DURHAM NC 27708, USA

E-mail address: `jianfeng@math.duke.edu`

(C.S.) DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 21218, USA

E-mail address: `sogge@jhu.edu`

(S.S.) DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CT 06511, USA

E-mail address: `stefan.steinerberger@yale.edu`